Stability of Model Predictive Control

Riccardo Scattolini
Dipartimento di Elettronica e Informazione
Consider the system

\[ x(k + 1) = f(x(k), u(k)) \]

where \( f \) is continuously differentiable with respect to its arguments and \( f(0,0)=0 \). The state and control variables must satisfy the following constraints

\[ x \in X, \quad u \in U \]

where \( X \) and \( U \) contain the origin as an interior point.

**The problem is to design MPC algorithms guaranteeing that the origin of the closed-loop system is an asymptotically stable equilibrium point.**
The auxiliary control law

Assume to know an auxiliary control law

\[ u = \kappa_a(x) \]

and a positively invariant set \( X_f \subset X \) containing the origin such that, for the closed-loop system

\[ x(k + 1) = f(x(k), \kappa_a(x(k))) \]

and for any \( x(\bar{k}) \in X_f \) one has

\[ x(k) \in X_f \quad , \quad k \geq \bar{k} \]
\[ u(k) = \kappa_a(x(k)) \in U \quad , \quad k \geq \bar{k} \]
MPC problem: at any time $k$ find the sequence

$$u(k), u(k + 1), ..., u(k + N - 1)$$

minimizing the cost function ($Q > 0$, $R > 0$)

$$J(x(k), u(\cdot), N) = \sum_{i=0}^{N-1} \left( \|x(k + i)\|_Q^2 + \|u(k + i)\|_R^2 \right) + V_f(x(k + N))$$

subject to

$$x(k + i) \in X, \quad u(k + i) \in U$$

$$x(k + N) \in X_f$$

The RH solution implicitly defines the MPC time-invariant control law

$$u = \kappa_{RH}(x)$$
Theorem

Let $X^{RH}(N)$ be the set of states where a solution of the optimization problem exists.

If, for any $x \in X_f$ the condition

$$V_f \left( f(x(k), \kappa_a(x(k))) \right) - V_f(x(k)) + \left( \|x(k)\|_Q^2 + \|\kappa_a(x(k))\|_R^2 \right) \leq 0$$

is fulfilled and

$$V_f(x) \leq \alpha_f(\|x\|)$$

where $\alpha_f(\|x\|)$ is a class $K$ function, then the origin of the closed-loop system with the MPC-RH control law is an asymptotically stable equilibrium point with region of attraction $X^{RH}(N)$. Moreover, if $\alpha_f(\|x\|) = b \|x\|^2$ and $X_f = X^{RH}(N)$ then the origin is exponentially stable in $X^{RH}(N)$. 
**Proof**

Let \( x(k) \in X^{RH}(N) \) and \( U^o(k, N) = \begin{bmatrix} u^o_k(k) & u^o_k(k + 1) & \cdots & u^o_k(k + N - 2) & u^o_k(k + N - 1) \end{bmatrix} \)

be the optimal solution at \( k \) with prediction horizon \( N \). Then, at time \( k+1 \)

\[ \tilde{U}(k + 1, N) = [u^o_k(k + 1) \cdots u^o_k(k + N - 2)u^o_k(k + N - 1) \kappa_a(x(k + N))] \]

is a feasible solution, so that \( x(k + 1) \in X^{RH}(N) \)

Moreover,

\[ V(x, N) := J(x, \kappa_{RH}(x), N) \geq \|x\|_Q^2 \]

so that the condition \( V(x, N) \geq w(\|x\|) \) is verified in \( X^{RH}(N) \)
At time $k$, the sequence

$$\tilde{U}(k, N + 1) = [U^o(k, N), \kappa_a(x(k + N))]$$

is feasible for the MPC problem with horizon $N+1$ and

$$J(x, \tilde{U}(k, N + 1), N + 1) = V(x, N) - V_f(x(k + N)) + V_f(x(k + N + 1)) + \|x(k + N)\|^2_Q + \|\kappa_a(x(k + N))\|^2_R \leq V(x, N)$$

so that we have the monotonicity property (with respect to $N$)

$$V(x, N + 1) \leq V(x, N), \forall x \in X^{RH}(N)$$

with $V(x, 0) = V_f(x), \forall x \in X_f$

Then $V(x, N + 1) \leq V(x, N) \leq V_f(x) \leq \alpha_f \|x(k)\|, \forall x \in X_f$

and the condition $V(x, N) \leq \psi(\|x\|), \forall x \in X_f$ is satisfied.
Finally

\[ V(x, N) = \|x\|^2_Q + \|\kappa_{RH}(x)\|^2_R + J(f(x, \kappa_{RH}(x)), u^o(k + 1, N - 1), N - 1) \]
\[ = \|x\|^2_Q + \|\kappa_{RH}(x)\|^2_R + V(f(x, \kappa_{RH}(x)), N - 1) \]
\[ \geq \|x\|^2_Q + \|\kappa_{RH}(x)\|^2_R + V(f(x, \kappa_{RH}(x)), N) \]
\[ \geq \|x\|^2_Q + V(f(x, \kappa_{RH}(x)), N), \quad \forall x \in X^{RH}(N) \]

and also the condition \( \Delta V(x) \leq -r(\|x\|), \forall x \in X^{RH}(N) \)

is satisfied.

