# Game semantics for untyped $\lambda\beta\eta$-calculus

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**Abstract.** We study extensional models of the untyped lambda calculus in the setting of game semantics. In particular, we show that, somewhat unexpectedly and contrary to what happens in ordinary categories of domains, all reflexive objects in the category of games $\mathcal{G}$, introduced by Abramsky, Jagadeesan and Malacaria, induce the same $\lambda$-theory. This is $\mathcal{H}^*$, the maximal theory induced already by the classical CPO model $D_\infty$, introduced by Scott in 1969. This results indicates that the current notion of game carries a very specific bias towards *head reduction*.

## Introduction

$\lambda$-theories are congruences over $\lambda$-terms, which extend pure $\beta$-conversion. Their interest lies in the fact that they correspond to the possible *operational* (*observational*) semantics of $\lambda$-calculus. Although researchers have mainly focused on only three such operational semantics, namely those given by head reduction, head lazy reduction or call-by-value reduction, the class of $\lambda$-theories is, in effect, unfathomly rich, see *e.g.* [6,14,13,9] for interesting examples of this complexity. Brute force, purely syntactical techniques are usually extremely difficult to use in the study of $\lambda$-theories. Therefore, since the seminal work of Dana Scott on $D_\infty$ in 1969 [18], semantical tools have been extensively investigated.

A large number of mathematical models for $\lambda$-calculus, arising from syntax-free constructions, have been introduced, since then, in various categories of domains (see e.g. [19,7,10,6,12,14,5,9]). And a rich host of *different* $\lambda$-theories now have a "fully abstract" syntax-free model, *i.e.* a model which induces precisely those identities which hold in the given theory. However, the denotational semantics supported by these models do not match all the possible operational semantics of $\lambda$-calculus.

For example, in most existing categories of domains, $\lambda$-models have too many functions, and hence many interesting $\lambda$-theories, such as those arising from observing termination under some natural sequential reduction strategy (see e.g. [13]), do not have fully abstract denotational models [14,5,8]. An example of such a strategy is the one which tries non-deterministically to reduce a term to a closed term. In the case of CPOs, the sequentiality embedded in these strategies clashes with the necessary existence of Scott-continuous "parallel" functions. On the other hand, in the case of coherent spaces, and stable functions, the presence of so called "parasitic" functions, prevents other kinds of identities deriving from monotonicity.

In this paper we explore the methodology for giving denotational semantics based on games, recently introduced by Abramsky, Jagadeesan, Malacaria, and Hyland, Ong (see [3,16]). This methodology has been extremely successful in modeling sequential languages [3,17]. It should be reasonable to expect, therefore, that one could obtain fully abstract game models, at least for those $\lambda$-theories mentioned above, which escape domain models. Of course, the very fact that game semantics faithfully captures *sequentiality*, should suggest also that even game semantics is not rich enough to provide fully abstract models for *all* $\lambda$-theories. It is possible to show, in fact, that there are $\lambda$-theories where, say, the behavior of an unsolvable term, *i.e.* a term with no head normal form, is that of a "parallel function", which checks if at least one of its arguments evaluates to a fixed term.

Somewhat surprisingly, however, it turns out that *all* reflexive objects, *i.e.* extensional $\lambda$-models, in the standard category of games of [3], determine $\lambda$-models which have the *same* theory. This is the *well known* maximal $\lambda$-theory $\mathcal{H}^*$ [6], already induced by Scott's $D_\infty$. We recall that, if $M, N$ are closed $\lambda$-terms (*i.e.* $M, N \in \Lambda^0$), and *HNF* denotes the set of $\lambda$-terms which have a *head normal form*, then $M =_{\mathcal{H}^*} N$ if and only if

$$\forall C[\;] \;.\; C[M], C[N] \in \Lambda^0 \Longrightarrow (C[M] \in \mathit{HNF} \iff C[N] \in \mathit{HNF})$$

Alternatively, this is the theory where two terms are equal if we cannot observe that head reduction terminates when one is placed in a given context, but does not terminate when the other is.

More specifically, in this paper we show that all reflexive objects in the Cartesian closed category of games $K_!(\mathcal{G})$ [3] determine $\lambda$-models which are isomorphic to models which can be constructed as special *non-initial* colimits in a category $\mathcal{G}^e$ of games and "embeddings", which mimics the traditional Scott's construction in CPOs and embedding-projection pairs. By extending the methodology of approximants originally introduced in [20,15,14] for the continuous case, to the setting of the game semantics, we study the fine structure of these models.

The paper [11] is a companion to the present one. Finitary logical descriptions of game models, in the spirit of [10,1], are introduced. The case of one of the models introduced in this paper is discussed in detail.

One can elaborate in various ways on the main result of this paper. In any case, we think that it provides a very clear indication that existing game semantics is more rigid than CPO semantics, which can model a very rich collection of $\lambda$-theories. Since the current notion of game appears to carry a very strong bias towards *head reduction*, a new notion of game seems to be necessary to model $\lambda$-theories different from $\mathcal{H}^*$. This appears to be rather problematic, since we feel that "head reduction" is intrinsic to games for which we can observe only *interactions with the environment*.

The present paper is organized as follows. In section 1, we introduce the categories of games that we shall utilize, namely $\mathcal{G}$ and $K_!(\mathcal{G})$. In Section 2 we discuss initial and non-initial solutions of recursive game equations. In Section 3 we introduce the special class of extensional $\lambda$-models $\mathcal{D}^*$, and we prove that all reflexive objects in $K_!(\mathcal{G})$ determine models belonging to $\mathcal{D}^*$. In Section 4 we study the fine structure of the models in $\mathcal{D}^*$ and prove that such models induce the theory $\mathcal{H}^*$. In Section 5 we give some concrete examples of extensional game $\lambda$-models, including the model arising from applying Scott's trick [19] to the game setting. Final remarks and directions for future work appear in section 6.

We assume the reader familiar with the basic notions and definitions of $\lambda$-calculus, see e.g. [6]. For the benefit of a reader coming from the $\lambda$-calculus community, this paper is self-contained as far as the theory of games, however the reader can refer to [2–4,16] for more details on this topic.

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## 1 Categories of games

In this section, we introduce two categories of games. Both are introduced by Abramsky, Jadgadeesan and Malacaria in 1993, [3]. Notice however that for our purposes the machinery of "questions and answers" *i.e.* the bracketing condition, seems unnecessary. One can safely, and more simply, focus only on the full and faithful sub-category of this category consisting of all those games all whose moves are labeled as questions.

We begin by giving the basic definitions.

**Definition 1 (Games).** A *game* has two participants: the *Player* and the *Opponent.* A game $A$ is a quadruple $(M_A, \lambda_A, P_A, \approx_A)$ where

- $M_A$ is the set of *moves* of the game.
- $\lambda_A : M_A \to \{O,P\} \times \{Q,A\}$ is the *labeling* function: it tells us if a move is taken by the Opponent or by the Player, and if it is a Question or an Answer. We can decompose $\lambda_A$ into $\lambda_A^{OP} : M_A \to \{O,P\}$ and $\lambda_A^{QA} : M_A \to \{Q,A\}$ and put $\lambda_A = \langle \lambda_A^{OP}, \lambda_A^{QA} \rangle$. We denote by $^-$ the function which exchanges Player and Opponent, i.e. $\overline{O} = P$ and $\overline{P} = O$. We also denote with $\overline{\lambda_A^{OP}}$ the function defined by $\overline{\lambda_A^{OP}}(a) = \overline{\lambda_A^{OP}(a)}$. Finally, we denote with $\overline{\lambda_A}$ the function $\langle \overline{\lambda_A^{OP}}, \lambda_A^{QA} \rangle$.
- $P_A$ is a non-empty and prefix-closed subset of the set $M_A^{\circledast}$ (written as $P_A \subseteq^{nepref} M_A^{\circledast}$), where $M_A^{\circledast}$ is the set of all sequences of moves which satisfy the following conditions:
  - $s = at \Rightarrow \lambda_A(a) = OQ$
  - $(\forall i : 1 \leq i \leq |s|)[\lambda_A^{OP}(s_{i+1}) = \overline{\lambda_A^{OP}(s_i)}]$
  - $(\forall\ t \sqsubseteq s)[|t \upharpoonright M_A^A| \leq |t \upharpoonright M_A^Q|]$

  where $M_A^A$ and $M_A^Q$ denote the subsets of game moves labeled respectively as Answers and as Questions, $s \upharpoonright M$ denotes the set of moves of $M$ which appear in $s$ and $\sqsubseteq$ is the substring relation. $P_A$ is called the set of *positions* of the game $A$.

