POLITECNICO DI MILANO

Implicit solution of the constrained IH-LQ control

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Some preliminary results on Lyapunov stability…

Definitions - 1

Given the system $x(k+1) = f(x(k)), x(0) = x_0$, where f is an arbitrary (discontinuous) function, $\bar{x}$ is an equilibrium point if $f(\bar{x}) = \bar{x}$

Letting $\bar{x}$ be and equilibrium and $X^o \subseteq R^n$ an open neighborhood of $\bar{x}$, then $\bar{x}$ is

1. ***stable*** if for any $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)$ such that

$$\|x_o - \bar{x}\| \leq \delta \Rightarrow \|x(k) - \bar{x}\| \leq \varepsilon \text{ for any } k \geq 0$$

2. ***attractive*** in X^o if

$$\lim_{k\to\infty} \|x(k) - \bar{x}\| = 0 \text{ for any } x_o \in X^o$$

3. ***asymptotically stable*** in X^o if it is stable and attractive in X^o
4. ***exponentially stable*** in X^o if there exist $\theta \geq 0, \lambda \in (0,1)$ such that

$$\|x(k) - \bar{x}\| \leq \theta \|x_o - \bar{x}\| \lambda^k \text{ for any } k \geq 0$$

Definitions - 2

$\varphi : R_+ \to R$ is a *K* function if it is continuous, strictly increasing with $\varphi(0) = 0$

$\varphi : R_+ \to R$ is a K_∞ function if it is continuous, strictly increasing with $\varphi(0) = 0$ and $\varphi(x) \to \infty$ for $x \to \infty$

Let $X^o \subseteq R^n$ be a positively invariant set for the system

$$x(k+1) = f(x(k))$$

containing a neighborhood $\mathcal{N}$ of the equilibrium $\bar{x} = 0$

Let w, ψ, r be class *K* functions and assume that there exists a nonnegative scalar function $V : X^o \to R_+, V(0) = 0$ such that

$$V(x) \geq w(\|x\|), \quad \forall x \in X^o$$
$$V(x) \leq \psi(\|x\|), \quad \forall x \in \mathcal{N}$$
$$\Delta V(x) \leq -r(\|x\|), \quad \forall x \in X^o$$

$$\Downarrow$$

then the origin is an asymptotically stable equilibrium in X^o.
Moreover, if $w(\|x\|) := a\|x\|^\sigma, \psi(\|x\|) := b\|x\|^\sigma, r(\|x\|) := c\|x\|^\sigma$
for some $a, b, c, \sigma > 0$ and $\mathcal{N} = X^o$ then the origin is exponentially stable in X^o

Proof - 1

Stability

Take $\eta > 0$ such that

$$\mathcal{B}_\eta := \{x \in R^n \mid \|x\| \leq \eta\} \subseteq \mathcal{N}$$

Then, for any $0 < \varepsilon \leq \eta$ one can choose $\delta \in (0, \varepsilon)$ such that $\psi(\delta) < w(\varepsilon)$.

For any $x_0 \in \mathcal{B}_\delta \subseteq X^o$ it follows that

$$\ldots \leq V(x(k+1)) \leq V(x(k)) \leq \ldots \leq V(x(0)) \leq$$
$$\psi(\|x_0\|) \leq \psi(\delta) < w(\varepsilon)$$

Since $V(x) > w(\varepsilon)$ for any $x \in X^o \backslash \mathcal{B}_\varepsilon$, it follows that $x \in \mathcal{B}_\varepsilon$ for any $k \geq 0$ and the origin is stable.

Proof - 2

Attractivity

Since $V(x)>0$ and $\Delta V(x) \leq 0$, there exists $lim_{k\to\infty} V(x(k)) = V_L$.

Then $lim_{k\to\infty} \Delta V(x(k)) = V_L - V_L = 0$.

