Gain scheduling controllers for LPV systems

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I. Introduction

Control specifications for a single dynamical process can be often fullled by designing a gain scheduling controller $R(s, w)$, where w is a time-varying parameter, so that the controller $R(s, \bar{w})$ is tailored to some system operating region (e.g. a proportional controller when the output error $e = y - r$ is large and an integral controller when the error is below a given threshold).

Such idea can be extended to deal with systems whose parameters depend linearly on a time-varying parameter $w(t)$, the so called Linear Parameter Varying (**LPV**) plants, and such parameter is known **at each time instant** and can be used to tune the controller.

The **first goal** which will be analyzed will be that of designing a gain scheduling controller such that for every constant (time frozen) value of the parameter $\bar{w}$ the closed-loop system is stable and, moreover, stability is preserved for the time-varying closed-loop system

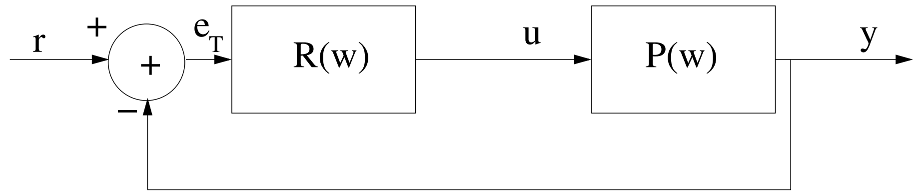
The **second goal** which will be achieved will be that of finding an ad-hoc state space realization $(\tilde{F}(w), \tilde{G}(w), \tilde{H}(w), \tilde{K}(w))$ of a given gain scheduling controller $R(s, w) = (F(w), G(w), H(w), K(w))$ so that closed-loop stability is preserved when the parameter varies and it is such that for every frozen valued $\bar{w}$

$$H(\bar{w})\,(sI - F(\bar{w}))^{-1}\,G(\bar{w}) + K(\bar{w}) = \tilde{H}(\bar{w})\left(sI - \tilde{F}(\bar{w})\right)^{-1}\tilde{G}(\bar{w}) + \tilde{K}(\bar{w})$$

This accompanying notes will consider both problems of synthesizing gain scheduling controller for **LPV** plants or finding ad hoc realizations of a given family of scheduling controllers.

The solution of both problems is based on Lyapunov arguments [1], [2], [3] and strongly relies on the Youla parametrization of stabilizing controllers for linear plants. To this aim, an overview of the Zhou-Doyle-Glover like parametrization [4] will also be presented.

By means of the above parameterization it will be shown how the presented gain scheduling problem for **LPV** systems can be solved by splitting it into simpler problems: state feedback, state estimation and realization of an ad-hoc chosen Youla parameter.



II. Youla coprime parameterization for SISO systems

The material here presented is extracted from [5].

Consider the proper SISO plant

$$P(s) = (A,B,C,D) = \frac{n_P(s)}{d_P(s)}$$

with

$$d_P(s) = \det(sI - A) \quad n_P(s) = C\mathrm{adj}\,(sI - A)\,B + Dd_P(s),$$

Theorem 2.1: Assume $P(s)$ is asymptotically stable. The set of all the controllers $C(s)$ for which the feedback loop is internally stable is

$$\left\{ \frac{Q}{1-PQ} \;:\; Q \in \mathcal{S} \right\}$$

When the plant is not stable, we first determine a **coprime** factorization over $\mathcal{S}$, the set of asymptotically stable, proper, real rational functions, of the plant $P(s)$. $\mathcal{S}$ is a commutative ring with identity, i.e. $F, G \in \mathcal{S} \Rightarrow F+G,\;\; FG \in \mathcal{S}$.

Two transfer functions M and N are coprime if there exists $X, Y \in \mathcal{S}$ such that

$$NX + MY = 1$$

(M, N are coprime iff they have no finite or infinite common zeros in $Re(s) \geq 0$.

Let $P = \frac{N}{M}$ be a coprime factorization and X, Y be a solution of $NX + MY = 1$.

Theorem 2.2: The set of all the controllers which internally stabilize $P(s) = \frac{N(s)}{M(s)}$ is

$$\left\{ \frac{X+MQ}{Y-NQ}, \quad Q \in \mathcal{S} \right\}$$

The resulting closed–loop transfer functions are

$$T(s) = \frac{CP}{1+CP} = \frac{\frac{X+MQ}{Y-NQ}\frac{N}{M}}{1+\frac{X+MQ}{Y-NQ}\frac{N}{M}} = \frac{NX+NMQ}{MY-MNQ+NX+NMQ} = N(X+MQ)$$

$$S(s) = \frac{1}{1+CP} = 1 - T(s) = M(Y-NQ)$$

A. *Coprime factorization SISO state space realization (optional)*

In this section we give a control feedback interpretation of the factorization. The presented material is introduced to give a state space interpretation of the coprime factorization, but it will not be used in the sequel since a direct state-space parametrization will be adopted. Given $P = (A,B,C,D)$ let J and L be such that $A+BJ$ and $A+LC$ are asymptotically stable and consider the estimated state feedback control law as in figure 1. The closed loop system dynamics becomes

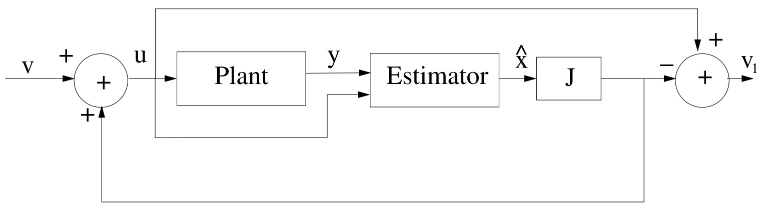


Fig. 1. Youla SISO parametrization

$$\begin{array}{rcl} \dot{x} &=& (A+BJ)x - BJe + Bv \\ \dot{e} &=& (A+LC)e \\ y &=& (C+DJ)x - DJe + Dv \\ u &=& Jx - Je + v \end{array}$$

