POLITECNICO DI MILANO

Stability of Model Predictive Control

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Stability of MPC - 1

Consider the system

$$x(k+1) = f(x(k), u(k))$$

where *f* is continuously differentiable with respect to its arguments and *f(0,0)=0*. The state and control variables must satisfy the following constraints

$$x \in X \quad , \quad u \in U$$

where *X* and *U* contain the origin as an interior point.

The problem is to design MPC algorithms guaranteeing that the origin of the closed-loop system is an asymptotically stable equilibrium point.

The auxiliary control law

Assume to know an auxiliary control law

$$u = \kappa_a(x)$$

and a positively invariant set $X_f \subset X$ containing the origin such that, for the closed-loop system

$$x(k+1) = f(x(k), \kappa_a(x(k)))$$

and for any $x(\bar{k}) \in X_f$

one has

$$x(k) \in X_f \quad , \quad k \geq \bar{k}$$

$$u(k) = \kappa_a(x(k)) \in U \quad , \quad k \geq \bar{k}$$

Stability of MPC - 3

MPC problem: at any time *k* find the sequence

$$u(k), u(k+1), ..., u(k+N-1)$$

minimizing the cost function (*Q>0, R>0*)

$$J(x(k), u(\cdot), N) = \sum_{i=0}^{N-1} \left(\|x(k+i)\|_Q^2 + \|u(k+i)\|_R^2 \right) + V_f(x(k+N))$$

subject to

$$x(k+i) \in X \quad , \quad u(k+i) \in U$$

$$x(k+N) \in X_f$$

The RH solution implicitly defines the MPC time-invariant control law

$$u = \kappa_{RH}(x)$$

Theorem

Let $X^{RH}(N)$ be the set of states where a solution of the optimization problem exists.

If, for any $x \in X_f$ the condition

$$V_f\left(f\left(x(k), \kappa_a(x(k))\right)\right) - V_f(x(k))) + \left(\|x(k)\|_Q^2 + \|\kappa_a(x(k))\|_R^2\right) \le 0$$

is fulfilled and

$$V_f(x) \le \alpha_f\left(\|x\|\right)$$

where $\alpha_f\left(\|x\|\right)$ is a class *K* function, then the origin of the closed-loop system with the MPC-RH control law is an asymptotically stable equilibrium point with region of attraction $X^{RH}(N)$. Moreover, if $\alpha_f\left(\|x\|\right) = b\,\|x\|^2$ and $X_f = X^{RH}(N)$ then the origin is exponentially stable in $X^{RH}(N)$.

Proof

Let $x(k) \in X^{RH}(N)$ and

$$U^o(k,N) = \left[\begin{array}{ccccc} u_k^o(k) & u_k^o(k+1) & \cdots & u_k^o(k+N-2) & u_k^o(k+N-1) \end{array}\right]$$

be the optimal solution at *k* with prediction horizon *N*. Then, at time *k+1*

$$\tilde{U}(k+1,N) = [u_k^o(k+1) \cdots \; u_k^o(k+N-2) u_k^o(k+N-1) \;\; \kappa_a(x(k+N))]$$

is a feasible solution, so that $x(k+1) \in X^{RH}(N)$

Moreover,

$$V(x,N) := J(x, \kappa_{RH}(x), N) \geq \|x\|_Q^2$$

so that the condition $V(x,N) \geq w(\|x\|)$ is verified in $X^{RH}(N)$

At time *k*, the sequence

$$\tilde{U}(k,N+1)=[U^o(k,N),\kappa_a(x(k+N))]$$

is feasible for the MPC problem with horizon *N+1* and

$$\begin{aligned}J(x,\tilde{U}(k,N+1),N+1)&=V(x,N)-V_f(x(k+N))+V_f(x(k+N+1))\\&\quad+\|x(k+N)\|_Q^2+\|\kappa_a(x(k+N))\|_R^2\\&\le V(x,N)\end{aligned}$$

so that we have the monotonicity property (with respect to *N)*

$$V(x,N+1)\le V(x,N),\ \forall x\in X^{RH}(N)$$

with $V(x,0)=V_f(x),\ \forall x\in X_f$

Then $V(x,N+1)\le V(x,N)\le V_f(x)\le\alpha_f\left(\|x(k)\|\right),\ \forall x\in X_f$

and the condition $V(x,N)\le\psi\left(\|x\|\right),\forall x\in X_f$ is satisfied.