Since $0 \leq r(\|x(k)\|) \leq -\Delta V(x(k)$, it follows that

$$lim_{k\to\infty} r(\|x(k)\|) = 0$$

Suppose that it is not true that $lim_{k\to\infty} \|x(k)\| = 0$, then there exist $\mu>0$ and a subsequence $\{x(k_n)\}$ such that $\|x(k_n)\| > \mu > 0$ for any $n \geq 0$. In turn, since r is monotone and positive, this implies that $r(\|x(k_n)\|) > r(\mu) > 0$ for any $n \geq 0$. This contradicts the fact that $r(\|x(k)\|)$ converges to zero. Therefore

$$lim_{k\to\infty} \|x(k)\| = 0$$

for any $x_0 \in X^o$ and the origin is asymptotically stable in X^o.

Proof - 3

Exponential stability

Recall that $X^0 = \mathcal{N}$ is positively invariant, therefore

$$V(x(k)) \leq \psi(\|x(k)\|) \quad , \quad \Delta V(x(k)) \leq -r(\|x(k)\|)$$

for any *k* and

$$V(f(x(k)) - V(x(k)) \leq -c\|x(k)\|^{\sigma} = -\frac{c}{b}\psi(\|x(k)\|) \leq -\frac{c}{b}V(x(k))$$

This implies $0 \leq V(f(x(k))) \leq \left(1 - \frac{c}{b}\right)V(x(k)), k \geq 0$ and

$$V(x(k)) \leq \left(1 - \frac{c}{b}\right)^k V(x_0), k \geq 0$$

Note that $\rho := 1 - \frac{c}{b} \in [0, 1)$ since *b,c> 0* and *V(x(k))≥0.*

Proof - 4

Moreover it holds that

$$0 \leq V(f(x(k)) \leq V(x(k)) - c\,\|x(k)\|^\sigma \leq \psi(\|x(k)\|) - c\,\|x(k)\|^\sigma = (b-c)\,\|x(k)\|^\sigma$$

and

$$a\,\|x(k)\|^\sigma \leq V(x(k)) \leq \rho^k V(x_0) \leq \rho^k b\,\|x_0\|^\sigma$$

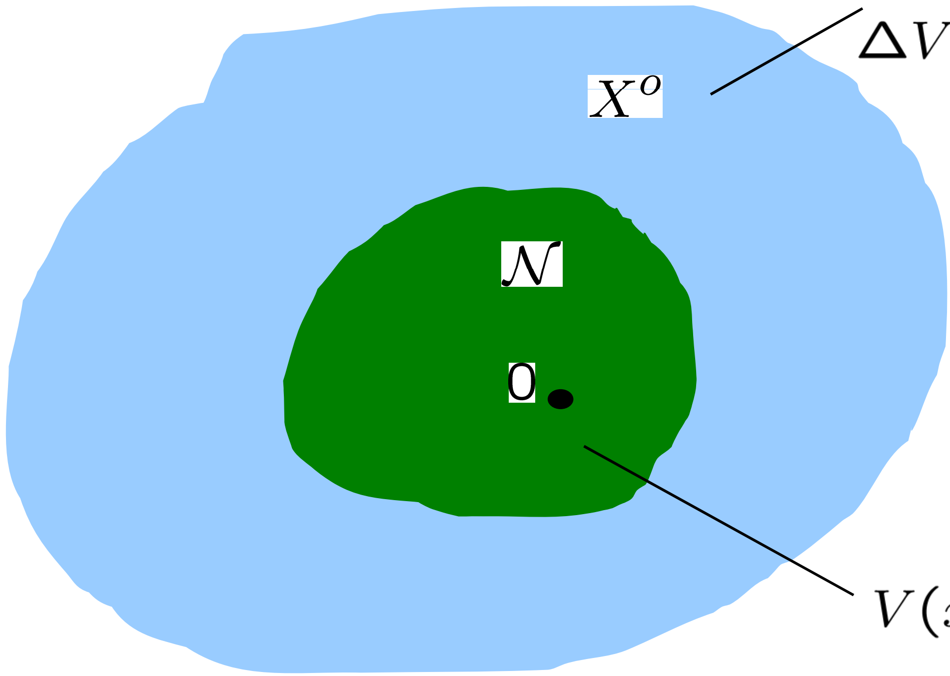
so that $\|x(k)\| \leq \theta\,\|x_0\|\,\lambda^k$ for any $x_0 \in X^0$ and $k \geq 0$

with $\theta := \left(\frac{b}{a}\right)^{\frac{1}{\sigma}} > 0, \lambda := \rho^{\frac{1}{\sigma}} \in [0,1)$