The transfer function from v to u is

$$M(s) := \left[\begin{array}{c|c} A+BJ & B \\ \hline J & 1 \end{array} \right]$$

and that from v to y is

$$N(s) := \left[\begin{array}{c|c} A+BJ & B \\ \hline C+DJ & D \end{array} \right]$$

so that

$$u = Mv, \qquad y = Nv \Rightarrow y = NM^{-1}u = \frac{N}{M}u, \quad \text{with } N, M \in \mathcal{S}$$

The transfer function from y and u to v_1 is described by

$$\begin{array}{rcl} \dot{\hat{x}} & = & (A + LC)\hat{x} + (B + LD)u - Ly \\ v_1 & = & -J\hat{x} + u \end{array}$$

so that, denoting by

$$X(s) := \left[\begin{array}{c|c} A + LC & -L \\ \hline -J & 0 \end{array} \right]$$

and by $Y(s)$

$$Y(s) := \left[\begin{array}{c|c} A + LC & B + LD \\ \hline -J & 1 \end{array} \right]$$

both in $\mathcal{S}$, we have that the transfer function from v to v_1 is

$$MX + NY = 1$$

III. System representations and operations

Given an n dimensional continuous-time plant

$$P(s):\ \left\{\begin{array}{rcl}\dot{x}(t) & = & Ax(t)+Bu(t)\\ y(t) & = & Cx(t)+Du(t)\end{array}\right. \quad = \left[\begin{array}{c|c} A & B \\ \hline C & D\end{array}\right]$$

with m inputs and p outputs, its $p\times m$ transfer matrix is

$$P(s) = C\,(sI-A)^{-1}\,B + D$$

Definition 3.1: The transfer matrix $\hat{P}(s)$ is a right (left) inverse of the transfer matrix $P(s)$ if $P(s)\hat{P}(s) = I$ $(\hat{P}(s)P(s) = I)$

Lemma 3.1: Let $D^\dagger$ denote a right $DD\dagger = I$ (or left $D\dagger D = I$) inverse of D. The right (left) inverse of P is the n dimensional p-inputs/m-outputs system

$$P^\dagger:\ \left[\begin{array}{c|c} A - BD^\dagger C & -BD^\dagger \\ \hline D^\dagger C & D^\dagger\end{array}\right]$$

Remark 3.1: A SISO system $P(s) = \frac{n(s)}{d(s)}$ admits a right=left inverse iff the degree of $n(s)$ is equal to the degree of $d(s)$

Example 3.1:

$$P(s) = \frac{s+1}{s+3} : \left\{\begin{array}{rcl}\dot{x} & = & -3x+u\\ y & = & -2x+u\end{array}\right.$$

$A=-3,\quad B=1,\quad ,C=-2\quad ,D=1$

$$P^{-1}(s) = \frac{s+3}{s+1} : \left\{\begin{array}{rcl}\dot{z} & = & -z-y\\ u & = & -2z+y\end{array}\right.$$

Definition 3.2: Given a transfer matrix $P(s)$ a state space realization (A,B,C,D) is **minimal** if A has the smallest possible dimension.

Theorem 3.1: A state space realization (A,B,C,D) of a transfer matrix

$$P(s) = \frac{N(s)}{d(s)}$$

where $d(s)$ is a monic scalar polynomial and $N(s)$ a polynomial matrix, is minimal iff (A,B) is controllable and (C,A) is observable.

Theorem 3.2: Given two minimal realizations (A_1,B_1,C_1,D_1) and (A_2,B_2,C_2,D_2) of the same transfer matrix $P(s)$ there exists a unique nonsingular matrix S such that

$$A_2 = SA_1S^{-1},\quad B_2 = SB_1,\quad C_2 = C_1S^{-1}$$

The next result is concerned on the determination of realizations whose system matrices share the same quadratic Lyapunov function.

Lemma 3.2: Assume a family of asymptotically stable transfer matrices $(A(w),B(w),C(w),D(w))$ (of the same dimension) which depend on a parameter $w\in\mathcal{W}$ is given. Then there exists a family of transformation matrices $S(w)$ such that, denoting by $\tilde{A}(w) = S(w)A(w)S^{-1}(w)$,

$$\tilde{A}(w)^T + \tilde{A}(w) < 0 \quad \forall \quad w\in\mathcal{W}$$

Proof: Since for every value of w $A(w)$ has negative eigenvalues, it is possible to compute the positive–definite solution $P(w)$ of the Lyapunov equation:

$$A^T(w)P(w) + P(w)A(w) = -I. \tag{1}$$

and every $P(w)$ admits a factorization

$$P(w) = S^T(w)S(w), \tag{2}$$

where $S(w)$ is an upper triangular matrix (Cholesky's decomposition). Then

$$\overbrace{S(w)A(w)S^{-1}(w)}^{\tilde{A}(w)} + \overbrace{S^{-T}(w)A(w)^TS^T(w)}^{\tilde{A}(w)^T} =$$
$$S^{-T}(w)\left(S^T(w)S(w)A(w) + A(w)^TS^T(w)S(w)\right)S^{-1}(w) = -S^{-T}(w)S^{-1}(w) < 0$$

■

Remark 3.2: The above result, which is at the basis of the development of the next sections and was first presented in [1], allows to determine, given a family of asymptotically stable transfer matrices, same dimensional equivalent realizations which share the **same** quadratic Lyapunov function, $V(x) = x^T x$. In simpler words it states that for any given a system $\dot{x} = A((w))x$, which is asymptotically stable for every constant $w \in \mathcal{W}$,there exist a transformation $S(w)$ such that the **LPV** system

$$\dot{x} = \tilde{A}(w)x$$

is asymptotically stable.

In the next section the solution of the gain scheduling problem will also pass through the determination of equivalent realizations of different dimensions, whose definition is reported next.

Definition 3.3: We say that two realizations (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) of $P(s)$ are equivalent if

$$C_1 (sI - A_1)^{-1} B_1 + D_1 = C_2 (sI - A_2)^{-1} B_2 + D_2$$

and

- (A_1, B_1) and (A_2, B_2) are stabilizable;
- (C_1, A_1) and (C_2, A_2) are detectable.