Finally

$$\begin{aligned} V(x,N) &= \|x\|_Q^2 + \|\kappa_{RH}(x)\|_R^2 + J(f(x,\kappa_{RH}(x)), u^o(k+1,N-1), N-1) \\ &= \|x\|_Q^2 + \|\kappa_{RH}(x)\|_R^2 + V(f(x,\kappa_{RH}(x)), N-1) \\ &\geq \|x\|_Q^2 + \|\kappa_{RH}(x)\|_R^2 + V(f(x,\kappa_{RH}(x)), N) \\ &\geq \|x\|_Q^2 + V(f(x,\kappa_{RH}(x)), N), \quad \forall x \in X^{RH}(N) \end{aligned}$$

and also the condition $\Delta V(x) \leq -r(\|x\|), \forall x \in X^{RH}(N)$

is satisfied.

In conclusion, *V(x,N)* is a Lyapunov function.

Moreover, if $\alpha_f(\|x\|) = b\|x\|^2, X_f = X^{RH}(N)$ the origin is exponentially stable

Remark 1

The main point is to prove that the cost function is decreasing. For this, it is not necessary to find the optimum, but just a sequence

$$\bar{U}(k) = \left[\begin{array}{ccccc} \bar{u}_k(k) & \bar{u}_k(k+1) & \cdots & \bar{u}_k(k+N-2) & \bar{u}_k(k+N-1) \end{array}\right]$$

such that

$$\bar{J}(x(k), \bar{U}(k), k) < \tilde{J}(x(k), \tilde{U}(k,N), k)$$

Remark 2

It is possible to conclude that

$$X^{RH}(N+1) \supseteq X^{RH}(N)$$

In fact, with longer horizons one has more degrees of freedom.

Remark 3

$$X^{RH}(N) \supseteq X_f$$

In fact, the auxiliary control law $u = \kappa_a(x)$ can be used by the optimization algorithm

Remark 4

There exists a value $\bar{N}$ such that $X^{RH}(\bar{N}) \supseteq \bar{X}_f$ where $\bar{X}_f$ is the maximum (unknown) positively invariant set associated to the auxiliary control law.

But, how to select the terminal cost and the terminal set?

Zero terminal constraint - 1

This is the first algorithm proposed, defined by

$$\kappa_a(x) = 0$$
$$X_f = \{0\}$$
$$V_f = 0$$

In fact, since *f(0,0)=0*, if at time *k* the optimal sequence is

$$U^o(k) = \left[\begin{array}{cccccc} u_k^o(k) & u_k^o(k+1) & \cdots & u_k^o(k+N-2) & u_k^o(k+N-1) \end{array}\right]$$

leading to $x^o(k+N) = 0$, at time *k+1* the sequence

$$U(k+1) = \left[\begin{array}{ccccc} u_k^o(k+1) & u_k^o(k+2) & \cdots & u_k^o(k+N-1) & 0 \end{array}\right]$$

is such that

$$x^o(k+N+1) = x^o(k+N) = 0$$

and the condition

$$V_f\left(f\left(x(k), \kappa_a(x(k))\right)\right) - V_f(x(k)) + \left(\|x(k)\|_Q^2 + \|\kappa_a(x(k))\|_R^2\right) \leq 0$$

is satisfied (all the terms are null).

Zero terminal constraint - 2

Remark 1

The terminal constraint *x(k+N)=0* is difficult to verify for nonlinear systems. For linear systems without constraints it is possible to compute the explicit solution (CRHPC algorithm).

Remark 2

When the input constraints are present, $X^{RH}(N)$ coincides with the constrained controllability set $X^{con}(N)$, which can be computed for linear systems.

Consider the linear system

$$x(k+1) = Ax(k) + Bu(k)$$

and the LQ control law (computed with the same *Q*, *R* matrices of the MPC cost function)

$$u(k) = -K_{LQ}x(k)$$

Define the matrix *P* solution of

$$\left(A - BK_{LQ}\right)' P \left(A - BK_{LQ}\right) - P = -\left(Q + K'_{LQ}RK_{LQ}\right)$$

and the terminal set

$$X_f = \left\{x \middle| \, x'Px \leq \alpha\right\} \subset X$$

where α is a sufficiently small value.