Interpretation

The result extends the Lyapunov theory by considering non continuous Lyapunov functions (the cost function in constrained MPC control)

$$V(x) \geq w(\|x\|), \qquad \forall x \in X^o$$
$$\Delta V(x) \leq -r(\|x\|), \qquad \forall x \in X^o$$



$$V(x) \leq \psi(\|x\|), \qquad \forall x \in \mathcal{N}$$

RH and IH-LQ control - 1

Consider again the linear system with measurable state

$$x(k+1) = Ax(k) + Bu(k)$$

and the performance index

$$J(x(k), u(\cdot)) = \sum_{i=0}^{N-1} \left(\|x(k+i)\|_Q^2 + \|u(k+i)\|_R^2 \right) + \|x(k+N)\|_S^2$$

where *R*>0 and (for simplicity) *Q>0*

If the RH approach is used and $S = \bar{P}$ where

$$\bar{P} = A'\bar{P}A + Q - A'\bar{P}B\left(R + B'\bar{P}B\right)^{-1} B'\bar{P}A$$

the resulting closed-loop control law coincides with the IH-LQ solution, which is stabilizing under the usual assumptions

RH and IH-LQ control - 2

In fact, recall that the RH solution is

$$u^o(k) = -K(0)x(k)$$

where

$$K(0) = \left(R + B'P(1)B\right)^{-1} B'P(1)A$$

$$P(i) = Q + A'P(i+1)A - A'P(i+1)B\left(R + B'P(i+1)B\right)^{-1} B'P(i+1)A$$

$$P(N) = S$$

Then $S = \bar{P} \longrightarrow P(i) = \bar{P}, i = 0, 1, \ldots, N-1, K(0) = K_{LQ}$

RH and IH-LQ control - 3

Interpretation of the result

From the IH-LQ control theory

$$x'(k+N)\bar{P}x(k+N) = \sum_{i=N}^{\infty}\left(\|x(k+i)\|_Q^2 + \left\|K_{LQ}x(k+i)\right\|_R^2\right)$$

Then, with the choice $S = \bar{P}$ the terminal cost is the "cost to go" from *N* to infinity when the ***auxiliary control law***

$$u(k) = -K_{LQ}x(k)$$

is assumed to be used from the end of the prediction horizon onwards. The resulting *J* is the classical IH-LQ cost function.

With a suitable choice of the terminal cost, the RH control law guarantees stability in the unconstrained case

The fake Riccati equation - 1

If $S \neq \overline{P}$, recalling that

$$u^o(k) = -K(0)x(k) \quad , \quad K(0) = \left(R + B'P(1)B\right)^{-1} B'P(1)A$$

P(1) can be seen as the solution of the stationary ***fake*** Riccati equation

$$P(1) = \underbrace{P(1) - P(0) + Q}_{\tilde{Q}} + A'P(1)A - A'P(1)B\left(R + B'P(1)B\right)^{-1} B'P(1)A$$

If $P(1) - P(0) \geq 0 \longrightarrow \tilde{Q} > 0$ and the RH control law can be seen as the IH-LQ solution of

$$\tilde{J}_\infty(x(k), u(\cdot)) = \sum_{i=0}^{\infty} \left(\|x(k+i)\|^2_{\tilde{Q}} + \|u(k+i)\|^2_R\right)$$

and stability is still guaranteed under the usual assumptions

The fake Riccati equation - 2

When $P(1) - P(0) \geq 0$?

