Antiwindup for Stable Linear Systems With Input Saturation: An LMI-Based Synthesis

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***Abstract*—This paper considers closed-loop quadratic stability and $\mathcal{L}_2$ performance properties of linear control systems subject to input saturation. More specifically, these properties are examined within the context of the popular linear antiwindup augmentation paradigm. Linear antiwindup augmentation refers to designing a linear filter to augment a linear control system subject to a local specification, called the "unconstrained closed-loop behavior." Building on known results on $\mathcal{H}_\infty$ and LPV synthesis, the fixed order linear antiwindup synthesis feasibility problem is cast as a nonconvex matrix optimization problem, which has an attractive system theoretic interpretation: the lower bound on the achievable $\mathcal{L}_2$ performance is the maximum of the open and unconstrained closed-loop $\mathcal{L}_2$ gains. In the special cases of zero-order (static) and plant-order antiwindup compensation, the feasibility conditions become (convex) linear matrix inequalities. It is shown that, if (and only if) the plant is asymptotically stable, plant-order linear antiwindup compensation is always feasible for large enough $\mathcal{L}_2$ gain and that static antiwindup compensation is feasible provided a quasi-common Lyapunov function, between the open-loop and unconstrained closed-loop, exists. Using the solutions to the matrix feasibility problems, the synthesis of the antiwindup augmentation achieving the desired level of $\mathcal{L}_2$ performance is then accomplished by solving an additional LMI.**

***Index Terms*—Antiwindup analysis, antiwindup synthesis, control systems, cost optimal control, finite $\mathcal{L}_2$ gain, linear matrix inequalities (LMIs), linear parameter varying (LPV).**

I. Introduction

PERHAPS the first problem in nonlinear control is to design high performance feedback algorithms for linear systems with input saturation. This task is theoretically challenging and, since actuator saturation is ubiquitous, it is critical for practical applications. Over the last decade considerable attention has been given to controlling linear systems with input saturation and significant progress has been reported in the literature.

The control objective for linear systems with input saturation becomes even more difficult to obtain when the behavior of the feedback algorithm must match a given behavior in the absence of input saturation. For example, the controller may need to be a particular PID controller for initial conditions and disturbances that do not trigger input saturation. A local requirement like this can arise for many reasons. In flight control, handling qualities specifications dictate local controller attributes. In vibration attenuation problems, frequency domain specifications constrain the local design. In general, it is common to encounter control problems where many years of experience have gone into the development of a small signal controller and an augmentation of that controller is desired to handle the effects of input saturation that appear occasionally. Augmentation is necessary when the predetermined controller is ill suited for the input saturation nonlinearity. Among early control algorithms, those that were most seriously affected by input saturation were those that contained integral action, e.g., PI or PID controllers. It was observed that, due to input saturation, the state of the integrator would "wind up" to excessively large values, leading to sluggish performance of the closed-loop control system [18]. It is for this reason that the phrase "antiwindup augmentation" is used to describe the problem of synthesizing controllers, subject to a local specification (called the unconstrained controller), for linear systems with input saturation.

As first noted in [5], the most typical embodiment of antiwindup augmentation has the form shown in Fig. 1, where $\mathcal{P}$ represents the linear plant and $\mathcal{C}$ represents the local controller specification. Due to the complexity of the antiwindup problem, where strict requirements for the small-signal behavior of the augmented system are combined with global (large-signal) stability, early antiwindup schemes were mostly heuristics and lacked mathematical rigor. (see, e.g., [11] and [2] for surveys of these early schemes). Only in the last decade has the problem been addressed in a more formal way with stability guarantees and clear performance specifications.

In [7], the antiwindup compensator synthesis problem was approached in a framework relying on $\mathcal{H}_\infty$ optimal control. The main thrust of this method was to interpret the performance of the resulting antiwindup compensator during saturation as an $\mathcal{L}_2$ gain minimization problem. The importance and practicality of the $\mathcal{L}_2$ norm was also recognized in [20], where stable plants were considered and a possible optimization procedure was suggested in terms of the $\mathcal{H}_\infty$ norms of certain transfer functions. In

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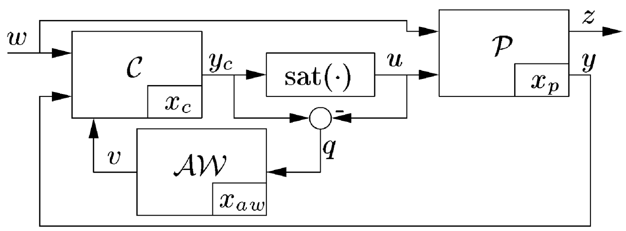

Fig. 1. Antiwindup augmentation scheme.

the work of [30], the induced $\mathcal{L}_2$ norm was linked directly with the behavior of the closed-loop system during saturation. Furthermore, various stability and performance tests for the closed loop system could be formulated as convex feasibility problems, for which efficient solvers are now available. In [29], a formal definition of the antiwindup problem was given. An important aspect of this definition was that recovery of linear performance (a concept also discussed in [6] and the references therein) was stated in terms of nonlinear $\mathcal{L}_2$ gains involving the unconstrained and the actual response of the system.

In recent years, several control applications started employing linear matrix inequalities (LMIs) [4] as a tool to exploit the (sometimes not evident) convexity of certain optimization problems in order to compute global optima in an extremely simplified way. Although many valid antiwindup constructions have been proposed, especially in the last decade that do not rely on LMIs (see, e.g., [10], [20], [27], [24], [15], and [13]), we will only focus here on LMI-based antiwindup designs.

While the control problem suggested by Fig. 1 is nonlinear, one way to tackle it is to treat it as a linear parameter varying (LPV) problem, where $\mathrm{sat}(y_c)$ is replaced by $\Theta(t)y_c$ and $\Theta(\cdot)$ is a measurable, matrix-valued function taking values in a set consistent with reproducing the saturation nonlinearity. Within this approach, special care has to be taken in assuring the well-posedness of the interconnection around the nonlinearity. This is not an issue in the general LPV framework because $\Theta(t)$ is only a function of time. However, in the control problem in Fig. 1, $\Theta(t)$ is actually better written as $\Theta(y_c(t))$, and might result undefined if the system's response $y_c(t)$ is not well defined. We address and solve this well-posedness problem, in this paper, by means of a global nonsmooth inverse function theorem. The great advantage provided by the LPV framework is that quadratic stability and performance by means of fixed order antiwindup augmentation can be addressed using the LMI-based LPV synthesis ideas in [1] and [3] which derive from a combination of [23] and [8] (see also [12]). These synthesis ideas were applied to the control of linear systems with input saturation in [26] and [32], but not to what we have called the antiwindup augmentation problem since the control is not designed to match a given local controller.

The goal of this paper is to construct fixed-order dynamic antiwindup compensators which guarantee a given level of performance using suitable finite $\mathcal{L}_2$ gains of the augmented system as the performance objective (this was also considered in [22]). The basis for the study is the LMI-based $\mathcal{H}_\infty$ controller characterization of [8] and [12], where both full and reduced order controllers meeting an $\mathcal{H}_\infty$ norm-bound are described in terms of a nonconvex feasibility problem, which reduce to a convex feasibility problem when a certain rank constraint becomes inactive. When viewed in this LMI-based framework, the antiwindup augmentation design with $\mathcal{L}_2$ performance objective leads to nice system theoretic interpretations: a lower bound on the $\mathcal{L}_2$ gain achievable by the augmented system is the maximum of the $\mathcal{L}_2$ gains of the open-loop plant (with zero control input) and that of the unconstrained closed-loop system. Moreover, when the antiwindup compensator order is zero (static) or equal to the order of the plant (plant-order), the nonconvex matrix constraints can be reformulated in terms of (convex) LMI constraints that can be easily solved, optimizing globally the performance and providing simple and effective constructions for the antiwindup augmentation. Finally, by way of these new tools, plant-order augmentation can be shown to be always feasible (for large enough $\mathcal{L}_2$ gain), while static augmentation is feasible if and only if there exists a quasi-common quadratic Lyapunov function between the open-loop plant and the unconstrained closed-loop system. Moreover, asymptotic stability of the plant is shown to be a necessary condition for the global $\mathcal{L}_2$ performance requirement of this paper to be attainable.

LMI tools have been brought to bear on the antiwindup framework in very recent years. One of the earliest papers where LMIs and antiwindup were combined is [19] where stability and $\mathcal{L}_2$ performance analysis of closed-loop systems with static antiwindup compensation is formulated as an LMI problem amounting to the determination of a "simultaneous quadratic Lyapunov function." Moreover, [19] formulates the associated synthesis problem in terms of bilinear matrix inequalities. In [17], the stability analysis of more general antiwindup closed-loop systems arising from known antiwindup constructions were formulated in terms of LMIs and a first attempt to transform these LMI stability analysis tools into controller synthesis tools was made by the same authors in [16], where the modified mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem was brought to bear in the static and dynamic antiwindup synthesis problem, noting that it was associated with nonlinear matrix inequalities. Only recently, a complete LMI formulation of the static antiwindup design problem, (namely, the case where the system $\mathcal{AW}$ in Fig. 1 is static, i.e., it has no dynamic state) was given in [22]. The result stops short of a system theoretic interpretation of the feasibility conditions for static antiwindup.

The main drawback of the static construction in [22] is that in several situations the LMI constraints are unfeasible. To address this problem, the same authors proposed an alternative static antiwindup design in [21], based on the approximate solution of nonlinear matrix inequalities, to relax the quadratic stability requirement to piecewise quadratic stability.

The rest of the paper is organized as follows. Section II gives a precise statement of the problem including a Lyapunov-based formulation of stability and performance. Section III gives the main results of this paper. In Section III-B, necessary and sufficient conditions for the existence of an antiwindup compensator guaranteeing stability and a given level of performance is given. Interesting connections between the existence of a suitable antiwindup compensator and properties of the open-loop plant and of the unconstrained closed-loop system are established based on this conditions. Furthermore, it is shown how, for some special values of the antiwindup compensator order, these conditions can be easily checked solving

LMIs based on the unconstrained controller and plant matrices. In these special cases, based on the LMI formulation, the minimization of the performance level can be carried out as a simple convex optimization problem that converges to a global minimum. Section III-A proposes a LMI to ascertain the performance of a given antiwindup compensator applied to a given system. In Section III-C, it is shown that, once the necessary and sufficient conditions have been verified, it is possible to construct the desired antiwindup compensator by solving another LMI which efficiently provides a state-space representation of the dynamics of such an antiwindup compensator. In Section IV, the proposed antiwindup construction method is applied to a simulation example taken from the literature and to an experimental system. The remaining Section V provides the necessary tools for the proof of the main contribution of this paper through the statement and proof of interesting intermediate results.

II. Problem Definition

A. Unconstrained Closed-Loop System

Consider a linear *plant* given by

$$\mathcal{P} \quad \begin{cases} \dot{x}_p = A_p x_p + B_{p,u} u + B_{p,w} w \\ y = C_{p,y} x_p + D_{p,yu} u + D_{p,yw} w \\ z = C_{p,z} x_p + D_{p,zu} u + D_{p,zw} w \end{cases} \tag{1}$$

where $x_p \in \mathbb{R}^{n_p}$ is the plant state, $u \in \mathbb{R}^{n_u}$ is the control input, $w \in \mathbb{R}^{n_w}$ is the exogenous input (possibly containing disturbance, reference and measurement noise), $y \in \mathbb{R}^{n_y}$ is the plant output available for measurement, $z \in \mathbb{R}^{n_z}$ is the performance output (possibly corresponding to a weighted tracking error) and $A_p, B_{p,u}, B_{p,w}, C_{p,y}, D_{p,yu}, D_{p,yw}, C_{p,z}, D_{p,zu}$, and $D_{p,zw}$ are matrices of suitable dimensions. The plant with $u \equiv 0$ will be referred to as the *open-loop plant*.

Assume also that, an *unconstrained controller* has been designed

$$\mathcal{C} \quad \begin{cases} \dot{x}_c = A_c x_c + B_{c,y} y + B_{c,w} w + v_1 \\ y_c = C_c x_c + D_{c,y} y + D_{c,w} w + v_2 \end{cases} \tag{2}$$

(where $x_c \in \mathbb{R}^{n_c}$ is the controller state, $y_c \in \mathbb{R}^{n_u}$ is the controller output, v_1 and v_2 are additional inputs that will be used for antiwindup augmentation and $A_c, B_{c,y}, B_{c,w}, C_c, D_{c,y}$, and $D_{c,w}$ are matrices of suitable dimensions) in such a way that its interconnection to the linear plant through the equations

$$u = y_c \quad v_1 = 0 \quad v_2 = 0 \tag{3}$$

is well-posed and guarantees internal stability of the arising closed-loop system. The interconnection of (1) and (2) via (3) corresponds to the block diagram in Fig. 2 which we will refer to as the *unconstrained closed-loop system*. By selecting the state $x_\ell := [x_p^T \ x_c^T]^T \in \mathbb{R}^{n_{\mathrm{CL}} \times n_{\mathrm{CL}}}$, where $n_{\mathrm{CL}} := n_p + n_c$, and focusing on the effect of the exogenous input w on the performance output z, we can write the dynamics of the unconstrained closed-loop system as a single linear system with state–space representation

$$\begin{aligned} \dot{x}_\ell &= A_{\mathrm{CL}} x_\ell + B_{\mathrm{CL},w} w \\ z &= C_{\mathrm{CL},z} x_\ell + D_{\mathrm{CL},zw} w \end{aligned} \tag{4}$$

where $A_{\mathrm{CL}}, B_{\mathrm{CL},w}, C_{\mathrm{CL},z}$, and $D_{\mathrm{CL},zw}$ are uniquely determined by the matrices in (1) and (2).

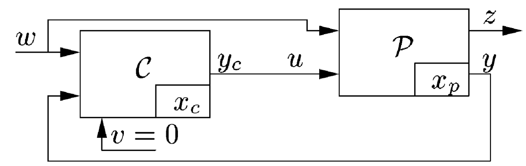

Fig. 2. Unconstrained closed-loop system.

B. Input Saturation and Antiwindup Augmentation

Instead of considering a particular plant input nonlinearity, we consider a class of input nonlinearities defined in Definition 2 (which requires the immediately following definition) in order to state necessary and sufficient conditions for stability and performance.

Definition 1: Given any symmetric positive–definite matrix $V_s \in \mathbb{R}^{n_u \times n_u}$ and two matrices $W_1, W_2 \in \mathbb{R}^{n_u \times r}$, define the *$V_s$-product* of W_1 and W_2 as

$$\langle W_1, W_2 \rangle_{V_s} := W_1^T V_s W_2.$$

A function $f : \mathbb{R}^{n_u} \to \mathbb{R}^{n_u}$ is said to *belong to the sector* $[0, I]_{V_s}$ if $\langle f(w), w - f(w) \rangle_{V_s} \geq 0$ for all $w \in \mathbb{R}^{n_u}$. A function $f : \mathbb{R}^{n_u} \to \mathbb{R}^{n_u}$ is said to *belong to the incremental sector* $[0, I]_{V_s}$ if $\langle Jf(y), I - Jf(y) \rangle_{V_s} \geq 0$ for almost all $y \in \mathbb{R}^{n_u}$, where $Jf(y)$ denotes the Jacobian of f evaluated at y. ○

Definition 2: A function $\phi : \mathbb{R}^{n_u} \to \mathbb{R}^{n_u}$ is said to belong to Φ_{V_s} if the function $\phi(\cdot)$ is locally Lipschitz, belongs to the incremental sector $[0, I]_{V_s}$ and $\phi(0) = 0$. ○

Remark 1: If $\phi(\cdot)$ belongs to Φ_{V_s} then $\phi(\cdot)$ belongs to the sector $[0, I]_{V_s}$. Also, when $V_s = I$, the V_s-product $\langle W_1, W_2 \rangle_{V_s}$ coincides with the standard product $W_1^T W_2$. Furthermore, the sector $[0, I]_I$ property coincides with the sector $[0, I]$ property defined in [14, p. 403]. ○

Suppose the control input of the plant is subject to a nonlinearity, namely

$$u = \phi(y_c) \tag{5}$$

where $\phi(\cdot)$ belongs to Φ_{V_s}.

Remark 2: The $\phi(\cdot)$ in (5) could be a decentralized saturation function, namely

$$\mathrm{sat}(y_c) := [\mathrm{sat}_1(y_{c1}) \quad \mathrm{sat}_2(y_{c2}) \quad \cdots \quad \mathrm{sat}_{n_u}(y_{cn_u})]^T$$

where[1]

$$\mathrm{sat}_i(y_{ci}) := \frac{y_{ci}}{\max\left\{1, \frac{|y_{ci}|}{M_i}\right\}}$$

$M_i \in \mathbb{R}, M_i > 0$ for $i = 1, \ldots, n_u$. Such decentralized saturation functions belong to Φ_{V_s} if V_s is a diagonal positive–definite matrix. ○

Given an integer $n_{\mathrm{aw}} \geq 0$, we address the problem of designing an order n_{aw} linear *antiwindup compensator*

$$\mathcal{AW} \quad \begin{cases} \dot{x}_{\mathrm{aw}} = \Lambda_1 x_{\mathrm{aw}} + \Lambda_2 (y_c - u) \\ v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \Lambda_3 x_{\mathrm{aw}} + \Lambda_4 (y_c - u) \end{cases} \tag{6}$$

[1]For the purpose of this paper, decentralized saturation can denote the larger set of decentralized functions where $\mathrm{sat}_i(\cdot)$ is locally Lipschitz, $\mathrm{sat}_i(0) = 0$ and $(d/(ds))\,\mathrm{sat}_i(s) \in [0, 1]$ almost everywhere for $i = 1, \ldots, n_u$.

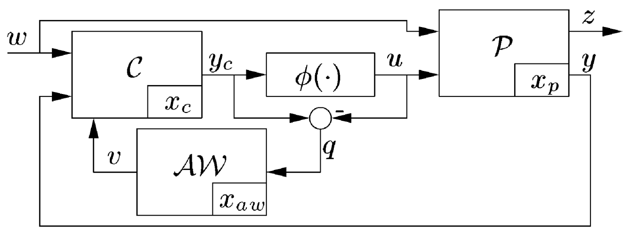

Fig. 3. Antiwindup closed-loop system.

(where $x_{\text{aw}} \in \mathbb{R}^{n_{\text{aw}}}$ is the antiwindup state, $v \in \mathbb{R}^{n_v}$ (with $n_v := n_c + n_u$) is the antiwindup output, and the matrices $\Lambda_1, \Lambda_2, \Lambda_3$, and Λ_4 are of suitable dimensions) that guarantees a desirable $\mathcal{L}_2$ relationship between the exogenous input w and the performance output z for all $\phi(\cdot)$ that belong to Φ_{V_s}. The interconnection (1), (2), (5), (6) will henceforth be called the *antiwindup closed-loop system* and is shown in Fig. 3.

C. Lyapunov Characterization of Stability and Performance

A desirable stability and performance property for the antiwindup closed-loop system will be presented in terms of Lyapunov analysis tools.

Definition 3: Given the linear plant $\mathcal{P}$ in (1) and the unconstrained controller $\mathcal{C}$ in (2), *a linear antiwindup compensator* (6) *of order* n_{aw} *guarantees well-posedness and quadratic performance of level* γ if the augmented antiwindup closed-loop system (1), (2), (5), (6) is such that, for all $\phi(\cdot)$ that belong to Φ_{V_s}

1) the interconnection (1), (2), (5), (6) is well-posed;
2) there exists a scalar $\epsilon > 0$ and a quadratic Lyapunov function $V(x) = x^T P x$ (with $x := [x_p^T \ x_c^T \ x_{\text{aw}}^T]^T$ and $P = P^T > 0$) such that its time derivative $\dot{V}$ along the dynamics of (1), (2), (5), (6) satisfies

$$\dot{V} < -\epsilon x^T x - \frac{1}{\gamma} z^T z + \gamma w^T w \qquad \forall (x, w) \neq 0. \quad (7)$$

∘

Remark 3: Definition 3 entails (sufficient) conditions for internal stability of the antiwindup closed-loop system and for finite $\mathcal{L}_2$ gain γ from w to z for all $\phi(\cdot)$ that belong to Φ_{V_s}. Indeed, since the interconnection (5) is well-posed [as guaranteed by item 1)], item 2) guarantees

i) *quadratic stability*, derived by rewriting (7) with $w = 0$, which implies

$$\dot{V} \leq -\epsilon |x|^2;$$

ii) $\mathcal{L}_2$ *gain from* w *to* z *smaller than* γ. Indeed, inequality (7) can be integrated on both sides from 0 to t (assuming zero initial conditions) to obtain

$$0 \leq V(t) + \epsilon \int_0^t |x|^2 \, dt' \leq \frac{-1}{\gamma} \int_0^t |z|^2 \, dt' + \gamma \int_0^t |w|^2 \, dt'$$

which implies the finite $\mathcal{L}_2$ gain γ from w to z:

$$||z||_2 \leq \gamma ||w||_2.$$

∘

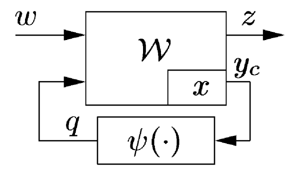

Fig. 4. Compact antiwindup closed-loop system.

III. LMI-Based Antiwindup Analysis and Synthesis

The main contribution of this paper is presented in three parts. In Section III-A, we will provide tools for performance analysis when the antiwindup augmentation (6) is preassigned. In Section III-B, we provide nonlinear matrix conditions whose feasibility is necessary and sufficient to guarantee the existence of an antiwindup compensator that guarantees stability and performance in the sense of Definition 3. For special cases, these nonlinear matrix conditions are transformed into a set of LMIs. Finally, in Section III-C, we will give a procedure to construct antiwindup compensators that induce the performance levels guaranteed by suitable solutions to the matrix conditions in Section III-B.

A. LMI-Based Antiwindup Performance Analysis

Assume that the plant $\mathcal{P}$ in (1), the controller $\mathcal{C}$ in (2) and the linear antiwindup compensator $\mathcal{AW}$ in (6) are given. Then, for analysis purposes, the level of performance can be determined by solving an LMI eigenvalue problem[2].