Consider also the terminal weight

$$V_f(x) = x'Px$$

These choices fulfill the stability condition

$$V_f\left(f\left(x(k),\kappa_a(x(k))\right)\right) - V_f(x(k)) + \left(\|x(k)\|_Q^2 + \|\kappa_a(x(k))\|_R^2\right) \leq 0$$

In fact

$$\begin{aligned}
\Gamma(x(k)) := \quad & V_f\left(f\left(x(k), -K_{LQ}x(k)\right)\right) - V_f\left(x(k)\right) + \left(\|x(k)\|_Q^2 + \|\kappa_a(x(k))\|_R^2\right) \\
= \quad & x'(k)\Big\{\left(A - BK_{LQ}\right)' P\left(A - BK_{LQ}\right) - P + \left(Q + K_{LQ}'RK_{LQ}\right)\Big\}x(k) \\
= 0 &
\end{aligned}$$

Moreover, X_f is positively invariant for the auxiliary control law, since its boundary coincides with a level line of the Lyapunov function associated to the closed-loop system.

Finally, with continuity arguments it can be concluded that in the neighborhood of the origin (i,.e. for a sufficiently small α) one has

$$u = -K_{LQ}x \in U$$

Note that the terminal cost can be interpreted as the “cost to go” of a classical LQ-IH approach.

First assume that the system is linearizable at the origin

$$f(x,u) = \left.\frac{\partial f}{\partial x}\right|_{x=u=0} \delta x + \left.\frac{\partial f}{\partial u}\right|_{x=u=0} \delta u + \phi(\delta x, \delta u)$$
$$= A\delta x + B\delta u + \phi(\delta x, \delta u)$$

where

$$\lim_{\|(\delta x, \delta u)\| \to 0} \sup \frac{\|\phi(\delta x, \delta u)\|}{\|(\delta x, \delta u)\|} = 0$$

For the linearized system compute with the same *Q, R* matrices the LQ control law

$$\delta u(k) = -K_{LQ} \delta x(k)$$

which will be used as the auxiliary control law.

For the corresponding nonlinear controlled system one has

$$f(x,u) = (A - BK_{LQ})\delta x + \phi(\delta x, -K_{LQ}\delta x)$$

and

$$\lim_{\|\delta x\| \to 0} \sup \frac{\left\| \phi(\delta x, -K_{LQ}\delta x) \right\|}{\|(\delta x, \delta u)\|} = 0$$

Now solve the Lyapunov equation

$$\left(A - BK_{LQ}\right)' P \left(A - BK_{LQ}\right) - P = -\beta \left(Q + K'_{LQ} R K_{LQ}\right) \qquad \beta > 1$$

and consider again the terminal cost

$$V_f(x) = x' P x$$

Quasi Infinite Horizon MPC – nonlinear systems - 3

In the neighborhood of the origin,

$$\Gamma(x(k)) := V_f\left(f\left(x(k), -K_{LQ}x(k)\right)\right) - V_f\left(x(k)\right) + \left(\|x(k)\|_Q^2 + \|\kappa_a(x(k))\|_R^2\right)$$

$$= f\left(x(k), -K_{LQ}x(k)\right)' Pf\left(x(k), -K_{LQ}x(k)\right)$$

$$- x'(k)Px(k) + x'(k)Qx(k) + x'(k)K_{LQ}'RK_{LQ}x(k)$$

$$\Big\downarrow \quad \bar{\phi}(x) = \phi(x, -K_{LQ}x)$$

$$\begin{aligned}\Gamma(x(k)) = \; & x'(k)\left(A - BK_{LQ}\right)' P\left(A - BK_{LQ}\right)x(k) + 2x'(k)P\bar{\phi}(x(k)) + \\ & -x'(k)Px(k) + \bar{\phi}'(x)P\bar{\phi}(x) + x'(k)\left(Q + K_{LQ}'RK_{LQ}\right)x(k) \\ = \; & x'(k)\Big\{\left(A - BK_{LQ}\right)' P\left(A - BK_{LQ}\right) - P + \left(Q + K_{LQ}'RK_{LQ}\right)\Big\}x(k) + \\ & +2x'(k)P\bar{\phi}(x(k)) + \bar{\phi}'(x)P\bar{\phi}(x) \\ = \; & x'(k)\left(1 - \beta\right)\left(Q + K_{LQ}'RK_{LQ}\right)x(k) + 2x'(k)P\bar{\phi}(x(k)) + \bar{\phi}'(x)P\bar{\phi}(x)\end{aligned}$$

Letting $$L_{\bar{\phi}} = \sup \frac{\left\|\bar{\phi}(x)\right\|}{\|x\|}$$

one has $$2x'(k)P\bar{\phi}(x(k)) \leq 2\,\|P\|\,L_{\bar{\phi}}\,\|x(k)\|^2$$

and $$\bar{\phi}'(x)P\bar{\phi}(x) \leq \|P\|\,L_{\bar{\phi}}^2\,\|x(k)\|^2$$

Since $L_{\bar{\phi}} \to 0 \, for \, \|x(k)\| \to 0$ then $\Gamma(x(k)) \leq 0$ in a sufficiently small neighborhood of the origin, so that the decreasing condition is satisified.