(i.e., the unobservable/uncontrollable subspaces of each realization are asymptotically stable)

Definition 3.4: Given a $(p_1 + p_2) \times (m_1 + m_2)$ transfer matrix

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \tag{3}$$

and the $m_2 \times p_2$ and $m_1 \times p_1$ transfer matrices

$$\Delta_l, \qquad \Delta_u$$

consider the connection as in figure 2 corresponding to the dynamic systems

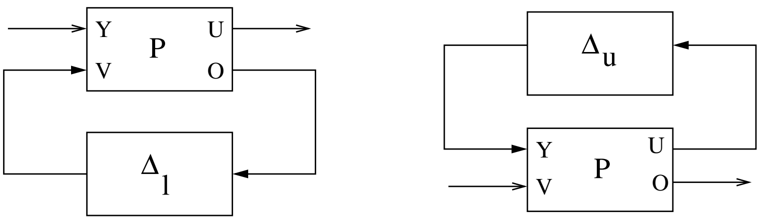


Fig. 2. Lower (left) and upper (right) linear fractional transformations (LFT)

$$\begin{aligned} \begin{bmatrix} U(s) \\ O(s) \end{bmatrix} &= \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} Y(s) \\ V(s) \end{bmatrix} \\ V(s) &= \Delta_l O(s) \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} U(s) \\ O(s) \end{bmatrix} &= \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} Y(s) \\ V(s) \end{bmatrix} \\ Y(s) &= \Delta_u U(s) \end{aligned}$$

the lower and upper linear fractional transformations (**LFT**) are defined as

$$\begin{aligned} \mathcal{F}_l(P, \Delta_l) &= P_{11} + P_{12}\Delta_l (I - P_{22}\Delta_l)^{-1} P_{21} \\ \mathcal{F}_u(P, \Delta_u) &= P_{22} + P_{21}\Delta_u (I - P_{11}\Delta_u)^{-1} P_{12} \end{aligned}$$

Lemma 3.3: Let $G = \mathcal{F}_l(P, K)$. Then, if P and G are proper, $\det P(\infty) \neq 0$, $\det\left(P - \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix}\right) \neq 0$, P_{12} and P_{21} square and invertible, K is proper and

$$K = \mathcal{F}_u(P^{-1}, G)$$

IV. GAIN SCHEDULING CONTROLLER FOR SISO STABLE LPV PLANTS

In this section we will consider the gain scheduling problem for SISO stable **LPV** plants. We start from a couple of motivating examples and subsequently we will provide a solution based on the Youla parametrization for stable SISO plants.

Example 4.1: Consider the simple fluid plant represented in Fig. 3. Its (stable) first-order model is

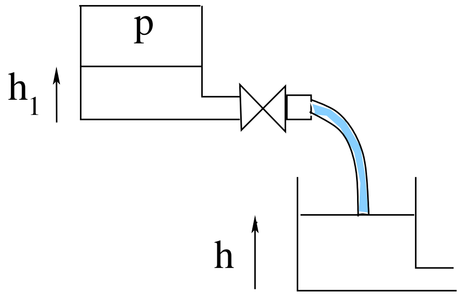


Fig. 3. Simple fluid–flow plant.

$$\dot{y}(t) = -\alpha\, y(t) + w(t)\, u(t), \;\; \alpha > 0 \tag{4}$$

where $y(t) = h(t) - \bar{h}$ is the deviation of the actual level $h(t)$ in the reservoir from the equilibrium level $\bar{h}$, $u(t)$ is the valve opening, $p(t)$ is the (measured) pressure in the tank, $h_1(t)$ is the level of the fluid in the tank and $w(t) = \rho\sqrt{p(t)/\gamma + h_1(t)}$, with γ denoting the fluid density and with ρ constant. Assume that bounds of the form $0 < w^- \leq w(t) \leq w^+$, $\forall t$ are given and that the controller transfer function is

$$\frac{\kappa(w)}{s+\beta}, \;\; \beta > 0, \tag{5}$$

with

$$\kappa(w) = -\frac{\kappa_0}{w}, \;\; \kappa_0 > 0\,, \tag{6}$$

to compensate for pressure variations.
When w is constant, the feedback control system is internally stable with characteristic polynomial:

$$d(s) = s^2 + (\beta + \alpha)s + \alpha\beta + \kappa_0. \tag{7}$$

The controller (5) admits the two realizations:

$$\Sigma_1(w) = \left[\begin{array}{c|c} -\beta & 1 \\ \hline -\kappa_0/w & 0 \end{array}\right], \; \Sigma_2(w) = \left[\begin{array}{c|c} -\beta & -\kappa_0/w \\ \hline 1 & 0 \end{array}\right], \tag{8}$$

yielding, respectively, the closed–loop system matrices:

$$A_1(w) = \begin{bmatrix} -\alpha & -\kappa_0 \\ 1 & -\beta \end{bmatrix}, \; A_2(w) = \begin{bmatrix} -\alpha & w \\ -\kappa_0/w & -\beta \end{bmatrix}. \tag{9}$$

$\Sigma_1(w)$ leads to an asymptotically stable feedback system independently of how w varies with time, whereas $\Sigma_2(w)$ may not. Indeed, if α and β are small enough and w varies over a sufficiently large range, the system may be unstable [6].

Example 4.2: Let $P(s)$ in Fig. 4 be

$$P(s) = \frac{1}{s+\mu}. \tag{10}$$

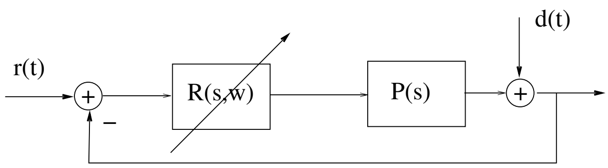


Fig. 4. LTI plant with LPV compensator.