To formulate suitably the corresponding LMIs, we need to introduce additional notation which corresponds to representing the antiwindup closed-loop system in a compact way, as in Fig. 4. In particular define $\psi(\cdot) : \mathbb{R}^{n_u} \rightarrow \mathbb{R}^{n_u}$ with output $q \in \mathbb{R}^{n_u}$ as

$$q = \psi(y_c) := y_c - \phi(y_c). \quad (8)$$

Next, define the overall state variable $x \in \mathbb{R}^n$, where $n := n_p + n_c + n_{\text{aw}}$, as

$$x := \begin{bmatrix} x_p^T & x_c^T & x_{\text{aw}}^T \end{bmatrix}^T$$

which allows the linear dynamics of the plant, controller and antiwindup compensator to be combined and written as

$$\mathcal{W} \quad \begin{cases} \dot{x} = Ax + B_q q + B_w w \\ y_c = C_y x + D_{yq} q + D_{yw} w \\ z = C_z x + D_{zq} q + D_{zw} w \end{cases} \quad (9)$$

where the matrices $A, B_q, B_w, C_y, D_{yq}, D_{yw}, C_z, D_{zq}$, and D_{zw} are of appropriate dimensions and are uniquely determined by the matrices in (1), (2), and (6).

After a suitable change of coordinates the interconnection between (8) and (9), named the *compact antiwindup closed-loop system* and shown in Fig. 4, corresponds to the antiwindup closed-loop system (1), (2), (5), (6).

Theorem 1: Given the antiwindup closed-loop system (8), (9) and a scalar $\tilde{\gamma}$, the antiwindup closed-loop system is well-posed and guarantees quadratic performance of level $\tilde{\gamma}$ if and

[2]The LMI eigenvalue problem (see, e.g., [4, p. 10]) is to minimize a linear function subject to an LMI constraint (or to determine that the constraint is unfeasible).

only if there exists a solution (Q,δ,γ) to the following LMI problem:

$$\begin{bmatrix} QA^T+AQ & B_qU+QC_y^T & B_w & QC_z^T \\ UB_q^T+C_yQ & D_{yq}U+UD_{yq}^T-2U & D_{yw} & UD_{zq}^T \\ B_w^T & D_{yw}^T & -\gamma I & D_{zw}^T \\ C_zQ & D_{zq}U & D_{zw} & -\gamma I \end{bmatrix}<0 \tag{10a}$$

$$Q=Q^T>0 \tag{10b}$$

$$U=\delta V_s^{-1}>0 \tag{10c}$$

$$\gamma\le\tilde{\gamma}. \tag{10d}$$

Proof: See Section V-A. ■

Remark 4: Convex Performance Analysis: Given a plant, controller and antiwindup compensator that make up an antiwindup closed-loop system, the greatest lower bound on performance $\tilde{\gamma}^*$ can be obtained by solving in the unknowns (Q,δ,γ) the convex LMI eigenvalue problem $\tilde{\gamma}^* := \inf(\gamma)$ subject to (10a)–(10c). ○

Remark 5: If $\phi(\cdot)$ belongs to Φ_{V_s} and V_s^{-1} is linearly parameterized, then extra degrees of freedom can be exploited when solving the LMIs (10). This is the case for decentralized saturation functions introduced in Remark 2. Observe that δV_s^{-1} is linearly parameterized over the family of diagonal positive definite matrices. Hence, in the decentralized case, (10c) can be replaced by $U=\mathrm{diag}(u_1,\dots u_{n_u})>0$ where u_i are unknown, thus allowing extra degrees of freedom in the minimization of γ. ○

Although Theorem 1 provides a useful tool for analysis purposes, it can not easily be used for antiwindup synthesis because the unknown antiwindup compensator matrices multiply the unknown Q, thus making the matrix inequality (10a) nonlinear. In the sequel, suitable procedures are given to construct antiwindup compensators that guarantee well-posedness and quadratic performance.

B. Feasibility of the Antiwindup Synthesis Problem

To assist in the system theoretic interpretation of the matrix inequalities that will follow, recall the well-known LMI formulation of the bounded real lemma for continuous time systems (for a complete proof see, e.g., [25, p. 82]).

Lemma 1 (Bounded Real Lemma): The following statements are equivalent.

1) $\|D+C(sI-A)^{-1}B\|_\infty<\gamma$ and A is Hurwitz.
2) There exists a symmetric positive–definite solution X to the LMI

$$\begin{bmatrix} XA^T+AX & B & XC^T \\ B^T & -\gamma I & D^T \\ CX & D & -\gamma I \end{bmatrix}<0.$$

The following definition will be useful to simplify the notation throughout this paper.

Definition 4: Given the plant $\mathcal{P}$ in (1), the controller $\mathcal{C}$ in (2), an integer $n_{\mathrm{aw}}\ge 0$ and a scalar $\tilde{\gamma}$, define the matrix conditions $\mathrm{MC}(\mathcal{P},\mathcal{C},n_{\mathrm{aw}},\tilde{\gamma})$ as the following set of matrix conditions in the unknowns (R,S,γ):

$$\begin{bmatrix} R_{11}A_p^T+A_pR_{11} & B_{p,w} & R_{11}C_{p,z}^T \\ B_{p,w}^T & -\gamma I & D_{p,zw}^T \\ C_{p,z}R_{11} & D_{p,zw} & -\gamma I \end{bmatrix}<0 \tag{11a}$$

$$\begin{bmatrix} SA_{\mathrm{CL}}^T+A_{\mathrm{CL}}S & B_{\mathrm{CL},w} & SC_{\mathrm{CL},z}^T \\ B_{\mathrm{CL},w}^T & -\gamma I & D_{\mathrm{CL},zw}^T \\ C_{\mathrm{CL},z}S & D_{\mathrm{CL},zw} & -\gamma I \end{bmatrix}<0 \tag{11b}$$

$$R=R^T=\begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22}\end{bmatrix}>0 \tag{11c}$$

$$S=S^T>0 \tag{11d}$$

$$R-S\ge 0 \tag{11e}$$

$$\mathrm{rank}\,(R-S)\le n_{\mathrm{aw}} \tag{11f}$$

$$\gamma\le\tilde{\gamma}. \tag{11g}$$

Moreover, $\mathrm{MC}(\mathcal{P},\mathcal{C},n_{\mathrm{aw}},\tilde{\gamma})$ is said to be *feasible* if there exists a solution (R,S,γ) that satisfies (11). ○

The following theorem, representing our main result, provides necessary and sufficient conditions for the existence of an antiwindup compensator that guarantees well-posedness and quadratic performance of level $\tilde{\gamma}$ in terms of the matrix conditions $\mathrm{MC}(\mathcal{P},\mathcal{C},n_{\mathrm{aw}},\tilde{\gamma})$.

Theorem 2: Given the plant $\mathcal{P}$ in (1), the unconstrained controller $\mathcal{C}$ in (2), an integer $n_{\mathrm{aw}}\ge 0$ and scalar $\tilde{\gamma}$, there exists a linear antiwindup compensator of order n_{aw} that guarantees well-posedness and quadratic performance of level $\tilde{\gamma}$ if and only if $\mathrm{MC}(\mathcal{P},\mathcal{C},n_{\mathrm{aw}},\tilde{\gamma})$ is feasible.

Proof: See Section V. ■

Remark 6: The Greatest Lower Bound on Achievable Performance: The goal of optimal antiwindup design is to construct an antiwindup compensator that guarantees a performance level as small as possible. Based on Theorem 2, the greatest lower bound on achievable performance $\tilde{\gamma}^*$ such that $\mathrm{MC}(\mathcal{P},\mathcal{C},n_{\mathrm{aw}},\tilde{\gamma}^*)$ is feasible can, in principle, be determined by solving in the unknowns (R,S,γ) the nonconvex optimization problem $\tilde{\gamma}^* := \inf(\gamma)$ subject to (11a)–(11f). ○

Remark 7 Lower Bounds on Performance Level: Using Lemma 1, (11a) and (11b) have a system theoretic interpretation. In particular, observe that (11a) constrains γ to be no less than the $\mathcal{H}_\infty$ norm of the plant $\mathcal{P}$ with $u\equiv 0$, input w and output z or equivalently, no less than the $\mathcal{L}_2$ gain from w to z associated with the open-loop plant. Similarly, (11b) constrains γ to be no less than the $\mathcal{L}_2$ gain of the unconstrained closed-loop system (4). While these two LMIs provide lower bounds for the $\mathcal{L}_2$ gain achievable by the antiwindup closed-loop system, (11e) and (11f) establish a nonlinear coupling between the two conditions. ○

Based on the previous remark, it is evident that for condition (11a) to be feasible the plant (1) needs to be asymptotically stable. Since Theorem 2 also establishes the necessity of (11) for antiwindup feasibility, asymptotic stability of the plant is shown there to be necessary if one wants to guarantee the global properties of Definition 3. One of the reasons that it is necessary for A_p to be Hurwitz is that we are asking for global quadratic stability in the absence of inputs. Even if we didn't insist on quadratic stability, with appropriate detectability and stabilizability conditions from w to z, it is a straightforward consequence of the classical small gain theorem that finite gain $\mathcal{L}_2$ stabilizability by bounded controls implies that A_p is Hurwitz. In the more general case of non asymptotically stable linear plants (which is not addressed in this paper), the global properties of Definition 3 should be relaxed to be able to guarantee useful results.

In the next section, we will show that the nonlinear condition (11f) can be transformed into a linear one, in some special cases.

1) LMI Formulations of the Feasibility Condition: An appealing property of Theorem 2 is that all but one of the conditions in $\mathrm{MC}(\mathcal{P},\mathcal{C},n_{\mathrm{aw}},\tilde{\gamma})$ are linear with respect to the unknowns (R,S,γ), the exception being (11f)—the rank condition. Paralleling the necessary and sufficient conditions for reduced order $\mathcal{H}_\infty$ control synthesis (see, e.g., [8, eq. (26)], when considering the *full order* case $n_{\mathrm{aw}} = n_p + n_c$, the rank condition is trivially satisfied and the optimization of the performance level and the determination of the corresponding solution (R,S,γ) reduces to a convex LMI eigenvalue problem, for which numeric algorithms are readily available (see, e.g., [9]).

For the full-order case, the rank condition is guaranteed satisfied and the optimal performance level $\tilde{\gamma}^*$ such that $\mathrm{MC}(\mathcal{P},\mathcal{C},n_p+n_c,\tilde{\gamma}^*)$ is feasible can be determined by solving in the unknowns (R,S,γ) the LMI eigenvalue problem $\tilde{\gamma}^* := \inf(\gamma)$ subject to (11a)–(11e). However, when considering antiwindup compensation of *reduced-order* $(n_{\mathrm{aw}} < n_p + n_c)$, the rank condition needs to be satisfied and the conditions $\mathrm{MC}(\mathcal{P},\mathcal{C},n_{\mathrm{aw}},\tilde{\gamma})$ become nonlinear. By exploiting the special structure of the antiwindup design problem, in the following Propositions 1 and 2 we will show how to replace the nonlinear rank condition with equivalent linear conditions, for the special reduced order cases $n_{\mathrm{aw}} = 0$ and $n_{\mathrm{aw}} \geq n_p$, respectively. In these two special cases, all the matrix inequalities are linear in the unknowns, and the minimization problem for $\tilde{\gamma}$ becomes a convex LMI eigenvalue problem.

Proposition 1 $(n_{\mathrm{aw}} = 0)$*:* Given the plant $\mathcal{P}$ in (1), the controller $\mathcal{C}$ in (2) and a scalar $\tilde{\gamma}$, $\mathrm{MC}(\mathcal{P},\mathcal{C},0,\tilde{\gamma})$ is feasible if and only if there exists a solution (R,γ) to the following LMI conditions:

$$\begin{bmatrix} R_{11}A_p^T + A_pR_{11} & B_{p,w} & R_{11}C_{p,z}^T \\ B_{p,w}^T & -\gamma I & D_{p,zw}^T \\ C_{p,z}R_{11} & D_{p,zw} & -\gamma I \end{bmatrix} < 0 \quad (12a)$$

$$\begin{bmatrix} RA_{\mathrm{CL}}^T + A_{\mathrm{CL}}R & B_{\mathrm{CL},w} & RC_{\mathrm{CL},z}^T \\ B_{\mathrm{CL},w}^T & -\gamma I & D_{\mathrm{CL},zw}^T \\ C_{\mathrm{CL},z}R & D_{\mathrm{CL},zw} & -\gamma I \end{bmatrix} < 0 \quad (12b)$$

$$R = R^T = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix} > 0 \quad (12c)$$

$$\gamma \leq \tilde{\gamma}. \quad (12d)$$

Proof: If $n_{\mathrm{aw}} = 0$, (11f) is satisfied if and only if $R = S$; thus (11e) is satisfied and (11d) is redundant. Hence, the proof follows by rewriting the remaining inequalities in (11) with $R = S$. ■

Proposition 2 $(n_{\mathrm{aw}} \geq n_p)$*:* Given the plant $\mathcal{P}$ in (1), the controller $\mathcal{C}$ in (2), an integer $n_{\mathrm{aw}} \geq n_p$ and a scalar $\tilde{\gamma}$, $\mathrm{MC}(\mathcal{P},\mathcal{C},n_{\mathrm{aw}},\tilde{\gamma})$ is feasible if and only if there exists a solution (R_{11},S,γ) to the following LMI problem:

$$\begin{bmatrix} R_{11}A_p^T + A_pR_{11} & B_{p,w} & R_{11}C_{p,z}^T \\ B_{p,w}^T & -\gamma I & D_{p,zw}^T \\ C_{p,z}R_{11} & D_{p,zw} & -\gamma I \end{bmatrix} < 0 \quad (13a)$$

$$\begin{bmatrix} SA_{\mathrm{CL}}^T + A_{\mathrm{CL}}S & B_{\mathrm{CL},w} & SC_{\mathrm{CL},z}^T \\ B_{\mathrm{CL},w}^T & -\gamma I & D_{\mathrm{CL},zw}^T \\ C_{\mathrm{CL},z}S & D_{\mathrm{CL},zw} & -\gamma I \end{bmatrix} < 0 \quad (13b)$$

$$R_{11} = R_{11}^T > 0 \quad (13c)$$

$$S = S^T = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} > 0 \quad (13d)$$

$$R_{11} - S_{11} > 0 \quad (13e)$$

$$\gamma \leq \tilde{\gamma}. \quad (13f)$$

Proof [Feasibility of (13) ⇒ Feasibility of (11)]: Given a solution (R_{11},S,γ) to (13), take $R_{12} = S_{12}$ and $R_{22} = S_{22}$. Then R and S trivially satisfy the rank constraint (11f) since, by (13e), $R_{11} > S_{11}$, then $R \geq S > 0$. Hence, R is positive definite and (R,S,γ) satisfies Conditions (11) with $n_{\mathrm{aw}} \geq n_p$.

[*Feasibility of* (11) ⇒ *Feasibility of* (13)]. Suppose (11) is satisfied by a solution (R,S,γ). Then (11e) guarantees $R_{11} - S_{11} \geq 0$. Then there exists a symmetric positive–definite matrix $\bar{R}_{11}$, such that with $\bar{R}_{11} - S_{11} > 0$, (13a) is satisfied. ([To show this, take $\epsilon > 0$ such that $\bar{R}_{11} = R_{11} + \epsilon I_{n_p}$ satisfies (13a). Moreover, $R_{11} - S_{11} \geq 0 \Rightarrow \bar{R}_{11} - \epsilon I_{n_p} - S_{11} \geq 0 \Rightarrow \bar{R}_{11} - S_{11} > 0$, as desired]. Finally, (13) is satisfied by $(\bar{R}_{11},S,\gamma)$. ■

Based on Theorem 2 and Propositions 1 and 2, the following theorem gives suitable conditions for the feasibility of the conditions $\mathrm{MC}(\mathcal{P},\mathcal{C},n_{\mathrm{aw}},\tilde{\gamma})$ in Definition 4.

Theorem 3: The following properties hold.

1) There exists a scalar $\tilde{\gamma}$ such that $\mathrm{MC}(\mathcal{P},\mathcal{C},0,\tilde{\gamma})$ is feasible if and only if there exists a matrix R that is a solution to the LMI problem

$$\begin{gathered} R_{11}A_p^T + A_pR_{11} < 0 \\ RA_{\mathrm{CL}}^T + A_{\mathrm{CL}}R < 0 \\ R = R^T = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix} > 0. \end{gathered} \quad (14)$$

2) There exists a scalar $\tilde{\gamma}$ such that $\mathrm{MC}(\mathcal{P},\mathcal{C},n_p,\tilde{\gamma})$ is feasible if and only if A_p is Hurwitz.
3) If $\mathrm{MC}(\mathcal{P},\mathcal{C},n_1,\tilde{\gamma})$ is feasible and $n_1 \leq n_2$, then $\mathrm{MC}(\mathcal{P},\mathcal{C},n_2,\tilde{\gamma})$ is feasible.
4) If $\mathrm{MC}(\mathcal{P},\mathcal{C},n_1,\tilde{\gamma})$ is feasible and $n_1 \geq n_p$, then $\mathrm{MC}(\mathcal{P},\mathcal{C},n_p,\tilde{\gamma})$ is feasible.

Proof:

Item 1) If $\mathrm{MC}(\mathcal{P},\mathcal{C},0,\tilde{\gamma})$ is feasible then by Proposition 1 there exists a matrix R that satisfies (12a) and (12b) with $\gamma = \tilde{\gamma}$. Since each block on the main diagonal of both (12a) and (12b) is negative definite, then the top left block diagonal entries which correspond to the inequalities (14), are negative definite as well.

Assume there exists a symmetric positive definite matrix R that satisfies (14). Since (14) corresponds to the top left block diagonal entries of Conditions (12a) and (12b), then there exists a large enough $\gamma = \bar{\gamma}$ such that (12a) and (12b) are satisfied. The result follows from Proposition 1 picking $\tilde{\gamma} = \bar{\gamma}$.

Item 2) First, note that there exists a matrix $R_{11} = R_{11}^T > 0$ such that $R_{11}A_p^T + A_pR_{11} < 0$ if and only if A_p is Hurwitz. Moreover, since the unconstrained closed-loop system is exponentially stable, $A_{\rm CL}$ is Hurwitz and there exists a matrix $\bar{S} = \bar{S}^T > 0$ such that $\bar{S}A_{\rm CL}^T + A_{\rm CL}\bar{S} < 0$. Since $R_{11} > 0$, there exists a sufficiently small $\epsilon > 0$ such that $R_{11} - \epsilon\bar{S}_{11} > 0$. Take $S = \epsilon\bar{S}$. Then there exists a large enough $\gamma = \bar{\gamma}$ such that (R_{11}, S, γ) satisfies (13). The proof is completed by applying Proposition 2 picking $\tilde{\gamma} = \bar{\gamma}$.

Item 3) The result is a direct consequence of Definition 4 since if the rank condition (11f) holds for $n_{\rm aw} = n_1$ then it also holds for $n_{\rm aw} = n_2 \geq n_1$.

Item 4) The result is a direct consequence of Proposition 2 since Conditions (13) are independent of $n_{\rm aw}$. ■

An important implication of Theorem 3 is that not only does the antiwindup construction always admit a solution choosing $n_{\rm aw} = n_p$, but also given the optimal performance γ^* achievable by a solution of any order $n_{\rm aw} \geq n_p$, then by item 4 of the theorem, this same performance is achievable by an antiwindup compensator of order n_p. Hence, the restriction that the antiwindup compensator order is n_p does not restrict the minimum achievable performance level.

Moreover, item 1) of Theorem 3 implies that, in many situations, static antiwindup compensation does not provide a feasible solution to this antiwindup problem, regardless of the performance level $\tilde{\gamma}$. Indeed, condition (14) corresponds to requiring the existence of a quasi-common quadratic Lyapunov function between the open-loop plant and the unconstrained closed-loop system. In particular, if the unconstrained controller is static ($R_{11} = R$), it exactly requires a common quadratic Lyapunov function. In the general case of a dynamic unconstrained controller, it is a generalization of this requirement based on the fact that the size of the unconstrained closed-loop system is larger than the size of the open-loop plant.

Remark 8: Greatest Lower Bound on Achievable Performance via Convex Optimization: Remark 6 provides a method to determine the greatest lower bound on performance by solving a nonconvex optimization problem. In the light of Propositions 1 and 2, the greatest lower bound on performance can be determined by solving a convex optimization problem when considering static or at least plant-order antiwindup compensation. In particular, the greatest lower bound on achievable performance, $\tilde{\gamma}_s^*$, using a static antiwindup compensator can be determined by solving, in the unknowns (R, γ), the convex LMI eigenvalue problem: $\tilde{\gamma}_s^* := \inf(\gamma)$ subject to (12a)–(12c). Similarly, the greatest lower bound on achievable performance, $\tilde{\gamma}_{n_p}^*$, using an antiwindup compensator of order greater than or equal to the order of the plant can be determined by solving, in the unknowns (R_{11}, S, γ), the convex LMI eigenvalue problem: $\tilde{\gamma}_{n_p}^* := \inf(\gamma)$ subject to (13a)–(13e). ◦

C. LMI-Based Antiwindup Synthesis

Although the results in Section III-B provide natural conditions for the existence of an antiwindup compensator achieving a certain performance level for the closed-loop system in Fig. 3, they do not provide tools for the construction of such a compensator. In this section, based on a solution (R, S, γ) to $\mathrm{MC}(\mathcal{P}, \mathcal{C}, n_{\rm aw}, \tilde{\gamma})$ arising from Theorem 2 or Proposition 1 or 2, we give a procedure to construct a state-space representation of an antiwindup compensator that guarantees well-posedness and quadratic performance of level $\tilde{\gamma}$. The effectiveness of the procedure is then formally stated in Theorem 4.

To suitably describe the procedure for the construction of the antiwindup compensator, we will first introduce an equivalent representation for the antiwindup closed-loop system (1), (2), (5), (6) represented in Fig. 3. By stacking the plant and the controller states into a single state vector $x_{\rm CL} := [x_p^T \ x_c^T]^T \in \mathbb{R}^{n_{\rm CL}}$, with $n_{\rm CL} := n_p + n_c$, the antiwindup closed-loop system can be written as shown in Fig. 5. The dynamics of the subsystem $\mathcal{H}$ in Fig. 5 is given by

$$\mathcal{H} \quad \begin{cases} \dot{x}_{\rm CL} = A_{\rm CL}x_{\rm CL} + B_{{\rm CL},w}w + B_{{\rm CL},q}q + B_{{\rm CL},v}v \\ z = C_{{\rm CL},z}x_{\rm CL} + D_{{\rm CL},zw}w + D_{{\rm CL},zq}q + D_{{\rm CL},zv}v \\ y_c = C_{{\rm CL},y}x_{\rm CL} + D_{{\rm CL},yw}w + D_{{\rm CL},yq}q + D_{{\rm CL},yv}v \end{cases} \tag{15}$$

where the matrices $A_{\rm CL}$, $B_{{\rm CL},w}$, $B_{{\rm CL},q}$, $B_{{\rm CL},v}$, $C_{{\rm CL},z}$, $D_{{\rm CL},zw}$, $D_{{\rm CL},zq}$, $D_{{\rm CL},zv}$, $C_{{\rm CL},y}$, $D_{{\rm CL},yw}$, $D_{{\rm CL},yq}$, and $D_{{\rm CL},yv}$ are of appropriate dimensions and only depend on the matrices of the plant (1) and of the controller (2).