Theorem *(not proven)*

If $P(i+1) \geq P(i)$ for any *i>0,* then $P(i+1-k) \geq P(i-k), i > k \geq 0$

Then, if at any iteration of the Riccati equation the solution is decreasing, it is decreasing in all the following iterations. In view of this result, if

$$S = P(N) \geq P(N-1)$$

The RH control law is stabilizing for any positive value of *N.*

The choice of S satisfying the above condition is not trivial

Consider now the following IH constrained problem

$$\mathcal{P}_\infty = \min J_\infty(x(k), u(\cdot)) = \sum_{i=0}^{\infty} \left(\|x(k+i)\|_Q^2 + \|u(k+i)\|_R^2 \right)$$

$$x(k+i) \in X \quad , \quad i \geq 0$$

$$u(k+i) \in U \quad , \quad i \geq 0$$

where *U* and *X* are closed sets containing the origin, *Q>0, R>0*

The solution of this problem can not be computed with the HJB equation or with the "open-loop" solution in view of the infinite number of constraints to be considered

Constrained IH-LQ control - 2

An important result (to be proven later):

The solution of the stated optimization problem can be found by solving, with a sufficiently long prediction horizon N and with the RH strategy , the FH optimal control problem

$$\mathcal{P}_N = \min J_N(x(k), u(\cdot)) = \sum_{i=0}^{N-1} \left(\|x(k+i)\|_Q^2 + \|u(k+i)\|_R^2 \right) + \|x(k+N)\|_{\bar{P}}^2$$

$$x(k+i) \in X \quad , \quad i = 0, \dots, N-1$$

$$u(k+i) \in U \quad , \quad i = 0, \dots, N-1$$

An intuitive proof:

For N sufficiently long, by solving problem P_N the state at the end of the prediction horizon is near the origin, where the state and control constraints are not active

Constrained IH-LQ control - 3

It must be assumed that *x(k)* belongs to the positively invariant admissible set for $\mathcal{P}_\infty$, that is the set of states which can be satisfied by fulfilling the state and control constraints.

$$\bar{X} = \{x(k) \,|\exists u(\cdot) \in U : x(k+i) \in X, \; i \geq 0, \; and \; J^o_\infty < \infty\}$$

Define now the positively invariant admissible set $\bar{X}_{LQ}$ associated to the IH-LQ control law $u(k) = -K_{LQ}x(k)$

$$x(k) \in \bar{X}_{LQ} \Longrightarrow \begin{cases} u(k+i) = -K_{LQ}x(k+i) \in U \quad , \quad i \geq 0 \\ x(k+i) = \left(A - BK_{LQ}\right)^i x(k) \in \bar{X}_{LQ} \quad , \quad i \geq 0 \end{cases}$$

How to compute $\bar{X}_{LQ}$?

First note that $x'\bar{P}x - c = 0, c > 0$ is a level line of the Lyapunov function $V(x) = x'\bar{P}x$ for the closed-loop system with the IH-LQ control law. Therefore, in the unconstrained case

$$X_c = \left\{ x \,\middle|\, x'\bar{P}x \leq c \right\}$$

is a positively invariant set for the closed-loop system with IH-LQ control. Now, if it possible to find a set

$$\Gamma = \left\{ x \,\middle|\, x \in X \;\; and \;\; u = -K_{LQ}x \in U \right\}$$

