POLITECNICO DI MILANO





Introduction to Model Predictive Control

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Finite horizon optimal control

Consider the system

$$x(k+1) = Ax(k) + Bu(k) \qquad\qquad x \in R^n,\ u \in R^m$$

At time *k* we want to compute the sequence of future control variables

$$u(k),\ u(k+1),\ \ldots,\ u(k+N-1)$$

minimizing the performance index

$$J(x(k),u(\cdot),k) = \sum_{i=0}^{N-1} \left(\|x(k+i)\|_Q^2 + \|u(k+i)\|_R^2 \right) + \|x(k+N)\|_S^2$$

where

$$Q = Q' \geq 0, R = R' > 0,\ S = S' \geq 0$$

and *N* is the so-called *prediction horizon*

Finite horizon (FH) optimal control closed-loop solution

The optimal solution is given by the state-feedback control law

$$u^o(k+i) = -K(i)x(k+i), \qquad i = 0, 1, ..., N-1$$

where the gain *K(i)* is

$$K(i) = \left(R + B'P(i+1)B\right)^{-1} B'P(i+1)A$$

and *P(i)* is the solution of the difference Riccati equation

$$\begin{aligned} P(i) &= Q + A'P(i+1)A + \\ &- A'P(i+1)B\left(R + B'P(i+1)B\right)^{-1} B'P(i+1)A \end{aligned}$$

with initial condition

$$P(N) = S$$

Infinite horizon (IH) LQ control

Consider the IH performance index

$$J(x(k),u(\cdot),k)=\sum_{i=0}^{\infty}\left(\|x(k+i)\|_Q^2+\|u(k+i)\|_R^2\right)$$

$$Q=Q'=C'C\geq 0,\; R=R'>0$$

If the pair *(A,B)* is reachable and the pair *(A,C)* is observable, the optimal control law is

$$u^o(k)=-Kx(k)$$

where

$$K=(R+B'PB)^{-1}B'PA$$

P is the unique positive definite solution of the algebraic Riccati equation

$$P=Q+A'PA-A'PB(R+B'PB)^{-1}B'PA$$

and the closed-loop system is asymptotically stable

Recall the Lagrange equation

$$x(k+i) = A^i x(k) + \sum_{j=0}^{i-1} A^{i-j-1} B u(k+j), \qquad i > 0$$

and define

$$X(k) = \begin{bmatrix} x(k+1) \\ x(k+2) \\ \vdots \\ x(k+N-1) \\ x(k+N) \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^{N-1} \\ A^N \end{bmatrix} \qquad U(k) = \begin{bmatrix} u(k) \\ u(k+1) \\ \vdots \\ u(k+N-2) \\ u(k+N-1) \end{bmatrix}$$

$$\mathcal{B} = \begin{bmatrix} B & 0 & 0 & \cdots & 0 & 0 \\ AB & B & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A^{N-2}B & A^{N-3}B & A^{N-4}B & \cdots & B & 0 \\ A^{N-1}B & A^{N-2}B & A^{N-3}B & \cdots & AB & B \end{bmatrix}$$

Then the future state variables are given by

$$X(k) = \mathcal{A}x(k) + \mathcal{B}U(k)$$

Finite horizon (FH) optimal control open-loop solution - 2

Moreover define

$$\mathcal{Q} = \begin{bmatrix} Q & 0 & \cdots & 0 & 0 \\ 0 & Q & \cdots & 0 & 0 \\ \cdots & \cdots & \ddots & \cdots & \cdots \\ 0 & 0 & \cdots & Q & 0 \\ 0 & 0 & \cdots & 0 & S \end{bmatrix}, \qquad \mathcal{R} = \begin{bmatrix} R & 0 & \cdots & 0 & 0 \\ 0 & R & \cdots & 0 & 0 \\ \cdots & \cdots & \ddots & \cdots & \cdots \\ 0 & 0 & \cdots & R & 0 \\ 0 & 0 & \cdots & 0 & R \end{bmatrix}$$

and note that the problem consists of minimizing with respect to *U(k)* the performance index

$$\bar{J}(x(k), u(\cdot), k) = X'(k)\mathcal{Q}X(k) + U'(k)\mathcal{R}U(k)$$

where, with respect to the original cost function, the term $x'(k)Qx(k)$ has been ignored