– $\approx_A$ is an equivalence relation on $P_A$ which satisfies the following properties:
  - $s \approx_A s' \Rightarrow |s| = |s'|$
  - $sa \approx_A s'a' \Rightarrow s \approx_A s'$
  - $s \approx_A s' \ \wedge \ sa \in P_A \Rightarrow (\exists a')[sa \approx_A s'a']$

In the above $s$, $s'$, $t$ and $t'$ range over sequences of moves, while $a$, $a'$, $b$ and $b'$ range over moves. The empty sequence is written $\epsilon$.

**Definition 2 (Strategies).** A strategy for the Player in a game $A$ is a non-empty set $\sigma \subseteq P_A^{even}$ of positions of even length such that $\overline{\sigma} = \sigma \cup dom(\sigma)$ is prefix-closed, where $dom(\sigma) = \{t \in P_A^{odd} \mid (\exists a)[ta \in \sigma]\}$, and $P_A^{odd}$ and $P_A^{even}$ denote the sets of positions of odd and even length respectively.

A strategy can be seen as a set of rules which tells the Player which move to take after the last move by the Opponent.

In this paper we shall consider only *history-free* strategies, *i.e.* strategies which depend only on the *last* move by the Opponent.

**Definition 3 (History-free strategies).** A strategy $\sigma$ for a game $A$ is history-free if it satisfies the following properties:

1. $sab, tac \in \sigma \Rightarrow b = c$
2. $sab, t \in \sigma, ta \in P_A \Rightarrow tab \in \sigma$

The equivalence relation on positions $\approx_A$ can be extended to strategies in the following way.

**Definition 4.** Let $\sigma, \tau$ be strategies, $\sigma \approx \tau$ if and only if

1. $sab \in \sigma, s'a'b' \in \tau, sa \approx_A s'a' \Rightarrow sab \approx_A s'a'b'$
2. $s \in \sigma, s' \in \tau, sa \approx_A s'a' \Rightarrow (\exists b)[sab \in \sigma]$ iff $(\exists b')[s'a'b' \in \tau]$

Such an extension is not in general an equivalence relation since it might lack reflexivity. If $\sigma$ is a strategy for a game $A$ such that $\sigma \approx \sigma$, we write $\sigma : A$.

**Definition 5 (Tensor product).** Given games $A$ and $B$ the tensor product $A \otimes B$ is the game defined as follows:

– $M_{A\otimes B} = M_A + M_B$
– $\lambda_{A\otimes B} = [\lambda_A, \lambda_B]$
– $P_{A\otimes B} \subseteq M_{A\otimes B}^{\circledast}$ is the set of positions, $s$, which satisfy the following conditions:
  1. the projections on each component (written as $s \upharpoonright A$ or $s \upharpoonright B$) are positions for the games $A$ and $B$ respectively;
  2. every answer in $s$ must be in the same component game as the corresponding question.
– $s \approx_{A\otimes B} s' \iff s \upharpoonright A \approx_A s \upharpoonright A', s \upharpoonright B \approx_B s \upharpoonright B', (\forall i)[s_i \in M_A \Leftrightarrow s'_i \in M_A]$

Here $+$ denotes disjoint union of sets, that is $A + B = \{\mathrm{in}_l(a) \mid a \in A\} \cup \{\mathrm{in}_r(b) \mid b \in B\}$, and $[-,-]$ is the usual (unique) decomposition of a function defined on disjoint unions.

It is easy to see that in such a game only the Opponent can switch component.

**Definition 6 (Unit).** The unit element for the tensor product is given by the empty game $I = (\varnothing, \varnothing, \{\epsilon\}, \{(\epsilon, \epsilon)\})$.

**Definition 7 (Linear implication).** Given games $A$ and $B$ the compound game $A \multimap B$ is defined as follows:

- $M_{A \multimap B} = M_A + M_B$
- $\lambda_{A \multimap B} = [\overline{\lambda_A}, \lambda_B]$
- $P_{A \otimes B} \subseteq M^{\circledast}_{A \otimes B}$ is the set of positions, $s$, which satisfy the following conditions:
  1. the projections on each component are positions for the games $A$ and $B$ respectively;
  2. every answer in $s$ must be in the same component game as the corresponding question.
- $s \approx_{A \multimap B} s' \iff s \upharpoonright A \approx_A s \upharpoonright A', s \upharpoonright B \approx_B s \upharpoonright B', (\forall i)[s_i \in M_A \Leftrightarrow s'_i \in M_A]$

It is easy to see that in such a game only the Player can switch component.

**Definition 8 (Exponential).** Given a game $A$ the game $!A$ is defined by:

- $M_{!A} = \omega \times M_A = \sum_{i \in \omega} M_A$
- $\lambda_{!A}(\langle i, a \rangle) = \lambda_A(a)$
- $P_{!A} \subseteq M^{\circledast}_{!A}$ is the set of positions, $s$, which satisfy the following conditions:
  1. $(\forall i \in \omega)[s \upharpoonright A_i \in P_{A_i}]$;
  2. every answer in $s$ is in the same index as the corresponding question.
- $s \approx_{!A} s' \iff \exists$ a permutation of indexes $\alpha \in S(\omega)$ such that:
  - $\pi_1^*(s) = \alpha^*(\pi_1^*(s'))$
  - $(\forall i \in \omega)[\pi_2^*(s \upharpoonright \alpha(i)) \approx \pi_2^*(s \upharpoonright i)]$

  where $\pi_1$ and $\pi_2$ are the projections of $\omega \times M_A$ and $s \upharpoonright i$ is an abbreviation of $s \upharpoonright A_i$.

One can easily see that the following definition is well posed and that the objects introduced in Definitions 5, 6 provide indeed a categorical tensor product and its unit.

**Definition 9 (The category of games $\mathcal{G}$).** The category $\mathcal{G}$ has as objects games and as morphisms, between games $A$ and $B$, the equivalence classes, for the relation $\approx_{A \multimap B}$, of history-free strategies $\sigma : A \multimap B$. We denote the equivalence class of $\sigma$ by $[\sigma]$.

The identity for each game $A$ is given by the (equivalence class) of the *copy-cat* strategy $id_A = \{s \in P_{A' \multimap A''} \mid s \upharpoonright A' = s \upharpoonright A''\}$ where the superscripts are introduced to distinguish between the two different occurrences of the game $A$.

Composition is given by the extension on equivalence classes of the following composition of strategies. Given strategies $\sigma : A \multimap B$ and $\tau : B \multimap C$, $\tau \circ \sigma : A \multimap C$ is defined by

$$\tau \circ \sigma = \{s \upharpoonright (A,C) \mid s \in (M_A + M_B + M_C)^* \ \wedge\ s \upharpoonright (A,B) \in \overline{\sigma}, s \upharpoonright (B,C) \in \overline{\tau}\}^{even}$$

Throughout this paper, without loss of generality, we shall restrict ourselves to "irredundant" games, *i.e.* to games such that every move appears in at least one position. Any redundant game is in fact categorically isomorphic to an irredundant one.

One can easily see that the constructions introduced in Definitions 5, 7 and 8 can be made to be functorial.

The category $\mathcal{G}$ is a monoidal closed category [3] but it is not Cartesian closed.

**Definition 10 (A Cartesian closed category of games).** The category $K_!(\mathcal{G})$ is the category obtained by taking the co-Kleisli category over $\mathcal{G}$ over the comonad $(!, \mathtt{der}, \delta)$ [3], where for each game $A$ the strategies $\mathtt{der}_A : !A \multimap A$ and $\delta_A : !A \multimap !!A$ are defined as follows:

- $\mathtt{der}_A = [\{s \in P_{!A \multimap A} \mid s \upharpoonright\ (!A)_0 = s \upharpoonright A\}]$
- $\delta_A = [\{s \in P_{!A \multimap\ !!A} \mid s \upharpoonright (!A)_{p(i,j)} = s \upharpoonright\ (!(!A)_i)_j\}]$ where $p : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a pairing function

The category $K_!(\mathcal{G})$ has as objects games and as morphisms between games $A$ and $B$ the equivalence class of history-free strategies for the game $!A \multimap B$. Moreover it is Cartesian closed.