for any *c* such that

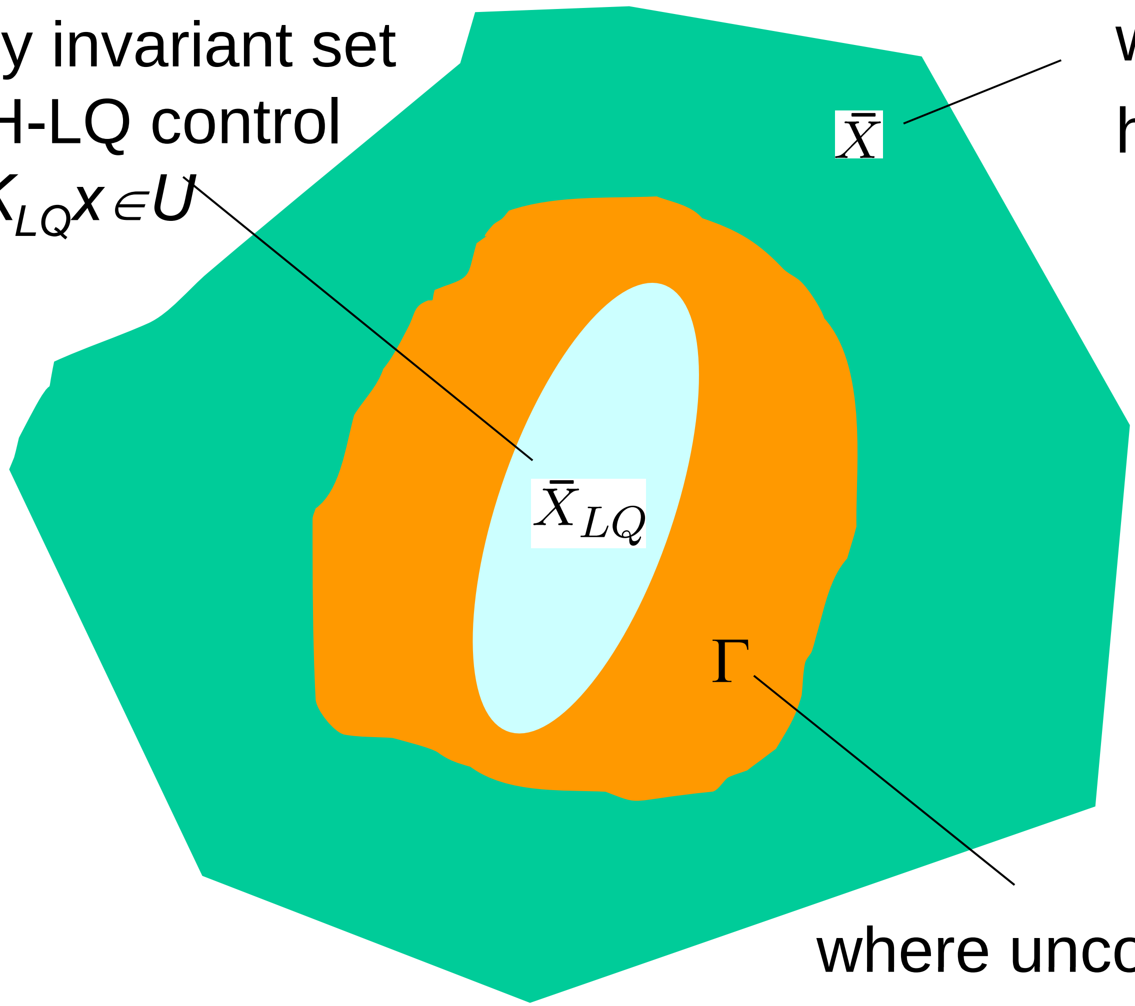
$$X_c \subseteq \Gamma$$

it follows that X_c is the required set $\bar{X}_{LQ}$

Constrained IH-LQ control - 5

Positively invariant set for the IH-LQ control law $u=-K_{LQ}x\in U$

where $\mathcal{P}_\infty$ has a solution



where unconstrained IH-LQ is feasible ($u=-K_{LQ}x\in U$)

Constrained IH-LQ control - 6

Theorem

Let $x(k) \in \bar{X}$. Then, given a set $\bar{X}_{LQ}$ there exists a (computable) sufficiently long prediction horizon *N* such that the solution of the associated problem P_N is such that

$$x(k+N) \in \bar{X}_{LQ}$$

The computation of an upper bound of *N* can be performed with results available in the literature. Assuming that this value has been determined, in view of the dynamic programming approach the solution of $\mathcal{P}_\infty$ coincides with the solution of $\mathcal{P}_N$. In fact, the terminal cost of $\mathcal{P}_N$ is the cost to go of the IH problem.