which does not depend on *U(k)*

The performance index is a quadratic function of *U(k)*

$$\bar{J}(x(k),u(\cdot),k) =$$
$$(\mathcal{A}x(k)+\mathcal{B}U(k))'\,\mathcal{Q}\,(\mathcal{A}x(k)+\mathcal{B}U(k))+U'(k)\mathcal{R}U(k)$$
$$= x'(k)\mathcal{A}'\mathcal{Q}\mathcal{A}x(k)+2x'(k)\mathcal{A}'\mathcal{Q}\mathcal{B}U(k)+U'(k)\left(\mathcal{B}'\mathcal{Q}\mathcal{B}+\mathcal{R}\right)U(k)$$

It's minimum easily turns out to be

$$U^o(k) = -\left(\mathcal{B}'\mathcal{Q}\mathcal{B}+\mathcal{R}\right)^{-1}\mathcal{B}'\mathcal{Q}\mathcal{A}x(k)$$

which is the sequence of future control variables computed at *k*.

Finite horizon (FH) optimal control open-loop solution - 4

Note that, letting

$$\mathcal{K} = (\mathcal{B}'\mathcal{Q}\mathcal{B} + \mathcal{R})^{-1}\mathcal{B}'\mathcal{Q}\mathcal{A} = \begin{bmatrix} \mathcal{K}(0) \\ \mathcal{K}(1) \\ \vdots \\ \mathcal{K}(N-1) \end{bmatrix}, \quad \mathcal{K}(i) \in R^{m,n}$$

The optimal sequence of future control moves is

$$U^o(k) = -\begin{bmatrix} \mathcal{K}(0) \\ \mathcal{K}(1) \\ \vdots \\ \mathcal{K}(N-1) \end{bmatrix} x(k)$$

or equivalently

$$u^o(k+i) = -\mathcal{K}(i)x(k), \quad i = 0, 1, \dots, N-1$$

which is an open-loop solution for *i>0*

Comments

- In the nominal case, the closed-loop solution

$$u^o(k+i) = -K(i)x(k+i), \quad i = 0, 1, \ldots, N-1$$

and the open-loop one

$$u^o(k+i) = -\mathcal{K}(i)x(k), \quad i = 0, 1, \ldots, N-1$$

coincide

- The open-loop solution has been computed by showing that the future states depend on

1. The current state *x(k)*, known at time *k*
2. The future values *U(k)* of the control variables

Constrained problems - 1

What happens when there are constraints on the control and/or state variables? Consider for example classical saturations due to actuators (row-by-row inequalities)

$$u_m \leq u(k+i) \leq u_M, \quad i = 0, ..., N-1$$

The closed-loop solution is not available, while letting

$$U_m = \begin{bmatrix} u_m \\ u_m \\ \vdots \\ u_m \end{bmatrix}, \quad U_M = \begin{bmatrix} u_M \\ u_M \\ \vdots \\ u_M \end{bmatrix}$$

while the open-loop one can be reformulated as a mathematical programming problem

Constrained problems - 2

Performance index to be minimized with respect to *U(k)*

$$min \bar{J}(x(k), u(\cdot), k) =$$
$$(\mathcal{A}x(k) + \mathcal{B}U(k))' \, \mathcal{Q} \, (\mathcal{A}x(k) + \mathcal{B}U(k)) + U'(k)\mathcal{R}U(k)$$

with constraints

$$X(k) = \mathcal{A}x(k) + \mathcal{B}U(k)$$
$$U_m \leq U(k) \leq U_M$$

This problem can be easily solved by means of a *QP* method with reduced computational time (which obviously depends on the size)