Based on the linear system (15), we can formalize a procedure for the construction of the antiwindup compensator.

1) Procedure 1 (Construction of the Antiwindup Compensator):

Step 1) *Solve the feasibility conditions.*

Given the plant $\mathcal{P}$, the controller $\mathcal{C}$, an integer $n_{\rm aw} \geq 0$ and a scalar $\tilde{\gamma}$, determine a solution (R, S, γ) that satisfies the conditions $\mathrm{MC}(\mathcal{P}, \mathcal{C}, n_{\rm aw}, \tilde{\gamma})$.

Step 2) *Construct the matrix Q.*

Using the solution (R, S, γ) from Step 1, define the matrix $N \in \mathbb{R}^{n_{\rm CL} \times n_{\rm aw}}$ as a solution of the following equation:

$$RS^{-1}R - R = NN^T. \tag{16}$$

Since R and S are invertible and Conditions (11e) and (11f) of Definition 4 are satisfied, then $RS^{-1}R - R$ is positive semidefinite and of rank $n_{\rm aw}$, so there always exists a matrix N satisfying (16). Define the matrix $M \in \mathbb{R}^{n_{\rm aw} \times n_{\rm aw}}$ as

$$M := I + N^TR^{-1}N. \tag{17}$$

Finally, define the matrix $Q \in \mathbb{R}^{(n_{\rm CL}+n_{\rm aw}) \times (n_{\rm CL}+n_{\rm aw})}$ as

$$Q := \begin{bmatrix} R & N \\ N^T & M \end{bmatrix}. \tag{18}$$

Step 3) *Construct other required matrices.*

Construct the matrices $A_\circ \in \mathbb{R}^{n \times n}$, $B_{q\circ} \in \mathbb{R}^{n \times n_u}$, $C_{y\circ} \in \mathbb{R}^{n_u \times n}$, $D_{yq\circ} \in \mathbb{R}^{n_u \times n_u}$, $C_{z\circ} \in \mathbb{R}^{n_z \times n}$, $D_{zq\circ} \in \mathbb{R}^{n_z \times n_u}$, $H_1^T \in \mathbb{R}^{n \times (n_{\rm aw}+n_v)}$,

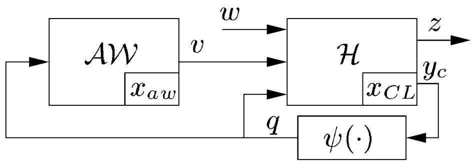

Fig. 5. Equivalent representation of the antiwindup closed-loop system.

$G_1 \in \mathbb{R}^{(n_{\rm aw}+n_u)\times n}, G_2 \in \mathbb{R}^{(n_{\rm aw}+n_u)\times n_u}, H_2^T \in \mathbb{R}^{n_u\times(n_{\rm aw}+n_v)}, H_3^T \in \mathbb{R}^{n_z\times(n_{\rm aw}+n_v)}, B_w \in \mathbb{R}^{n\times n_w}, D_{zw} \in \mathbb{R}^{n_z\times n_w}$, and $D_{yw} \in \mathbb{R}^{n_u\times n_w}$ as follows:

$$A_\circ = \begin{bmatrix} A_{\rm CL} & 0 \\ 0 & 0 \end{bmatrix} \quad B_{q\circ} = \begin{bmatrix} B_{{\rm CL},q} \\ 0 \end{bmatrix}$$
$$C_{y\circ} = [C_{{\rm CL},y} \quad 0] \tag{19a}$$
$$D_{yq\circ} = D_{{\rm CL},yq} \quad C_{z\circ} = [C_{{\rm CL},z} \quad 0] \quad D_{zq\circ} = D_{{\rm CL},zq}$$
$$H_1^T = \begin{bmatrix} 0 & B_{{\rm CL},v} \\ I_{n_{\rm aw}} & 0 \end{bmatrix} \quad G_1 = \begin{bmatrix} 0 & I_{n_{\rm aw}} \\ 0 & 0 \end{bmatrix}$$
$$G_2 = \begin{bmatrix} 0 \\ I_{n_u} \end{bmatrix}$$
$$H_2^T = [0 \quad D_{{\rm CL},yv}] \quad H_3^T = [0 \quad D_{{\rm CL},zv}] \tag{19b}$$
$$B_w = \begin{bmatrix} B_{{\rm CL},w} \\ 0 \end{bmatrix} \quad D_{zw} = D_{{\rm CL},zw} \quad D_{yw} = D_{{\rm CL},yw} \tag{19c}$$

Step 4) *Construct and solve the antiwindup compensator LMI.*

Stack the matrices of the antiwindup compensator (6) in a single matrix $\Lambda \in \mathbb{R}^{(n_{\rm aw}+n_v)\times(n_{\rm aw}+n_u)}$ as follows:

$$\Lambda := \begin{bmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_3 & \Lambda_4 \end{bmatrix}. \tag{20}$$

Choose any $\delta \in \mathbb{R}, \delta > 0$ and define $U = \delta V_s^{-1}$. Based on the matrices determined in Steps 2) and 3) of this procedure, construct the matrices $\Psi \in \mathbb{R}^{(n+n_u+n_w+n_z)\times(n+n_u+n_w+n_z)}, H \in \mathbb{R}^{(n_{\rm aw}+n_v)\times(n+n_u+n_w+n_z)}$, and $G \in \mathbb{R}^{(n_{\rm aw}+n_u)\times(n+n_u+n_w+n_z)}$ as shown in (21a)–(21c) at the bottom of the page. Finally, compute the matrix Λ associated with the desired antiwindup compensator by solving the LMI

$$\Psi + G^T\Lambda^T H + H^T \Lambda G < 0. \tag{22}$$

Theorem 4: Given the plant $\mathcal{P}$, the controller $\mathcal{C}$, an integer $n_{\rm aw}$, a scalar $\tilde\gamma$ and a solution (R,S,γ) to $\mathrm{MC}(\mathcal{P},\mathcal{C},n_{\rm aw},\tilde\gamma)$, the LMI (22) constructed according to Procedure 1 is guaranteed to be solvable for Λ. Furthermore, the solution Λ defines the matrices of a linear antiwindup compensator (6) of order $n_{\rm aw}$ that guarantees well-posedness and quadratic performance of level $\tilde\gamma$.

Proof: See Section V. ■

Remark 9: To overcome implementation problems, it might be desirable for the antiwindup compensator arising from Procedure 1 to be strictly proper. At least for the case when the controller (2) is strictly proper (namely, $D_{c,y} = 0$ and $D_{c,w} = 0$), this is possible without increasing the performance level $\tilde\gamma$ but increasing the dimension of the antiwindup compensator (6) by adding n_u states. Indeed, the conditions of Theorem 2 hold for a given $\tilde\gamma$ if and only if they hold for some $\bar\gamma = \tilde\gamma - \delta_\gamma$, with δ_γ sufficiently small. Then, following a singular perturbation approach (see, e.g., [14, Sec. 9.4]), it can be shown that there exists a sufficiently small constant $\mu > 0$ such that the same antiwindup compensator augmented with the filter

$$\mu\dot q = -q + u_f$$

located at its input (namely, choosing $u_f = \psi(y_c)$) still guarantees well-posedness and quadratic performance of level $\tilde\gamma$. Indeed, defining the new state variable $\xi := q - \psi(y_c)$, a singular perturbation argument allows us to prove a relation similar to (7) for the new antiwindup closed-loop system. In particular, taking any $d \in (0,1)$ a new (Lipschitz) Lyapunov function $\bar V(x,\xi) := (1-d)V(x) + d\xi^2$ can be shown to satisfy (7) for a smaller ϵ but the same original value for $\tilde\gamma$ (this is possible by the preliminary insertion of the margin δ_γ). ∘

Remark 10: When the saturation function is decentralized (consequently, by Remark 5, U can be selected as a diagonal positive definite unknown), the static antiwindup construction in Procedure 1 (with $n_{\rm aw} = 0$) corresponds to the optimal static antiwindup construction proposed in [22], where the matrix U is an unknown diagonal positive–definite matrix (therein U^{-1} is referred to as the "stability multiplier") and the parameter $Q = R$, instead of being determined in Step 2), is undetermined and considered as an extra unknown variable in the inequality (22). Indeed, due to the simpler structure of the problem when $n_{\rm aw} = 0$ (causing $G_1 = 0$), inequality (22) turns out to be linear in the unknowns $Q, U, \Lambda_4 U$, and γ, hence being solvable through a single-step solution, where γ can be once again minimized in a convex way. Although the stability multiplier was employed in [22] to improve the antiwindup performance, an interesting implication of Theorem 2 is that since the conditions

$$\Psi = \begin{bmatrix} QA_\circ^T + A_\circ Q & B_{q\circ}U + QC_{y\circ}^T & B_w & QC_{z\circ}^T \\ UB_{q\circ}^T + C_{y\circ}Q & D_{yq\circ}U + UD_{yq\circ}^T - 2U & D_{yw} & UD_{zq\circ}^T \\ B_w^T & D_{yw}^T & -\gamma I & D_{zw}^T \\ C_{z\circ}Q & D_{zq\circ}U & D_{zw} & -\gamma I \end{bmatrix} \tag{21a}$$
$$H = [H_1 \quad H_2 \quad 0 \quad H_3] \tag{21b}$$
$$G = [G_1 Q \quad G_2 U \quad 0 \quad 0]. \tag{21c}$$

$\mathrm{MC}(\mathcal{P}, \mathcal{C}, n_{\mathrm{aw}}, \tilde{\gamma})$ are independent of V_s, then the minimum achievable performance level does not depend on the stability multiplier. ○

IV. Application Examples

In this section, the antiwindup construction proposed in Section III-C is applied to two linear windup-prone control systems. The first one is a simulation example that illustrates the effectiveness of the construction in the nontrivial case of a multiple-input–multiple-output system. The second one is an experimental application that shows the success of our algorithms when applied to practical control problems. In particular, the application that we have chosen exhibits a difficult windup problem for which static antiwindup is not even capable of guaranteeing quadratic stability (this is verified by checking the conditions in Theorem 3) and the more sophisticated plant-order dynamic antiwindup compensation scheme is necessary.

Example 1 (The Longitudinal Dynamics of an F8 Aircraft [13], [19]): Consider a fourth-order linear model of the longitudinal dynamics of the F8 aircraft and the eighth order linear unconstrained controller introduced in [13]. The two inputs to the plant are the elevator angle and the flaperon angle, each one limited between ± 25 degrees and the two outputs of the plant are the pitch angle and the flight path angle. The controller input is the difference between the plant output and the reference input. The authors of [13] observe a substantial performance loss when the plant input is subject to saturation and propose a reference governor scheme for antiwindup purposes. We will compare their result to the antiwindup compensators designed using the methods in this paper.

The methods in this paper depend on the realization of the unconstrained controller. Using the matrices $\mathbf{A_a}, \mathbf{B_a}, \mathbf{C_a}, \mathbf{H}$, and $\mathbf{G}$ defined in [13], choose the realization of the controller according to

$$A_c = \begin{bmatrix} \mathbf{A_a} + \mathbf{B_a}\mathbf{G} - \mathbf{H}\mathbf{C_a} & 0 \\ \mathbf{G} & 0 \end{bmatrix} \quad -B_{c,w} = B_{c,y} = \begin{bmatrix} \mathbf{H} \\ 0 \end{bmatrix}$$
$$C_c = \begin{bmatrix} 0 & I \end{bmatrix}$$

and $D_{c,y}$ and $D_{c,w}$ are zero matrices of appropriate dimensions. By selecting the performance output $z = y - w$ where w denotes the reference input, a static antiwindup compensator can be constructed using Procedure 1 with $n_{\mathrm{aw}} = 0$ which guarantees performance level $\gamma = 22.19$ and the resulting antiwindup compensator consists of the gain is as shown in the equation at the bottom of the page. Similarly, a plant-order antiwindup compensator can be constructed using the same performance output and Procedure 1 with $n_{\mathrm{aw}} = n_p$, resulting in an antiwindup compensator with guaranteed performance level $\gamma = 19.39$. To save space, the constructed matrices are not written here. The antiwindup closed-loop system response, and the other responses discussed thus far, are shown in Fig. 6, where the bold solid line is the unconstrained trajectory, the dotted line is the saturated trajectory, the dashed line is the antiwindup response with the method of [13], the dash-dotted line is our static antiwindup response, and the thin solid line is our plant-order antiwindup response. Both of the antiwindup closed-loop system responses have significant overshoot and are, perhaps, undesirable.

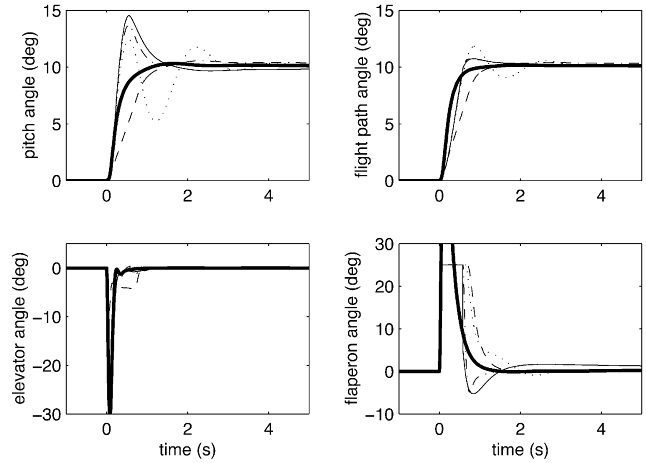

Fig. 6. Example 1. Comparison of the unconstrained response (bold solid) and of the saturated response (dotted) to the static (dash-dotted) and dynamic (thin solid) antiwindup designs with $z = y - w$ and to the scheme of Kapasouris *et al.* (dashed).

Next, we will show that the antiwindup trajectories can be significantly improved by selecting a different performance output. We observe the most substantial degradation in performance of the saturated closed-loop trajectories is the large overshoot and settling time of the pitch angle. For this reason, we select the performance objective to be composed of the pitch angle error and the angular acceleration due to the plant state on the pitch angle. In particular, we will define the performance output via the matrices

$$C_{p,z} = \begin{bmatrix} 0 & 0 & 0 & \frac{3}{4} \\ -.8 & -.0006 & -12 & 0 \end{bmatrix}$$
$$D_{p,zu} = 0_{2\times 2}$$
$$D_{p,zw} = \begin{bmatrix} -\frac{3}{4} & 0 \\ 0 & 0 \end{bmatrix}.$$

A static antiwindup compensator can now be constructed using Procedure 1 with $n_{\mathrm{aw}} = 0$ which guarantees performance level $\gamma = 26.18$ and the resulting antiwindup compensator consists of the gain as shown in the the equation at the bottom of the page. Similarly, a plant-order antiwindup compensator can be constructed using the same pitch angle performance output and Procedure 1 with $n_{\mathrm{aw}} = n_p$, resulting in an antiwindup compensator with guaranteed performance level $\gamma = 22.91$. To save space, the constructed matrices are not written here. The antiwindup closed-loop system response, and some of the other

$$\Lambda_4 = \begin{bmatrix} 6.1077 & 10.113 & -5.1947 & -1267 & -0.176\,47 & 0.893\,73 & 6.2456 & 11.053 & -0.906\,67 & 1.5318 \\ -1.9882 & 9.6373 & -3.8543 & -566.8 & -0.241\,58 & 0.317\,19 & -1.9261 & 10.102 & 0.059\,48 & 0.019\,87 \end{bmatrix}^T.$$

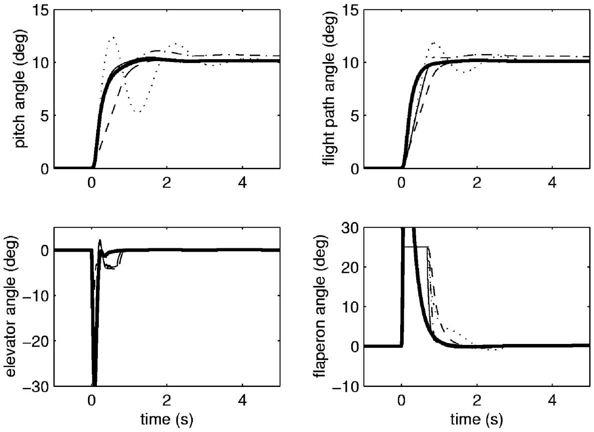

Fig. 7. Example 1. Comparison of the unconstrained response (bold solid) and of the saturated response (dotted) to the static (dash-dotted) and dynamic (thin solid) antiwindup designs with pitch angle performance output and to the scheme of Kapasouris *et al.* (dashed).

responses discussed previously, are shown in Fig. 7 where the bold solid line is the unconstrained trajectory, the dotted line is the saturated trajectory, the dashed line is the antiwindup response with the method of [13], the dash-dotted line is our static antiwindup response, and the thin solid line is our plant-order antiwindup response. The trajectories of this antiwindup closed-loop system designed using the pitch angle performance output, particularly with plant-order antiwindup, are highly desirable and are a marked improvement over the scheme proposed in [13].

Example 2 (An Experimental Example): The cart-spring-pendulum system shown in Figs. 8 and 9 (which is available at the Control and Computation Laboratory at the University of California, Santa Barbara) consists of a cart restricted to motion on a straight and level track which is attached via a spring to a fixed wall. A pendulum is suspended from the cart by a hinge so as to be constrained to the vertical plane defined by the track. The cart is equipped with a DC motor that exerts a torque to a small toothed wheel which, in turn, applies a force on the cart. The system will be disturbed by a sharp tap on the pendulum that comes from a human hand. For the purpose of deriving a model, the experimental system will be considered to be composed of a massless spring attached to a frictionless cart from which a slender rod freely hangs.

The output of the system is the position p of the cart, in meters, relative to the spring's equilibrium point and the angular position θ of the pendulum, in radians, relative to the vertical; both positions are measured with optical encoders. The physical inputs of the system are the voltage u applied to the armature of the dc motor, in Volts, and a disturbance force w, in Newtons. The force from the motor f, in Newtons, is modeled as $f = k_1 u - k_2 \dot{p}$. The operating range of the control input is constrained by the range of the D/A converter, $[-5, 5]$ Volts (which, incidentally, nearly covers the entire operating range of

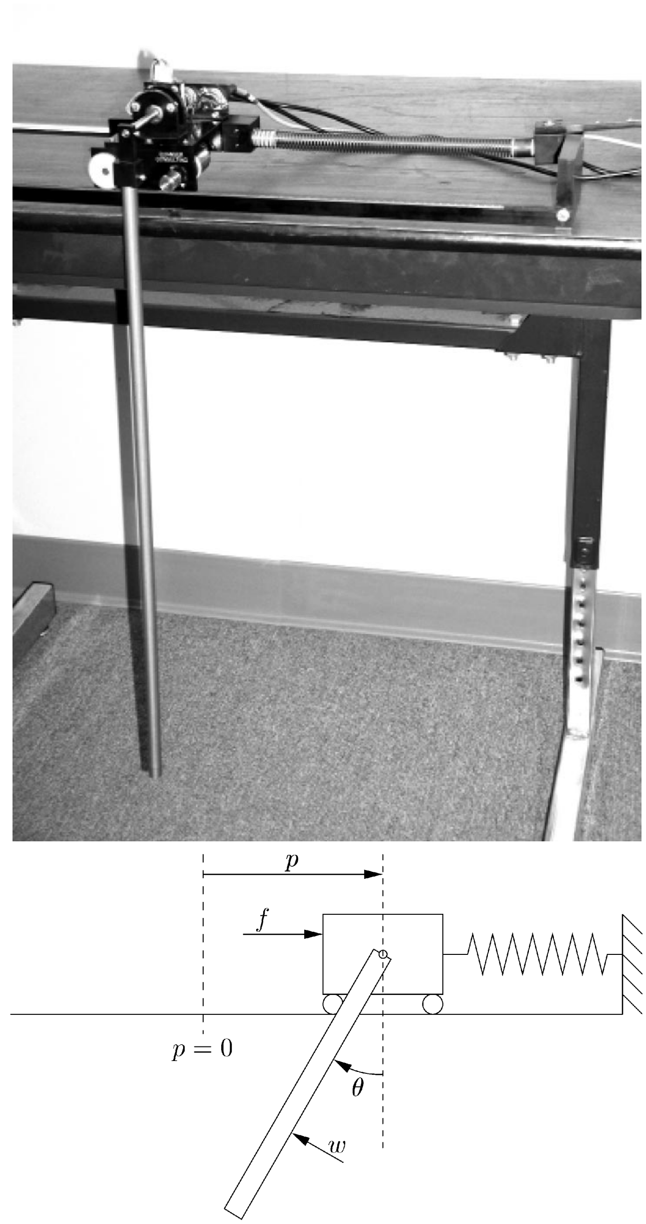

Fig. 8. Damped mass-spring-pendulum system in Example 2.

the DC motor, $[-6, 6]$ Volts). The disturbance w is a force in the plane of motion orthogonal to the pendulum of length $2l$ and acts at a distance of $(4/3)l$ from the cart-pendulum hinge. A nonlinear model of the system can be derived by applying standard Euler–Lagrange techniques. Moreover, defining the plant state as $x_p := [p \ \dot{p} \ \theta \ \dot{\theta}]^T$, a linearized model around the origin is given by (1) and

$$A_p = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -330.46 & -12.15 & -2.44 & 0 \\ 0 & 0 & 0 & 1 \\ -812.61 & -29.87 & -30.10 & 0 \end{bmatrix}$$

$$\Lambda_4 = \begin{bmatrix} 14.339 & -55.258 & -0.10926 & -7.6946 & 0.01282 & 0.30214 & 14.406 & -55.168 & -0.97438 & -0.019895 \\ -76.926 & 528.52 & 2.4325 & 5.9347 & 0.54967 & 0.69361 & -76.628 & 527.19 & -0.13474 & -0.76157 \end{bmatrix}^T.$$

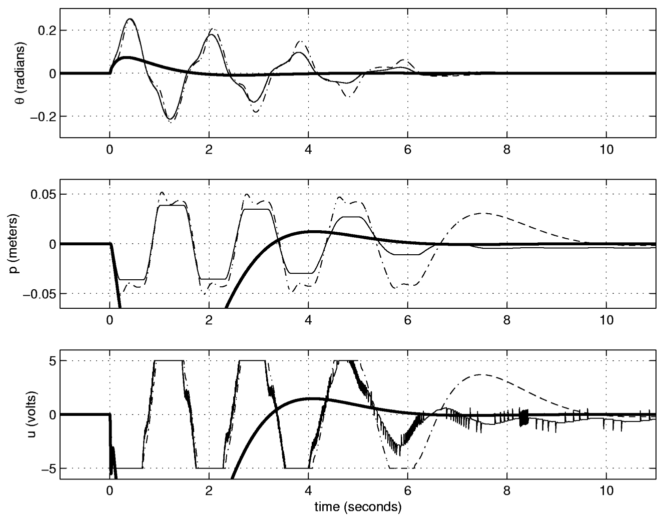

Fig. 9. Example 2. Response to the larger pendulum tap. Simulated unconstrained response (bold solid); simulated saturated response (dash-dotted); experimental saturated response (thin solid).

$$B_{p,u} = \begin{bmatrix} 0 \\ 2.717\,62 \\ 0 \\ 6.682\,68 \end{bmatrix} \quad B_{p,w} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 15.61 \end{bmatrix}$$
$$C_{p,y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad D_{p,yu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$D_{p,yw} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where A_p is Hurwitz.