**Definition 11 (Cartesian product).** Given games $A$ and $B$ the Cartesian product $A \ \wedge\ B$ is the game defined as follows:

- $M_{A \ \wedge\ B} = M_A + M_B$
- $\lambda_{A \ \wedge\ B} = [\lambda_A, \lambda_B]$
- $P_{A \ \wedge\ B} = P_A + P_B$
- $\approx_{A \ \wedge\ B} = \approx_A + \approx_B$

### 1.1 Order-enrichment

Following [3] we can enrich each homset of $\mathcal{G}$ with a partial order structure, as follows

**Definition 12.** Given strategies $\sigma$ and $\tau$ we write $\sigma \lesssim \tau$ iff

$$(\forall s, s', a, b, a')[sab \in \sigma \wedge\ s' \in \tau \wedge\ sa \approx s'a' \Longrightarrow \exists b'.(s'a'b' \in \tau\ \wedge\ sab \approx s'a'b')]$$

**Definition 13.** Given a game $A$ and strategies $\sigma : A$ and $\tau : A$ we define

$$[\sigma] \sqsubseteq_A [\tau] \Longleftrightarrow \sigma \lesssim \tau$$

Given a game $A$ let $\hat{A}$ be the set of equivalence classes of history-free strategies for $A$. $\sqsubseteq_A$ is a partial order over $\hat{A}$. The least element in this partial order is $[\{\epsilon\}]$.

We now prove that this partial order is not complete. This answers a question raised in [3] page 21.

**Definition 14 (Game $\mathfrak{N}$).** The game $\mathfrak{N}$ is defined as follows:

- $M_{\mathfrak{N}} = \{q, !\} \cup \{n, \overline{n} \mid n \in \mathbb{N}\}$
- $\lambda_{\mathfrak{N}}(q) = \lambda_{\mathfrak{N}}(\overline{n}) = OQ$
  $\lambda_{\mathfrak{N}}(!) = \lambda_{\mathfrak{N}}(n) = PQ$
- $P_{\mathfrak{N}} = \{qn(\overline{n-1})(n-1)(\overline{n-2})\ldots\overline{0}0q!q!q!\ldots \mid n \in \mathbb{N}\}^{nepref}$
- $s \approx_{\mathfrak{N}} t \Leftrightarrow |s| = |t|$

**Theorem 15.** $(\hat{\mathfrak{N}}, \sqsubseteq_{\mathfrak{N}})$ *is not a complete partial order.*

*Proof.* Consider the following strategies indexed by $n \geq 1$:

$$\sigma_n = \{qn(\overline{n-1})(n-1)(\overline{n-2})\ldots 1\}^{nepref}$$

It is easy to check that $\sigma_n \lesssim \sigma_m$ for $n \leq m$. The chain $[\sigma_0], [\sigma_1], \ldots, [\sigma_n], \ldots$ has no lub, since there is no infinite history-free strategy in $\mathfrak{N}$. $\square$

Since $\hat{\mathfrak{N}} \simeq \mathcal{G}(I, \mathfrak{N})$ we have that

**Corollary 16.** *The categories $\mathcal{G}$ and $K_!(\mathcal{G})$ are not cpo-enriched categories under the order relation on morphisms of Definition 13.*

## 2 Solution of recursive games equations

The categories of games $\mathcal{G}$ and $K_!(\mathcal{G})$ allow for the existence of *recursive* objects, *i.e.* objects that are fixed points of particular functors. In this section we analyze and elaborate the method proposed by Abramsky and McCusker in [4], for defining recursive games. In a simple-minded, well-founded setting, this method allows to define only *initial* fixed points of functors. However in order to model non-trivially $\lambda\beta\eta$-calculus, it is well known that we need to define models which arise from *non*-initial fixed points of a suitable functor. To this end we have to change the functor altogether and use some form of encoding or, equivalently, generalize the method of [4] and consider games "up to" isomorphisms, or consider *non-well-founded* sets. In this section we shall explore the first two alternatives.

### 2.1 Initial fixed points

We start by discussing briefly the method of Abramsky and McCusker [4] in a well-founded setting. This method follows the pattern used for building initial fixed points in the context of information systems. First a complete partial order $\trianglelefteq$ on games is introduced.

**Definition 17.** Let $A, B$ be games, $A$ is a *sub-game* of $B$ ($A \trianglelefteq B$) iff

- $M_A \subseteq M_B$;
- $\lambda_A = \lambda_B \upharpoonright M_A$;
- $P_A = P_B \cap M_A^{\circledast}$;
- $s \approx_A s'$ iff $s \approx_B s'$ and $s \in P_A$.

One can easily see that the sub-game relation defines a complete partial order on games. Hence a functor $F$ which is continuous with respect to $\trianglelefteq$ has a (minimal) fixed point $D = F(D)$ given by $\bigsqcup_{\trianglelefteq} F^n(I)$. Notice that we have indeed an identity between $D$ and $F(D)$.

In domain theory, non-initial fixed points for a functor $F$ are usually obtained by carrying out the above construction starting from some object $A$, different from the initial one (*i.e.* $I$ in this case), such that $A \trianglelefteq F(A)$. However one can prove that for functors $F$ obtained from constant functors by composition of the basic functors $\&$, $\otimes$, $\multimap$, $(\ )_\perp$, and $!$, and for every game $A$, whose moves are well-founded sets, if $A \trianglelefteq F(A)$ then $\exists n \in \mathbb{N}$ s.t. $A \trianglelefteq F^n(I)$. Hence, eventually, *only* initial fixed points can be obtained using this technique in well-founded Set Theory.

As remarked earlier, even if no non-trivial model of $\lambda\beta\eta$-calculus can be obtained applying this technique directly to the functor $!D \multimap D$, nevertheless using Scott's trick (see [19]) we can still define models of $\lambda\beta\eta$. What we need is a non-trivial game which satisfies the equivalence $D \simeq D\&D$. To see this consider the initial fixed point, $E$, of the functor $F(X) = X \to D$. This is clearly non trivial. One can easily see that the following chain of equivalences holds $E = E \to D = (E \to D) \to D \simeq (E \to (D \times D)) \to D \simeq ((E \to D) \times (E \to D)) \to D = ((E \to D) \times E) \to D \simeq (E \to D) \to (E \to D) = E \to E$. We shall present this model in Section 5.

### 2.2 Non-initial fixed points

In order to obtain a non-initial fixed point of a functor, without having to deal with the subtleties of non-well-founded sets, or with indirect encodings, we present a generalization of the method proposed in [4], "up to isomorphism".

The basic idea is to obtain a fixed point of a functor $F$ as a limit of a chain of approximations $D_0, D_1, D_2, \ldots$ where, not necessarily $D_n \trianglelefteq D_{n+1}$, but only a weaker relation between $D_n$ and $D_{n+1}$ holds. We simply ask that each $D_n$ is isomorphic to a sub-game $B$ of $D_{n+1}$. In order to formalize our construction we need to introduce a new category $\mathcal{G}^e$. A similar category was introduced also in [2] for other purposes.

**Definition 18.** Given games $A$ and $B$ an *embedding* $f : A \rightarrowtail B$ is a total injective function $f : M_A \to M_B$ such that:

- $\lambda_A = \lambda_B \circ f$
- $f^*(P_A) = P_B \cap (f^*(M_A))^{\circledast}$
- $s \approx_A s'$ iff $f^*(s) \approx_B f^*(s')$

In the above we have used the notation $f^*$ to denote the natural extension of $f$ both to sequences and sets of sequences.

**Definition 19.** The category of games $\mathcal{G}^e$ has as objects games and as morphisms embeddings.

**Proposition 20.** *The category $\mathcal{G}^e$ is $\omega$-cocomplete.*

Each embedding $f : A \rightarrowtail B$ in $\mathcal{G}^e$ induces two morphisms $f^+ : A \multimap B$ and $f_- : B \multimap A$ in $\mathcal{G}$ defined as follows.