The Receding Horizon (RH) principle - 1

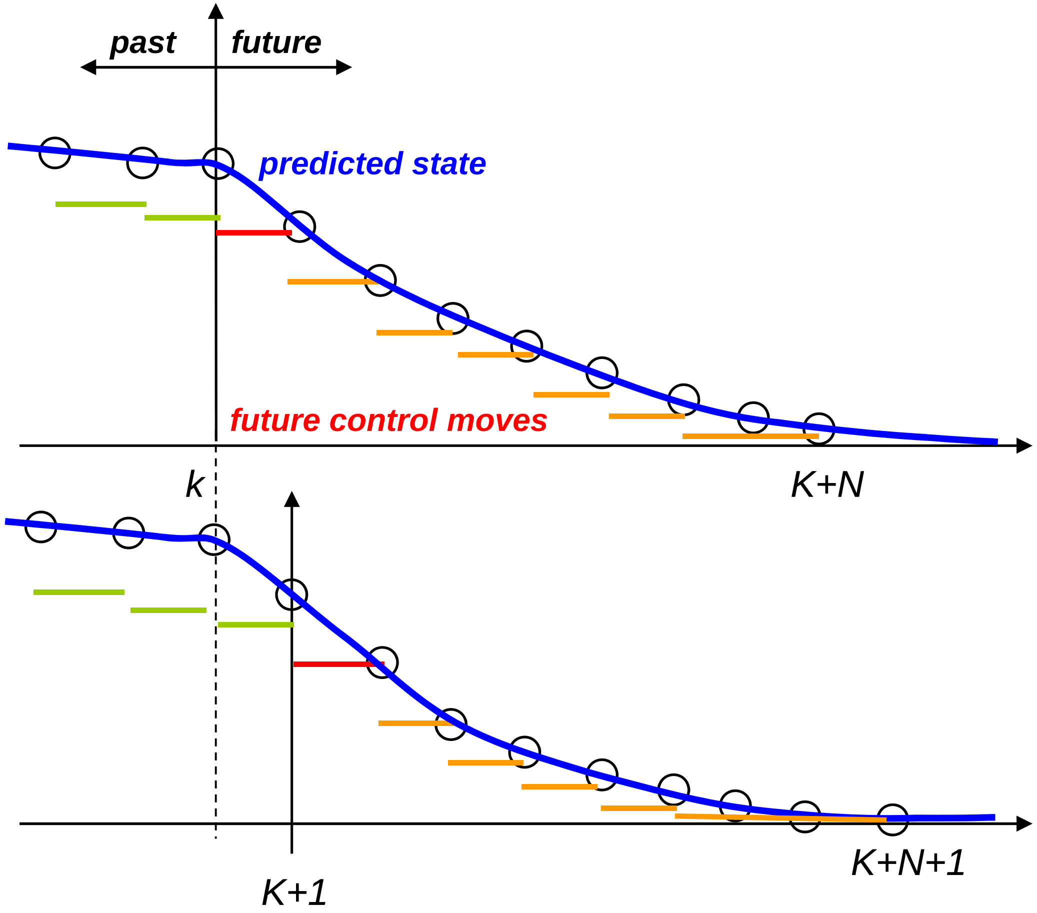
In any case (open- or closed-loop solutions) at time *k* the sequence of optimal control values

$$u^o(k),\ ...,\ u^o(k+N-1)$$

is computed over the prediction horizon. This is a time-varying control law defined over a finite horizon. How to obtain a time-invariant control law?

The Receding Horizon (moving horizon) principle: at any time *k* solve the optimization problem over the prediction horizon $[k,k+N]$ and apply only the first input $u^o(k)$ of the optimal sequence $U^o(k)$. At time $k+1$ repeat the optimization over the prediction horizon $[k,k+N+1]$

Receding Horizon



The RH control law

The RH principle allows one to obtain the state-feedback time-invariant control law

$$u = \kappa_{RH}(x)$$

For constrained systems this is implicitly defined, while in the unconstrained case it coincides with the first element of the open-loop solution

$$u^o(k) = -\mathcal{K}(0)x(k)$$

and with the first element of the closed-loop solution

$$u^o(k) = -K(0)x(k) \qquad K(0) = \left(R + B'P(1)B\right)^{-1} B'P(1)A$$

obtained by iterating the Riccati equation backwards from

$$P(N) = S$$

RH and stability

It is not a-priori guaranteed that the RH control law stabilizes the closed-loop. Consider the system

$$x(k+1) = 2x(k) + u(k)$$

and the performance index

$$J = \sum_{i=0}^{N-1} (x^2(k+i) + u^2(k+i))$$

Therefore $A = 2, B = 1, Q = 1, R = 1, S = P(N) = 0$

and

$$P(N) = 0, K(N-1) = 0, A - BK(N-1) = 2$$

$$P(N-1) = 1, K(N-2) = 1, A - BK(N-2) = 1$$

$$P(N-2) = 3, K(N-3) = 1.5, \quad A - BK(N-3) = 0.5$$

Stability is achieved only for *N>2*

Reference signals and disturbances - 1

Consider the system with disturbances

$$x(k+1) = Ax(k) + Bu(k) + Md(k)$$
$$y(k) = Cx(k) + d(k)$$

and the cost function penalizing the tracking error with respect to the reference signal y^o

$$J(x(k), u(\cdot), k) = \sum_{i=0}^{N-1} \left(\left\| y^0(k+i) - y(k+i) \right\|_Q^2 + \| u(k+i) \|_R^2 \right) +$$
$$+ \left\| y^0(k+N) - y(k+N) \right\|_S^2$$