To (partly) suppress a disturbance $d(t)$ with a (dominant) sinusoidal component of frequency ω_0, the controller transfer function can be chosen as

$$R(s) = \frac{\kappa\omega^2(s+\alpha)}{s^2 + 2\xi\omega s + \omega^2} \tag{11}$$

with $\omega = \omega_0$ and ξ small.
If the disturbance frequency varies in a range $\omega^- \leq \omega_0 \leq \omega^+$ (as is often the case in practice), the parameter ω in (11) must be adjusted accordingly (which, of course, requires measuring the disturbance frequency) and the control system becomes an LPV system whose stability depends on the controller realization. By realizing the controller in companion form, the state, input and output matrices of the resulting closed–loop system turn out to be:

$$\left[\begin{array}{c|c} A_{CL} & B_{CL} \\ \hline C_{CL} & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -\mu & \kappa\omega^2\alpha & \kappa\omega^2 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -\omega^2 & -2\xi\omega & 1 \\ \hline 1 & 0 & 0 & 0 \end{array}\right]. \tag{12}$$

Assume for simplicity $\alpha = \mu$ in (11) to cancel the pole of (10). Then the realization is nonminimal and, indeed, equivalent to

$$\left[\begin{array}{c|c} A_{CL} & B_{CL} \\ \hline C_{CL} & 0 \end{array}\right] = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -(\kappa+1)\omega^2 & -2\xi\omega & 1 \\ \hline \kappa\omega^2 & 1 & 0 \end{array}\right]. \tag{13}$$

It is known that this system is unstable for ξ small and w varying with time.

From the above examples it is apparent that the closed-loop stability does not depend on the controller only, but on its realization too.

Let us formulate the problem and its solution in the general setting.

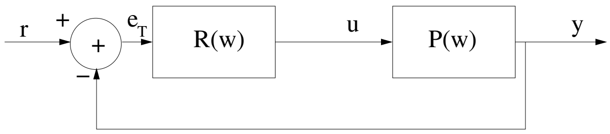


Fig. 5. SISO feedback control loop

Theorem 4.1: Given a SISO **LPV** family of stable plants $P(s,w) = (A(w), B(w), C(w))$ and a gain scheduling controller $R(s,w) = (F(w), G(w), H(w), K(w))$ such that the closed-loop system is stable for any fixed value of the parameter $w \in \mathcal{W}$, there exists an equivalent gain scheduling controller $\tilde{R}(s,w) = (\tilde{F}(w), \tilde{G}(w), \tilde{H}(w), \tilde{K}(w))$ such that the resulting **LPV** closed-loop system is asymptotically stable.

Proof: Since the closed loop system is stable for any constant value of w, it is possible to rewrite the gain scheduling controller as

$$R(s,w) = \frac{Q(s,w)}{1 - Q(s,w)P(s,w)} \tag{14}$$

Solving the above in $Q(s,w)$ yields

$$Q(s,w) = \frac{R(s,w)}{1 + R(s,w)P(s,w)}$$

(the above is the control sensitivity function, say the transfer function from r to u in figure 5) whose state space realization is

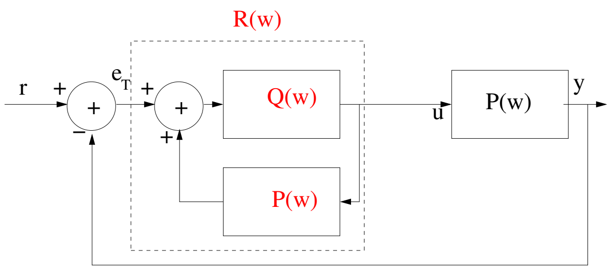


Fig. 6. Controller parametrization

$$Q(s,w): \left\{ \begin{array}{rcl} \dot{z} & = & \overbrace{\begin{bmatrix} A(w) - B(w)K(w)C(w) & B(w)H(w) \\ -G(w)C(w) & F(w) \end{bmatrix}}^{A_{CL}(w)} z(t) + \overbrace{\begin{bmatrix} B(w)K(w) \\ G(w) \end{bmatrix}}^{B_{CL}(w)} r(t) \\ u(t) & = & \underbrace{\begin{bmatrix} -K(w)C(w) & H(w) \end{bmatrix}}_{C_{CL}(w)} z(t) + \underbrace{K(w)}_{D_{CL}(w)} r(t) \end{array} \right.$$

Since the family $A_{CL}(w)$ is asymptotically stable for every value of w there exists a family of transformations $S(w)$ such that the **LPV** system

$$\begin{array}{rcl} \dot{z} & = & \overbrace{S(w)A_{CL(w)}S^{-1}(w)}^{\tilde{A}_{CL}(w)}\, z + \overbrace{S(w)B_{CL}(w)}^{\tilde{B}_{CL}(w)}\, r(t) \\ u & = & \underbrace{C_{CL}(w)S^{-1}(w)}_{\tilde{C}_{CL}(w)}\, z + \underbrace{D_{CL}(w)}_{\tilde{D}_{CL}(w)}\, r(t) \end{array}$$

is asymptotically stable for every trajectory of $w \in \mathcal{W}$. Let $\tilde{Q}(s,w) = (\tilde{A}_{CL}, \tilde{B}_{CL}, \tilde{C}_{CL}, \tilde{D}_{CL})$.

Clearly, as in (14), for every w the controller

$$\tilde{R}(w) = \frac{\tilde{Q}(w)}{1 - \tilde{Q}(w)P(w)}$$

is equivalent to $R(w)$. Its state space realization (w dropped for clarity) is

$$\tilde{R}: \left\{ \begin{array}{rcl} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} & = & \begin{bmatrix} \tilde{A}_{CL} & \tilde{B}_{CL}C \\ B\tilde{C}_{CL} & A + B\tilde{D}_{CL}C \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} \tilde{B}_{CL} \\ B\tilde{D}_{CL} \end{bmatrix} e_T \\ u & = & \begin{bmatrix} \tilde{C}_{CL} & \tilde{D}_{CL}C \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \tilde{D}_{CL} e_T \end{array} \right.$$