**Definition 21.** Given an embedding $f : A \rightarrowtail B$, put

$$f^+ = \{t \in P_{A \multimap B} \mid t \in s_f\}$$

$$f_- = \{t' \in P_{B \multimap A} \mid t' \in s_f\}$$

where $s_f$ is the least set satisfying:

$$s_f = \{t\, a\, f(a) \mid t \in s_f,\ a \in M_A\} \cup \{t' f(a)\, a \mid t' \in s_f,\ a \in M_A\} \cup \{\epsilon\}.$$

One can easily see that $(g \circ f)^+ = g^+ \circ f^+$ and $(g \circ f)_- = f_- \circ g_-$. The category $\mathcal{G}^e$ is indeed isomorphic to a sub-category of $\mathcal{G}$ and to a sub-category of $\mathcal{G}^{op}$. Now, using the well-known machinery, we can obtain fixed points of any continuous functor $F$ in $\mathcal{G}^e$.

**Theorem 22.** *Given a game $D$ and an embedding $f : D \rightarrowtail F(D)$, let $\langle D_\infty, \mu_n \rangle_{n \in \omega}$ be the colimit of the chain $\langle (F)^n(D), (F)^n(f) \rangle_{n \in \omega}$. Then, the game $D_\infty$ is the fixed point of the functor $F$. The isomorphic embeddings $\varphi : D_\infty \rightarrowtail F(D_\infty)$ and $\psi : F(D_\infty) \rightarrowtail D_\infty$ are given by $\varphi = \bigsqcup_{n \in \omega} F(\mu_n) \circ \mu_n^{-1}$ and $\psi = \bigsqcup_{n \in \omega} \mu_n \circ F(\mu_n)^{-1}$ respectively, where the lubs are taken in the category of* partial *embeddings.*

The following Proposition will be useful in the sequel.

**Proposition 23.** *Given a game $D$ and an embedding $f : D \rightarrowtail F(D)$ let $\langle D_\infty, \mu_n \rangle_{n \in \omega}$ be the fixed point of the functor $F$. For each $n \in \mathbb{N}$ let $p_n : D_\infty \multimap D_\infty = (\mu_n)^+ \circ (\mu_n)_-$. Then we have:*

1. *for each game $A$ and for each strategy $\sigma : A \multimap D_\infty$ the following equality holds: $p_n \circ \sigma \;=\; \{s \mid s \in (\sigma \cap (in_l(M_A) \cup in_r(\mu_n(M_{F^n(D)})))^{\circledast})\}$*
2. *for each $n \in \mathbb{N}$*
   (a) $p_n \sqsubseteq p_{n+1}$
   (b) $\bigsqcup_{n \in \omega} p_n = id$
   (c) $p_n \circ p_m = p_{min\{m,n\}}$.

Using the above machinery, given an endofunctor $F$ in $\mathcal{G}$ (either variant or covariant), one can obtain a fixed point of $F$ provided there exists a covariant continuous functor $F^e$ in $\mathcal{G}^e$, which coincides with $F$ on objects.

One can easily see that this is the case for constant functors, the functors &, $\otimes$, $\multimap$, !, $(\ )_\perp$ and their compositions.

## 3 Extensional $\lambda$-models in $K_!(\mathcal{G})$

As it is well known, a model for $\lambda\beta\eta$-calculus is a pair $\mathsf{D} = \langle D, f\rangle$, where $D$ is an *extensional reflexive object* in a Cartesian closed category, *i.e.* an object $D$ such that $D$ isomorphic to $D \to D$, and $f : D \to [D \to D]$ is an isomorphism. Two models $\mathsf{D} = \langle D, f\rangle$, $\mathsf{D}' = \langle D', f'\rangle$ are isomorphic if there exists an isomorphism $g : D \to D'$ such that $f' \circ g = [g^{-1} \to g] \circ f$. This implies that the two "applicative structures" are the same, *i.e.* for each $\sigma, \tau : A \to D$ we have that $g \circ ev \circ \langle f \circ \sigma, \tau\rangle = ev \circ \langle f' \circ (g \circ \sigma), g \circ \tau\rangle$.

In this section, using the techniques outlined in Section 2, we define a subclass, $\mathcal{D}^*$, of extensional models in $K_!(\mathcal{G})$, and prove the crucial result, namely, that each extensional model in $K_!(\mathcal{G})$ is isomorphic to a model in $\mathcal{D}^*$. In Section 4 we will prove that all models in $\mathcal{D}^*$ induce the $\lambda$-theory $\mathcal{H}^*$.

The endofunctor *Fun* on the category $\mathcal{G}^e$ is defined as follows:

- $\mathit{Fun}(D) = [D \to D] = (!D \multimap D)$;
- $f : A \rightarrowtail B$, then: $\mathit{Fun}(f) = [!f, f]$ where $!f(\langle i, a\rangle) = \langle i, f(a)\rangle$.

One can easily see that *Fun* is continuous.

**Definition 24.** Let $\mathcal{D}^*$ be the class of $\lambda$-models $\mathsf{D} = \langle D, f\rangle$ where $D$ is the limit of a chain generated by iterating the functor *Fun* on an initial game $D_0$, using an initial embedding $f^* : D_0 \rightarrowtail \mathit{Fun}(D_0)$, such that for each $m \in M_{D_0}$, $f^*(m) = \mathrm{in}_r(m')$ for some $m' \in M_{D_0}$. And where the isomorphism $f : D \to \mathit{Fun}(D)$ in $\mathcal{G}$ is $\varphi^+ \circ \mathtt{der}_D$, where $\varphi$ is the isomorphic embedding given by the colimit construction.

Isomorphisms in $K_!(\mathcal{G})$ can be reduced to isomorphisms in $\mathcal{G}^e$. In fact we have:

**Proposition 25.** – *For each isomorphism $\sigma : A \to B$ in $K_!(\mathcal{G})$, there exists an isomorphic strategy $\sigma' : A \multimap B$ such that $\sigma = \sigma' \circ \mathtt{der}_A$.*
– *For each isomorphic strategy $\sigma : A \multimap B$, there exists an isomorphic embedding $f_\sigma : A \rightarrowtail B$ such that $\sigma = (f_\sigma)^+$.*

*Proof.* The proof of the first part of the lemma is straightforward.

As shown in [3], a history-free strategy $\sigma : A \multimap B$ can be described as a map $g_\sigma$ from Opponent's moves to Player's moves in the game $A \multimap B$. If $\sigma$ is an isomorphism, with inverse $\sigma^{-1}$ then $g_\sigma$ maps each Player's move of $A$ in a Player's move of $B$, and each Opponent's move in $B$ to a Opponent's move in $A$. In fact, suppose by contradiction, that an Opponent's move $b \in M_B$ is such that $g_\sigma(b) = b'$ is a Player's move in $B$, then $g_{\sigma^{-1}\circ\sigma}(b) = b'$ and therefore $\sigma^{-1} \circ \sigma$ is not the copy-cat strategy. By a similar argument one can prove that $g_\sigma$ and $g_{\sigma^{-1}}$ are one the inverse of the other.

The function $f_\sigma$ is then defined as

$$f_\sigma(a) = \begin{cases} g_{\sigma(a)} & \text{if } \lambda_B^{OP}(a) = O \\ g_{\sigma^{-1}}(a) & \text{if } \lambda_A^{OP}(a) = P \end{cases}$$

By an analysis similar to the one above, and using the bracketing condition it is possible to prove that $f_\sigma$ preserves the labeling and that $f^*_\sigma(P_A) = P_B$.

In order to establish the main result of this section, we need a new definition, and prove a technical lemma.

**Definition 26.** Given a game $A$ and a move $a \in M_A$, the rank of $a$, $r(a)$, is the smallest integer $n$ such that there exists a sequence of moves $a_1, \ldots, a_n$ such that $a_1, \ldots, a_n, a \in P_A$.