Again, define

$$Y^o(k) = \begin{bmatrix} y^o(k+1) \\ y^o(k+2) \\ \vdots \\ y^o(k+N-1) \\ y^o(k+N) \end{bmatrix} \qquad Y(k) = \begin{bmatrix} y(k+1) \\ y(k+2) \\ \vdots \\ y(k+N-1) \\ y(k+N) \end{bmatrix}, \qquad \mathcal{A}_c = \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^{N-1} \\ CA^N \end{bmatrix},$$

Reference signals and disturbances - 2

$$\mathcal{B}_c = \begin{bmatrix} CB & 0 & 0 & \cdots & 0 & 0 \\ CAB & CB & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ CA^{N-2}B & CA^{N-3}B & CA^{N-4}B & \cdots & CB & 0 \\ CA^{N-1}B & CA^{N-2}B & CA^{N-3}B & \cdots & CAB & CB \end{bmatrix} \qquad D(k) = \begin{bmatrix} d(k) \\ d(k+1) \\ \vdots \\ d(k+N-2) \\ d(k+N-1) \\ d(k+N) \end{bmatrix}$$

$$\mathcal{M}_c = \begin{bmatrix} CM & I & 0 & \cdots & 0 & 0 & 0 \\ CAM & CM & I & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ CA^{N-2}M & CA^{N-3}M & CA^{N-4}M & \cdots & CM & I & 0 \\ CA^{N-1}M & CA^{N-2}M & CA^{N-3}M & \cdots & CAM & CM & I \end{bmatrix}$$

Then, the future outputs are

$$Y(k) = \mathcal{A}_c x(k) + \mathcal{B}_c U(k) + \mathcal{M}_c D(k)$$

and the problem is equivalent to minimize the cost function

$$\bar{J}(x(k), u(\cdot), k) = (Y^o(k) - Y(k))' \mathcal{Q} (Y^o(k) - Y(k)) + U'(k) \mathcal{R} U(k)$$

Reference signals and disturbances - 3

In the unconstrained case the optimal solution is

$$U^o(k) = \left(\mathcal{B}_c' \mathcal{Q} \mathcal{B}_c + \mathcal{R}\right)^{-1} \mathcal{B}_c' \mathcal{Q} \left(Y^o(k) - \mathcal{A}_c x(k) - \mathcal{M}_c D(k)\right)$$

which depends on the future reference signals $Y^o(k)$ and on the future disturbances $D(k)$. For this reason, MPC can "anticipate" future reference variations or the effect of known disturbances.

When the future disturbance is unknown, it is a common practice to set

$$d(k+i) = d(k), i > 0$$

Comments - 1

Concerning the RH control law obtained from

$$U^o(k) = (\mathcal{B}_c' \mathcal{Q} \mathcal{B}_c + \mathcal{R})^{-1} \mathcal{B}_c' \mathcal{Q} \left(Y^o(k) - \mathcal{A}_c x(k) - \mathcal{M}_c D(k) \right)$$

- In all the considered cases the state *x(k)* has been assumed to be measurable. Otherwise an observer can be used, also to estimate the disturbance *d(k).*
- No integral action has been forced in the feedback control law, so that (provided that closed-loop stability can be assumed), no steady state zero error regulation can be achieved for constant reference signal.