Suppose the system is allowed to come to rest before it is disturbed and we are interested in the response of the system due to two test pendulum taps, one small[3] and one five times larger. Suppose further the objective is to return the pendulum and cart quickly and gently to their equilibrium after the smaller taps and gracefully handle the larger taps to the pendulum. Following an LQG construction, an observer based controller of the form (2) is designed where

$$K = [64.81 \quad 213.12 \quad 1242.27 \quad 85.82]$$
$$L = \begin{bmatrix} 64 & 2054 & -8 & -1432 \\ -8 & -280 & 142 & 10\,169 \end{bmatrix}^T$$

$A_c := A_p - B_{p,u}K - LC_{p,y}, B_{c,y} := L, C_c := -K$, and $B_{c,w}, D_{c,y}$, and $D_{c,w}$ are zero matrices of appropriate dimensions.

For the simulations reported here, we have used the linearized model of the plant. Indeed, the resulting trajectories are almost the same as the corresponding ones with the nonlinear Euler–Lagrange model, thus confirming the appropriateness of the linear approximation for our operating conditions. For the smaller pendulum tap, the plant input does not saturate and the unconstrained response is deemed desirable, both in simulation and in experiment. The settling time for the pendulum is approximately 1.5 s and for the cart, it is 3 s. The larger pendulum taps, however, give rise to undesirable closed-loop behavior, i.e., the settling time is severely deteriorated. In Fig. 9, the bold solid curve represents the simulated (ideal) unconstrained response, the dash–dotted curve represents the simulation of the saturated response and the thin solid curve represents the corresponding experiment.[4] The noticeable mismatch between the thin solid and the dash-dotted curves is cause by unmodeled effects of the experimental device: mainly backlash and stiction affecting the movement of the cart on the track. Besides these unmodeled phenomena (which cause significant differences, especially on the tails of the responses), the fourth order model represents sufficiently well the dynamics of our experimental system.

Based on the antiwindup construction proposed in Procedure 1, the undesired behavior of Fig. 9 can be mitigated by augmenting the experimental control system according to the diagram in Fig. 3. To determine an optimal selection of the antiwindup compensator matrices we first choose a performance output z. By inspecting Fig. 9, we see that for the larger pendulum taps, the pendulum swings wildly causing the cart to chase after the pendulum, almost in vain. To reduce quickly the magnitude of θ, we choose the matrices related to the performance output z as follows: $C_{p,z} = [0 \;\; 0 \;\; 1 \;\; 0], D_{p,zu} = 0, D_{p,zw} = 0$. A first antiwindup design attempt is carried out by selecting $k = 0$ to explore feasibility of static antiwindup compensation. Unfortunately, for this system, the associated LMIs (12) in Proposition 1 are unfeasible.[5] As a further step, we move to dynamic antiwindup compensation of order $k = n_p$, which, based on the asymptotic stability of the plant, is guaranteed to be feasible by Theorem 3. To construct this compensator, Procedure 1 is applied with $n_{\mathrm{aw}} = n_p$ and the following compensation matrices are obtained, which guarantee a performance level of $\gamma = 181.82$:

$$\Lambda_1 = \begin{bmatrix} -65.02 & 198.43 & 98.11 & -66.75 \\ 223.94 & -697.09 & -347.39 & 247.24 \\ 41.17 & -98.10 & -47.56 & 55.25 \\ -121.39 & 309.97 & 138.31 & -131.52 \end{bmatrix}$$
$$\Lambda_2 = \begin{bmatrix} 0.0688 \\ -0.2620 \\ -0.0637 \\ 0.1559 \end{bmatrix}$$
$$\Lambda_3 = \begin{bmatrix} 41.22 & -160.42 & -106.41 & 82.03 \\ -3469.09 & 8318.57 & 3423.87 & -2388.49 \\ -162.51 & 386.26 & -35.56 & 71.07 \\ -4584.37 & 9490.06 & -11\,350.16 & 11\,407.08 \\ 587.11 & -1687.16 & -821.25 & 632.86 \end{bmatrix}$$
$$\Lambda_4 = \begin{bmatrix} -0.0622 \\ 2.9070 \\ 0.2338 \\ 5.5623 \\ 0 \end{bmatrix}.$$

[3]For simulation purposes, the smaller pendulum tap is modeled a constant force of 1.588 Newton with duration 0.01 s.

[4]Although a continuous time controller has been designed, it is implemented in discrete time. We allow Quanser Consulting Inc. software, WinCon 3.1, to convert our continuous time controller to discrete time using the Runge–Kutta fixed-step solver with sampling time 0.0005 s.

[5]Unfeasibility was determined due the inability of the MATLAB LMI Control Toolbox to find a feasible solution to the LMIs (12).

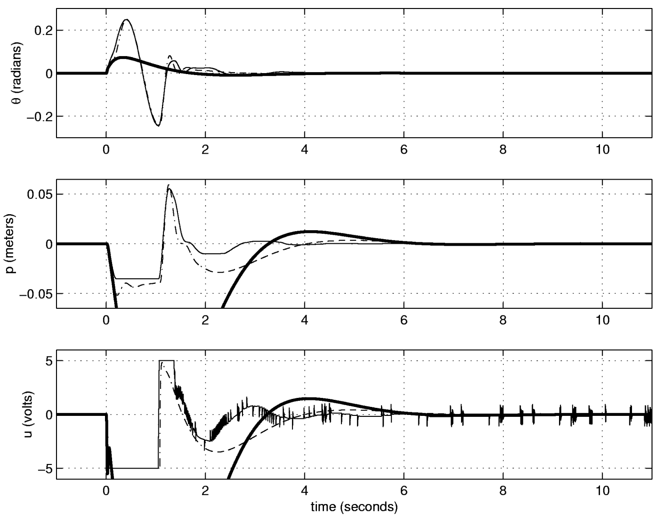

Fig. 10. Example 2. Response to the larger pendulum tap. Simulated unconstrained response (bold solid); simulated response with antiwindup (dash-dotted); experimental response with antiwindup (thin solid).

The thin solid curve in Fig. 10 represents the experimental response of the closed-loop system with dynamic antiwindup compensation to the same disturbance that generates the undesirable response represented in Fig. 9. Similarly to Fig. 9, the dash–dotted curve represents a simulation of the closed-loop with the linear plant model, while the bold solid curve represents a simulation of the unconstrained closed-loop system's response. A comparison between the thin solid responses in Figs. 10 and 9 illustrates that the insertion of the antiwindup compensator greatly improves the experimental response to the larger pendulum taps, while structurally preserving the desirable performance of the (previously designed) unconstrained controller for the smaller pendulum taps. It should be recognized that the tails of the simulated responses are quite different from the experimental ones because of the unmodeled effects commented above. Nevertheless, the plant model is mostly accurate in the operating conditions where the plant input is close to the saturation limits. These are the operating conditions of interest for the antiwindup action, hence a more accurate model of the plant does not seem to be necessary for the antiwindup design.

V. Proof of the Main Result

A. *Proof of Theorem 1*

To prove Theorem 1, the following lemmas will be useful. The proofs of Lemmas 2 and 3 can be carried as in [33].

Lemma 2: Consider a locally Lipschitz function F: $\mathbb{R}^n \mapsto \mathbb{R}^n$ and assume that the Jacobian of F satisfies

$$JF(x) \in \mathcal{M}, \qquad \text{for almost all } x \in \mathbb{R}^n$$

where the set $\mathcal{M}$ is compact, convex, and each matrix in $\mathcal{M}$ is nonsingular. Then there exists a (unique) globally Lipschitz function G: $\mathbb{R}^n \mapsto \mathbb{R}^n$ such that $F(G(x)) = x$ for all $x \in \mathbb{R}^n$. Equivalently, F is a homeomorphism with globally Lipschitz inverse.

Lemma 3: Given two square matrices D and $V = V^T > 0$, if $-2V + VD + D^TV < 0$ then $I - D\Delta$ is nonsingular for all Δ such that the linear map $z \mapsto \Delta z$ belongs to the sector $[0, I]_V$.

Lemma 4: Given any symmetric positive-definite matrix V_s, the function $\phi(\cdot)$ belongs to Φ_{V_s} if and only if the function $\psi(y_c) := y_c - \phi(y_c)$ belongs to Φ_{V_s}.

Proof (Sufficiency): Assume $\phi(\cdot)$ belongs to Φ_{V_s}. Clearly, $\psi(\cdot)$ is globally Lipschitz. Moreover, since $\langle \phi(y) - y, \phi(y)\rangle_{V_s} \le 0$ for all y, then $\langle -\psi(y), y - \psi(y)\rangle_{V_s} \le 0$ for all y, namely $\psi(\cdot)$ belongs to sector $[0, I]_{V_s}$. Moreover, since $J\psi(y_c) = I - J\phi(y_c)$ whenever $J\phi(y_c)$ exists it follows that $\langle J\psi(y), I - J\psi(y)\rangle_{V_s} \ge 0$ for almost all $y \in \mathbb{R}^{n_u}$. Thus $\psi(\cdot)$ belongs to Φ_{V_s}. The necessity can be proven by swapping the functions $\phi(\cdot)$ and $\psi(\cdot)$ in the previous proof. ∎

The following facts will also be useful for the proof of Theorem 1.

Fact 1: By noting that y_c and z are linear functions of x, q, and w, writing the upcoming (23) in matrix inequality form and taking its Schur complement [4, p. 7], it can be shown that given $P = P^T > 0$, and $V = x^TPx$, where its derivative along the dynamics of the system (8), (9) is $\dot{V} = 2x^TP(Ax + B_q q + B_w w)$, then

$$\dot{V} + \frac{1}{\gamma}z^Tz - \gamma w^Tw + 2\tau q^TW(y_c - q) < 0 \qquad \forall (x, q, w) \ne 0 \quad (23)$$

if and only if the equation shown at the bottom of the page holds.

Fact 2: By employing the $\mathcal{S}$-procedure [4, p. 24], it is shown that given any symmetric positive definite matrix W and (as in Fact 1) $\dot{V} = 2x^TP(Ax + B_q q + B_w w)$, if

1) there exists a scalar $\tau \ge 0$ such that

$$\dot{V} + \frac{1}{\gamma}z^Tz - \gamma w^Tw + 2\tau q^TW(y_c - q) < 0 \quad \forall (x, q, w) \ne 0$$

then

2)

$$\dot{V} + \frac{1}{\gamma}z^Tz - \gamma w^Tw < 0 \quad (25a)$$

for all $(x, q, w) \ne 0$ such that

$$q^TW(y_c - q) = q^TW(C_yx + D_{yq}q + D_{yw}w - q) \ge 0. \quad (25b)$$

$$\begin{bmatrix} A^TP + PA & PB_q + \tau C_y^TW & PB_w & C_z^T \\ B_q^TP + \tau WC_y & \tau(WD_{yq} + D_{yq}^TW - 2W) & \tau WD_{yw} & D_{zq}^T \\ B_w^TP & \tau D_{yw}^TW & -\gamma I & D_{zw}^T \\ C_z & D_{zq} & D_{zw} & -\gamma I \end{bmatrix} < 0.$$

In addition, if there exists at least one selection (x^*, q^*, w^*) such that

$$q^{*T} W(C_y x^* + D_{yq} q^* + D_{yw} w^* - q^*) > 0 \tag{26}$$

then item 2) implies item 1).

Fact 3: There exists a selection $(x^\star, q^\star, w^\star)$ that satisfies (26).

Proof: If there exist $x^\star, w^\star$ such that $[C_y x^\star + D_{yw} w^\star] \neq 0$, then pick $q^\star = \varepsilon[C_y x^\star + D_{yw} w^\star]$ with ε sufficiently small to satisfy (26). Conversely, if $C_y x + D_{yw} w = 0$ for all (x, w), then the controller $\mathcal{C}$ is identically zero. In this trivial case, $u = 0$ for all times. Namely, since the saturation never activates, the antiwindup problem is nonexistent. From a more system theoretic viewpoint, in this case the optimal performance γ is the $\mathcal{L}_2$ gain of the open-loop plant, and an antiwindup compensator that achieves this performance level is the identically zero antiwindup compensator. ■

Proof of Theorem 1:

Necessity Assume that for a given plant, controller and antiwindup compensator of order n_{aw}, well-posedness and quadratic performance of level $\tilde{\gamma}$ are guaranteed in the sense of Definition 3. Lemma 4 guarantees $\psi(\cdot)$ belongs to Φ_{V_s}, and therefore $\langle q, y_c - q\rangle_{V_s} \geq 0$. Hence, by inequality (7), there exists a quadratic Lyapunov function $V(x) = x^T P x$ where $P = P^T > 0$ such that item 2 in Fact 2 is satisfied with $W = V_s$ and $\gamma = \tilde{\gamma}$. Combined with Fact 3, Fact 2 implies that there exists a constant $\tau \geq 0$ that satisfies (24). Finally, by Fact 1, (27), shown at the bottom of the page, holds. Moreover, since all block diagonal terms in (27) must be nonzero, then $\tau \neq 0$. Defining $Q := P^{-1}$ and $U := \tau^{-1} V_s^{-1}$ and then premultiplying and postmultiplying (27) by the symmetric block diagonal matrix $\operatorname{diag}(Q, U, I, I)$, it follows that there exists $Q = Q^T > 0$ and $\delta := \tau^{-1} > 0$ that satisfy (10a), as desired.

Sufficiency If there exist Q, γ, and $\delta > 0$ that satisfy (10), define $P := Q^{-1}$ and $\tau := \delta^{-1}$ and premultiply and postmultiply (10a) by the symmetric block diagonal matrix $\operatorname{diag}(P, \tau V_s, I, I)$. The resulting inequality guarantees (27) because $\tilde{\gamma} \geq \gamma$. Then, Fact 1 and Fact 2 guarantee that the function $V(x) = x^T P x$ satisfies item 2 in Fact 2 with $W = V_s$. Since $q = \psi(y_c)$ and $\psi(\cdot)$ belongs to $[0, I]_{V_s}$, inequality (25b) is always satisfied by the trajectories of the closed-loop system (1), (2), (5), (6). Hence, since the inequality in (25a) is strict, there exists a small enough $\epsilon > 0$ such that inequality (7) in item 2 of Definition 3 is guaranteed.

To show well-posedness in item 1) of Definition 3, rewrite the interconnection of (8) and the middle equation of (9) as

$$H(y_c) := y_c - D_{yq}\psi(y_c) = C_y x + D_{yw} w$$

where $H(\cdot)$ is globally Lipschitz. Since, by Lemma 4, the function $\psi(\cdot)$ belongs to Φ_{V_s}, then almost everywhere, $\Delta = J\psi(y_c)$ is such that $\langle \Delta, I - \Delta\rangle_{V_s} \geq 0$. This can be rewritten as

$$-2\Delta^T V_s \Delta + V_s \Delta + \Delta^T V_s \geq 0 \tag{28}$$

where $\Delta = J\psi(y_c)$ almost everywhere. Then, for almost all y_c, the Jacobian of $H(y_c)$ satisfies

$$JH(y_c) \in \{(I - D_{yq}\Delta), \Delta : -2\Delta^T V_s \Delta + V_s \Delta + \Delta^T V_s \geq 0\} =: J\mathcal{H}$$

where the set $J\mathcal{H}$ is compact by the boundedness of Δ and because the inequality in (28) is nonstrict. The set $J\mathcal{H}$ is also convex because, by Schur complement, inequality (28) can be written as an LMI in Δ. Furthermore, since the diagonal entries of (10a) are negative definite, then $-2V_s + V_s D + D^T V_s < 0$ and, by Lemma 3, each matrix in the set $J\mathcal{H}$ is nonsingular. Then, by Lemma 2 there exists a (unique) globally Lipschitz function $\zeta(\cdot)$ such that $y_c = \zeta(C_y x + D_{yw} w)$. Finally, the Lipschitz property of the right-hand side of (9) guarantees the existence and uniqueness of solutions, thus proving well-posedness of the interconnection between (8) and (9). •

B. Proof of Theorems 2 and 4

A key step in the proof of Theorems 2 and 4 is the connection between the matrix conditions $\mathrm{MC}(\mathcal{P}, \mathcal{C}, n_{\text{aw}}, \tilde{\gamma})$ in Definition 4, the LMIs for analysis (10) in Theorem 1, and the LMI (22) in the final step of Procedure 1. The LMIs (10a) and (22) coincide but are in different unknowns; the LMI (10a) is in the unknown Q and the LMI (22) is in the unknown Λ. Indeed, since the system (9) represented by the diagram in Fig. 4 coincides with the system (6), (15) represented in Fig. 5, the matrices in (9) can be expressed in terms of the matrices in (6), (15). Within this equivalence, it is easy to check that the matrices B_w, D_{yw}, and D_{zw} in (9) coincide with those defined in (19c) and the remaining matrices in (9) satisfy

$$\begin{aligned} A &= A_\circ + H_1^T \Lambda G_1 \quad C_y = C_{y\circ} + H_2^T \Lambda G_1 \\ C_z &= C_{z\circ} + H_3^T \Lambda G_1 \\ B_q &= B_{q\circ} + H_1^T \Lambda G_2 \quad D_{yq} = D_{yq\circ} + H_2^T \Lambda G_2 \\ D_{zq} &= D_{zq\circ} + H_3^T \Lambda G_2. \end{aligned} \tag{29}$$

The following theorem establishes the equivalence between the feasibility of the matrix conditions $\mathrm{MC}(\mathcal{P}, \mathcal{C}, n_{\text{aw}}, \tilde{\gamma})$ in Definition 4 and the feasibility of the matrix constraints (10) and (22).

Theorem 5:

1) Given the plant $\mathcal{P}$ in (1), controller $\mathcal{C}$ in (2), integer $n_{\text{aw}} \geq 0$ and scalar $\tilde{\gamma}$, there exist matrices Q, Λ and scalars γ, δ

$$\begin{bmatrix} A^T P + PA & PB_q + \tau C_y^T V_s & PB_w & C_z^T \\ B_q^T P + \tau V_s C_y & \tau\left(-2V_s + V_s D_{yq} + D_{yq}^T V_s\right) & \tau V_s D_{yw} & D_{zq}^T \\ B_w^T P & \tau D_{yw}^T V_s & -\tilde{\gamma} I & D_{zw}^T \\ C_z & D_{zq} & D_{zw} & -\tilde{\gamma} I \end{bmatrix} < 0. \tag{27}$$

satisfying (10) (with the definitions (19), (29)) if and only if the matrix conditions $\mathrm{MC}(\mathcal{P}, \mathcal{C}, n_{\mathrm{aw}}, \tilde{\gamma})$ are feasible.

2) Given a feasible solution (R, S, γ) to $\mathrm{MC}(\mathcal{P}, \mathcal{C}, n_{\mathrm{aw}}, \tilde{\gamma})$, the matrix Q constructed in (16), (17), (18) guarantees that the LMI (22) in the unknowns $(\Lambda, \delta, \gamma)$ is solvable and the arising solution $(Q, \Lambda, \delta, \gamma)$ also satisfies (10) [with the definitions (19) and (29)]. ∎

Proof: See Section V-B. ∎

Proof of Theorem 2: The composition of Theorem 1 and item 1 in Theorem 5 imply Theorem 2.

Proof of Theorem 4: Step 1) of Procedure 1 is assumed to be solvable. Steps 2) and 3) are constructive. For Step 4), the matrices (21) can always be constructed based on the matrices computed at the preceding steps. Moreover, by item 2) in Theorem 5, the matrix Q constructed in Step 2 guarantees that the LMI (22) is solvable for Λ and any feasible solution $(\Lambda, \delta, \gamma)$ to the LMI (22) is such that $(Q, \Lambda, \delta, \gamma)$ satisfies (10). Hence, by Theorem 1, the antiwindup closed-loop system (8), (9) corresponding to Λ is well-posed and guarantees quadratic performance of level $\tilde{\gamma}$. •

The following lemmas, proven in [8], [12] and [23], respectively, will be useful for the proof of Theorem 5.

Lemma 5 (Projection Lemma [8, Lemma 3.1]): Given a symmetric matrix $\Psi \in \mathbb{R}^{m \times m}$ and two matrices G, H of column dimension m, consider the problem of finding some matrix Λ of compatible dimensions such that

$$\Psi + G^T \Lambda^T H + H^T \Lambda G < 0. \tag{30}$$

Denote by W_G, W_H any matrices whose columns form bases of the null space of G and H, respectively. Then (30) is solvable for Λ if and only if

$$W_H^T \Psi W_H < 0 \tag{31a}$$

$$W_G^T \Psi W_G < 0. \tag{31b}$$

Lemma 6 [23]: Let $R, Z \in \mathbb{R}^{n \times n}$ be symmetric positive definite matrices. Then the two conditions

$$Z - R^{-1} \geq 0$$
$$\mathrm{rank}[Z - R^{-1}] \leq n_{\mathrm{aw}}$$

hold if and only if there exist $N \in \mathbb{R}^{n \times n_{\mathrm{aw}}}$ and $M \in \mathbb{R}^{n_{\mathrm{aw}} \times n_{\mathrm{aw}}}$, with $M = M^T > 0$ such that

$$\begin{bmatrix} R & N \\ N^T & M \end{bmatrix} > 0 \quad \begin{bmatrix} R & N \\ N^T & M \end{bmatrix}^{-1} = \begin{bmatrix} Z & ? \\ ? & ? \end{bmatrix}$$

where $?$ denotes matrix entries that we do not care about.