Proof

If, for a given $\bar{N}$, one has $x(k+\bar{N}) \notin \bar{X}_{LQ}$, then $x(k+\tilde{N}) \notin \bar{X}_{LQ}$ for $\tilde{N} < \bar{N}$. Now take *p* and *q* such that

$$0 < q \leq q_m = \inf_{x \notin \bar{X}_{LQ}} \left\{ x'Qx \right\}$$

$$0 < p \leq p_m = \inf_{x \notin \bar{X}_{LQ}} \left\{ x'\bar{P}x \right\}$$

Then

$$\tilde{J}_N^o(x(k)) = \sum_{i=0}^{N-1} \left(\|x(k+i)\|_Q^2 + \|u^o(k+i)\|_R^2 \right) + \|x(k+N)\|_{\bar{P}}^2 \geq$$

$$\geq \sum_{i=0}^{N-1} \|x(k+i)\|_Q^2 + \|x(k+N)\|_{\bar{P}}^2 \geq Nq + p$$

and, for $N \to \infty$, one has $\tilde{J}_N^o \to \infty$, which contradicts $x(k) \in \bar{X}$.

Constrained IH-LQ control - 7

Main ingredients of the solution:

- a stabilizing auxiliary control law $u(k) = -K_{LQ}x(k)$;
- a terminal cost (the cost to go to infinity) $\|x(k+N)\|_P^2$;
- a terminal positively invariant set $\bar{X}_{LQ}$ for the auxiliary control law where the control constraints are satisfied;
- a terminal constraint $x(k+N) \in \bar{X}_{LQ}$, which can be automatically fulfilled with a suitable choice of the prediction horizon *N*.

Constrained IH-LQ control - 8

Instead of using a long prediction horizon *N* which automatically fulfills the terminal constraint $x(k+N) \in \bar{X}_{LQ}$, it is possible to explicitly force it at the price of obtaining a non optimal (in the LQ sense) solution.

The new problem consists of solving with respect to the sequence $u(k), u(k+1), \ldots, u(k+N-1)$ the optimization problem

$$\mathcal{P} = \min J(x(k), u(\cdot)) = \sum_{i=0}^{N-1} \left(\|x(k+i)\|_Q^2 + \|u(k+i)\|_R^2 \right) + \|x(k+N)\|_P^2$$

$$\begin{aligned} x(k+i) &\in X \quad , \quad i = 0, \ldots, N-1 \\ u(k+i) &\in U \quad , \quad i = 0, \ldots, N-1 \\ x(k+N) &\in \bar{X}_{LQ} \end{aligned}$$

Theorem

If at a given time *k* there exists a feasible solution for problem *P*, the origin of the closed-loop system obtained with the corresponding *RH* solution is an asymptotically stable equilibrium point.

The proof hinges on a classical way of reasoning in predictive control

which is the stability region?

Constrained IH-LQ control - 10

Proof

Let

$$U^o(k) = \begin{bmatrix} u_k^o(k) & u_k^o(k+1) & \cdots & u_k^o(k+N-2) & u_k^o(k+N-1) \end{bmatrix}$$

be the optimal solution at t time *k* and $J^o(x(k))$ the corresponding value of the cost function. In view of the RH principle, only $u_k^o(k)$ is applied.

At *k+1* the new state is

$$x^o(k+1) = Ax(k) + Bu_k^o(k)$$

and

$$\bar{U}(k) = \begin{bmatrix} u_k^o(k+1) & u_k^o(k+2) & \cdots & u_k^o(k+N-1) & -K_{LQ}x^o(k+N) \end{bmatrix}$$

is a feasible solution for *P* (in fact $x^o(k+N) \in \bar{X}_{LQ}$).

However, the value $\bar{J}(x(k+1),k+1)$ of the cost function corresponding to $\bar{U}(k)$ is not the minimal one (this is not the optimal solution $J^o(x^o(k+1))$) and

$$J^o(x^o(k+1)) \leq \bar{J}(x^o(k+1)$$

From the definition of the cost function one has

$$\begin{aligned}
\bar{J}(x^o(k+1)) - J^o(x^o(k)) \leq & \|x(k+N+1)\|_{\bar{P}}^2 - \|x^o(k+N)\|_{\bar{P}}^2 \\
& + \left(\|x^o(k+N)\|_Q^2 + \left\| -K_{LQ} x^o(k+N) \right\|_R^2 \right) \\
& - \left(\|x(k)\|_Q^2 + \left\| -K_{LQ} x(k) \right\|_R^2 \right)
\end{aligned}$$