**Lemma 27.** *For each game $A$, for each embedding $f : A \rightarrowtail Fun(A)$ and for each move $a \in M_A$, if $f(a) = in_l(\langle n, a' \rangle)$ then $r(a') < r(a)$; if $f(a) = in_r(a')$ then $r(a') \leq r(a)$.*

*Proof.* Let $s_a$ be a minimal position with end point $a$. The projection of $f^*(s_a)$ on the left component must still be a position in $P_{!A}$. Its length is strictly smaller than that of $s_a$, since the initial move of $s_a$ has to be mapped onto a move on the right component. $\square$

**Theorem 28.** *Each extensional model in $K_!(\mathcal{G})$ is isomorphic to a model in $\mathcal{D}^*$.*

*Proof.* Let $\langle D, \sigma \rangle$ be an extensional model in $K_!(\mathcal{G})$. Then, by Proposition 25 there exists an isomorphic embedding $f : D \rightarrowtail Fun(D)$, such that $\sigma = f^+ \circ \mathtt{der}_A$. Let $M_{D_0}$ be the largest subset of $M_D$ such that $\forall d \in M_{D_0}\ \exists d' \in M_{D_0}$ such that $f(d) = \mathrm{in}_r(d')$. Alternatively, with a slight abuse of notation, we can define $M_{D_0} = \{d \in D \mid \forall n \in \mathbb{N} \,.\, (\mathrm{in}_r{}^{-1} \circ f)^n(d) \text{ is defined}\}$.

It is immediate to verify that the quadruple $D_0 = (M_{D_0}, \lambda_D \restriction M_{D_0}, P_D \cap M^{\circledast}_{D_0}, \approx_D \cap (M_{D_0} \times M_{D_0})\ )$ is indeed a sub-game of $D$. By the construction of $D_0$, it follows that $f_0 = f_{|_{D_0}}$ is an embedding from $D_0$ to $Fun(D_0)$.

Let $D^*$ be the limit of the $\omega$-chain $\langle Fun^n(D_0), Fun^n(f_0) \rangle_{n \in \mathbb{N}}$, and let $f^* : D^* \rightarrowtail Fun(D^*)$ be the isomorphic embedding induced by the limit construction. We will prove that there exists an isomorphic embedding $f' : D^* \rightarrowtail D$ such that $f \circ f' = Fun(f') \circ f^*$.

The isomorphism $f'$ is defined as follows: given $d \in Fun^n(D_0)$ $f'([d]_\equiv) = f^{-1}_{0,n}(d)$, where $f_{0,n} : D \rightarrowtail Fun^n(D)$ is the isomorphism $Fun^{n-1}(f) \circ \ldots \circ Fun(f) \circ f$. Since, for each $n \in \mathbb{N}$, $M_{Fun^n(D_0)} \subseteq M_{Fun^n(D)}$, $f'$ is a well defined function from $M_{D^*}$ to $M_D$. Moreover it is not difficult to verify that $f'$ is an embedding.

We need to prove that $f'$ is surjective. This can be done by induction on the rank of the moves in $D$. Formally, we will prove that for each move $d \in M_D$ there exists a move $d'$ in $M_{D^*}$ such that $d = f'(d')$.

- Basic step. this follows form the fact that all initial moves (*i.e.* moves of rank 0) are in $M_{D_0}$.
- Induction step. Let $d \in M_D$ be a move of rank $n+1$, two possible cases arise. Either $d \in D_0$, and therefore $d = f'([d]_\equiv)$, or there exist $p, i \in \mathbb{N}$ and $d' \in M_D$ such that $\forall m \leq p, (\mathrm{in}_r{}^{-1} \circ f)^m(d)$ is defined and $f((\mathrm{in}_r{}^{-1} \circ f)^p(d)) = \mathrm{in}_l(\langle i, d' \rangle)$. By Lemma 27 the rank of $d'$ is less than $n+1$ and, hence, by

induction hypothesis, there exists $k \in \mathbb{N}$ and $d'' \in \mathit{Fun}^k(D_0)$ such that $d' = f'([d'']_\equiv)$. Let $d''' = \mathrm{in}_r{}^p \circ \mathrm{in}_l(\langle i, d'' \rangle) \in \mathit{Fun}^{k+p}(D_0)$, it is not difficult to verify that $d = f'([d''']_\equiv)$.

Moreover, it is straightforward to verify that $f \circ f' = \mathit{Fun}(f') \circ f^*$, and from the fact that: $[(f^+ \circ \mathtt{der}_D)^{-1} \to (f^+ \circ \mathtt{der}_D)] = \mathit{Fun}(f)^+ \circ \mathtt{der}_{D \to D}$, the theorem follows straightforwardly. $\square$

## 4 The fine structure of models in $\mathcal{D}^*$

In order to analyze the equational theories induced by the models in $\mathcal{D}^*$, we establish an *Approximation Theorem*, in the style of [20,14]. Using this result we will be able to characterize the meaning of a term in the model as the lub of the set of the meanings of the syntactical *approximants* of the term.

To our knowledge this is the first time such a theorem is proved for models in "non-concrete" categories such as game models.

As usual it is convenient to consider $\Lambda(\Omega)$, an extension of $\lambda$-calculus with a constant to denote divergence, and its indexed version $\Lambda(\Omega)^{\mathbb{N}}$.

**Definition 29.** 1. The set of $\lambda\Omega$-terms, $\Lambda(\Omega)(\ni M)$ is defined from a set of variables $Var(\ni x)$ as follows:

$$M ::= x \mid MM \mid \lambda x.M \mid \Omega.$$

2. The set of (possibly) *indexed terms* $\Lambda(\Omega)^{\mathbb{N}}(\ni M)$ is the superset of $\Lambda(\Omega)$ defined as follows:

$$M ::= x \mid MM \mid \lambda x.M \mid \Omega \mid M^n.$$

3. A term is *truly indexed* if it is of the shape $M^n$. A term is *completely indexed* if all its subterms of the shape constant, variable, abstraction, and application are immediate subterms of truly indexed terms.

The intended meaning of an indexed term $M^n$ is the $n$-th projection of the interpretation of the term $M$. Hence the interpretation of $\lambda$-terms in $K_!(\mathcal{G})$ is defined as follows.

**Definition 30.** Let $\mathsf{D} = \langle D, \varphi \rangle$ be a model in $\mathcal{D}^*$. The interpretation of a term $M \in \Lambda(\Omega)^{\mathbb{N}}$ (whose free variables are among the list $\Delta = \{x_1, \ldots, x_n\}$) in the model is the strategy $[\![M]\!]^{\mathsf{D}}_{\Delta} : !(\overbrace{D \,\&\, \ldots \,\&\, D}^{|\Delta|}) \multimap D$ defined inductively as follows:

$[\![x_i]\!]^{\mathsf{D}}_{\Delta} = \pi_i^{\Delta}$;
$[\![MN]\!]^{\mathsf{D}}_{\Delta} = ev \circ \langle(\varphi \circ [\![M]\!]^{\mathsf{D}}_{\Delta})\ ,\ [\![N]\!]^{\mathsf{D}}_{\Delta}\rangle$;
$[\![\lambda x.M]\!]^{\mathsf{D}}_{\Delta} = \psi \circ \Lambda(\ [\![M]\!]^{\mathsf{D}}_{\Delta,x})$;
$[\![M^n]\!]^{\mathsf{D}}_{\Delta} = p_n \circ [\![M]\!]^{\mathsf{D}}_{\Delta}$;

$[\![\Omega]\!]^{\mathsf{D}}_{\Delta} = \sigma_\epsilon;$

where $\pi_i^{\Delta}$ are the canonical projection morphisms, $ev$ and $\Lambda$ denote "evaluation" and "abstraction" in the Cartesian closed category $K_!(\mathcal{G})$, $\sigma_\epsilon = [\{\epsilon\}]$, $\psi = \varphi^{-1}$ and the $p_n$ are the strategies defined in Proposition 23.

Given strategies $\sigma, \tau$ with codomain $D$, we use the abbreviation $\sigma \cdot \tau$ to denote the strategy $ev \circ \langle(\varphi \circ \sigma),\ \tau\rangle$, and we will denote with $(D)^n$ the game $\overbrace{D \,\&\ldots\&\, D}^{n}$.

The main result of this section is Theorem 43. In order to establish it we need several preliminary results.

**Lemma 31.** *For each model $\mathsf{D} = \langle D^*, \varphi\rangle$ in $\mathcal{D}^*$, for each game $A$ and pair of strategies $\sigma, \tau : !A \multimap D^*$, we have:*

1. $(p_0 \circ \sigma) \cdot \tau = (p_0 \circ \sigma) \cdot \sigma_\epsilon = p_0 \circ (\sigma \cdot \sigma_\epsilon)$
2. $(p_{n+1} \circ \sigma) \cdot \tau \sqsubseteq p_{n+1} \circ (\sigma \cdot (p_n \circ \tau))\ \forall n \in \mathbb{N}$

Notice that in the statement of Lemma 31.2, we have not taken equality but only inequality. This is done in order to be able to deal simultaneously not only with models in $\mathcal{D}^*$, but also, in Section 5, with models obtained using the trick of Scott outlined in Section 2.