In conclusion, \( V(x, N) \) is a Lyapunov function.

Moreover, if \( \alpha_f(\|x\|) = b\|x\|^2, X_f = X^{RH}(N) \) the origin is exponentially stable
Remark 1

The main point is to prove that the cost function is decreasing. For this, it is not necessary to find the optimum, but just a sequence

\[ \bar{U}(k) = \begin{bmatrix} \bar{u}_k(k) & \bar{u}_k(k + 1) & \cdots & \bar{u}_k(k + N - 2) & \bar{u}_k(k + N - 1) \end{bmatrix} \]

such that

\[ \bar{J}(x(k), \bar{U}(k), k) < \bar{J}(x(k), \bar{U}(k, N), k) \]

Remark 2

It is possible to conclude that

\[ X^{RH}(N + 1) \supseteq X^{RH}(N) \]

In fact, with longer horizons one has more degrees of freedom.
**Remark 3**

\[ X^{RH}(N) \supseteq X_f \]

In fact, the auxiliary control law \( u = \kappa_a(x) \) can be used by the optimization algorithm.

**Remark 4**

There exists a value \( \bar{N} \) such that \( X^{RH}(\bar{N}) \supseteq \bar{X}_f \) where \( \bar{X}_f \) is the maximum (unknown) positively invariant set associated to the auxiliary control law.

**But, how to select the terminal cost and the terminal set?**
This is the first algorithm proposed, defined by

\[ \kappa_\alpha(x) = 0 \]
\[ X_f = \{0\} \]
\[ V_f = 0 \]

In fact, since \( f(0,0) = 0 \), if at time \( k \) the optimal sequence is

\[ U^o(k) = \begin{bmatrix} u_k^o(k) & u_k^o(k+1) & \cdots & u_k^o(k+N-2) & u_k^o(k+N-1) \end{bmatrix} \]

leading to \( x^o(k+N) = 0 \), at time \( k+1 \) the sequence

\[ U(k+1) = \begin{bmatrix} u_k^o(k+1) & u_k^o(k+2) & \cdots & u_k^o(k+N-1) & 0 \end{bmatrix} \]

is such that

\[ x^o(k+N+1) = x^o(k+N) = 0 \]

and the condition

\[ V_f(f(x(k), \kappa_\alpha(x(k)))) - V_f(x(k))) + \left( \|x(k)\|_Q^2 + \|\kappa_\alpha(x(k))\|_R^2 \right) \leq 0 \]

is satisfied (all the terms are null).
**Remark 1**

The terminal constraint $x(k+N)=0$ is difficult to verify for nonlinear systems. For linear systems without constraints it is possible to compute the explicit solution (CRHPC algorithm).

**Remark 2**

When the input constraints are present, $X^{RH}(N)$ coincides with the constrained controllability set $X^{con}(N)$, which can be computed for linear systems.
Consider the linear system

\[ x(k + 1) = Ax(k) + Bu(k) \]

and the LQ control law (computed with the same \( Q, R \) matrices of the MPC cost function)

\[ u(k) = -K_{LQ}x(k) \]

Define the matrix \( P \) solution of

\[ \left( A - BK_{LQ} \right)' P \left( A - BK_{LQ} \right) - P = - \left( Q + K_{LQ}' R K_{LQ} \right) \]

and the terminal set

\[ X_f = \left\{ x | x'Px \leq \alpha \right\} \subset X \]

where \( \alpha \) is a sufficiently small value.
Consider also the terminal weight

\[ V_f(x) = x'Px \]

These choices fulfill the stability condition

\[ V_f \left( f(x(k), \kappa(x(k))) \right) - V_f(x(k)) + \left( \|x(k)\|_Q^2 + \|\kappa(x(k))\|_R^2 \right) \leq 0 \]

In fact

\[
\Gamma(x(k)) := V_f \left( f \left( x(k), -K_{LQ}x(k) \right) \right) - V_f(x(k)) + \left( \|x(k)\|_Q^2 + \|\kappa(x(k))\|_R^2 \right) \\
= x'(k) \left\{ \left( A - BK_{LQ} \right)' P \left( A - BK_{LQ} \right) - P + \left( Q + K_{LQ}' R K_{LQ} \right) \right\} x(k) \\
= 0
\]
Moreover, $X_f$ is positively invariant for the auxiliary control law, since its boundary coincides with a level line of the Lyapunov function associated to the closed-loop system.