Comments - 2

For constant reference signals y^o, assuming that there exists a pair $(\bar{x}, \bar{u})$ such that

$$\bar{x} = A\bar{x} + B\bar{u}$$
$$y^o = C\bar{x}$$

a more significant performance index is

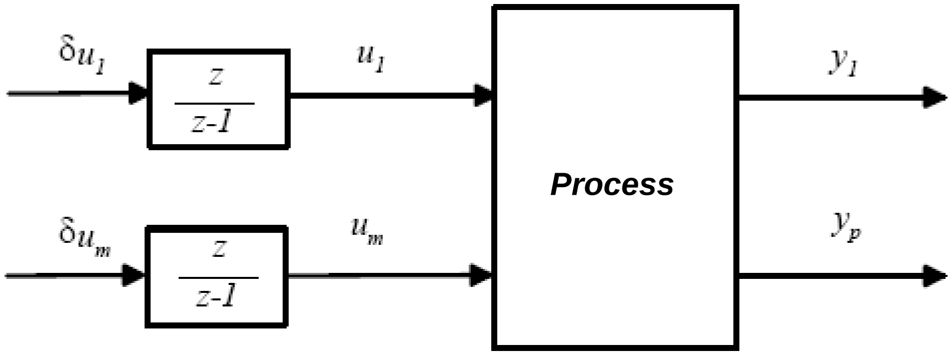
$$J(x(k), u(\cdot), k) = \sum_{i=0}^{N-1} \left(\left\| y^0 - y(k+i) \right\|_Q^2 + \| u(k+i) - \bar{u} \|_R^2 \right) + \\ + \left\| y^0 - y(k+N) \right\|_S^2$$

which penalizes the control deviation with respect to the desired equilibrium point.

Note also that these performance indices do not penalize the state, so that a observability (detectability) assumption is advisable.

Insertion of an integral action - 1

It is a common practice in MPC to plug an integral action at the inputs



Integrators

$$v(k+1) = v(k) + \delta u(k)$$
$$u(k) = v(k) + \delta u(k)$$

A tricky problem with the integral action - 1

System + integrators (no disturbances)

$$\begin{bmatrix} x(k+1) \\ v(k+1) \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \begin{bmatrix} x(k) \\ v(k) \end{bmatrix} + \begin{bmatrix} B \\ I \end{bmatrix} \delta u(k)$$

$$y(k) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ v(k) \end{bmatrix}$$

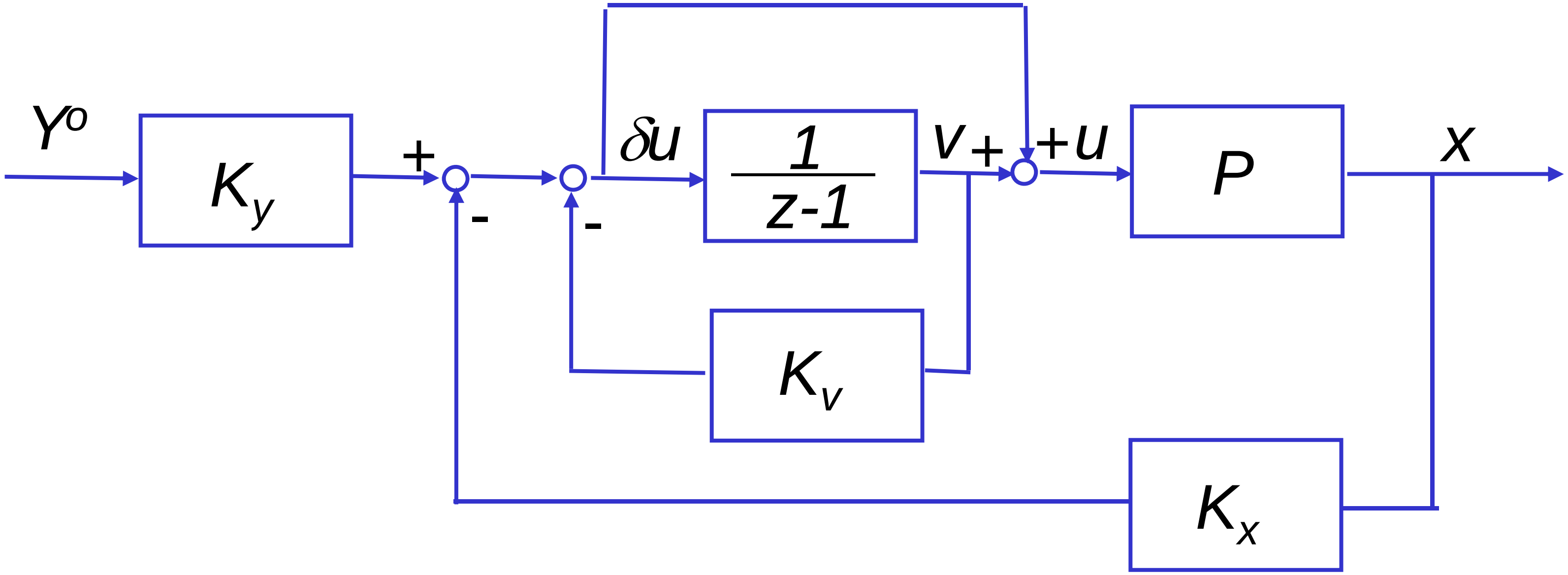
Performance index with tracking error and control variations