To check that the so derived realization results in closed-loop stability for the **LPV** system it is sufficient to rewrite the closed-loop system dynamics:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \tilde{A}_{CL} & \tilde{B}_{CL}C & -\tilde{B}_{CL}C \\ B\tilde{C}_{CL} & (A + B\tilde{D}_{CL}C) & -B\tilde{D}_{CL}C \\ B\tilde{C}_{CL} & B\tilde{D}_{CL}C & A - B\tilde{D}_{CL}C \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ x \end{bmatrix} + \begin{bmatrix} \tilde{B}_{CL} \\ B\tilde{D}_{CL} \\ B\tilde{D}_{CL} \end{bmatrix} r$$

$$y = \begin{bmatrix} 0 & 0 & C \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ x \end{bmatrix}$$

with the new state variables

$$z_1 = x \quad z_2 = q_1 \quad z_3 = q_2 - x$$

which yields (w back in place for final show)

$$\begin{array}{rcl} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} & = & \begin{bmatrix} A(w) & B(w)\tilde{C}_{CL}(w) & B(w)\tilde{D}_{CL}(w)C(w) \\ 0 & \tilde{A}_{CL}(w) & \tilde{B}_{CL}(w)C(w) \\ 0 & 0 & A(w) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} B(w)\tilde{D}_{CL}(w) \\ \tilde{B}_{CL}(w) \\ 0 \end{bmatrix} r(k) \\ y & = & \begin{bmatrix} C(w) & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \end{array}$$

■

In the general case, when dealing with MIMO or **LPV** unstable systems, a complete parametrization of the controllers is needed. To this aim, the next section will briefly present the Youla state parametrization which will be subsequently used to solve the gain scheduling problem in the general case.

V. STATE SPACE YOULA PARAMETRIZATION

Theorem 5.1: Given the MIMO plant $(A(w), B(w), C(w))$ let $J(w)$ and $L(w)$ be such that $A(w)+B(w)J(w)$ and $A(w)+L(w)C(w)$ are asymptotically stable. Then, for every transfer matrix (Youla parameter) $T(s,w)$ with no poles in the closed right half plane the y-to-u transfer function obtained from

$$\left\{\begin{array}{rcl} \dot{\hat{x}} & = & (A(w)+B(w)J(w)+L(w)C(w))\hat{x} - L(w)y + B(w)v \\ u & = & J(w)\hat{x} + v \\ o & = & C(w)\hat{x} - y \end{array}\right.$$
$$V(s) = T(s,w)O(s)$$

or, equivalently, defining

$$Q_0(s,w) = \left[\begin{array}{c|cc} A(w)+B(w)J(w)+L(w)C(w) & -L(w) & B(w) \\ \hline J(w) & 0 & I \\ C(w) & -I & 0 \end{array}\right] = \left[\begin{array}{cc} Q_0^{11}(w) & Q_0^{12}(w) \\ Q_0^{21}(w) & Q_0^{22}(w) \end{array}\right]$$
$$R(s,w) = \mathcal{F}_l(Q_0(w,s), T(w,s))$$

is a (positive feedback stabilizing) controller $R(w,s)$. Such controller can be written as a lower LFT

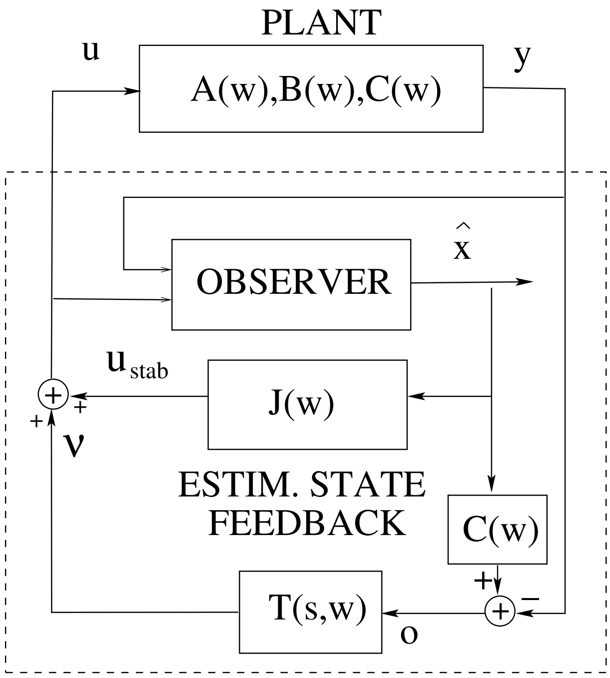


Fig. 7. Stabilizing controller

Remark 5.1: Remember that for every choice of $T(w,s)$ there is obviously an extra degree of freedom in the choice of its state representation $(F(w), G(w), H(w), K(w))$

Theorem 5.2: Given the plant $P(s,w) = (A(w), B(w), C(w))$, let $L(w)$ and $J(w)$ be such that $A(w)+B(w)J(w)$ and $A(w)+L(w)C(w)$ are asymptotically stable. Then, for any (positive feedback) stabilizing controller

$$R(s,w) = (F(w), G(w), H(w), K(w))$$

there exists an asymptotically stable Youla parameter $T(s,w)$ such that

$$R(s,w) = \mathcal{F}_l\left(Q_0(s,w), T(s,w)\right)$$

The Youla parameter can be computed as the upper LFT

$$T(s,w) = \mathcal{F}_u\left(Q_0^{-1}(s,w), R(s,w)\right)$$

and its state space representation is

$$\begin{array}{rcl} \left[\begin{array}{c} \dot{q}_1 \\ \dot{q}_2 \end{array}\right] & = & \left[\begin{array}{cc} A(w)+B(w)K(w)C(w) & -B(w)H(w) \\ -G(w)C(w) & F(w) \end{array}\right]\left[\begin{array}{c} q_1 \\ q_2 \end{array}\right] + \left[\begin{array}{c} B(w)K(w)-L(w) \\ -G(w) \end{array}\right] o \\ v & = & \left[\begin{array}{cc} J(w)-K(w)C(w) & H(w) \end{array}\right]\left[\begin{array}{c} q_1 \\ q_2 \end{array}\right] - K(w)o \end{array} \quad (15)$$

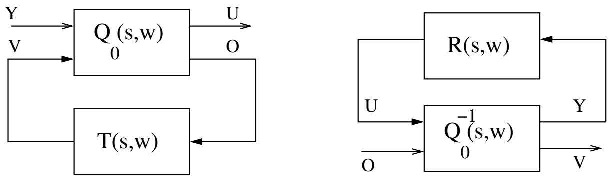


Fig. 8. Lower (left) and upper (right) linear fractional transformations (LFT)