Finally note that $X_f = \left\{x \middle| \, x'Px \leq \alpha\right\} \subset X$ is positively invariant for the auxiliary LQ control law, as it coincides with a level line of the Lyapunov function associated to the linearized system. Moreover, in a neighborhood of the origin $u = -K_{LQ}x \in U$.

Prediction and control horizons - 1

For nonlinear systems it can be difficult to compute the largest terminal set where the stability and feasibility conditions are verified for the auxiliary control law.

If a small X_f is used (smaller that the largest and unknown one associated to the terminal set), it could be necessary to use a very large prediction horizon N. This in turn means that the number of optimization variables can become very high, with significant computational burden.

To avoid this problem, it is useful to use different prediction (N_p) and control (N_c) horizons.

Prediction and control horizons - 2

The problem can be reformulated as follows.

Solve with respect to the sequence $u(k), u(k+1), ..., u(k+N_c-1)$

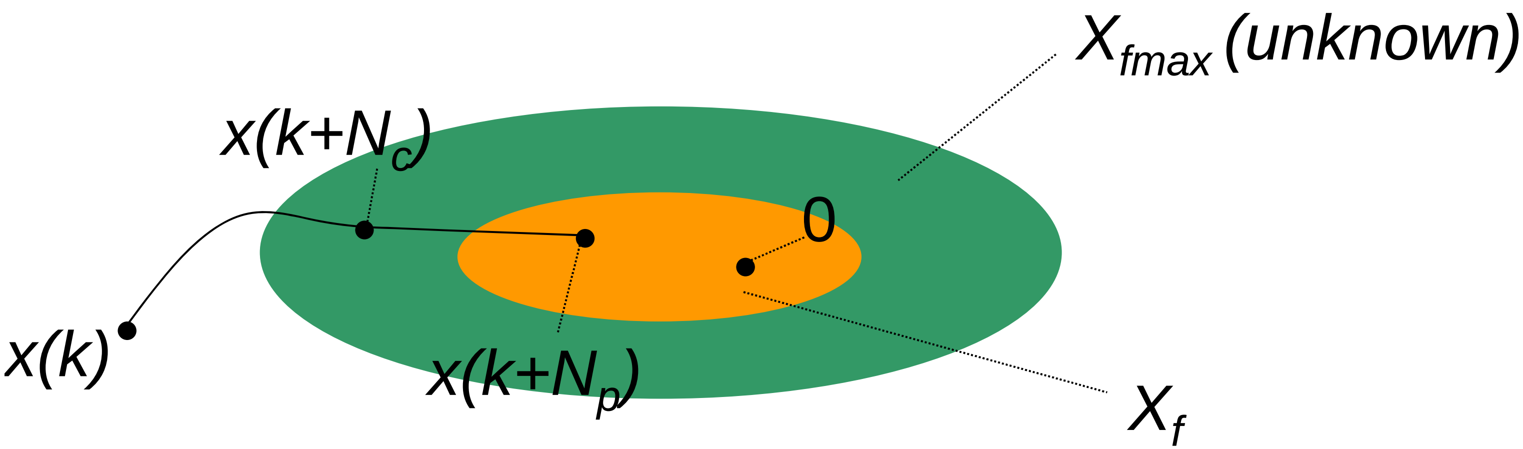
The following optimization problem

$$J(x(k), u(\cdot), N_p) = \sum_{i=0}^{N_p-1} \left(\|x(k+i)\|_Q^2 + \|u(k+i)\|_R^2 \right) + V_f(x(k+N_p))$$

$$x(k+i) \in X \quad , \quad u(k+i) \in U$$

$$u(k+i) = \kappa_a(x(k+i)) \quad , \quad i \in [N_c, N_p)$$

$$x(k+N_p) \in X_f$$



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