$$J(x(k), u(\cdot), k) = \sum_{i=0}^{N-1} \left(\left\| y^0(k+i) - y(k+i) \right\|_Q^2 + \|\delta u(k+i)\|_R^2 \right)$$
$$+ \left\| y^0(k+N) - y(k+N) \right\|_S^2$$

Unconstrained case: the RH control law is linear

$$\delta u(k) = \mathcal{K}_y Y^o(k) - \mathcal{K}_x x(k) - \mathcal{K}_v v(k)$$

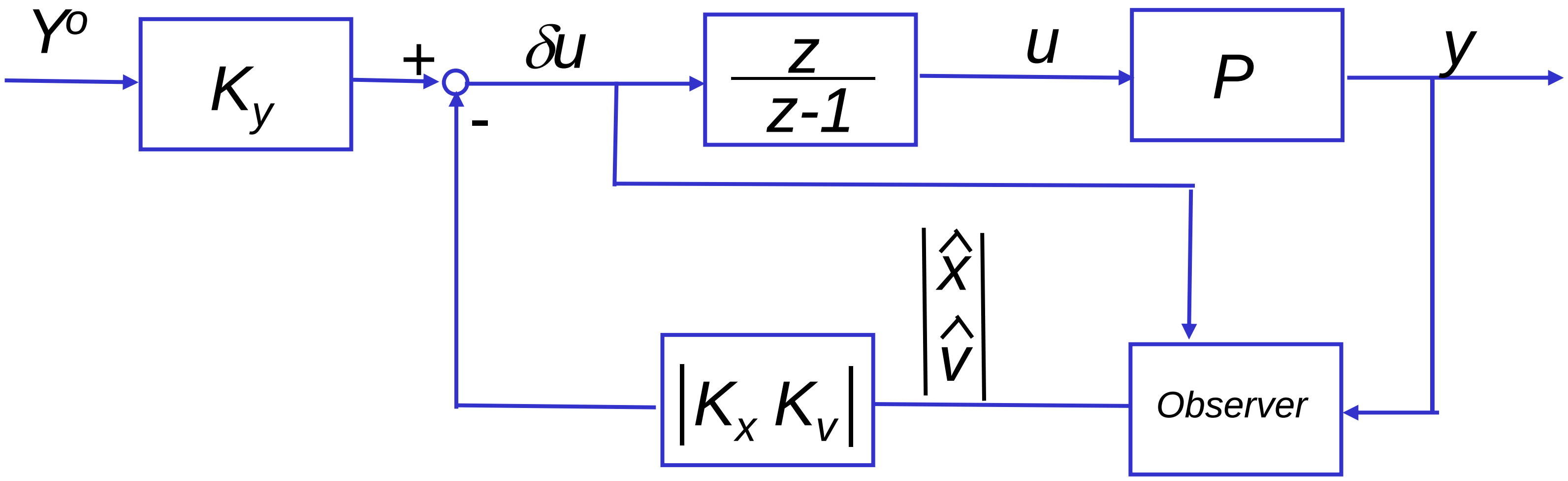
A tricky problem with the integral action - 1



The integrator disappears due to the feedback term on *v*

A tricky problem with the integral action - 2

If an observer is used to estimate also the state of the integrator (a-priori known), the integral action is preserved



This problem can be avoided with other formulations of MPC

An alternative formulation - 1

Consider the system

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k)$$

and write it as

$$x(k+1) - x(k) = A(x(k) - x(k-1)) + B(u(k) - u(k-1))$$
$$y(k+1) = y(k) + C(x(k+1) - x(k))$$

or

$$\delta x(k+1) = A\delta x(k) + B\delta u(k)$$
$$y(k+1) = y(k) + CA\delta x(k) + CB\delta u(k)$$

and in final form with integral action

$$\begin{bmatrix} \delta x(k+1) \\ y(k+1) \end{bmatrix} = \begin{bmatrix} A & 0 \\ CA & I \end{bmatrix} \begin{bmatrix} \delta x(k) \\ y(k) \end{bmatrix} + \begin{bmatrix} B \\ CB \end{bmatrix} \delta u(k)$$
$$y(k) = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} \delta x(k) \\ y(k) \end{bmatrix}$$