Proof: Recall that the inverse of a transfer function matrix with state space realization $O(s) = (A_o, B_o, C_o, D_o)$ is

$$O^{-1}(s) : (A_o - B_o D_o^{-1} C_o, -B_o D_o^{-1}, D_o^{-1} C_o, D_o^{-1})$$

Since for the transfer function $Q_0(s,w)$ the above matrices (w dropped for clarity) are

$$A_o = A + BJ + LC \quad B_o = \begin{bmatrix} -L & B \end{bmatrix}$$
$$C_o = \begin{bmatrix} J \\ C \end{bmatrix} \qquad D_o = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

the matrices needed for the computation of the inverse are

$$D_o^{-1} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$
$$A_o - B_o D_o^{-1} C_o, -B_o D_o^{-1} = A + BJ + LC - \begin{bmatrix} -L & B \end{bmatrix} \begin{bmatrix} J \\ C \end{bmatrix} = A$$
$$-B_o D_o^{-1} = \begin{bmatrix} -B & -L \end{bmatrix}$$
$$D_o^{-1} C_o = \begin{bmatrix} -C \\ J \end{bmatrix}$$

and thus the state space realization of $Q_0^{-1}(s)$ is

$$Q_0^{-1}(s,w) : \quad \left(A, \begin{bmatrix} -B & -L \end{bmatrix}, \begin{bmatrix} -C \\ J \end{bmatrix}, \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \right)$$

The upper LFT of $Q_0(s,w)$ and $R(s,w)$ state representation is thus

$$\begin{array}{rcl} \dot{q}_1 & = & A(w)q_1 - B(w)u - L(w)o \\ y & = & -C(w)q_1 - o \\ v & = & J(w)q_1 + u \\ \dot{q}_2 & = & F(w)q_2 + G(w)y \\ u & = & H(w)q_2 + K(w)y \end{array}$$

which, by proper rearrangement, results in the state representation (15). Finally, to show that such state space system is asymptotically stable it is sufficient to check that

$$\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} A(w) + B(w)K(w)C(w) & -B(w)H(w) \\ -G(w)C(w) & F(w) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} = \begin{bmatrix} A(w) + B(w)K(w)C(w) & B(w)H(w) \\ G(w)C(w) & F(w) \end{bmatrix}$$

which is the closed-loop system matrix with the adopted stabilizing controller. ■

Remark 5.2: Note that, for a given $T(s,w)$ of order n_T, the order of the controller obtained via Youla parametrization is $n_T + n_P$, being n_P the order of the plant. Viceversa, for a given controller of minimal form of order n_K, the corresponding Youla parameter has dimension $n_K + n_P$. This in turn implies that if one goes from the controller to the Youla parameter and then back to it via Youla parameterization, the so obtained controller has order equal to $2n_P + n_K$.

VI. Gain scheduling controllers, general case

In the general case, when dealing with MIMO/unstable **LPV** plants, the first problem one has to deal with is that of checking whether there is a gain scheduling controller and the second is that, given a gain scheduling controller, to rewrite it so that the overall **LPV** systems is asymptotically stable, as formalized next.

A. Problem setting

Consider the linear parameter varying n-states, m-inputs and p-outputs plant

$$\begin{array}{rcl} \dot{x}(t) & = & A(w)\,x(t) + B(w)\,u(t), \\ y(t) & = & C(w)\,x(t), \end{array} \tag{16}$$

and $w \in \mathcal{W}$, with $\mathcal{W}$ compact. For this system we consider two problems.

Problem 1: **(Stabilizability)** Design a gain scheduling controller

$$\begin{array}{rcl} \dot{z}(t) & = & F(w)\,z(t) + G(w)\,y(t), \\ \xi(t) & = & H(w)\,z(t) + K(w)\,y(t), \end{array} \tag{17}$$

such that **closed-loop stability is preserved for any admissible trajectory** $w(t) \in \mathcal{W}$

Problem 2: **(Parametrization)** Given a gain scheduling controller

$$\begin{array}{rcl} \dot{z}(t) & = & F(w)\,z(t) + G(w)\,y(t), \\ \xi(t) & = & H(w)\,z(t) + K(w)\,y(t), \end{array}$$

find an equivalent realization

$$\begin{array}{rcl} \dot{z}(t) & = & \tilde{F}(w)\,z(t) + \tilde{G}(w)\,y(t), \\ \xi(t) & = & \tilde{H}(w)\,z(t) + \tilde{K}(w)\,y(t), \end{array}$$

such that

- $H(w)\,(sI - F(w))^{-1}\,G(w) + K(w) = \tilde{H}(w)\left(sI - \tilde{F}(w)\right)^{-1}\tilde{G}(w) + \tilde{K}(w)$
  for every constant value $w \in \mathcal{W}$
- closed-loop stability is preserved for any admissible trajectory $w \in \mathcal{W}$.

B. Stabilizability conditions

To check the closed-loop system beahviour, different notions of stability can be adopted. In the following we will mainly focus on quadratic stabilizability conditions (i.e., which can be checked by means of quadratic Lyapunov functions) and on polyhedral stabilizability.

Theorem 6.1: The LPV system (16) is **quadratically stabilizable** by means of a compensator of the form (17) if and only if there exist two positive–definite matrices P and Q, and two matrices $U(w)$ and $Y(w)$ dependent on parameter w such that

$$PA(w)^T + A(w)P + B(w)U(w) + U(w)^T B(w)^T < 0\,, \tag{18}$$

$$A(w)^T Q + QA(w) + Y(w)C(w) + C^T(w)Y(w)^T < 0\,, \tag{19}$$

Proof: If (16) is quadratically stabilized by (17), then by definition there exists a symmetric positive definite matrix P such that $\left(A(w)^{cl}\right) P + P\left(A(w)^{cl}\right)^T < 0$. After a proper partition, we write

$$\begin{bmatrix} A(w)+B(w)K(w)C(w) & B(w)H(w) \\ G(w)C(w) & F(w) \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \begin{bmatrix} A(w)+B(w)K(w)C(w) & B(w)H(w) \\ G(w)C(w) & F(w) \end{bmatrix}^T < 0$$

If we take the left upper block of the previous expression, denoting by $U(w) \doteq K(w)C(w)P_{11} + H(w)P_{12}^T$, we have

$$\begin{aligned} & (A(w)+B(w)K(w)C(w))P_{11} + (B(w)H(w))P_{12}^T + P_{11}(A(w)+B(w)K(w)C(w))^T + P_{12}(B(w)H(w))^T = \\ = \quad & A(w)P_{11} + P_{11}A(w)^T + B(w)U(w) + U(w)^T B(w) < 0 \end{aligned}$$

which is eq. (18). The dual equation (19) is derived exactly in the same way.