Constrained IH-LQ control - 12

or

$$\bar{J}(x^o(k+1)) - J^o(x^o(k)) \leq \left\|Ax^o(k+N) - BK_{LQ}x^o(k+N)\right\|_{\bar{P}}^2 - \|x^o(k+N)\|_{\bar{P}}^2 + \left(\|x^o(k+N)\|_Q^2 + \left\|-K_{LQ}x^o(k+N)\right\|_R^2\right) - \left(\|x(k)\|_Q^2 + \|u^o(k)\|_R^2\right)$$

Now note that

$$\left\|Ax^o(k+N) - BK_{LQ}x^o(k+N)\right\|_{\bar{P}}^2 - \|x^o(k+N)\|_{\bar{P}}^2 + \left(\|x^o(k+N)\|_Q^2 + \left\|-K_{LQ}x^o(k+N)\right\|_R^2\right) =$$

$$x^{o\prime}(k+N)\left[Q + \left(A - BK_{LQ}\right)' \bar{P}\left(A - BK_{LQ}\right) - \bar{P} + K_{LQ}' R K_{LQ}\right] x^o(k+N) = 0$$

Therefore

$$\bar{J}(x^o(k+1)) - J^o(x^o(k)) \leq -\left(\|x(k)\|_Q^2 + \|u^o(k)\|_R^2\right)$$

Constrained IH-LQ control - 13

Recalling that

$$J^o(x^o(k+1)) \leq \bar{J}(x^o(k+1)$$

one has

$$J^o(x^o(k+1)) - J^o(x^o(k)) \leq -\left(\|x(k)\|_Q^2 + \|u^o(k)\|_R^2\right)$$

Since by assumption $Q > 0, J^o(x,k) > 0$ and

$$J^o(x^o(k+1)) - J^o(x^o(k)) \leq 0$$

for any $x \neq 0$. In conclusion, J^o is a Lyapunov function and the result follows (***continuity of the Lyapunov function?***).

Explicit constrained IH-LQ - 1

It has been shown that, for a sufficiently large value of *N*, the IH constrained LQ control is equivalent to

$$\mathcal{P}_N = \min_{u(k+i)} J_N(x(k), u(\cdot)) = \sum_{i=0}^{N-1} \left(\|x(k+i)\|_Q^2 + \|u(k+i)\|_R^2 \right) + \|x(k+N)\|_P^2$$

$$\begin{aligned} y_{\min} \leq \ & Cx(k+i) \leq y_{\max} \quad , \quad i = 0, \ldots, N-1 \\ u_{\min} \leq \ & u(k+i) \leq u_{\max} \quad , \quad i = 0, \ldots, N-1 \end{aligned}$$

Moreover, with standard arguments, the problem can be formulated as

$$J_N(x(k), u(\cdot)) = x'(k)\underbrace{\mathcal{A}'\mathcal{Q}\mathcal{A}}_{Y}x(k) + 2x'(k)\underbrace{\mathcal{A}'\mathcal{Q}\mathcal{B}}_{F}U(k) + U'(k)\underbrace{\left(\mathcal{B}'\mathcal{Q}\mathcal{B} + \mathcal{R}\right)}_{H}U(k)$$

$$GU(k) \leq W + Ex(k)$$

where *detH≠0* and *W* contains the values $u_{min}, u_{max}, y_{min}, y_{max}$

By neglecting the terms which do not depend on *U(k)*, the problem can be reformulated as follows

$$\mathcal{P}_N = \min_{u(k+i)} \left\{ U'(k)HU(k) + 2x'(k)FU(k) \right\}$$

$$GU(k) \leq W + Ex(k)$$

Now define the auxiliary variable

$$z(k) = U(k) + H^{-1}F'x(k)$$

which is a linear function of *U* and *x*.