Finally, with continuity arguments it can be concluded that in the neighborhood of the origin (i.e. for a sufficiently small $\alpha$) one has

$$u = -K_{LQ}x \in U$$

Note that the terminal cost can be interpreted as the “cost to go” of a classical LQ-IH approach.
First assume that the system is linearizable at the origin

\[
\begin{align*}
f(x, u) &= \frac{\partial f}{\partial x} \bigg|_{x=u=0} \delta x + \frac{\partial f}{\partial u} \bigg|_{x=u=0} \delta u + \phi(\delta x, \delta u) \\
&= A\delta x + B\delta u + \phi(\delta x, \delta u)
\end{align*}
\]

where

\[
\lim_{\|(\delta x, \delta u)\| \to 0} \sup_{\|(\delta x, \delta u)\|} \|\phi(\delta x, \delta u)\| = 0
\]

For the linearized system compute with the same \(Q, R\) matrices the LQ control law

\[
\delta u(k) = -K_{LQ} \delta x(k)
\]

which will be used as the auxiliary control law.
For the corresponding nonlinear controlled system one has

\[ f(x, u) = (A - BK_{LQ})\delta x + \phi(\delta x, -K_{LQ}\delta x) \]

and

\[
\lim_{\|\delta x\| \to 0} \sup_{\|\delta x, \delta u\|} \left\| \phi(\delta x, -K_{LQ}\delta x) \right\| = 0
\]

Now solve the Lyapunov equation

\[
(A - BK_{LQ})' P (A - BK_{LQ}) - P = -\beta (Q + K_{LQ}'RK_{LQ}) \quad \beta > 1
\]

and consider again the terminal cost

\[ V_f(x) = x'Px \]
In the neighborhood of the origin,

\[
\Gamma(x(k)) := V_f \left( f \left( x(k), -K_{LQ}x(k) \right) \right) - V_f (x(k)) + \left( \|x(k)\|_Q^2 + \|\kappa_a(x(k))\|_R^2 \right) \\
= f \left( x(k), -K_{LQ}x(k) \right)' P f \left( x(k), -K_{LQ}x(k) \right) \\
- x'(k)P x(k) + x'(k)Q x(k) + x'(k)K'_{LQ}R K_{LQ} x(k) \\
\bar{\phi}(x) = \phi(x, -K_{LQ}x)
\]

\[
\Gamma(x(k)) = x'(k) \left( A - BK_{LQ} \right)' P \left( A - BK_{LQ} \right) x(k) + 2x'(k) P \bar{\phi}(x(k)) + \\
- x'(k)P x(k) + \bar{\phi}(x) P \bar{\phi}(x) + x'(k) \left( Q + K'_{LQ}R K_{LQ} \right) x(k) \\
= x'(k) \left\{ \left( A - BK_{LQ} \right)' P \left( A - BK_{LQ} \right) - P + \left( Q + K'_{LQ}R K_{LQ} \right) \right\} x(k) + \\
+ 2x'(k) P \bar{\phi}(x(k)) + \bar{\phi}'(x) P \bar{\phi}(x) \\
= x'(k) \left( 1 - \beta \right) \left( Q + K'_{LQ}R K_{LQ} \right) x(k) + 2x'(k) P \bar{\phi}(x(k)) + \bar{\phi}'(x) P \bar{\phi}(x)
\]
Letting \( L_\Phi = \sup \frac{\| \Phi(x) \|}{\| x \|} \)

one has \( 2x'(k)P\Phi(x(k)) \leq 2 \| P \| L_\Phi \| x(k) \|^2 \)

and \( \Phi'(x)P\Phi(x) \leq \| P \| L_\Phi^2 \| x(k) \|^2 \)

Finally note that is positively invariant

Since \( L_\Phi \rightarrow 0 \) for \( \| x(k) \| \rightarrow 0 \) then \( \Gamma(x(k)) \leq 0 \) in a sufficiently small neighborhood of the origin, so that the decreasing condition is satisfied.

Finally note that \( X_f = \{ x \mid x'Px \leq \alpha \} \subset X \) is positively invariant for the auxiliary LQ control law, as it coincides with a level line of the Lyapunov function associated to the linearized system. Moreover, in a neighborhood of the origin \( u = -K_{LQ}x \in U \).
For nonlinear systems it can be difficult to compute the largest terminal set where the stability and feasibility conditions are verified for the auxiliary control law.

If a small $X_f$ is used (smaller that the largest and unknown one associated to the terminal set), it could be necessary to use a very large prediction horizon $N$. This in turn means that the number of optimization variables can become very high, with significant computational burden.

*To avoid this problem, it is useful to use different prediction ($N_p$) and control ($N_c$) horizons.*
The problem can be reformulated as follows. Solve with respect to the sequence \( u(k), u(k + 1), \ldots, u(k + N_c - 1) \). The following optimization problem

\[
J(x(k), u(\cdot), N_p) = \sum_{i=0}^{N_p-1} \left( \|x(k + i)\|_Q^2 + \|u(k + i)\|_R^2 \right) + V_f(x(k + N_p))
\]

\( x(k + i) \in X \), \( u(k + i) \in U \)

\( u(k + i) = \kappa_a(x(k + i)) \), \( i \in [N_c, N_p) \)

\( x(k + N_p) \in X_f \)
Essential references