To show that the two conditions imply quadratic stabilizability, assume that (18) and (19) hold and take the linear gains $J(w) = U(w)P^{-1}$ and $L(w) = Q^{-1}Y(w)$. Consider now the observer–based feedback compensator

$$\begin{aligned} \dot{\hat{x}}(t) &= (A(w)+L(w)C(w)+B(w)J(w))\hat{x}(t) - L(w)y(t) + B(w)v(t) \\ u(t) &= J(w)\hat{x}(t) + v(t) \end{aligned} \tag{20}$$

and apply the change of variables $x_1(t) = x(t)$ and $x_2(t) = \hat{x}(t) - x(t)$ to achieve the closed loop system matrix

$$\begin{bmatrix} A(w)+B(w)J(w) & B(w)J(w) \\ 0 & A(w)+L(w)C(w) \end{bmatrix} \tag{21}$$

Both diagonal blocks are quadratically stable with Lyapunov matrices given by P and Q, respectively, since

$$\begin{aligned} P(A(w)+B(w)J(w))^T + (A(w)+B(w)J(w))P = PA(w)^T + U(w)^T B(w)^T + A(w)P + B(w)U(w) &< 0 \\ (A(w)+L(w)C(w))^T Q + Q(A(w)+L(w)C(w)) = A(w)^T Q + C(w)^T Y(w)^T + QA(w) + Y(w)C(w) &< 0 \end{aligned}$$

Since the system is in upper triangular form with **LPV** diagonal blocks, the overall systems is **LPV** stable. ■

If less conservative conditions are required, one can resort to polyhedral Lyapunov functions, as per the next result.

Theorem 6.2: The LPV system (16) is **polyhedrally stabilizable** by means of a compensator of the form (17) if and only if there exist a full row rank $n \times \mu$ matrix X, a full column rank $\nu \times n$ matrix R, as well as an $m \times \mu$ matrix $U(w)$, a $\nu \times p$ matrix $L(w)$ and $\mathcal{M}$ matrices $P(w)$ and $Q(w)$ dependent on w such that

$$A(w)X + B(w)U(w) = XP(w) \qquad \bar{1}^T P(w) \le -\beta\bar{1}^T, \tag{22}$$

$$RA(w) + L(w)C(w) = Q(w)R \qquad Q(w)\bar{1} \le -\beta\bar{1} \tag{23}$$

Proof: If a closed-loop polyhedral function exists then

$$\begin{bmatrix} A(w)+B(w)K(w)C(w) & B(w)H(w) \\ G(w)C(w) & F(w) \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix} = \begin{bmatrix} X \\ Z \end{bmatrix} P(w) \qquad \bar{1}^T P(w) \le -\beta\bar{1}^T \tag{24}$$

whose upper block row yields

$$A(w)X + B(w)K(w)C(w)X + B(w)H(w)Z = A(w)X + B(w)U(w) = XP(w), \tag{25}$$

where $U(w) = K(w)C(w)X + H(w)Z$, which proves (22). Condition (23) can be proven in a similar way by duality.

To show that (22) and (23) imply polyhedral stabilizability, consider the compensator of order $\nu + \mu - n$ described by

$$\begin{aligned} \dot{r}(t) &= Q(w)r(t) - L(w)y(t) + RB(w)u(t), && (26)\\ \hat{x}(t) &= Mr(t), && (27)\\ \dot{z}(t) &= F_{SF}(w)z(t) + G_{SF}(w)\hat{x}(t), && (28)\\ u(t) &= H_{SF}(w)z(t) + K_{SF}(w)\hat{x}(t) + v(t), && (29)\\ v(t) &= 0, && (30) \end{aligned}$$

where M is any left inverse of R, i.e.,

$$MR = I, \tag{31}$$

and $F_{SF}(w), G_{SF}(w), H_{SF}(w), K_{SF}(w)$ can be computed from

$$\begin{bmatrix} K_{SF}(w) & H_{SF}(w) \\ G_{SF}(w) & F_{SF}(w) \end{bmatrix} = \begin{bmatrix} U(w) \\ V(w) \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix}^{-1}, \tag{32}$$

where Z is any complement of X that makes the square matrix $\begin{bmatrix} X \\ Z \end{bmatrix}$ invertible and

$$V(w) = ZP(w). \tag{33}$$

Letting

$$s(t) \doteq R\, x(t) - r(t) \tag{34}$$

and choosing $[x^T \; z^T \; s^T]^T$ as the state vector, after simple manipulations the following closed–loop matrix is obtained:

$$\left[\begin{array}{cc|c} A(w) + B(w)K_{SF}(w) & B(w)H_{SF}(w) & -B(w)K_{SF}(w)M \\ G_{SF}(w) & F_{SF}(w) & -G_{SF}(w)M \\ \hline 0 & 0 & Q(w) \end{array}\right]. \tag{35}$$

The above system is in diagonal form and both diagonal blocks are **LPV** stable (see [3]). ∎

C. Gain scheduling parametrization: quadratic case

In this section we will show that if the **LPV** system $(A(w), B(w), C(w), D(w))$ is quadratically stabilizable, then every gain scheduling controller $(F(w), G(w), H(w), K(w))$ which is such that the closed loop system is asymptotically stable for every **frozen** value of w can be reparametrized by means of the Youla results as to obatain an **LPV** stable closed–loop system.