The following Lemmata and Definitions follow closely the pattern of [20,14], and they amount essentially to the game theoretic version of the corresponding "continuous result".

**Definition 32.** The erasing function $\mathcal{R} : \Lambda(\Omega)^{\mathbb{N}} \to \Lambda(\Omega)$ is inductively defined as follows:

1. $\mathcal{R}(x) = x;\ \mathcal{R}(\Omega) = \Omega$
2. $\mathcal{R}(PQ) = \mathcal{R}(P)\mathcal{R}(Q)$
3. $\mathcal{R}(\lambda x.P) = \lambda x.\mathcal{R}(P)$
4. $\mathcal{R}(M^n) = \mathcal{R}(M)$

**Lemma 33.** *For each model $\mathsf{D} = \langle D^*, \varphi\rangle$ in $\mathcal{D}^*$, for each term $M \in \Lambda(\Omega)$ whose free variables are in $\Delta$, given a finite strategy $\sigma : !(D^*)^{|\Delta|} \multimap D^*$ s.t. $\sigma \sqsubseteq [\![M]\!]^{\mathsf{D}}_{\Delta}$ there exists a natural number $n$ s.t. $\sigma \sqsubseteq [\![M^n]\!]^{\mathsf{D}}_{\Delta}$.*

**Lemma 34.** *For each model $\mathsf{D} = \langle D^*, \varphi\rangle$ in $\mathcal{D}^*$, for each term $M \in \Lambda(\Omega)$ whose free variables are in $\Delta$, given a finite strategy $\sigma : !(D^*)^{|\Delta|} \multimap D^*$ s.t. $\sigma \sqsubseteq [\![M]\!]^{\mathsf{D}}_{\Delta}$ there exists a completely indexed term $Q \in \Lambda(\Omega)^{\mathbb{N}}$ such that $\mathcal{R}(Q) = M$ and $\sigma \sqsubseteq [\![Q]\!]^{\mathsf{D}}_{\Delta}$.*

**Lemma 35.** *Let $A$ be a game and $\sigma : A$ a strategy. $\sigma = \bigsqcup\{\tau : A \mid \tau$ finite and $\tau \sqsubseteq \sigma\}$.*

From the above Lemmata it follows that

**Proposition 36.** *For each model* $D = \langle D^*, \varphi \rangle$ *in* $\mathcal{D}^*$, *for each term* $M \in \Lambda$, $[\![M]\!]^D_\Delta = \bigsqcup\{[\![Q]\!]^D_\Delta \mid Q$ *is a completely indexed term s.t.* $\mathcal{R}(Q) = M\}$.

**Definition 37.** 1. The following reduction rules are definable on $\Lambda(\Omega)$:

$$(\Omega_1) \quad \lambda x.\Omega \to \Omega$$

$$(\Omega_2) \quad \Omega M \to \Omega$$

2. The following reduction rules are definable on completely indexed terms of $\Lambda(\Omega)^{\mathbb{N}}$:

$$(\Omega_1^n) \quad \lambda x.\Omega^n \to \Omega^0$$

$$(\Omega_2^n) \quad \Omega^n M \to \Omega^0$$

$$(\beta_I) \quad ((\lambda x.P^n)^{m+1} Q^p)^h \to (P[x/Q^a])^b$$

where $b = min\{n, m+1, h\}, a = min\{m, p\}$

$$(\beta_0) \quad ((\lambda x.P)^0 Q)^h \to (P[x/\Omega])^0$$

$$(\beta_{i,j}) \quad (M^i)^j \to M^{min\{i,j\}}$$

Notice again that the above definition of the $(\beta_I)$ indexed reduction rule and the statement of the following Theorem are not formulated as in [20], but are relaxed so as to take care of the model $D^{\mathbb{N}}$ (see Section 5).

**Theorem 38 (Validity of indexed reduction).** *For each model* $D = \langle D^*, \varphi \rangle$ *in* $\mathcal{D}^*$, *the rules* $(\Omega_1^n), (\Omega_2^n), (\beta_I), (\beta_0)$ *and* $(\beta_{i,j})$ *are valid in the following sense: let* $P, Q \in \Lambda(\Omega)^{\mathbb{N}}$ *then*

$$(P \twoheadrightarrow_{\Omega_1^n \Omega_2^n \beta_0 \beta_I \beta_{i,j}} Q) \Longrightarrow [\![P]\!]^D_\Delta \sqsubseteq [\![Q]\!]^D_\Delta.$$

**Lemma 39.** *Let* $Q \in \Lambda(\Omega)^{\mathbb{N}}$ *be a completely indexed term. Then* $Q$ *is* $\Omega_1^n \Omega_2^n \beta_0 \beta_I \beta_{i,j}$*-normalizing.*

From the above lemma it is immediate to state the following.

**Lemma 40.** *For each model* $D = \langle D^*, \varphi \rangle$ *in* $\mathcal{D}^*$, *for each term* $M \in \Lambda$, $[\![M]\!]^D_\Delta = \bigsqcup\{[\![N]\!]^D_\Delta \mid \exists Q$ *completely indexed term such that* $\mathcal{R}(Q) = M$ *and* $N$ *is the* $\Omega_1^n \Omega_2^n \beta_0 \beta_I \beta_{i,j}$*-normal form of* $Q\}$.

**Definition 41.** The *direct approximant* of a $\lambda$-term $M \in \Lambda$ is a normal form $A \in \Lambda(\Omega)$ obtained from $M$ by replacing each redex in $M$ by $\Omega$, and performing all the $\Omega_1^n \Omega_2^n$-reductions.

**Definition 42.** The set of *approximants* of $M$ is the set $\mathcal{A}(M) = \{A \mid \exists M', M \twoheadrightarrow_{\beta\eta} M'$ and $A$ is the direct approximant of $M'\}$.

**Theorem 43 (Approximation theorem).** *For each model* $\mathsf{D} = \langle D^*, \varphi\rangle$ *in* $\mathcal{D}^*$, *for each term* $M \in \Lambda$, $[\![M]\!]^{\mathsf{D}}_{\Delta} = \bigsqcup\{[\![A]\!]^{\mathsf{D}}_{\Delta} \mid A \in \mathcal{A}(M)\}$.

**Theorem 44.** *For each model* $\mathsf{D} = \langle D^*, \varphi\rangle$ *in* $\mathcal{D}^*$, $Th(\mathsf{D}) = \mathcal{H}^*$.

*Proof.* Using Theorem 38 and standard techniques (see e.g. [6] Section 19.2), one can prove that the structure of the approximants in Theorem 43 as well as the partial order induced by the semantics over them coincide precisely with those of the approximants of Scott's $D_\infty$ model as in the continuous case. In particular their corresponding Böhm trees satisfy the $\eta^\infty$-equivalence. Hence the argument for the continuous case goes through even in the game setting, and we have that the theory of $\mathsf{D}$, just as in the continuous case, is $\mathcal{H}^*$. $\square$

## 5 Some examples of concrete game models for $\lambda\beta\eta$-calculus

In this Section, by way of example, we introduce four extensional $\lambda$-models in $K_!(\mathcal{G})$. The first two belong to $\mathcal{D}^*$, the third does not, while the fourth one is the model obtained by carrying the analogous of Scott's trick as outlined in Section 2.

We start by introducing three initial embeddings for the functor *Fun*:

**Definition 45.**
1. Let $D_0^\circ = (\{*\}, \lambda(*) = OQ, \{\epsilon, *\}, id)$ and define $f_\circ : D_0^\circ \rightarrowtail (!D_0^\circ \multimap D_0^\circ)$ by $f_\circ(*) = \mathrm{in}_r(*)$;
2. let $D_0^{*\circ} = (\{*, \circ\}, (\lambda(*) = OQ, \lambda(\circ) = PQ), \{\epsilon, *, *\circ\}, id)$ and define $f_{*\circ} : D_0^{*\circ} \rightarrowtail (!D_0^{*\circ} \multimap D_0^{*\circ})$ by $f_{*\circ}(*) = \mathrm{in}_r(*)$ and $f_{*\circ}(\circ) = \mathrm{in}_r(\circ)$;
3. let $D_0^{**} = (\{*, \circ\}, (\lambda(*) = OQ, \lambda(\circ) = PQ), \{\epsilon, *, *\circ\}, id)$ and define $f_{**} : D_0^{**} \rightarrow (!D_0^{**} \multimap D_0^{**})$ by $f_{**}(*) = \mathrm{in}_r(*)$ and $f_{**}(\circ) = \mathrm{in}_l(\langle 0, *\rangle)$.