Proof of Theorem 5: We first prove the necessity part of item 1. According to the definitions (19), (20), (21), and (29), (10a) coincides with (22) as shown in (33) at the bottom of the page. We will apply Lemma 5 to inequality (33) [which coincides with (10a)] to show that there exists a feasible solution $(Q, \Lambda, \gamma, \delta)$ to (10) if and only if the conditions $\mathrm{MC}(\mathcal{P}, \mathcal{C}, n_{\mathrm{aw}}, \tilde{\gamma})$ in Definition 1 are feasible. In particular, we will show that (31a) is equivalent to (11a) and that (31b) is equivalent to (11b), that the coupling between (11a) and (11b) through Ψ can be rewritten as (11e), (11f).

Condition (11a): According to (19b), (21b) and the explicit expressions for the matrices in (15), H can be written as (34), shown at the bottom of the page, where $\Delta_{yp} := (I - D_{p,yu} D_{c,y})^{-1}$ and $\Delta_{yc} := (I - D_{c,y} D_{p,yu})^{-1}$ are well defined (namely the matrices in parentheses are invertible) by the well-posedness of the unconstrained interconnection. According to this special structure, a matrix that spans the null space of H is

$$W_H = \begin{bmatrix} I_{n_p} & 0 & 0 & -B_{p,u} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_w} & 0 \\ 0 & 0 & 0 & -D_{p,zu} & 0 & I_{n_z} \end{bmatrix}^T. \tag{35}$$

Indeed, by the assumption of well-posedness of the unconstrained closed-loop system, Δ_{yc} is full rank, hence, according to the (34), the dimension of the null space of H is necessarily $n_p + n_z + n_w$. Moreover, the rank of W_H is $n_p + n_z + n_w$ and it can be verified by computation that $HW_H = 0$.

$$\begin{aligned}
&\begin{bmatrix} QA^T + AQ & B_q U + QC_y^T & B_w & QC_z^T \\ UB_q^T + C_y Q & D_{yq} U + UD_{yq}^T - 2U & D_{yw} & UD_{zq}^T \\ B_w^T & D_{yw}^T & -\gamma I & D_{zw}^T \\ C_z Q & D_{zq} U & D_{zw} & -\gamma I \end{bmatrix} \\
&= \Psi + \begin{bmatrix} H_1^T \Lambda G_1 Q + QG_1^T \Lambda^T H_1 & H_1^T \Lambda G_2 U + QG_1^T \Lambda^T H_2 & 0 & QG_1^T \Lambda^T H_3 \\ H_2^T \Lambda G_1 Q + UG_2^T \Lambda^T H_1 & H_2^T \Lambda G_2 U + UG_2^T \Lambda^T H_2 & 0 & UG_2^T \Lambda^T H_3 \\ 0 & 0 & 0 & 0 \\ H_3^T \Lambda G_1 Q & H_3^T \Lambda G_2 U & 0 & 0 \end{bmatrix} \\
&= \Psi + H^T \Lambda G + G^T \Lambda^T H \leq 0.
\end{aligned} \tag{33}$$

$$H = \begin{bmatrix} 0 & 0 & I_{n_{\mathrm{aw}}} & 0 & 0 & 0 \\ 0 & I_{n_c} & 0 & 0 & 0 & 0 \\ \Delta_{yc}^T B_{p,u}^T & D_{p,yu}^T \Delta_{yp}^T B_{c,y}^T & 0 & \Delta_{yc}^T & 0 & \Delta_{yc}^T D_{p,zu}^T \end{bmatrix} \tag{34}$$

Assume that, according to (18), the matrix Q is partitioned as

$$Q = \begin{bmatrix} R & N \\ N^T & M \end{bmatrix}$$

where

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix}.$$

Then, inequality (31a) can be computed explicitly based on (35) and (21a) with (19a), (19c). After some computations it follows that $W_H^T \Psi W_H < 0$ coincides with the inequality in (11a), as desired.

Condition (11b): According to (21c), the matrix G can be factored as follows:

$$\begin{aligned} G = G_O\bar{T} &= [G_1 Q \quad G_2 U \quad 0 \quad 0] \\ &= \underbrace{[G_1 \quad G_2 \quad 0 \quad 0]}_{G_O} \underbrace{\mathrm{diag}(Q, U, I, I)}_{\bar{T}} \end{aligned}$$

where $G_O \in \mathbb{R}^{(n_{\mathrm{aw}}+n_u)\times(n+n_u+n_z+n_w)}$ and $\bar{T} \in \mathbb{R}^{(n+n_u+n_z+n_w)\times(n+n_u+n_z+n_w)}$. Since $\bar{T}$ is invertible (indeed, $Q > 0$ and $U > 0$ by assumption), we can write

$$W_G^T \Psi W_G = W_G^T \bar{T} \underbrace{\bar{T}^{-1} \Psi \bar{T}^{-1}}_{\bar{\Phi}} \underbrace{\bar{T} W_G}_{W_{G_O}} = W_{G_O}^T \bar{\Psi} W_{G_O}$$

where W_{G_O} spans the null space of G_O and, according to the definitions $P = Q^{-1}$ and $U = W^{-1}$ (36) shown at the bottom of the page holds. Based on (19b), we can write explicitly the entries of G_O as

$$G_O = [G_1 \quad G_2 \quad 0 \quad 0] = \begin{bmatrix} 0 & 0 & I_{n_{\mathrm{aw}}} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_u} & 0 & 0 \end{bmatrix}.$$

Hence, a matrix $W_{G_O} \in \mathbb{R}^{(n_p+n_c+n_{\mathrm{aw}}+n_u+n_z+n_w)\times(n_p+n_c+n_z+n_w)}$ that spans the null space of G_O is

$$W_{G_O} := \begin{bmatrix} I_{n_p} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{n_c} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_z} & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n_w} \end{bmatrix}^T. \tag{37}$$

Using the partition of the matrix

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$$

we can compute explicitly the inequality (31b) based on the definitions (36) and (37) and substituting (19a) and (19c) into the entries of $\bar{\Psi}$. After some computations it follows that $W_G^T \Psi W_G < 0$ coincides with the inequality in (11b), as desired.

Conditions (11e) and (11f): Since $P = Q^{-1}$, and $S = P_{11}^{-1}$, then from the partitions of P and Q we have

$$Q = \begin{bmatrix} R & N \\ N^T & M \end{bmatrix} > 0 \quad \text{and} \quad Q^{-1} = P = \begin{bmatrix} S^{-1} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$$

which can be rewritten as follows:

$$\begin{bmatrix} R & N \\ N^T & M \end{bmatrix} > 0 \quad \begin{bmatrix} R & N \\ N^T & M \end{bmatrix}^{-1} = \begin{bmatrix} S^{-1} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}. \tag{38}$$

By virtue of Lemma 6 expressions (38) are equivalent to

$$S^{-1} - R^{-1} \geq 0 \tag{39a}$$

$$\mathrm{rank}[S^{-1} - R^{-1}] \leq n_{\mathrm{aw}}. \tag{39b}$$

Premultiplying and postmultiplying the matrices in (39b) by S and R, respectively and performing a Cholesky factorization (see, e.g., [28, p. 195],) on (39a), we get Conditions (11e) and (11f), thus completing the proof of the necessity part of item 1). To prove the sufficiency in item 1), the aforementioned reasoning can be reversed. In particular, conditions (11e) and (11f) imply (39), which by Lemma 6 imply the existence of M, N satisfying (38). Finally, (11a) and (11b) hold with $\gamma \leq \tilde{\gamma}$, hence, by Lemma 5, inequality (30) holds too. This, in turn, implies that (10) is solvable.

Finally, we prove item 2) of the theorem. Since (22) coincides with (10) with the selection for Q (16)–(18), then provided the matrix Q satisfies expression (38), the proof of the sufficiency of item 1) can be followed verbatim to show that (22) is solvable with (16)–(18). To show that the construction (16)–(18) for Q satisfies (38), note that by the formulae for the inversion of block matrices [31, p. 23], the upper left block of P needs to satisfy

$$P_{11} = S^{-1} = R^{-1} + R^{-1}N(M - N^T R^{-1} N)^{-1} N^T R^{-1}$$

which, when premultiplied and postmultiplied by R and substituting the selection (17) for M, becomes

$$R + NN^T = RS^{-1}R$$

which, by (16), is always satisfied. •

VI. Conclusion

The problem of synthesizing fixed-order antiwindup compensators which meet an $\mathcal{L}_2$ performance bound has been addressed. The main results have demonstrated how a Lyapunov formulation of this problem can be expressed as a nonconvex optimization problem which closely resembles the LMI formulation of $\mathcal{H}_\infty$ controller synthesis. For certain antiwindup compensator state dimensions, the optimization problem is actually convex and hence can be solved using standard methods, which allow the construction of an optimal compensator that achieves a maximum performance level globally, via convex optimization.

$$\bar{\Psi} = \begin{bmatrix} A_\circ^T P + PA_\circ & PB_{q\circ} + C_{y\circ}^T W & PB_w & C_{z\circ}^T \\ B_{q\circ}^T P + WC_{y\circ} & -2W + WD_{yq\circ} + D_{yq\circ}^T W & WD_{yw} & D_{zq\circ}^T \\ B_w^T P & D_{yw}^T W & -\gamma I & D_{zw}^T \\ C_{z\circ} & D_{zq\circ} & D_{zw} & -\gamma I \end{bmatrix}. \tag{36}$$

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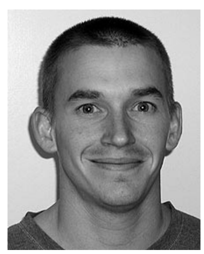

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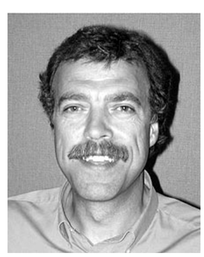

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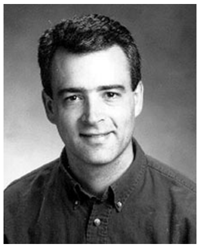

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Brief paper

Anti-windup synthesis for linear control systems with input saturation: Achieving regional, nonlinear performance[☆]

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Abstract

In this paper, we present LMI-based synthesis tools for regional stability and performance of linear anti-windup compensators for linear control systems. We consider both static and dynamic compensators. Algorithms are developed that minimize the upper bound on the regional $\mathscr{L}_2$ gain for exogenous inputs with $\mathscr{L}_2$ norm bounded by a given value, and that minimize this upper bound with a guaranteed reachable set or domain of attraction. Based on the structure of the optimization problems, it is shown that for systems whose plants have poles in the closed left-half plane, plant-order dynamic anti-windup can achieve semiglobal exponential stability and finite $\mathscr{L}_2$ gain for exogenous inputs with $\mathscr{L}_2$ norm bounded by any finite value. The problems are studied in a general setting where the only requirement on the linear control system is well-posedness and internal stability. The effectiveness of the proposed techniques is illustrated with an example.

Keywords: Anti-windup synthesis; Nonlinear $\mathscr{L}_2$ gain; Reachable set; Domain of attraction

1. Introduction

Anti-windup compensators are intended to maintain the performance of a linear control system in the local operating range, while guaranteeing global stability or minimizing the degradation of the global performance in the presence of actuator saturation. Earlier anti-windup compensators were constructed heuristically from experience and simulations (e.g., Lozier, 1956). Since the early 1990s, systematic approaches have been proposed, for instance, the reference/command governor scheme (e.g., Angeli & Mosca, 1999; Gilbert & Kolmanovsky, 1999; Shamma, 2000), the approaches based on H_∞ optimal control (e.g., Crawshaw & Vinnicombe, 2002; Edwards & Postlethwaite, 1999; Miyamoto & Vinnicombe, 1996), and the linear matrix inequality (LMI) based techniques, (e.g., Cao, Lin, & Ward, 2002; Gomes da Silva & Tarbouriech, 2005; Grimm et al., 2003; Grimm, Teel, & Zaccarian, 2004; Marcopoli & Phillips, 1996; Mulder, Kothare, & Morari, 2001; Zaccarian & Teel, 2002). For more comprehensive overviews of modern anti-windup approaches, see the works in Turner and Zaccarian (2006), Glattfelder, Ohta, Mosca, and Weiland (2000), and Kothare, Campo, Morari, and Nett (1994).

Among the LMI-based techniques, Mulder et al. (2001) studied the general case where the controller is dynamic, the exogenous input directly enters the actuator and there is an important correction term in the output equation of the controller. In Mulder et al. (2001), static anti-windup compensators were constructed for global stabilization and reduced $\mathscr{L}_2$ gain performance. These synthesis problems were first cast as convex optimization problems with LMI constraints for the general case. The recent work of Grimm et al. (2003) reached further by constructing dynamic anti-windup compensators for reduced

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global $\mathscr{L}_2$ gain via LMI optimization. In addition to showing the advantage of dynamic anti-windup over static anti-windup by numerical examples, Grimm et al. (2003) justified the original intention of introducing anti-windup compensation through rigorous theoretical analysis. It was concluded that, for a configuration with an exponentially stable plant and a stabilizing linear controller, the global $\mathscr{L}_2$ gain can always be made finite by designing the dynamic anti-windup compensator. This conclusion promises global stability before the anti-windup compensator is constructed, even before the linear controller is designed, thus giving full confidence in designing a linear controller for the best local performance.

While a finite global $\mathscr{L}_2$ gain gives a guaranteed global closed-loop performance, it might be conservative for practical situations where the $\mathscr{L}_2$ norm of the exogenous input is bounded below a known value. On the other hand, for plants which are not exponentially stable, the global $\mathscr{L}_2$ gain does not exist and it is necessary to determine the $\mathscr{L}_2$ gain for a class of norm-bounded inputs. These situations motivated us to estimate the nonlinear $\mathscr{L}_2$ gain for general linear saturated systems with anti-windup augmentation and to design an anti-windup compensator to minimize the regional $\mathscr{L}_2$ gain.

In our recent work, Hu, Teel, and Zaccarian (2006), we developed regional analysis tools for characterization of the nonlinear $\mathscr{L}_2$ gain and the reachable set for general saturated systems. The regional analysis results were based on two forms of parameterized differential inclusions: the polytopic differential inclusion and the norm-bounded differential inclusion (NDI). The parameter in each inclusion reflects the regional property and can be incorporated into the LMI-based optimization problems.

In this paper, we propose design methods for the construction of dynamic/static anti-windup compensators to optimize the regional performance evaluated in Hu et al. (2006) via the NDI description. The only assumptions on the original control systems are well-posedness and local stability. Our design methods will be theoretically justified via some semiglobal results for control systems whose plants are not exponentially unstable.

The paper is organized as follows. Section 2 describes the design problems and recalls some recent results on regional performance analysis. Section 3 addresses the three design problems and provides feasibility conditions for the existence of the respective solutions. Provided the corresponding feasibility conditions are satisfied, the anti-windup compensator synthesis are carried out following the procedure in Section 4. Section 5 illustrates the results of the paper through a numerical example.

Notation. Given a square matrix X we denote $\operatorname{He} X := X + X^{\mathrm{T}}$. For $P = P^{\mathrm{T}} > 0$, denote $\mathscr{E}(P) := \{x : x^{\mathrm{T}} P x \leqslant 1\}$. We call a linear system "marginally stable/unstable" if it has poles on the imaginary axis but not in the open right half plane.

2. Problem statement and preliminary results

2.1. Anti-windup configuration and design objectives

Consider a linear plant,

$$\mathscr{P}\begin{cases}\dot{x}_{\mathrm{p}} = A_{\mathrm{p}}x_{\mathrm{p}} + B_{\mathrm{p},u}u + B_{\mathrm{p},w}w,\\ y = C_{\mathrm{p},y}x_{\mathrm{p}} + D_{\mathrm{p},yu}u + D_{\mathrm{p},yw}w,\\ z = C_{\mathrm{p},z}x_{\mathrm{p}} + D_{\mathrm{p},zu}u + D_{\mathrm{p},zw}w,\end{cases} \tag{1}$$

where $x_{\mathrm{p}} \in \mathbb{R}^{n_{\mathrm{p}}}$ is the state, $u \in \mathbb{R}^{n_u}$ the control input, $w \in \mathbb{R}^{n_{\mathrm{w}}}$ the exogenous input (possibly containing disturbance, reference and measurement noise), $y \in \mathbb{R}^{n_y}$ the measurement output and $z \in \mathbb{R}^{n_z}$ the performance output. Assume that a linear controller is designed,

$$\mathscr{C}\begin{cases}\dot{x}_{\mathrm{c}} = A_{\mathrm{c}}x_{\mathrm{c}} + B_{\mathrm{c},y}y + B_{\mathrm{c},w}w + v_1,\\ y_{\mathrm{c}} = C_{\mathrm{c}}x_{\mathrm{c}} + D_{\mathrm{c},y}y + D_{\mathrm{c},w}w + v_2,\end{cases} \tag{2}$$

where $x_{\mathrm{c}} \in \mathbb{R}^{n_{\mathrm{c}}}$ is the controller state and $y_{\mathrm{c}} \in \mathbb{R}^{n_u}$ is the controller output, v_1 and v_2 will be used for anti-windup augmentation. In the absence of actuator saturation, the *unconstrained closed-loop* is formed by setting

$$u = y_{\mathrm{c}}, \quad v_1 = 0, \quad v_2 = 0. \tag{3}$$

Throughout the paper we assume the following.

Assumption 1. The unconstrained closed-loop system (1)–(3) is well posed and internally stable.

In the presence of actuator saturation, the relation between u and y_{c} is described as $u = \mathrm{sat}(y_{\mathrm{c}})$, where $\mathrm{sat}(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is the symmetric decentralized saturation function with its ith component depending only on the ith input component $y_{\mathrm{c}i}$ as follows $u_i := \mathrm{sign}(y_{\mathrm{c}i})\min\{|y_{\mathrm{c}i}|, \bar{u}_i\}$, $i = 1, \ldots, n_u$.

To minimize performance degradation caused by saturation, the closed-loop system can be augmented with the following anti-windup compensator:

$$\mathscr{AW}\begin{cases}\dot{x}_{\mathrm{aw}} = A_{\mathrm{aw}}x_{\mathrm{aw}} + B_{\mathrm{aw}}(\mathrm{sat}(y_{\mathrm{c}}) - y_{\mathrm{c}}),\\ v = C_{\mathrm{aw}}x_{\mathrm{aw}} + D_{\mathrm{aw}}(\mathrm{sat}(y_{\mathrm{c}}) - y_{\mathrm{c}}),\end{cases} \tag{4}$$

where $v = [v_1^{\mathrm{T}} \;\; v_2^{\mathrm{T}}]^{\mathrm{T}}$ is used as the anti-windup correction term in (2), and the unconstrained interconnection (3) is replaced by

$$u = \mathrm{sat}(y_{\mathrm{c}}). \tag{5}$$

The resulting nonlinear closed-loop (1), (2), (4), (5) is depicted in Fig. 1 and will be denoted *anti-windup closed-loop* henceforth.

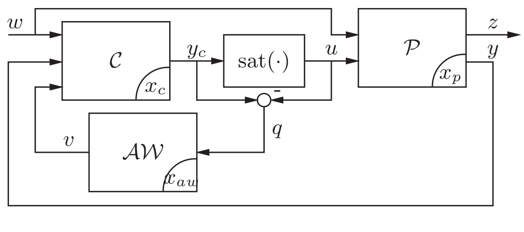

Fig. 1. The anti-windup closed-loop system.

The objective of this paper is to address the following synthesis problems:

Problem S1. *Consider the plant* (1), *the controller* (2) *and a bound s on* $\|w\|_2$. *Design an anti-windup compensator* (4) *such that the relation*

$$\|z\|_2 \leqslant \gamma \|w\|_2, \quad \forall w \text{ such that } \|w\|_2 \leqslant s, \quad x(0)=0 \tag{6}$$

is satisfied with a minimal γ.

Problem S2. *Consider the plant* (1), *the controller* (2) *and a bound s on* $\|w\|_2$. *Given a desired reachable set* $\mathscr{R}_\mathrm{p} \subset \mathbb{R}^{n_\mathrm{p}}$. *Design an anti-windup compensator* (4) *such that with* $x(0)=0$ *and* $\|w\|_2 \leqslant s$, *we have* $x_\mathrm{p}(t) \in \mathscr{R}_\mathrm{p}$ *for all* $t>0$ *while* (6) *is satisfied with a minimal* γ.

Problem S3. *Consider the plant* (1), *the controller* (2) *and a bound s on* $\|w\|_2$. *Given a desired stability region* $\mathscr{S}_\mathrm{p} \subset \mathbb{R}^{n_\mathrm{p}}$. *Design an anti-windup compensator such that the anti-windup closed-loop system is exponentially stable with stability region including* $\mathscr{S}_\mathrm{p}$ *in the plant directions, namely for every* $x_\mathrm{p}(0) \in \mathscr{S}_\mathrm{p}$, *there exist* $x_\mathrm{c}(0) \in \mathbb{R}^{n_\mathrm{c}}$, $x_\mathrm{aw}(0) \in \mathbb{R}^{n_\mathrm{aw}}$ *to make* $\lim_{t\to\infty} x(t)=0$ *while* (6) *is satisfied with a minimal* γ.

2.2. Preliminary results on performance analysis

In this section, we summarize the analysis results from Gomes da Silva and Tarbouriech (2005) and Hu et al. (2006) derived from a modified sector condition satisfied by the dead-zone function $\mathrm{d}z(y_\mathrm{c}) := y_\mathrm{c} - \mathrm{sat}(y_\mathrm{c})$. In particular, for the function $\mathrm{d}z(\cdot)$, given any $r \in \{r : -\bar{u}_i \leqslant r_i \leqslant \bar{u}_i,\ \forall i=1,\ldots,n_u\}$, the inequality $2\mathrm{d}z(y_\mathrm{c})^\mathrm{T} U^{-1}(\mathrm{sat}(y_\mathrm{c})+r) \geqslant 0$ holds for any positive definite diagonal matrix $U \in \mathbb{R}^{n_u \times n_u}$. From this fact, LMI conditions can be directly obtained when characterizing the directional derivative of a quadratic Lyapunov function along the flow equation of the saturated system. To this aim, the anti-windup closed-loop system of Fig. 1 can be represented in the following compact form:

$$\begin{aligned}
\dot{x} &= Ax + B_q \mathrm{d}z(y_\mathrm{c}) + B_w w,\\
y_\mathrm{c} &= C_y x + D_{yq}\mathrm{d}z(y_\mathrm{c}) + D_{yw} w,\\
z &= C_z x + D_{zq}\mathrm{d}z(y_\mathrm{c}) + D_{zw} w,
\end{aligned} \tag{7}$$

where $x := [x_\mathrm{p}^\mathrm{T}\ x_\mathrm{c}^\mathrm{T}\ x_\mathrm{aw}^\mathrm{T}]^\mathrm{T} \in \mathbb{R}^n$, $n := n_\mathrm{p} + n_\mathrm{c} + n_\mathrm{aw}$ and, by Assumption 1, the matrices appearing in (7) are uniquely determined from the matrices for the plant, the controller and the anti-windup in (1), (2), (4).