Theorem 6.3: If the stabilizability conditions in theorem 6.1 hold then for any w-frozen gain scheduling stabilizing controller $R(s,w) = (F(w), G(w), H(w), K(w))$ there exists an equivalent realization $\tilde{R}(s,w) = (\tilde{F}(w), \tilde{G}(w), \tilde{H}(w), \tilde{K}(w))$ such that the closed-loop **LPV** system is asymptotically stable.

Proof: From the results in the previous sections it is known that for any stabilizing controller the corresponding Youla parameter $T(s,w)$ can be computed as

$$T(s,w) = \mathcal{F}_u\left(Q_0(s,w)^{-1}, R(s,w)\right)$$

Its asymptotically state space realization is

$$\begin{array}{rcl} \left[\begin{array}{c} \dot{q}_1 \\ \dot{q}_2 \end{array}\right] & = & \overbrace{\left[\begin{array}{cc} A(w)+B(w)K(w)C(w) & -B(w)H(w) \\ -G(w)C(w) & F(w) \end{array}\right]}^{A_{You}(w)} \left[\begin{array}{c} q_1 \\ q_2 \end{array}\right] + \overbrace{\left[\begin{array}{c} B(w)K(w)-L(w) \\ -G(w) \end{array}\right]}^{B_{You}(w)} o \\ v & = & \underbrace{\left[\begin{array}{cc} J(w)-K(w)C(w) & H(w) \end{array}\right]}_{C_{You}(w)} \left[\begin{array}{c} q_1 \\ q_2 \end{array}\right] \underbrace{-K(w)}_{D_{You}(w)} o \end{array}$$

Since the family of transfer matrices $T(s,w)$ is stable for every w, there exists a family of transformation matrices $S(w)$ such that $\tilde{T}(s,w) = (\tilde{A}_{You}(w), \tilde{B}_{You}(w), \tilde{C}_{You}(w).\tilde{D}_{You}(w)) = (S(w)A_{You}(w)S^{-1}(w).S(w)B_{You}(w), C_{You}(w)S(w)^{-1}, D_{You}(w))$ is **LPV** stable. The stabilizing controller $\tilde{R}(s,w) = \mathcal{F}_l(Q_0(s,w), \tilde{T}(s,w))$ is equivalent to the original one. Its state space realization, obtained as the LFT of the state estimator and the Youla parameter, is

$$\begin{array}{rcl} \dot{\hat{x}} & = & (A(w)+B(w)J(w)+L(w)C(w))\,\hat{x} - L(w)y + B(w)v \\ o & = & C(w)\tilde{x} - y \\ \dot{x}_T & = & \tilde{A}_{You}(w)x_T + \tilde{B}_{You}(w)o \\ v & = & \tilde{C}_{You}(w)x_T + \tilde{D}_{You}(w)o \\ u & = & J\hat{x} + v \end{array}$$

To obtain the resulting closed loop system dynamics it is sufficient to add to the previous equations the state update one

$$\begin{array}{rcl} \dot{x} & = & A(w)x + B(w)u \\ y & = & C(w)x \end{array}$$

To verify the **LPV** stability it is then sufficient to rewrite the above with the change of variables

$$x_1 = x, \qquad x_2 = x_T, \qquad x_3 = x - \hat{x}$$

so as to obtain

$$\left[\begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{array}\right] = \left[\begin{array}{ccc} A(w)+B(w)J(w) & B(w)\tilde{C}_{You}(w) & -B(w)(J(w)+\tilde{D}_{You}(w)C(w)) \\ 0 & \tilde{A}_{You}(w) & \tilde{B}_{You}(w)C(w) \\ 0 & 0 & A(w)+L(w)C(w) \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right]$$

which is in upper triangular form with **LPV** asymptotically stable diagonal blocks. ■

D. Gain scheduling parametrization: polyhedral case

The polyhedral case is skipped here for brevity. The only difference with respect to the quadratic case is that the estimation dynamics are not given by $A(w) + L(w)C(w)$ but by the matrix $Q(w)$ and the state feedback dynamics $A(w) + B(w)J(w)$ have to be replaced by the dynamic observer as seen in the "Stabilizability conditions" section (see equations (21) and (35))

VII. Summary: gain scheduling controller realization, quadratic case

The realization of the gain-scheduling controller can be done as follows. Let $\mathcal{P}(w) = (A(w), B(w), C)(w)$ and let $R(s, w) = (F(w), G(w), H(w), K(w))$ be a positive feedback stabilizing controllers.

Step 1 Find matrices $J(w)$ and $L(w)$ such that $A(w) + B(w)J(w)$ and $A(w) + L(w)C(w)$ are asymptotically stable (hard if all the involved matrices are dependent on w), compute the observer-based controller $Q_0(s, w)$.

Step 2 Compute the Youla parameter corresponding to $R(s, w)$, $T(s, w) = \mathcal{F}_u(Q_0^{-1}, R(s, w))$.

Step 3 Compute an **LPV** stable realization of $Q(s, w)$: $(\tilde{A}_{You}(w), \tilde{B}_{You}(w), \tilde{C}_{You}(w), \tilde{D}_{You}(w))$

The (positive feedback) gain scheduling controller realization which guarantees **LPV** stability is given by

$$\begin{aligned}
\begin{bmatrix} \dot{\hat{x}} \\ \dot{x}_T \end{bmatrix} &= \begin{bmatrix} A(W) + B(w)J(w) + L(w)C(w) + B(w)\tilde{D}_{You}(w)C(w) & B(w)\tilde{C}_{You}(w) \\ \tilde{B}_{You}(w)C(w) & \tilde{A}_{You}(w) \end{bmatrix} \begin{bmatrix} \hat{x} \\ x_t \end{bmatrix} + \begin{bmatrix} L(w) + B(w)\tilde{D}_{You}(w) \\ \tilde{B}_{You}(w) \end{bmatrix} \\
u &= \begin{bmatrix} J(w) + \tilde{D}_{You}(w)C(w) & \tilde{C}_{You}(w) \end{bmatrix} \begin{bmatrix} \hat{x} \\ x_T \end{bmatrix} - \tilde{D}_{You}(w)y
\end{aligned}$$

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