Finally, using Theorem 22 we give the following definition:

**Definition 46.** The models $D_\infty^\circ, D_\infty^{*\circ}, D_\infty^{**}$ are determined by the limits of the chains generated by iterating the functor *Fun* on the embeddings $f_*, f_{*\circ}, f_{**}$, and by the corresponding injection $\varphi$ respectively.

The last $\lambda$-model we introduce, defined using Scott's trick, is $D^{\mathbb{N}}$.

**Definition 47.** Let $A_{\mathbb{N}} = (\mathbb{N}, \lambda n.OQ, \{\epsilon\} \cup \mathbb{N}, id)$. The model $D^{\mathbb{N}}$ is the one naturally induced by the least fixed point of the functor $F(D) = \,!D \multimap A_{\mathbb{N}}$ where the following chain holds for every $n \in \mathbb{N}$: $D^{\mathbb{N}}_{n+1} \simeq \,!D^{\mathbb{N}}_n \multimap A_{\mathbb{N}} \simeq \,!D^{\mathbb{N}}_n \multimap (A_{\mathbb{N}} \,\&\, A_{\mathbb{N}}) \simeq (!D^{\mathbb{N}}_n \multimap A_{\mathbb{N}}) \,\&\, (!D^{\mathbb{N}}_n \multimap A_{\mathbb{N}}) \simeq D^{\mathbb{N}}_{n+1} \,\&\, D^{\mathbb{N}}_{n+1}$. Hence we have $D^{\mathbb{N}}_{n+1} \simeq \,!D^{\mathbb{N}}_n \multimap A_{\mathbb{N}} \simeq \,!(D^{\mathbb{N}}_n \,\&\, D^{\mathbb{N}}_n) \multimap A_{\mathbb{N}} \simeq \,!D^{\mathbb{N}}_n \multimap (!D^{\mathbb{N}}_n \multimap A_{\mathbb{N}}) \simeq \,!D^{\mathbb{N}}_n \multimap D^{\mathbb{N}}_{n+1}$.

One can easily see that $D^{\mathbb{N}}$ is a $\lambda$-model since any bijection $p : \mathbb{N} + \mathbb{N} \rightarrow \mathbb{N}$, induces an isomorphism between $A_{\mathbb{N}}$ and $A_{\mathbb{N}} \,\&\, A_{\mathbb{N}}$.

## 6 Conclusions and Final Remarks

In this paper we have shown that all extensional $\lambda$-models in the category $K_!(\mathcal{G})$ of [3] induce the same $\lambda$-theory, this is the well-known theory $\mathcal{H}^*$. It is natural to conjecture, therefore, that also there is only one non-extensional *sensible $\lambda$-theory* which can be modeled using games. We recall that a sensible $\lambda$-theory is a theory where all unsolvable terms are equated. This would be the theory $\mathcal{B}$ of Böhm trees, and would be the theory of any reflexive object in $K_!(\mathcal{G})$, $\langle D, f\rangle$, for which $f$ maps the undefined strategy on $!D \multimap D$ on the undefined strategy on $D$, but it is not an isomorphism.

Our results clearly indicate that game models are *even* more rigid than continuous models. But is this really a "surprise", or a "bad surprise"? Definitely there must be some intrinsic feature of games, as they are currently defined, that is intimately related with head reduction. It is difficult to imagine at this stage what this is. Probably it is not the fact that we have considered only "history-free" strategies, more likely it has to do with the "strict" protocol of alternation of moves between Opponent and Player. We feel however that when the appropriate constraint will be relaxed, the perspicuous analytic power of *games* will become applicable also to other reduction strategies, besides head reduction.

We end this paper with two more technical notes.

It is worth noticing that in our models of untyped $\lambda$-calculus it is not necessary to take at the very end an extensional quotient, as is done for typed $\lambda$-calculus [3] or the lazy $\lambda$-calculus [4].

The essential ingredient in the proof of Theorem 44 is Lemma 31. This has a bearing also on models in the category of CPOs. Namely $\mathcal{H}^*$ is the theory of all inverse limits obtained starting from an initial injection where all points are mapped onto constant functions, as well as that of the "Scott's trick model" presented in [19].

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# A Proofs

**Proof of Proposition 20**

Given an $\omega$-chain $\langle D_n, f_n\rangle$ with $f_n : D_n \rightarrowtail D_{n+1}$ its colimit is $\langle D_\infty, \mu_n\rangle$ where $D_\infty$ is the game:

- $M_{D_\infty} = (\bigcup_{n\in\omega} M_{D_n})/_\equiv$
  where $\equiv$ is the least equivalence relation such that

$$\forall n \in \mathbb{N}\ \forall a \in D_n\ \forall b \in D_{n+1}.\ f_n(a) = b \ \Rightarrow\ a \equiv b.$$

- $\lambda_{D_\infty}([a]_\equiv) = \lambda_{D_n}(a)$ if $a \in D_n$,
- $P_{D_\infty} = \bigcup_{n\in\omega}\{[a_1]_\equiv[a_2]_\equiv\dots[a_p]_\equiv \mid a_1a_2\dots a_p \in P_{D_n}\}$
- $\approx_{D_\infty} = \bigcup_{n\in\omega}\ \{([a_1]_\equiv[a_2]_\equiv\dots[a_p]_\equiv, [a'_1]_\equiv[a'_2]_\equiv\dots[a'_p]_\equiv) \mid$
  $(a_1a_2\dots a_p, a'_1a'_2\dots a'_p) \in \approx_{D_n}\}$

The colimit functions $\mu_n : D_n \rightarrowtail D_\infty$ are defined by $\mu_n(a) = [a]_\equiv$. $\square$

**Proof of Lemma 31**

1. From Proposition 23 we have
   $p_0 \circ \sigma = \{s \in \sigma \mid s \in \sigma \ \wedge \ s \in (\mathrm{in}_l(M_{!A}) \cup \mathrm{in}_r(\mu_0(M_{D_0})))\}$. Let $\varphi' : D \multimap (!D \multimap D)$ the strategy such that $\varphi = \varphi' \circ \mathtt{der}_D$. It follows that
   $\varphi' \circ p_0 \circ \sigma : !A \multimap (!D^* \multimap D^*)$
   $= \{s \in \sigma \mid s \in \sigma \ \wedge \ s \in (\mathrm{in}_l(M_{!A}) \cup \mathrm{in}_r(\mathrm{in}_r(\mu_0(M_{D_0}))))\}$. Hence the strategy $\varphi' \circ (p_0 \circ \sigma)$ never interacts with its second input. We have then $(p_0 \circ \sigma) \cdot \tau = ev \circ \langle (\varphi \circ (p_0 \circ \sigma)), \tau \rangle = (p_0 \circ \sigma) \cdot \sigma_\epsilon$. In the same way we get $p_0 \circ (\sigma \cdot \sigma_\epsilon) = (p_0 \circ \sigma) \cdot \sigma_\epsilon$.
2. From Proposition 23, and Theorem 22 we have
   $\varphi' \circ (p_{n+1} \circ \sigma) : !A \multimap (!D^* \multimap D^*)$
   $= \{s \in \sigma \mid s \in \sigma \ \wedge \ s \in (\mathrm{in}_l(M_{!A}) \cup \mathrm{in}_r(\mathrm{in}_l(\mu_n(M_{Fun^n(D_0)})))$
   $\cup\, \mathrm{in}_r(\mathrm{in}_r(\mu_n(M_{Fun^n(D_0)}))))\}$.
   Hence the strategy $\varphi' \circ (p_{n+1} \circ \sigma)$ considers only the moves in $\mu_n(M_{Fun(D_0)})$ of its second component and returns only moves in $\mu_n(M_{Fun(D_0)})$ as output. It follows that $(p_{n+1} \circ \sigma) \cdot \tau = p_n \circ (\sigma \cdot (p_n \circ \tau))$, and from this $(p_{n+1} \circ \sigma) \cdot \tau \sqsubseteq p_{n+1}(\sigma \cdot (p_n \circ \tau))$.