Proposition 1. *Given* $Q \in \mathbb{R}^{n\times n}$, $Q = Q^\mathrm{T} > 0$. *Consider system* (7).

1. *If there exist* $Y \in \mathbb{R}^{n_u \times n}$ *and a diagonal* $U>0$ *satisfying* (Y_i *denotes the ith row of* Y)
$$\begin{bmatrix} \bar{u}_i^2/s^2 & Y_i \\ Y_i^\mathrm{T} & Q \end{bmatrix} \geqslant 0, \quad i=1,\ldots,n_u, \tag{8}$$
$$\mathrm{He}\begin{bmatrix} AQ & B_q U \\ C_y Q - Y & -U + D_{yq} U \end{bmatrix} < 0, \tag{9}$$
then the origin of system (7) *is exponentially stable with stability region containing the set* $\mathscr{E}((s^2 Q)^{-1})$.
2. *Given* $s>0$. *If there exist* $Y \in \mathbb{R}^{n_u\times n}$ *and a diagonal* $U>0$ *satisfying* (8) *and*
$$\mathrm{He}\begin{bmatrix} AQ & B_q U & B_w \\ C_y Q - Y & -U + D_{yq}U & D_{yw} \\ 0 & 0 & -\dfrac{I}{2} \end{bmatrix} \leqslant 0, \tag{10}$$
then for $x(0)=0$ *and* $\|w\|_2 \leqslant s$, *we have* $x(t) \in \mathscr{E}((s^2 Q)^{-1})$ *for all* $t \geqslant 0$.
3. *Given* $\gamma, s>0$. *If there exist* $Y \in \mathbb{R}^{n_u\times n}$ *and a diagonal* $U>0$ *satisfying* (8) *and*
$$\mathrm{He}\begin{bmatrix} AQ & B_q U & B_w & 0 \\ C_y Q - Y & -U + D_{yq}U & D_{yw} & 0 \\ 0 & 0 & -\dfrac{I}{2} & 0 \\ C_z Q & D_{zq} U & D_{zw} & -\dfrac{\gamma^2}{2} I \end{bmatrix} < 0, \tag{11}$$
then for $x(0)=0$ *and* $\|w\|_2 \leqslant s$, $\|z\|_2 \leqslant \gamma \|w\|_2$.

The three items in Proposition 1 can be, respectively, used for the estimation of the domain of attraction, of the reachable set and of the nonlinear $\mathscr{L}_2$ gain.

3. Regional anti-windup synthesis: feasibility

In this section, we present a set of feasibility conditions for solving Problems S1–S3. As with the global results in Grimm et al. (2003), the regional results would involve nonconvex conditions for a generic order of the anti-windup compensator but reduce to convex conditions when the anti-windup compensator is static or of the same order as that of the plant.

To present the synthesis results, we pull out the anti-windup dynamics (4) from (7), as in the block diagram of Fig. 2. The resulting dynamics for the block $\mathscr{H}$ is

$$\mathscr{H}\begin{cases} \dot{x}_\mathrm{cl} = A_\mathrm{cl} x_\mathrm{cl} + B_{\mathrm{cl},w} w + B_{\mathrm{cl},q} q + B_{\mathrm{cl},v} v,\\ z = C_{\mathrm{cl},z} x_\mathrm{cl} + B_{\mathrm{cl},zw} w + B_{\mathrm{cl},zq} q + B_{\mathrm{cl},zv} v,\\ y_\mathrm{c} = C_{\mathrm{cl},y} x_\mathrm{cl} + B_{\mathrm{cl},yw} w + B_{\mathrm{cl},yq} q + B_{\mathrm{cl},yv} v,\end{cases} \tag{12}$$

where $q := \mathrm{d}z(y_\mathrm{c})$ and $x_\mathrm{cl} := [x_\mathrm{p}^\mathrm{T}\ x_\mathrm{c}^\mathrm{T}]^\mathrm{T}$ and all the matrices are determined by those of the plant (1) and the controller (2).

3.1. Optimal $\mathscr{L}_2$ gain for norm-bounded inputs

The following theorem establishes feasibility conditions corresponding to Problem S1.

Theorem 1. *Consider the plant* (1) *and the controller* (2) *satisfying Assumption* 1. *Assume that* $x_\mathrm{p}(0)=0$, $x_\mathrm{c}(0)=0$ *and* $\|w\|_2 \leqslant s$:

(1) *an optimal plant-order anti-windup compensator solving Problem* S1 *can be constructed based on the optimal solution* (R_{11}, S, Z, γ^2) *to the following LMI-optimization*

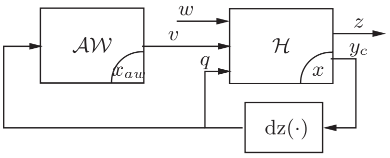

Fig. 2. Compact closed-loop representation separating out the anti-windup dynamics.

problem in the variables $R_{11} = R_{11}^{\mathrm{T}} > 0$, $S = S^{\mathrm{T}} := \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^{\mathrm{T}} & S_{22} \end{bmatrix} > 0$, $\gamma^2 > 0, Z$, *whenever the following constraints are feasible*:

$$\min_{Z, R_{11}, S, \gamma^2} \gamma^2, \textit{ subject to} \tag{13a}$$

$$\mathrm{He} \begin{bmatrix} A_{\mathrm{p}} R_{11} + B_{\mathrm{p},u} Z & B_{\mathrm{p},w} & 0 \\ 0 & -I/2 & 0 \\ C_{\mathrm{p},z} R_{11} + D_{\mathrm{p},zu} Z & D_{\mathrm{p},zw} & -\gamma^2 I/2 \end{bmatrix} < 0, \tag{13b}$$

$$\mathrm{He} \begin{bmatrix} A_{\mathrm{cl}} S & B_{\mathrm{cl},w} & 0 \\ 0 & -I/2 & 0 \\ C_{\mathrm{cl},z} S & D_{\mathrm{cl},zw} & -\gamma^2 I/2 \end{bmatrix} < 0, \tag{13c}$$

$$R_{11} - S_{11} > 0, \tag{13d}$$

$$\begin{bmatrix} \bar{u}_i^2 / s^2 & Z_i \\ Z_i^{\mathrm{T}} & R_{11} \end{bmatrix} \geqslant 0, \quad i = 1, \ldots, n_u, \tag{13e}$$

where Z_i *denotes the* i*th row of* Z;

(2) *an optimal static anti-windup compensator solving Problem* S1 *can be constructed based on the optimal solution* (R, Z, γ^2) *to the following LMI-optimization problem in the variables* $R = R^{\mathrm{T}} := \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^{\mathrm{T}} & R_{22} \end{bmatrix} > 0$, $\gamma^2 > 0$, Z, *whenever the following constraints are feasible*:

$$\min_{Z, R, \gamma^2} \gamma^2, \textit{ subject to} \tag{14a}$$

$$(13\mathrm{b}), \quad (13\mathrm{e}) \tag{14b}$$

$$\mathrm{He} \begin{bmatrix} A_{\mathrm{cl}} R & B_{\mathrm{cl},w} & 0 \\ 0 & -I/2 & 0 \\ C_{\mathrm{cl},z} R & D_{\mathrm{cl},zw} & -\gamma^2/2 \end{bmatrix} < 0. \tag{14c}$$

Proof. The proof is adapted from the proof of Theorem 2 and Proposition 2 in Grimm et al. (2003) with $n_{\mathrm{aw}} = n_{\mathrm{p}}$. To avoid overlap, we will outline the main idea for the case $n_{\mathrm{aw}} = n_{\mathrm{p}}$ and point out the differences between the situation in this paper and that in Grimm et al. (2003).

The proof is carried out by establishing that conditions (13b)–(13d) ensure the feasibility of (11) and (13e) ensures the feasibility of (8).

Feasibility of (11): Let Ψ, H and G be formed as in Grimm et al. (2003, p. 1516) and Y be the same variable as in (11). Let

$$\Psi_Y = \mathrm{He} \begin{bmatrix} 0 & 0 & 0 & 0 \\ -Y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} A_{\mathrm{aw}} & B_{\mathrm{aw}} \\ C_{\mathrm{aw}} & D_{\mathrm{aw}} \end{bmatrix},$$

where the blocks in Ψ_Y have the same dimensions as those in (11). Similar to the computation in Grimm et al. (2003), it can be verified that the left-hand side of (11) coincides with the following matrix:

$$\Phi = \Psi + \Psi_Y + G^{\mathrm{T}} \Lambda^{\mathrm{T}} H + H^{\mathrm{T}} \Lambda G. \tag{15}$$

Using Lemma 5 in Grimm et al. (2003) (projection lemma), the existence of Λ satisfying $\Phi < 0$ is equivalent to the feasibility of

$$W_H^{\mathrm{T}} (\Psi + \Psi_Y) W_H < 0, \tag{16}$$

$$W_G^{\mathrm{T}} (\Psi + \Psi_Y) W_G < 0, \tag{17}$$

where W_G, W_H can be any matrices whose columns form the bases of the null space of G and H.

A special W_H constructed in Grimm et al. (2003) is

$$W_H = \begin{bmatrix} I_{n_{\mathrm{p}}} & 0 & 0 & -B_{\mathrm{p},u} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_{\mathrm{w}}} & 0 \\ 0 & 0 & 0 & -D_{\mathrm{p},zu} & 0 & I_{n_z} \end{bmatrix}^{\mathrm{T}},$$

where the first 3×3 blocks add up to a total dimension of $(n_{\mathrm{p}} + n_{\mathrm{w}} + n_z) \times n$. Assume that Q is partitioned as

$$Q = \begin{bmatrix} R & N \\ N^{\mathrm{T}} & M \end{bmatrix}, \quad R = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^{\mathrm{T}} & R_{22} \end{bmatrix}, \tag{18}$$

where $R \in \mathbb{R}^{(n_{\mathrm{p}}+n_{\mathrm{c}}) \times (n_{\mathrm{p}}+n_{\mathrm{c}})}$, $R_{11} \in \mathbb{R}^{n_{\mathrm{p}} \times n_{\mathrm{p}}}$. It can be verified after tedious calculations that

$$W_H^{\mathrm{T}} (\Psi + \Psi_Y) W_H = \mathrm{He} \begin{bmatrix} A_{\mathrm{p}} R_{11} + B_{\mathrm{p},u} Z & B_{\mathrm{p},w} & 0 \\ 0 & -I/2 & 0 \\ C_{\mathrm{p},z} R_{11} + D_{\mathrm{p},zu} Z & D_{\mathrm{p},zw} & -\gamma^2/2 \end{bmatrix},$$

where $Z = Y \begin{bmatrix} I_{n_{\mathrm{p}}} \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^{n_u \times n_{\mathrm{p}}}$.

A special W_G is also given in Grimm et al. (2003) as $W_G = \bar{T}^{-1} W_{G\circ}$ with $\bar{T} = \mathrm{diag}\{Q, U, I, I\}$ and

$$W_{G\circ} = \begin{bmatrix} I_{n_{\mathrm{p}}} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{n_{\mathrm{c}}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_{\mathrm{w}}} & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n_z} \end{bmatrix}.$$

If we let $P = Q^{-1}$, and partition P as

$$P = \begin{bmatrix} S^{-1} & P_{12} \\ P_{12}^{\mathrm{T}} & P_{22} \end{bmatrix}$$

with $S \in \mathbb{R}^{(n_{\mathrm{p}}+n_{\mathrm{c}}) \times (n_{\mathrm{p}}+n_{\mathrm{c}})}$, it can be verified that $W_G^{\mathrm{T}}(\Psi + \Psi_Y) W_G < 0$ is equivalent to (13c).

Note that (13b) and (13c) are stated in terms of R_{11} and S, which are constrained by

$$\begin{bmatrix} R & N \\ N^{\mathrm{T}} & M \end{bmatrix} \begin{bmatrix} S^{-1} & P_{12} \\ P_{12}^{\mathrm{T}} & P_{22} \end{bmatrix} = QP = I. \tag{19}$$

For the case where the order of the anti-windup equals to the plant order, (19) is satisfied if $R_{11} - S_{11} > 0$. For the static case, it is satisfied if $R = S$.

Feasibility of (8): Assume that (13e) is satisfied and define $K_{\mathrm{p}i} := Z_i R_{11}^{-1}$. Then by Schur complement,

$$0 \leqslant \frac{\overline{u}_i^2}{s^2} - Z_i R_{11}^{-1} Z_i^{\mathrm{T}} = \frac{\overline{u}_i^2}{s^2} - K_{\mathrm{p}i} R_{11} K_{\mathrm{p}i}^{\mathrm{T}} = \frac{\overline{u}_i^2}{s^2} - [K_{\mathrm{p}i}\ \ 0\ \ 0] Q [K_{\mathrm{p}i}\ \ 0\ \ 0]^{\mathrm{T}} = \frac{\overline{u}_i^2}{s^2} - Y_i Q^{-1} Y_i^{\mathrm{T}},$$

where, according to the selection in step 4 of Procedure 1, $Y_i = [Z_i\, Z_i R_{11}^{-1} R_{12}\ \ Z_i R_{11}^{-1} N_1]$. Finally, by Schur complement, the last inequality above transforms into (8). □

For a system with an exponentially unstable plant, it is clear that there exists a norm bounded w such that $\|z\|_2$ is unbounded and thus (13) is feasible only for s bounded by a certain $\bar{s}$. On the other hand, if A_{p} is Hurwitz, then there exists a selection (R_{11}, S, Z, γ^2) with $Z = 0$ satisfying the LMIs in (13) for all s (this solution corresponds to the global construction proposed in Grimm et al., 2003).The critical case where A_{p} has eigenvalues on the imaginary axis and in the left half plane was not considered in Grimm et al. (2003) and can be addressed by the following proposition.

Proposition 2. *Consider the plant* (1) *and the controller* (2) *satisfying Assumption* 1. *Suppose that the plant is not exponentially unstable. Then for each* $s > 0$, *the constraints* ((13a)–(13d)) *are feasible with a finite* γ.

Proof. The following lemma is adapted from Teel (1995, Lemma 3.1) for the purposes of this proof.

Lemma 1. *Assume that* A_{p} *has eigenvalues with nonpositive real part and* $(A_{\mathrm{p}}, B_{\mathrm{p},u})$ *is stabilizable. Then there exists* $\gamma_0 > 0$ *such that for any* $k > 0$, *there exists* $R_{11} = R_{11}^{\mathrm{T}} > kI$ *satisfying*

$$\mathrm{He}\begin{bmatrix} A_{\mathrm{p}} R_{11} - \dfrac{\gamma_0^2}{2} B_{\mathrm{p},u} B_{\mathrm{p},u}^{\mathrm{T}} & B_{\mathrm{p},w} \\ 0 & -\dfrac{I}{2} \end{bmatrix} < 0. \tag{20}$$

Since the linear closed-loop system is exponentially stable, we can pick $S = S^{\mathrm{T}} > 0$ satisfying

$$\mathrm{He}\begin{bmatrix} A_{\mathrm{cl}} S & B_{\mathrm{cl},w} \\ 0 & -\dfrac{I}{2} \end{bmatrix} < 0. \tag{21}$$

Let $Z = -(\gamma_0^2/2) B_{\mathrm{p},u}^{\mathrm{T}}$. Then by Lemma 1, for any $k > 0$, there exists $R_{11} = R_{11}^{\mathrm{T}} > kI$ satisfying

$$\mathrm{He}\begin{bmatrix} A_{\mathrm{p}} R_{11} + B_{\mathrm{p},u} Z & B_{\mathrm{p},w} \\ 0 & -\dfrac{I}{2} \end{bmatrix} < 0. \tag{22}$$

Therefore, for any $s > 0$, we can choose R_{11} sufficiently large satisfying (22), $R_{11} - S_{11} > 0$ and

$$\begin{bmatrix} \dfrac{\bar{u}_i^2}{s^2} & Z_i \\ Z_i^{\mathrm{T}} & R_{11} \end{bmatrix} \geqslant 0, \quad i = 1, \ldots, n_u. \tag{23}$$

With R_{11} and Z fixed this way, we can determine a sufficiently large γ satisfying both (13b) and (13c). □

3.2. Optimal AW with guaranteed reachable set

In this section, we augment the synthesis problem of the previous section with the extra requirement that given a bound s on the $\mathscr{L}_2$ norm of w, the plant state does not exit a given desirable set $\mathscr{R}_{\mathrm{p}} \subset \mathbb{R}^{n_{\mathrm{p}}}$. We first consider

$$\mathscr{R}_{\mathrm{p}} = \mathscr{E}(R_{\mathrm{p}}^{-1}) = \{x_{\mathrm{p}} : x_{\mathrm{p}}^{\mathrm{T}} R_{\mathrm{p}}^{-1} x_{\mathrm{p}} \leqslant 1\}, \tag{24}$$

where $R_{\mathrm{p}} = R_{\mathrm{p}}^{\mathrm{T}} > 0$.

Theorem 2. *Consider the plant* (1) *and the controller* (2) *satisfying Assumption* 1. *Assume that* $x_{\mathrm{p}}(0) = 0$, $x_{\mathrm{c}}(0) = 0$ *and* $\|w\|_2 \leqslant s$. Then an optimal plant-order anti-windup compensator solving Problem S2 can be constructed based on the optimal solution (R_{11}, S, Z, γ^2) *to the following LMI-optimization problem in the variables* $R_{11} = R_{11}^{\mathrm{T}} > 0$, $S = S^{\mathrm{T}} := \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^{\mathrm{T}} & S_{22} \end{bmatrix} > 0$, $\gamma > 0, Z$, *whenever the corresponding constraints are feasible*:

$$\min_{Z,S,R_{11},\gamma^2} \gamma^2, \text{ subject to}$$

$$(13b),\quad (13c),\quad (13d),\quad (13e), \tag{25a}$$

$$s^2 R_{11} \leqslant R_{\mathrm{p}}. \tag{25b}$$

An optimal static anti-windup compensator solving Problem S2 *can be constructed if the problem* (14) *with the additional constraint* $s^2 R_{11} \leqslant R_{\mathrm{p}}$ *has a solution.*

Proof. The proof easily follows from that of Theorem 1 by observing that if (11) is satisfied, then by Proposition 1 the reachable set for the state of the plant is bounded by $\mathscr{E}((s^2 R_{11})^{-1})$. □

The main idea behind the results of Theorem 2 relies on the fact that if (25a) is satisfied, then there exists an anti-windup compensator such that the reachable set is bounded by $\mathscr{E}((s^2 R_{11})^{-1})$. Moreover, the constraint (25b) implies that this set is inside the desired reachability set given by (24), namely that $\mathscr{E}((s^2 R_{11})^{-1}) \subset \mathscr{E}(R_{\mathrm{p}}^{-1})$.

Based on Theorem 2, we can also formulate optimization problems to minimize the desirable reachable set $\mathscr{R}_{\mathrm{p}}$ under the constraints (25a) and (25b), with a guaranteed $\mathscr{L}_2$ gain γ (or without considering the $\mathscr{L}_2$ gain). The quantity to be minimized can be the trace of R_{p} or the determinant of R_{p}.

We may also take $\mathscr{R}_{\mathrm{p}}$ as the following unbounded set:

$$\mathscr{R}_{\mathrm{p}}(\alpha) = \{x_{\mathrm{p}} : |Cx| \leqslant \alpha\},$$

where $C \in \mathbb{R}^{1 \times n_{\mathrm{p}}}$ is a given row vector. Then $\mathscr{E}((s^2 R_{11})^{-1}) \subset \mathscr{R}_{\mathrm{p}}(\alpha)$ if and only if $C R_{11} C^{\mathrm{T}} \leqslant \alpha^2 / s^2$. Hence, if our objective is to minimize the maximum value of a particular output Cx_{p}, we may formulate the following optimization problem:

$$\min_{Z,S,R_{11},\alpha^2} \alpha^2, \text{ subject to}$$

$$(13b),\quad (13c),\quad (13d),\quad (13e), \tag{26a}$$

$$C R_{11} C^{\mathrm{T}} < \alpha^2 / s^2, \tag{26b}$$

where a desirable $\mathscr{L}_2$ gain can be incorporated in (26a). If there is no consideration for a desirable $\mathscr{L}_2$ gain, then the matrices in (13b) and (13c) can be simplified by removing the third block row and the third block column.

3.3. Optimal AW with guaranteed domain of attraction

We now augment the synthesis problem of Section 3.1 for the purpose of solving Problem S3 with a guaranteed domain of attraction in terms of the state of the plant:

$$\mathscr{S}_{\mathrm{p}} = \mathscr{E}(S_{\mathrm{p}}^{-1}) = \{x_{\mathrm{p}} : x_{\mathrm{p}}^{\mathrm{T}} S_{\mathrm{p}}^{-1} x_{\mathrm{p}} \leqslant 1\}, \tag{27}$$

where $S_{\mathrm{p}} = S_{\mathrm{p}}^{\mathrm{T}} > 0$.

Theorem 3. *Consider the plant* (1) *and the controller* (2) *satisfying Assumption* 1. *An optimal plant-order anti-windup compensator solving Problem* S3 *can be constructed based on the optimal solution* (R_{11}, S, Z, γ^2) *to the following LMI-optimization problem in the variables* $R_{11} = R_{11}^{\mathrm{T}} > 0$, $S = S^{\mathrm{T}} := \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^{\mathrm{T}} & S_{22} \end{bmatrix} > 0$, $\gamma > 0$, Z, *whenever the corresponding constraints are feasible*:

$$\min_{Z,S,R_{11},\gamma^2} \gamma^2, \text{ subject to}$$
$$(13\text{b}), \quad (13\text{c}), \quad (13\text{d}), \quad (13\text{e}), \tag{28a}$$

$$s^2 R_{11} \geqslant S_{\mathrm{p}}. \tag{28b}$$

An optimal static anti-windup compensator solving Problem S3 *can be constructed if the problem* (14) *with an additonal constraint* $s^2 R_{11} \geqslant S_{\mathrm{p}}$ *has a solution.*

Proof. The proof easily follows from that of Theorem 1 by observing that if (11) is satisfied, then for each $x_{\mathrm{p}}(0) \in \mathscr{E}((s^2 R_{11})^{-1})$, there exist $x_{\mathrm{c}}(0)$ and $x_{\mathrm{aw}}(0)$ such that $x(0) \in \mathscr{E}((s^2 Q)^{-1})$ and by Proposition 1, we have the desired exponential stability result. □

Based on Theorem 3, we can formulate various optimization problems to maximize the estimate of the domain of attraction (with respect to different measures of set size) with a guaranteed $\mathscr{L}_2$ gain (or without considering the $\mathscr{L}_2$ gain).