An alternative formulation - 2

For the system

$$\begin{bmatrix} \delta x(k+1) \\ y(k+1) \end{bmatrix} = \begin{bmatrix} A & 0 \\ CA & I \end{bmatrix} \begin{bmatrix} \delta x(k) \\ y(k) \end{bmatrix} + \begin{bmatrix} B \\ CB \end{bmatrix} \delta u(k)$$

$$y(k) = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} \delta x(k) \\ y(k) \end{bmatrix}$$

it is possible again to consider the performance index

$$J(x(k),u(\cdot),k) = \sum_{i=0}^{N-1} \left(\left\| y^0(k+i) - y(k+i) \right\|_Q^2 + \|\delta u(k+i)\|_R^2 \right) + \left\| y^0(k+N) - y(k+N) \right\|_S^2$$

In the unconstrained case and with proper redefinitions, the solution is

$$U^o(k) = \Gamma \left(Y^o(k) - \mathcal{A}_c \begin{bmatrix} \delta x(k) \\ y(k) \end{bmatrix} \right)$$

An alternative formulation - 3

In view of the system structure, for constant reference signals y^o

$$Y^o(k) - \mathcal{A}_c \begin{bmatrix} \delta x(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} y^o - \Phi_1 \delta x(k) - y(k) \\ y^o - \Phi_2 \delta x(k) - y(k) \\ \vdots \\ y^o - \Phi_N \delta x(k) - y(k) \end{bmatrix}$$

Then, letting

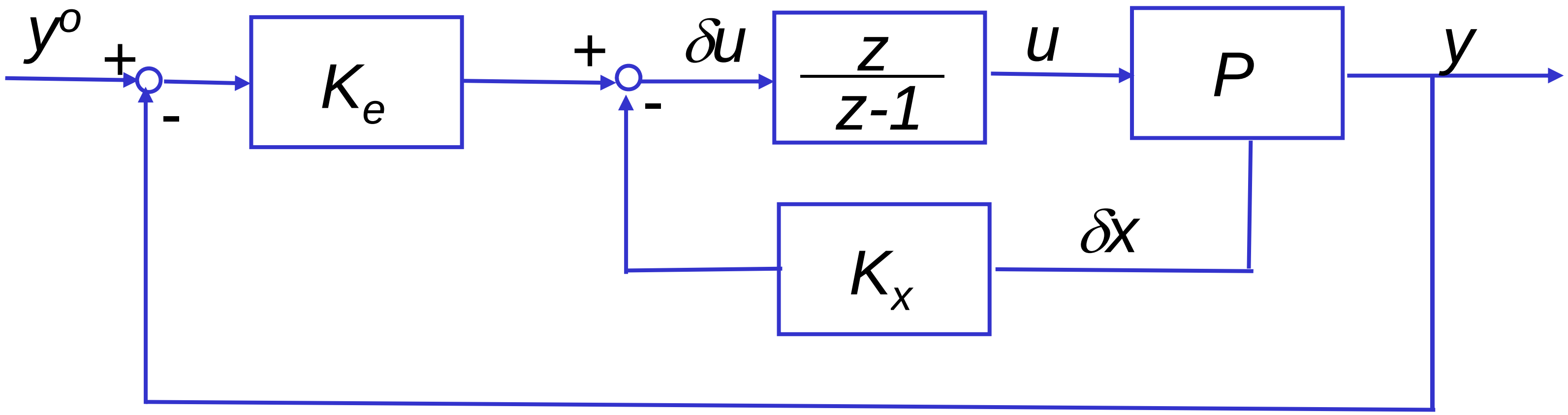
$$e(k) = y^o - y(k)$$

the unconstrained control law takes the form

$$\delta u(k) = K_e e(k) - K_x \delta x(k)$$

which clearly has an integral action on the error signal *e(k)*

An alternative formulation - 4



In steady-state conditions $\delta x=0$, so that $\delta u=0$ only for $e=0$

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