$\square$

**Proof of Lemma 33**

Since $\sigma \sqsubseteq [\![M]\!]^{\mathsf{D}}_{\Delta} : !(D^*)^{\Delta} \multimap D^*$ then each position of $\sigma$ has a corresponding position in $[\![M]\!]^{\mathsf{D}}_{\Delta}$; but $\sigma$ is finite, hence there exist an $n$ such that $\sigma$ involves only moves "in $D^*_n$". $\square$

**Proof of Lemma 34**

Induction on $M$.

1. $M \equiv x$. Follows from lemma 33.
2. $M \equiv \lambda x.P$. Given a finite $\sigma$ s.t. $\sigma \sqsubseteq [\![\lambda x.P]\!]^{\mathsf{D}}_{\Delta}$ then $\Lambda^{-1}(\varphi \circ \sigma) \sqsubseteq [\![P]\!]^{\mathsf{D}}_{\Delta,x}$. Since $\Lambda$ preserves finiteness of strategies and since $\varphi$ is an isomorphism, by induction hypothesis, there exists an indexed term $P^n$ s.t. $\Lambda^{-1}(\varphi \circ \sigma) \sqsubseteq [\![P^n]\!]^{\mathsf{D}}_{\Delta,x}$. Hence we get $\sigma \sqsubseteq [\![\lambda x.P^n]\!]^{\mathsf{D}}_{\Delta}$ by definition of interpretation of terms and hence $\sigma \sqsubseteq [\![(\lambda x.P^n)^m]\!]^{\mathsf{D}}_{\Delta}$ by Lemma 33.
3. $M \equiv PQ$. If $\sigma \sqsubseteq [\![PQ]\!]^{\mathsf{D}}_{\Delta}$ then $\sigma = ev \circ \langle \varphi \circ [\![P]\!]^{\mathsf{D}}_{\Delta}, [\![Q]\!]^{\mathsf{D}}_{\Delta} \rangle$. Since $ev$ is continuous there are finite strategies $\sigma_1, \sigma_2$ s.t. $\sigma_1 \sqsubseteq \varphi \circ [\![P]\!]^{\mathsf{D}}_{\Delta}$, $\sigma_2 \sqsubseteq [\![Q]\!]^{\mathsf{D}}_{\Delta}$ and $\psi \circ \sigma_1 \sqsubseteq [\![P]\!]^{\mathsf{D}}_{\Delta}$ since $\psi$ preserve finiteness of strategies. Hence, by induction hypothesis, there exist completely indexed terms $P^n$ and $Q^m$ s.t. $\psi \circ \sigma_1 \sqsubseteq [\![P^n]\!]^{\mathsf{D}}_{\Delta}$ and $\sigma_2 \sqsubseteq [\![Q^m]\!]^{\mathsf{D}}_{\Delta}$. By Lemma 33 we can then conclude that there exists an $h$ s.t. $\sigma \sqsubseteq [\![(P^n Q^m)^h]\!]^{\mathsf{D}}_{\Delta}$.

$\square$

**Proof of Proposition 36**

Since the projection functions are smaller that the identity we have: $[\![Q]\!]^{\mathsf{D}}_{\Delta} \sqsubseteq [\![M]\!]^{\mathsf{D}}_{\Delta}$. From Lemma 35 and 34 we have $[\![M]\!]^{\mathsf{D}}_{\Delta} = \bigsqcup\{\sigma \mid \sigma \text{ finite and } \sigma \sqsubseteq [\![M]\!]^{\mathsf{D}}_{\Delta}\} \sqsubseteq \bigsqcup\{[\![Q]\!]^{\mathsf{D}}_{\Delta} \mid Q \text{ is an indexed term and } \mathcal{R}(Q) = M\} = [\![M]\!]^{\mathsf{D}}_{\Delta}$ since $\sigma \sqsubseteq [\![Q]\!]^{\mathsf{D}}_{\Delta} \sqsubseteq [\![M]\!]^{\mathsf{D}}_{\Delta}$. $\square$

**Proof of Theorem 38**

1. Rules $(\Omega_1^n)$ and $(\Omega_2^n)$. Since the interpretation of $\Omega^n$ is always $\sigma_\epsilon$, the empty strategy, the validity of the rules follows easily.
2. Rule $(\beta_I)$.

$$
\begin{aligned}
&[\![((\lambda x.P^n)^{m+1}Q^p)^h]\!]^{\mathsf{D}}_{\Delta}\\
&= p_h\circ [\![(\lambda x.P^n)^{m+1}Q^p]\!]^{\mathsf{D}}_{\Delta}\\
&= p_h \circ ([\![(\lambda x.P^n)^{m+1}]\!]^{\mathsf{D}}_{\Delta}\cdot [\![Q^p]\!]^{\mathsf{D}}_{\Delta})\\
&= p_h \circ ((p_{m+1}\circ [\![\lambda x.P^n]\!]^{\mathsf{D}}_{\Delta}) \cdot (p_q\circ [\![Q]\!]^{\mathsf{D}}_{\Delta}))\\
&\sqsubseteq p_h \circ p_{m+1} \circ ([\![\lambda x.P^n]\!]^{\mathsf{D}}_{\Delta} \cdot (p_m \circ p_q\circ [\![Q]\!]^{\mathsf{D}}_{\Delta}))\\
&\quad \text{(by lemma 31)}\\
&= p_{min\{h,m+1\}} \circ ([\![\lambda x.P^n]\!]^{\mathsf{D}}_{\Delta} \cdot (p_{min\{m,q\}}\circ [\![Q]\!]^{\mathsf{D}}_{\Delta}))\\
&= p_{min\{h,m+1\}} \circ ([\![\lambda x.P^n]\!]^{\mathsf{D}}_{\Delta}\cdot [\![Q^{min\{m,q\}}]\!]^{\mathsf{D}}_{\Delta})\\
&= p_{min\{h,m+1\}}\circ [\![P^n[x/Q^{min\{m,q\}}]]\!]^{\mathsf{D}}_{\Delta}\\
&= p_{min\{h,m+1\}} \circ p_n\circ [\![P[x/Q^{min\{m,q\}}]]\!]^{\mathsf{D}}_{\Delta}\\
&= p_b\circ [\![P[x/Q^a]\!]^{\mathsf{D}}_{\Delta}
\end{aligned}
$$

3. Rule $(\beta_0)$.

$$
\begin{aligned}
&[\![((\lambda x.P)^0Q)^h]\!]^{\mathsf{D}}_{\Delta}\\
&= p_h \circ ((p_0\circ [\![\lambda x.P]\!]^{\mathsf{D}}_{\Delta})\cdot [\![Q]\!]^{\mathsf{D}}_{\Delta})\\
&= p_h \circ (p_0 \circ ([\![\lambda x.P]\!]^{\mathsf{D}}_{\Delta} \cdot \sigma_\epsilon))\\
&\quad \text{(by lemma 31)}\\
&= p_h \circ (p_0 \circ ([\![\lambda x.P]\!]^{\mathsf{D}}_{\Delta}\cdot [\![\Omega]\!]^{\mathsf{D}}_{\Delta}))\\
&= p_0 \circ ([\![P[x/\Omega]]\!]^{\mathsf{D}}_{\Delta})
\end{aligned}
$$

4. Rule $(\beta_{i,j})$ follows immediately from Proposition 23.2.c.

**Proof of Theorem 43**

Since $[\![A]\!]^{\mathsf{D}}_{\Delta} \sqsubseteq [\![M]\!]^{\mathsf{D}}_{\Delta}$ for each approximant $A$ and $\mathcal{A}(M)$ is a strongly directed set of strategies we get $\bigsqcup\{[\![A]\!] \mid A \in \mathcal{A}(M)\} \sqsubseteq [\![M]\!]^{\mathsf{D}}_{\Delta}$. Since erasing the indexes from an indexed normal term $Q$ we get an approximant of $M$, see [20], by lemma 40 we get the equality. $\square$