If the plant is exponentially stable, then global asymptotic stability by dynamic anti-windup compensation is guaranteed by the finite global $\mathscr{L}_2$ gain (Grimm et al., 2003). If the plant is exponentially unstable, then only regional stability can be obtained. For a plant that is marginally stable/unstable, the following proposition ensures semi-global stabilization.

Proposition 3. *Consider the plant* (1) *and the controller* (2) *satisfying Assumption* 1. *Suppose that the plant is not exponentially unstable. Then, for any finite* S_{p} *and for any* $s > 0$, (28a) *and* (28b) *are feasible.*

Proof. The proof follows the same arguments as the proof of Proposition 2. □

4. Regional anti-windup synthesis: construction

In this section, we provide a constructive algorithm for determining the matrices of a plant-order dynamic anti-windup compensator. The construction of a static compensator is much simpler and will be suggested in square brackets.

The algorithm follows from the proof of Theorems 1–3. It is based on the solution (R_{11}, S, Z, γ^2) [respectively, (R, Z, γ^2)] to the plant-order [respectively, static] anti-windup feasibility conditions. Note that the construction of the anti-windup matrices is the same for all the optimization problems andd the algorithm is similar to the ones reported in Grimm et al. (2003) (except for the use of the variable Z). Note that by the exponential stability established in Theorem 3 and since the x_{aw} dynamics in (4) are driven by $\mathrm{d}z(y_{\mathrm{c}})$, the matrix A_{aw} determined in the next procedure will be necessarily Hurwitz. This can also be deduced from the structure of the arising closed-loop matrix given in Hu, Teel, and Zaccarian (2005, Eq. (9a)).

Procedure 1 (*Anti-windup synthesis*). *Step* 1: *Solve the feasibility LMIs*: Find a solution (R_{11}, S, Z, γ^2) [for the static case, (R, Z, γ^2)] to the feasibility LMIs listed in Section 3.

Step 2: *Construct the matrix* Q: Define $R := \begin{bmatrix} R_{11} & S_{12} \\ S_{12}^{\mathrm{T}} & S_{22} \end{bmatrix}$ and let $N \in \mathbb{R}^{(n_{\mathrm{p}}+n_{\mathrm{c}})\times n_{\mathrm{aw}}}$ be a solution of the following equation:

$$RS^{-1}R - R = NN^{\mathrm{T}}. \tag{29}$$

Since R and S are invertible and by the feasibility conditions, $RS^{-1}R - R$ is positive semidefinite and of rank n_{aw}. Hence there always exists a matrix N satisfying Eq. (29). Let $M = I + N^{\mathrm{T}} R^{-1} N$ and

$$Q := \begin{bmatrix} R & N \\ N^{\mathrm{T}} & M \end{bmatrix}. \tag{30}$$

(For the static case, let $Q = R$.)

Step 3: *Build necessary matrices*: Construct the matrices $A_0 \in \mathbb{R}^{n\times n}$, $B_0 \in \mathbb{R}^{n\times n_u}$, $C_{y0} \in \mathbb{R}^{n_u\times n}$, $D_{yq0} \in \mathbb{R}^{n_u\times n_u}$, $C_{z0} \in \mathbb{R}^{n_z\times n}$, $D_{zq0} \in \mathbb{R}^{n_z\times n_u}$, $B_w \in \mathbb{R}^{n\times n_{\mathrm{w}}}$, $D_{zw} \in \mathbb{R}^{n_z\times n_{\mathrm{w}}}$ and $D_{yw} \in \mathbb{R}^{n_u\times n_{\mathrm{w}}}$ as

$$\left[\begin{array}{c|c|c} A_0 & B_{q0} & B_w \\ \hline C_{y0} & D_{yq0} & D_{yw} \\ \hline C_{z0} & D_{zq0} & D_{zw} \end{array}\right] = \left[\begin{array}{cc|c|c} A_{\mathrm{cl}} & 0 & B_{\mathrm{cl},q} & B_{\mathrm{cl},w} \\ 0 & 0 & 0 & 0 \\ \hline C_{\mathrm{cl},y} & 0 & D_{\mathrm{cl},yq} & D_{\mathrm{cl},yw} \\ \hline C_{\mathrm{cl},z} & 0 & D_{\mathrm{cl},zq} & D_{\mathrm{cl},zw} \end{array}\right].$$

(For the static case, define the matrices above by removing the second block row and block column of zeros from the right-hand side of the above equation.)

Step 4: *Anti-windup compensator LMI*: Let $m = n + n_u + n_{\mathrm{w}} + n_z$. Based on Steps 2 and 3, construct the matrices $H \in \mathbb{R}^{(n_{\mathrm{aw}}+n_v)\times m}$, $\Psi_R \in \mathbb{R}^{m\times m}$ and $G_U \in \mathbb{R}^{(n_{\mathrm{aw}}+n_u)\times m}$ as follows:

$$\Psi_R = \mathrm{He}\begin{bmatrix} A_0 Q & B_{q0}U - Y^{\mathrm{T}} & B_w & QC_{z0}^{\mathrm{T}} \\ C_{y0}Q & D_{yq0}U - U & D_{yw} & UD_{zq0}^{\mathrm{T}} \\ 0 & 0 & -\dfrac{I}{2}I & D_{zw}^{\mathrm{T}} \\ 0 & 0 & 0 & -\dfrac{\gamma^2}{2}I \end{bmatrix},$$

$$H = \left[\begin{array}{cc|c|c|c} 0 & I_{n_{aw}} & 0 & 0 & 0 \\ B_{\mathrm{cl},v}^{\mathrm{T}} & 0 & D_{\mathrm{cl},yv}^{\mathrm{T}} & 0 & D_{\mathrm{cl},zv}^{\mathrm{T}} \end{array}\right]$$

$$G_U = \left[\begin{array}{cc|c|c|c} N^{\mathrm{T}} & M & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \end{array}\right],$$

,

where $Y \in \mathbb{R}^{n_u \times (n_{\mathrm{p}}+n_{\mathrm{c}}+n_{\mathrm{aw}})}$ is defined as $Y := [Z\ \ ZR_{11}^{-1}R_{12}\ \ ZR_{11}^{-1}N_1]$ (where N_1 is the upper block of the matrix N) [For the static case, define $Y := [Z\ \ ZR_{11}^{-1}R_{12}]$]. Finally, solve the LMI

$$\Psi_R + G_U^{\mathrm{T}} \Lambda_U^{\mathrm{T}} H + H^{\mathrm{T}} \Lambda_U G_U < 0, \tag{31}$$

in the unknowns $\Lambda_U \in \mathbb{R}^{(n_{\mathrm{aw}}+n_v)\times(n_{\mathrm{aw}}+n_u)}$ and $U \in \mathbb{R}^{n_u \times n_u}$, $U > 0$ diagonal, and compute the matrices of the anti-windup compensator (4) as follows:

$$\begin{bmatrix} A_{\mathrm{aw}} & B_{\mathrm{aw}} \\ C_{\mathrm{aw}} & D_{\mathrm{aw}} \end{bmatrix} = \Lambda_U \begin{bmatrix} I & 0 \\ 0 & U^{-1} \end{bmatrix}. \tag{32}$$

(For the static case, compute the static anti-windup gain as $D_{\mathrm{aw}} = \Lambda_U U^{-1}$).

5. An example

We adopt Example 2 from Grimm et al. (2003). The plant is a cart-spring-pendulum system with one control input, one disturbance input, four states and one measurement output. The plant state is $x_{\mathrm{p}} = [p\ \ \dot{p}\ \ \theta\ \ \dot{\theta}]$, where p is the horizontal displacement of the cart and θ is the angle of the pendulum. The plant and controller parameters can be found in Grimm et al. (2003). For this example, the closed-loop system without anti-windup compensation is not globally stable. Also, there exists no static anti-windup compensation to make the global $\mathscr{L}_2$ gain bounded. With dynamic anti-windup augmentation, an upper bound for the achievable global $\mathscr{L}_2$ gain is found to be 181.1424.

The achievable $\mathscr{L}_2$ gain for every $s > 0$ by using plant-order anti-windup can be determined with the algorithm based on Theorem 1. By choosing different s over $(0, \infty)$, the achievable performance can be obtained as a function of s. Fig. 3 plot this achievable performance in solid curve. For comparison, we also plot the achievable performance by using static anti-windup, which is the dashed-dotted curve in Fig. 3. Also plotted in Fig. 3 (dashed) is the upper bound for the nonlinear $\mathscr{L}_2$ gain under a particular plant-order anti-windup compensator.

Next, we use algorithm (26) to determine an achievable upper bound on the displacement of the cart x_{p1} (by plant-order anti-windup) for a given norm bound s on $\|w\|_2$. For this purpose, we choose $C = [1\ \ 0\ \ 0\ \ 0]$. The relation between s and the achievable bound α is plotted in Fig. 4. If we take $C = [0\ \ 0\ \ 1\ \ 0]$, then an achievable upper bound on the angle of the pendulum x_{p3} can be obtained.

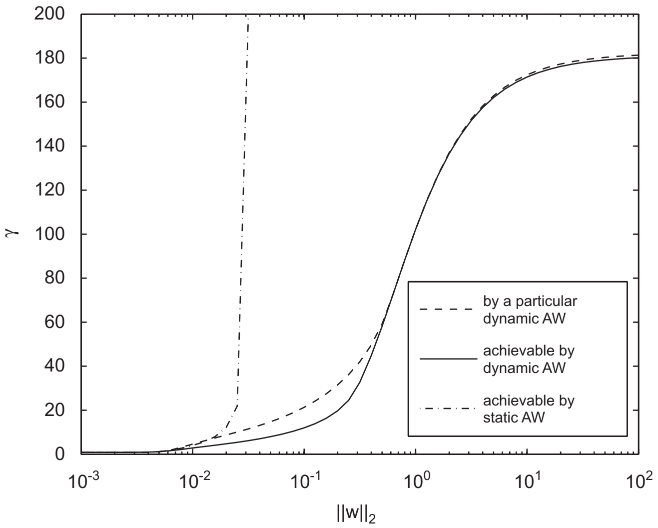

Fig. 3. Achievable nonlinear $\mathscr{L}_2$ gains.

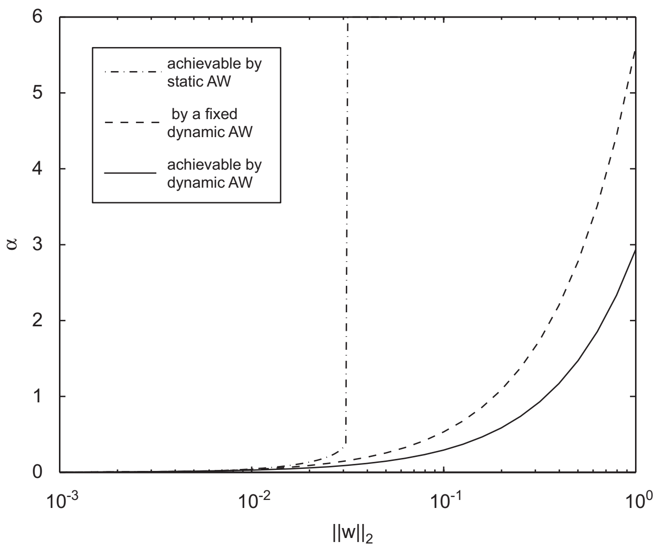

Fig. 4. Achievable bounds on x_{p1}.

6. Conclusions

This paper provided LMI-based tools for the construction of anti-windup compensators for general saturated linear control systems, where the only assumption on the closed-loop system is well-posedness and local stability. The design objectives are to optimize a few regional performance measures. Solutions to the problems have been presented through a set of convex optimization procedures based on LMI constraints. Furthermore, two semiglobal results have been established for the special case where the plant is marginally stable/unstable. An example has been given to illustrate the proposed tools.

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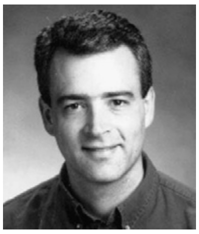

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Stability and Performance for Saturated Systems via Quadratic and Nonquadratic Lyapunov Functions

Tingshu Hu, Andrew R. Teel, *Fellow, IEEE*, and Luca Zaccarian

***Abstract*—In this paper, we develop a systematic Lyapunov approach to the regional stability and performance analysis of saturated systems in a general feedback configuration. The only assumptions we make about the system are well-posedness of the algebraic loop and local stability. Problems to be considered include the estimation of the domain of attraction, the reachable set under a class of bounded energy disturbances and the nonlinear $\mathcal{L}_2$ gain. The regional analysis is established through an effective treatment of the algebraic loop and the saturation/deadzone function. This treatment yields two forms of differential inclusions, a polytopic differential inclusion (PDI) and a norm-bounded differential inclusion (NDI) that contain the original system. Adjustable parameters are incorporated into the differential inclusions to reflect the regional property. The main idea behind the regional analysis is to ensure that the state remain inside the level set of a certain Lyapunov function where the PDI or the NDI is valid. With quadratic Lyapunov functions, conditions for stability and performances are derived as linear matrix inequalities (LMIs). To obtain less conservative conditions, we use a pair of conjugate non-quadratic Lyapunov functions, the convex hull quadratic function and the max quadratic function. These functions yield bilinear matrix inequalities (BMIs) as conditions for stability and guaranteed performance level. The BMI conditions cover the corresponding LMI conditions as special cases, hence the BMI results are guaranteed to be as good as the LMI results. In most examples, the BMI results are significantly better than the LMI results.**

***Index Terms*—Deadzone, domain of attraction, Lyapunov functions, nonlinear $\mathcal{L}_2$ gain, reachable set, saturation.**

I. Introduction

A. Background

SATURATION is an ubiquitous nonlinearity in engineering systems and is the most studied in the literature as compared with other types of nonlinearities. Intensified efforts have been devoted to control systems with saturation since the earlier 1990s due to a few notable breakthroughs (see, e.g., [36], [46], and [48]). Saturation exists in different parts of a control system, such as the actuator, the sensor, the controller and components within the plant. Most research has been devoted to addressing actuator saturation, which involves fundamental control problems such as constrained controllability and global/semi-global stabilization. These problems have been discussed in great depth, e.g., in [22], [36], [45], [46], [48], and [49] (among which, [22] considers exponentially unstable systems). Another significant problem arising from actuator saturation is anti-windup compensation, which has attracted tremendous attention over the past decade (see, e.g., [4]–[6], [8]–[10], [12], [16]–[18], [28], [34], [35], [39]–[41], [47], [50], [52], and [54]).

The approach that is adopted in most of the recent literature to address saturated systems can be categorized as a Lyapunov approach. In this approach, some quantitative measures of stability and performance, such as the size of the domain of attraction, the convergence rate, and the $\mathcal{L}_2$ gain, are characterized by using Lyapunov functions or storage functions. Then the design parameters (e.g., of a controller or of an antiwindup compensator) are incorporated into an optimization problem to optimize these quantitative measures for the closed-loop system. This approach is mostly fueled by the numerical success in solving convex optimization problems with linear matrix inequalities (LMIs) (e.g., see [2]). This is a general approach which can be applied to deal with systems with saturation and deadzone occurring at different locations. The first papers that use LMI-based methods to deal with saturated systems include [21], [35], [42], where [21], [42] consider state feedback design and [35] analyzes antiwindup systems. Since then, extensive LMI-based algorithms have been developed for analysis and design of saturated systems (see, e.g., [4]–[6], [10], [13], [16]–[18], [22], [25], [26], [39], [40], [47], and [54].)

There are mainly two steps involved in the Lyapunov approach. The first step is to include the saturation function or the deadzone function in a sector so that the original system can be cast into the general framework of absolute stability, or can be described with a linear differential inclusion (LDI). The second step applies available tools from absolute stability theory or from general Lyapunov approaches for LDIs, such as the circle criterion or the LMI characterizations of stability and performance in [2]. Roughly speaking, all the analysis tools used in the aforementioned works are obtained by applying quadratic Lyapunov/storage functions to the LDIs except that [39] used a piecewise quadratic function.

Because of the two-step framework, the effectiveness of a particular method depends on how the original system is transformed into LDIs and what kind of analysis tools for LDIs are used. In many works involving anti-windup compensation, global sectors are used to describe saturation/deadzone functions. It is well known that a global sector can be very

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conservative for regional analysis and can only be applied when the closed-loop system is globally stable or to detect global stability. In some other works, regional LDI descriptions (some based on local sectors) are derived to reduce the conservatism (see, e.g., [4], [5], [10], [13], [21], [25], [26], [35], and [42]). Along this direction, the regional LDI description introduced in [25], [26] has proved very effective and easy to manipulate. It has been used successfully for different configurations or for different purposes in [4], [5], [10], [13], [27], [28].

With an effective regional LDI description, there is yet more potential to be explored in the second step about the analysis of LDIs. It is now generally accepted that quadratic Lyapunov functions can be very conservative even for stability analysis of LDIs (see, e.g., [7], [11], [32], and [55]). For this reason, considerable attention has been paid to the construction and development of non-quadratic Lyapunov functions (e.g., see [1], [3], [7], [32], [33], [38], [53], and [55]).

Recently, a pair of conjugate Lyapunov functions have demonstrated great potential in the analysis of LDIs and saturated linear systems [14], [15], [23], [27]. One is called the convex hull quadratic function since its level set is the convex hull of a family of ellipsoids. The other is called max quadratic function since it is obtained by taking pointwise maximum over a family of quadratic functions and its level set is the intersection of a family of ellipsoids. Some conjugate relationships about these two functions were established in [14], [15]. Since these functions are natural extensions of quadratic functions, they can also be used to perform quantitative performance analysis beyond stability, such as to estimate the $\mathcal{L}_2$ gain, and the reachable set, for LDIs. A handful of dual bilinear matrix inequalities (BMIs) have been derived for these purposes in [14]. As compared with the corresponding LMIs resulting from quadratic Lyapunov functions, these BMIs contain extra degrees of freedom in the bilinear terms, which are injected through the nonquadratic functions. Experience with low order systems shows that these BMIs can be solved effectively with the path-following method in [20]. Although it is possible that numerical difficulties may arise for higher order systems, the great potential of these nonquadratic Lyapunov functions has been demonstrated in [14], [15], [27] through a set of numerical examples.

B. Problem Formulation

With the recent developments and effective tools mentioned in the previous section, we are now able to address more effectively some stability and performance problems for systems with saturation/deadzone in the following general form:

$$\begin{cases} \dot{x} &= Ax + B_q q + B_w w \\ y &= C_y x + D_{yq} q + D_{yw} w \\ z &= C_z x + D_{zq} q + D_{zw} w \\ q &= \mathrm{dz}(y) \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $q, y \in \mathbb{R}^m$, $w \in \mathbb{R}^r$, $z \in \mathbb{R}^p$. The deadzone function $\mathrm{dz}(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is defined as $\mathrm{dz}(y) := y - \mathrm{sat}(y)$, for all $y \in \mathbb{R}^m$, where $\mathrm{sat}(\cdot)$ is a vector saturation function with the saturation levels given by a vector $\bar{u} \in \mathbb{R}^m$, $\bar{u}_i > 0$, $i = 1, 2, \ldots, m$. In particular

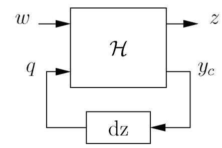

Fig. 1. Compact representation of a system with saturation/deadzone.

$$\mathrm{sat}(u_i) = \begin{cases} \bar{u}_i, & u_i \geq \bar{u}_i, \\ u_i, & u_i \in [-\bar{u}_i, \bar{u}_i], \\ -\bar{u}_i, & u_i \leq -\bar{u}_i. \end{cases} \quad \mathrm{sat}(u) = \begin{bmatrix} \mathrm{sat}(u_1) \\ \vdots \\ \mathrm{sat}(u_m) \end{bmatrix}.$$

In this paper, we consider symmetric saturation functions.[1] System (1) can be graphically depicted in block diagram form as in Fig. 1, where w is the exogenous input or disturbance and z is the output whose performance is under consideration. Many linear systems with saturation/deadzone components can be transformed into the aforementioned general form through a loop transformation. This general form has been used to study antiwindup systems in [16], [35], [40], [54]. When $D_{yq} = 0$, the system does not contain an algebraic loop, which can simplify the analysis and implementation. However, it was shown in [40] that the algebraic loop can be purposely introduced into the antiwindup configuration to reduce the global $\mathcal{L}_2$ gain. The importance of the parameter D_{yq} will also be illustrated in examples at the end of this paper.

We note that most of the previous works imposed various assumptions on the system, such as exponential stability of the original open-loop plant in an antiwindup configuration (e.g., [16], [40], and [54]). In these works, the global sector $[0, I]$ is used to describe the deadzone function. In some other works such as [4]–[6], [10], [13], [25]–[27], and [47] (among which [6] and [13] study the $\mathcal{L}_2$ gain), regional LDI descriptions are used to reduce the conservatism. In these works, the algebraic loop is absent ($D_{yq} = 0$) and the disturbance (in [6] and [13]) does not enter the deadzone function, i.e., $D_{yw} = 0$. In [31], the algebraic loop has a special structure, namely, D_{yq} is diagonal.

A recent attempt was made in [52] to perform regional analysis on the general form without the assumption on stability of the open-loop plant. The main idea, which had also been suggested in some other works, was to use a smaller sector $[0, K]$ with $K < I$ to bound the deadzone function. However, this idea would not work on the general form if $D_{yw} \neq 0$. As can be seen from the second equation in (1), y is not necessarily bounded in $\mathcal{L}_\infty$ norm when w is only bounded in the $\mathcal{L}_2$ norm. Hence, there exists no $K < I$ to bound the deadzone function even at $x = 0$. After all, as commented in [25] and [27], even in the absence of w, this kind of sector description is not only hard to manipulate, but also has a much restricted degree of freedom as compared with the regional LDI description initiated in [25] and [26].

In this paper, we will extend the regional LDI description in [25] and [26] to deal with the general situation where

[1]Asymmetric saturations can be treated with the methods developed here with some level of conservativeness by taking $\bar{u}_i$ as the minimum absolute value of the negative and positive saturation levels.

$D_{yq}, D_{yw} \neq 0$, and to address both stability and performance issues.

The only assumptions that we will make about the system (1) is its local stability (A is Hurwitz) and the well-posedness of the algebraic loop, which will be made precise in Section II. These were also the only assumptions made in [28] and they are clearly basic requirements for the system to be functional.