It is easy to show that

$$\frac{1}{2}z'(k)Hz(k) = \frac{1}{2}U'(k)HU(k) + x'(k)FU(k) + \frac{1}{2}x'(k)FH^{-1}F'x(k)$$

Moreover, constraint $GU(k) \le W + Ex(k)$ can be written as

$$Gz(k) \le W + Sx(k)$$

where $S = E + GH^{-1}F'$

Therefore, the original optimization problem is equivalent to the multiparametric optimization problem (it depends on the vector *x* which enters in the definition of *z*)

$$\mathcal{P}_z = \min_{z(k)} \frac{1}{2} z'(k) H z(k)$$
$$Gz(k) \le W + Sx(k)$$

It is possible to solve with respect to *z* as a function of *x* ($z=\kappa_z(x)$) and then to find *U* as a function of *x* (recall the definition $z(k) = U(k) + H^{-1}F'x(k)$). Finally, obtain $u=\kappa_x(x)$.

How to solve the multiparametric programming problem?

- Take a state x_0 belonging to the set X of admissible states (P_z has a solution).
- Given x_0, solve the QP problem and find z_0.
- Compute the subset of active constraints.

$$G^a z_o = S^a x_0 + W^a$$

where G^a, S^a, W^a are the (linearly independent) rows of G, S, W corresponding to the active constraints.

- Now, we want to find the region *CR0*, containing x_0, where these constraints are active for the optimal solution of P_z, as well as the value of z and of the Lagrange multipliers inside *CR0.*

The Lagrangian function associated to P_z is

$$\mathcal{L} = \frac{1}{2} z' H z + \lambda' \left(G z - W - S x \right)$$

Setting to zero the derivative respect to *z* one has

$$z = -H^{-1} G' \lambda$$

Moreover recall that

$$\lambda' \left(G z - W - S x \right) = 0$$

$$\lambda \geq 0$$

$$G z(k) \leq W + S x(k)$$

$$\lambda' \left(-G H^{-1} G' \lambda - W - S x \right) = 0$$

Explicit constrained IH-LQ - 6

$$\lambda' \left(-GH^{-1}G'\lambda - W - Sx\right) = 0$$

$$\Downarrow$$

Non active constraints: $\lambda^{(i)} = 0$

Active constraints: $-G^a H^{-1} G^{a\prime} \lambda^{(a)} - W^a - S^a x = 0 \quad , \quad \lambda^{(a)} > 0$

$$\Downarrow$$

$$\lambda^{(a)} = -\left(G^a H^{-1} G^{a\prime}\right)^{-1} \left(W^a + S^a x\right)$$

$$\Downarrow$$

$$z = -H^{-1}G'\lambda \longrightarrow z = H^{-1}G^{a\prime}\left(G^a H^{-1} G^{a\prime}\right)^{-1}\left(W^a + S^a x\right)$$

This is the required linear function $z=\kappa_z(x)$, which also allows one to compute the linear control law

$$u = K_{CR_0} x + \gamma_{CR_0}$$

which holds true inside *CR0.*

How to compute CR0?

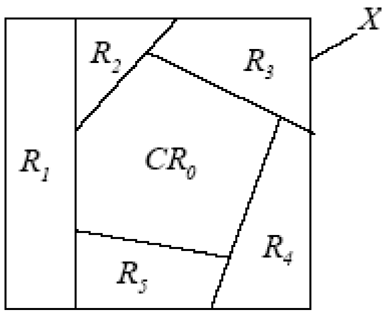
It is defined by the inequalities

$$-\left(G^a H^{-1} G^{a\prime}\right)^{-1}\left(W^a + S^a x\right) \geq 0 \quad \longleftrightarrow \quad \lambda^{(a)} > 0$$

$$G H^{-1} G^{a\prime}\left(G^a H^{-1} G^{a\prime}\right)^{-1}\left(W^a + S^a x\right) \leq \quad W + Sx(k)$$

$$Gz(k) \leq W + Sx(k)$$

Once *CR0* has been computed, the same procedure can be repeated outside it and new regions R_i can be found together with the associated linear control laws.



Comments

- The final number of regions can be very large, and it is necessary to compute on-line the “active” region (computationally very demanding).
- The same linear control law can be computed for different (adjacent) regions.
- The optimal cost function is continuous and piecewise quadratic. It can be used as a Lyapunov function.
- The method has been extended in many directions (tracking, disturbance rejection, …).
- The method can be used to compute gain scheduling controllers (PID type).
- Matlab SW is available.

Essential references

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