The objective of this paper is to carry out a systematic and comprehensive analysis of system (1) by using quadratic and nonquadratic Lyapunov functions. The following problems will be addressed.

1) Estimation of the domain of attraction (in the absence of w) by using invariant ellipsoids or invariant level sets of the nonquadratic Lyapunov functions.
2) With a given bound on the $\mathcal{L}_2$ norm of w, i.e, $\|w\|_2 \leq s$ for a given s, we would like to determine a set S as small as possible so that under the condition $x(0) = 0$, we have $x(t) \in S$ for all t. This set S will be considered as an estimate of the reachable set.
3) With $\|w\|_2 \leq s$ for a given s, we would like to determine a number $\gamma > 0$ as small as possible, so that under the condition $x(0) = 0$, we have $\|z\|_2 \leq \gamma\|w\|_2$. Performing this analysis for each $s \in (0, \infty)$, we obtain an estimate of the nonlinear $\mathcal{L}_2$ gain.

To address these problems systematically, we will first provide an effective treatment of the algebraic loop and the deadzone function in Section II. In particular, the necessary and sufficient condition for the well-posedness of the algebraic loop will be made explicit. Moreover, we will derive two forms of differential inclusions to describe the original system (1). The first one is a polytopic differential inclusion (PDI) involving a certain adjustable parameter or nonlinear function. This parameter or nonlinear function offers extra degrees of freedom associated with a local region under consideration. It will be optimized in conjunction with the Lyapunov functions in the final analysis problems. The second differential inclusion is a norm-bounded differential inclusion (NDI) which is derived from the PDI. The NDI is more conservative than the PDI but may be more numerically tractable for some cases.

In Section III, we will apply quadratic Lyapunov functions via the PDI and the NDI to characterize stability and performance of the original system (1). We note that quadratic functions have been used for these purposes in [4]–[6], [10], [13], [25], [26], [47] under the assumption that $D_{yq} = 0$ and $D_{yw} = 0$. In Section IV, we apply the convex hull quadratic function and the max quadratic function respectively via the PDI (It turns out that when these nonquadratics are applied to the NDI, they produce the same results as the quadratics). In Section V, we use a numerical example to demonstrate the effectiveness of this paper's results and the relationship between them. Section VI concludes this paper.

Notation:

— $|\cdot|_\infty$: For $u \in \mathbb{R}^m$, $|u|_\infty := \max_i |u_i|$.
— $\|\cdot\|_2$: For $u \in \mathcal{L}_2$, $\|u\|_2 := \left(\int_0^\infty u^T(t)u(t)dt\right)^{1/2}$.
— $I[k_1, k_2]$: For two integers $k_1, k_2, k_1 < k_2$, $I[k_1, k_2] := \{k_1, k_1 + 1, \ldots, k_2\}$.
— $\mathrm{sat}(\cdot)$: The symmetric saturation function with implicit saturation level given by $\bar{u} \in \mathbb{R}^m$.
— $\bar{U} := \mathrm{diag}\{\bar{u}_1, \ldots, \bar{u}_m\}$ where $\bar{u}_i > 0$ is the saturation level for the ith component of $\mathrm{sat}(\cdot)$.
— $\mathrm{dz}(u)$: The deadzone function, $\mathrm{dz}(u) := u - \mathrm{sat}(u)$.
— $\mathrm{co}S$: The convex hull of a set S.
— $\mathcal{K}$: The set of diagonal matrices with 0 or 1 at each diagonal element.
— $\mathrm{He}X$: For a square matrix X, $\mathrm{He}X := X + X^T$.
— $\mathcal{E}(P)$: For $P \in \mathbb{R}^{n\times n}$, $P = P^T \geq 0$, $\mathcal{E}(P) := \{x \in \mathbb{R}^n : x^TPx \leq 1\}$.
— $\mathcal{L}(H)$: For $H \in \mathbb{R}^{m\times n}$, $\mathcal{L}(H) := \left\{x \in \mathbb{R}^n : |Hx|_\infty \leq 1\right\}$.

About the relationship between $\mathcal{E}(P)$ and $\mathcal{L}(\bar{U}^{-1}H)$, for a given $s > 0$, we have (see, e.g., [25]),

$$s\mathcal{E}(P) \subseteq \mathcal{L}(\bar{U}^{-1}H) \Longleftrightarrow \begin{bmatrix} \frac{\bar{u}_\ell^2}{s^2} & H_\ell \\ H_\ell^T & P \end{bmatrix} \geq 0 \;\; \forall \ell \qquad (2)$$

where H_ℓ is the ℓth row of H and $\bar{u}_\ell$ is the ℓth diagonal element of $\bar{U}$.

II. Two Forms of Parameterized Differential Inclusions

Algebraic loops in linear systems can be easily solved (if they are well-posed). For system (1), the presence of the deadzone function makes the algebraic loop much harder to deal with. Theoretically, an explicit solution can be derived as a piecewise affine function, in terms of both x and w, by partitioning the vector space $\mathbb{R}^m$ into 3^m polytopic regions (see Remark 1). However, the complexity of the partition even for $m = 2$ or 3 makes the solution almost impossible to manipulate. In this paper, we would like to use convex sets to bound all the possible solutions. By doing that, we obtain differential inclusion descriptions for the original system (1) and make it more approachable with Lyapunov methods.

Recall that the deadzone function belongs to the $[0, I]$ sector, i.e., for each y there exists a diagonal $\Delta \in \mathbb{R}^{m\times m}$ satisfying $0 \leq \Delta \leq I$ and $\mathrm{dz}(y) = \Delta y$. Let $\mathcal{K}$ be the set of diagonal matrices whose diagonal elements are either 1 or 0. Then $\mathrm{co}\mathcal{K}$ is the set of diagonal Δ satisfying $0 \leq \Delta \leq I$. There are 2^m matrices in $\mathcal{K}$ and we number them as $K_i, i = 1, 2, \ldots, 2^m$. Then, we have $\mathcal{K} = \{K_i : i \in I[1, 2^m]\}$ and

$$\mathrm{dz}(y) \in \mathrm{co}\{K_iy : \; i \in I[1, 2^m]\}.$$

This relation holds for all $y \in \mathbb{R}^m$ but could be conservative over a local region where the system operates. In [25], [26], a flexible description was introduced for dealing with the saturated state feedback $\mathrm{sat}(Fx)$. This description can be easily adapted for the deadzone function. The main idea behind this description is the following simple fact.

Fact 1: Suppose $v_i \in [-\bar{u}_i, \bar{u}_i]$ (with $\bar{u}_i$ being the ith saturation level). For any $u_i \in \mathbb{R}$, we have $\mathrm{sat}(u_i) \in \mathrm{co}\{u_i, v_i\}$, i.e., $\mathrm{sat}(u_i) = \delta u_i + (1 - \delta)v_i$ for some $\delta \in [0, 1]$, and $\mathrm{dz}(u_i) \in \mathrm{co}\{0, u_i - v_i\}$, i.e., $\mathrm{dz}(u_i) = \delta(u_i - v_i)$ for some $\delta \in [0, 1]$.

This simple fact has also been used in [13] to analyze the nonlinear $\mathcal{L}_2$ gain for a special case of (1), where D_{yq}, D_{yw},

D_{zq} and D_{zw} are all zero. For the general case where D_{yq} may be nonzero, we have the following algebraic loop:

$$y = C_y x + D_{yq}\mathrm{dz}(y) + D_{yw}w. \tag{3}$$

This algebraic loop is said to be well-posed if there exists a unique solution y for each $C_y x + D_{yw}w$. A sufficient condition for the algebraic loop to be well-posed is the existence of a diagonal matrix $W > 0$ such that $2W - D_{yq}W - WD_{yq}^T > 0$ (see, e.g., [16], [44]). In the following claim, whose proof is reported in [30], we give a precise characterization of the well-posedness of the algebraic loop.

Claim 1: Assume that ϕ is the deadzone function or the saturation function. Then $y = D\phi(y) + v$ has a unique solution for every $v \in \mathbb{R}^m$ if and only if $\det(I - D\Delta) \neq 0$ for all $\Delta \in \mathrm{co}\mathcal{K}$.

Remark 1: If the algebraic loop $y = D\phi(y) + v$ is well-posed, then the solution y is a piecewise affine function of v with 3^m polytopic regions. To understand this, consider the function g: $y \mapsto v = y - D\phi(y)$. It is piecewise affine with 3^m polytopic partitions. If there is a unique solution y for each v, then each polytope in the domain of g is uniquely and affinely mapped to a polytope in the range of g. Hence, the inverse function of g, i.e., the solution of the algebraic loop, is also piecewise affine, with partition corresponding to that of the original g. ○

Based on Claim 1, we have the following criterion for the well-posedness of the algebraic loop (the proof can be found in [30]).

Claim 2: The algebraic loop (3) is well-posed if and only if the values of $\det(I - D_{yq}K_i), i \in I[1, 2^m]$, are all nonzero and have the same sign. In this case, we have

$$\{(I - \Delta D_{yq})^{-1}\Delta : \Delta \in \mathrm{co}\mathcal{K}\} \subseteq \mathrm{co}\{(I - K_i D_{yq})^{-1}K_i : i \in I[1, 2^m]\}. \tag{4}$$

The well-posedness condition in Claim 2 can be easily verified. The relation (4) will be used to bound the solution of the algebraic loop with a polytope.

Throughout this paper, we assume that this well-posedness condition is satisfied. For $i \in I[1, 2^m]$, denote

$$\begin{aligned} T_i &= (I - K_i D_{yq})^{-1}K_i \\ A_i &= A + B_q T_i C_y \quad B_i = B_w + B_q T_i D_{yw} \\ C_i &= C_z + D_{zq} T_i C_y \quad D_i = D_{zw} + D_{zq} T_i D_{yw}. \end{aligned} \tag{5}$$

Proposition 1: Let $h : \mathbb{R}^n \to \mathbb{R}^m$ be a given map and let h_ℓ be the ℓth component of h. Consider system (1). If $x \in \mathbb{R}^n$ satisfies $|h_\ell(x)| \le \bar{u}_\ell$ for all $\ell \in I[1, m]$, then

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} \in \mathrm{co}\left\{ \begin{bmatrix} A_i x + B_i w - B_q T_i h(x) \\ C_i x + D_i w - D_{zq} T_i h(x) \end{bmatrix} : i \in I[1, 2^m] \right\}. \tag{6}$$

Proof: Since $|h_\ell(x)| \le \bar{u}_\ell$ for all $\ell \in I[1, m]$, by Fact 1, we have

$$q = \mathrm{dz}(y) = \Delta(y - h(x))$$

for some $\Delta \in \mathrm{co}\mathcal{K}$. Recalling $y = C_y x + D_{yq}q + D_{yw}w$, we obtain $q = \Delta(C_y x + D_{yq}q + D_{yw}w - h(x))$. It follows that $q = (I - \Delta D_{yq})^{-1}\Delta(C_y x + D_{yw}w - h(x))$. By (4) and (5), we have

$$q \in \mathrm{co}\{T_i(C_y x + D_{yw}w - h(x)) : i \in I[1, 2^m]\}. \tag{7}$$

Applying this relation to the first and the third equations in (1), we obtain (6). □

By taking $h(x) = 0$ in (6), we obtain a polytopic linear differential inclusion (PLDI) representation which holds globally for the original system (1). A nonzero term $h(x)$ is used to inject additional degrees of freedom in some subset of the state space to reduce conservatism in regional analysis. When we use quadratic Lyapunov functions, we will choose $h(x) = Hx$ where H can be used as an optimizing parameter. When we use nonquadratic Lyapunov functions, a nonlinear $h(x)$ is more effective in general.

The PDI (6) involves 2^m vertices. This may present numerical difficulties when m is large (e.g., $m > 6$) and the order of the system is high. To reduce this computational burden, we may use a more conservative description; namely, to approximate the system (1) we may use an NDI, which is based on the following result, whose proof is in [30].

Claim 3: Let M be a positive diagonal matrix. Suppose that

$$2I - M^{-1}D_{yq}M - MD_{yq}^T M^{-1} = S^2$$

where S is symmetric and nonsingular. Then

$$\mathrm{co}\{(I - K_i D_{yq})^{-1}K_i : i \in I[1, 2^m]\} \subseteq \{M(S^{-2} + S^{-1}\Omega S^{-1})M^{-1} : \|\Omega\| \le 1\} \tag{8}$$

where $\|\Omega\|$ is the spectral norm of Ω (namely its largest singular value). Furthermore, each vertex of the left-hand side is on the boundary of the righthand side.

Proposition 2: Assume that there exist a diagonal $M > 0$ and a symmetric nonsingular S such that

$$S^2 = 2I - M^{-1}D_{yq}M - MD_{yq}^T M^{-1}.$$

Let $H \in \mathbb{R}^{m \times n}$ be given. For $\Omega \in \mathbb{R}^{m \times m}$, define

$$\begin{bmatrix} A_\Omega & B_\Omega \\ C_\Omega & D_\Omega \end{bmatrix} := \begin{bmatrix} A & B_w \\ C_z & D_{zw} \end{bmatrix} + \begin{bmatrix} B_q \\ D_{zq} \end{bmatrix} M(S^{-2} + S^{-1}\Omega S^{-1})M^{-1}\left[C_y - H \quad D_{yw} \right].$$

Consider system (1). If $x \in \mathbb{R}^n$ satisfies $|\bar{U}^{-1}Hx|_\infty \le 1$, then

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} \in \left\{ \begin{bmatrix} A_\Omega & B_\Omega \\ C_\Omega & D_\Omega \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} : \|\Omega\| \le 1 \right\}. \tag{9}$$

Proposition 2 can be proved like Proposition 1 by applying Claim 3 to (7) with $h(x) = Hx$ [note that $T_i = (I - K_i D_{yq})^{-1}K_i$)]. Then, we obtain

$$q \in \{M(S^{-2} + S^{-1}\Omega S^{-1}) \cdot M^{-1}((C_y - H)x + D_{yw}w) : \|\Omega\| \le 1\}.$$

Applying this to the original system (1), we obtain (9). We call (9) the NDI for (1). If $m = 1$, then the two sets in (8) are the same and the NDI is the same as the PDI. If $m > 1$, generally the NDI strictly contains the PDI. We also note that to obtain the NDI, there must exist a positive diagonal matrix M such that $2I - M^{-1}D_{yq}M - MD_{yq}^TM^{-1} > 0$, which is a stronger requirement than well-posedness.

III. Analysis With Quadratic Lyapunov Functions

A. Some General Results for Linear Differential Inclusions

In [2], extensive results were established for stability and performance analysis of LDIs by using quadratic Lyapunov functions. Consider the LDI

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} \in \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} : \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Phi \right\} \tag{10}$$

where Φ is a given convex set of matrices. The following lemma can be established like in the corresponding results in [2] by extending a polytopic Φ to a general Φ.

Lemma 1: Given $P = P^T > 0, \gamma > 0$, let $V(x) = x^TPx$ and denote by $\dot{V}(x,w)$ the derivative of V in any of the directions of the right-hand side of (10). The following holds.

1) $\dot{V}(x,w) < 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$ and $w = 0$, if

$$A^TP + PA < 0 \quad \forall A \in \left\{ [I \quad 0]X \begin{bmatrix} I \\ 0 \end{bmatrix} : X \in \Phi \right\}.$$

2) $\dot{V}(x,w) \le w^Tw$ for all $x \in \mathbb{R}^n, w \in \mathbb{R}^r$, if

$$\text{He} \begin{bmatrix} PA & PB \\ 0 & -\frac{I}{2} \end{bmatrix} \le 0 \qquad \forall \, [A \quad B] \in \{[I \quad 0]X : X \in \Phi\}.$$

3) $\dot{V}(x,w) + (1/\gamma^2)z^Tz \le w^Tw$ for all $x \in \mathbb{R}^n, w \in \mathbb{R}^r$, if

$$\text{He} \begin{bmatrix} PA & PB & 0 \\ 0 & -\frac{I}{2} & 0 \\ C & D & -\frac{\gamma^2 I}{2} \end{bmatrix} \le 0 \;\; \forall \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Phi. \tag{11}$$

The condition in item 1) guarantees that the ellipsoid $\mathcal{E}(P)$ is contractively invariant in the absence of w. It will be used for the estimation of the domain of attraction. The condition in item 2 guarantees that if $\|w\|_2 \le s$, then under the initial condition $x(0) = 0$, we will have $x(t) \in s\mathcal{E}(P)$ for all $t \ge 0$. This will be used to determine the reachable set under a class of bounded energy disturbances. Item 3) gives a condition for γ to be a bound for the $\mathcal{L}_2$ gain, i.e., $\|z\|_2 \le \gamma\|w\|_2$ for all w and $x(0) = 0$. The result in item 3) can also be found, e.g., in [19]. For the case where Φ is a polytope, we only need to verify the conditions at its vertices.

Combining Lemma 1 with the two differential inclusion descriptions, we will obtain different methods for the analysis of the original system (1). The crucial point is to guarantee that the PDI (6) [or the NDI (9)] is valid for all time under the class of disturbances and the set of initial $x(0)$'s under consideration. We are mainly concerned about the existence of a matrix H, such that $|\bar{U}^{-1}Hx(t)|_\infty \le 1$, i.e., $x(t) \in \mathcal{L}(\bar{U}^{-1}H)$, for all t. To ensure this property, we are going to construct a quadratic function $V(x) = x^TPx$, $P = P^T > 0$, and use Lemma 1 to guarantee that $x(t) \in s\mathcal{E}(P) \subseteq \mathcal{L}(\bar{U}^{-1}H)$ for all $t \ge 0$.

B. Analysis Based on the Polytopic Differential Inclusion

When $h(x) = Hx$, the PDI (6) can be written as

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} \in \text{co}\left\{ \begin{bmatrix} A_i - B_qT_iH & B_i \\ C_i - D_{zq}T_iH & D_i \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} : i \in I[1, 2^m] \right\}. \tag{12}$$

which corresponds to (10) with

$$\Phi = \text{co}\left\{ \begin{bmatrix} A_i - B_qT_iH & B_i \\ C_i - D_{zq}T_iH & D_i \end{bmatrix} : i \in I[1, 2^m] \right\}.$$

We will restrict our attention to a certain ellipsoid $s\mathcal{E}(P)$. For the purpose of presenting the results in terms of LMIs, we state the results using $Q = P^{-1}$ and $Y = HQ$. To apply the PDI description within the ellipsoid $s\mathcal{E}(P) = s\mathcal{E}(Q^{-1})$, we need to ensure that $s\mathcal{E}(P) \subseteq \mathcal{L}(\bar{U}^{-1}H)$ so that $|\bar{U}^{-1}Hx|_\infty \le 1$ (i.e., $|h_\ell(x)| \le \bar{u}_\ell$ for all ℓ) for all $x \in s\mathcal{E}(P)$, which is equivalent to [recall from (2)]

$$\begin{bmatrix} \frac{\bar{u}_\ell^2}{s^2} & H_\ell \\ H_\ell^T & P \end{bmatrix} \ge 0 \quad \forall \ell \in I[1, m]$$

where H_ℓ is the ℓth row of H and $\bar{u}_\ell$ is the ℓth diagonal element of $\bar{U}$. Multiplying on the left and the right by $\text{diag}\{1, Q\}$, we obtain the equivalent condition

$$\begin{bmatrix} \frac{\bar{u}_\ell^2}{s^2} & Y_\ell \\ Y_\ell^T & Q \end{bmatrix} \ge 0 \quad \forall \ell \in I[1, m]. \tag{13}$$

Theorem 1: Given $Q \in \mathbb{R}^{n\times n}$, $Q = Q^T > 0$. Let $V(x) = x^TQ^{-1}x$. Consider system (1).

1) If there exists $Y \in \mathbb{R}^{m\times n}$ satisfying (13) with $s = 1$ and

$$QA_i^T + A_iQ - Y^TT_i^TB_q^T - B_qT_iY < 0 \;\; \forall i \in I[1, 2^m] \tag{14}$$

then $\dot{V}(x,w) < 0$ for all $x \in \mathcal{E}(Q^{-1}) \setminus \{0\}$ and $w = 0$, i.e., $\mathcal{E}(Q^{-1})$ is a contractively invariant ellipsoid.

2) Let $s > 0$. If there exists $Y \in \mathbb{R}^{m\times n}$ satisfying (13) and

$$\text{He} \begin{bmatrix} A_iQ - B_qT_iY & B_i \\ 0 & -\frac{I}{2} \end{bmatrix} \le 0 \;\; \forall i \in I[1, 2^m] \tag{15}$$

then $\dot{V}(x,w) \le w^Tw$ for all $x \in s\mathcal{E}(Q^{-1}), w \in \mathbb{R}^r$. If $x(0) = 0$ and $\|w\|_2 \le s$, then $x(t) \in s\mathcal{E}(Q^{-1})$ for all $t \ge 0$.

3) Let $\gamma, s > 0$. If there exists $Y \in \mathbb{R}^{m\times n}$ satisfying (13) and

$$\text{He} \begin{bmatrix} A_iQ - B_qT_iY & B_i & 0 \\ 0 & -\frac{I}{2} & 0 \\ C_iQ - D_{zq}T_iY & D_i & -\frac{\gamma^2 I}{2} \end{bmatrix} \le 0 \qquad \forall i \in I[1, 2^m] \tag{16}$$

then $\dot{V}(x,w) + (1/\gamma^2)z^Tz \le w^Tw$ for all $x \in s\mathcal{E}(Q^{-1}), w \in \mathbb{R}^r$. If $x(0) = 0$ and $\|w\|_2 \le s$, then $\|z\|_2 \le \gamma\|w\|_2$.

Proof: Let $P = Q^{-1}$ and $H = YP$.

1) If we multiply (14) on the left and the right by P, we obtain $(A_i - B_qT_iH)^TP + P(A_i - B_qT_iH) < 0 \quad \forall i \in I[1, 2^m]$.

Applying item 1) of Lemma 1 to the LDI (12), this guarantees that $\dot{V}(x,w) < 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$ and $w = 0$ for (12). Because of (13) with $s = 1$, we have $\mathcal{E}(Q^{-1}) \subseteq \mathcal{L}(\bar{U}^{-1}H)$, i.e., $|\bar{U}^{-1}Hx|_\infty \le 1$ for all $x \in \mathcal{E}(P)$. By Proposition 1, system (1) satisfies (12) for all $x \in \mathcal{E}(Q^{-1})$. Hence, for system (1), we also have $\dot{V}(x,w) < 0$ for all $x \in \mathcal{E}(Q^{-1}) \setminus \{0\}$.

2) If we multiply (15) on the left and the right by $\mathrm{diag}\{P, I\}$, we obtain

$$\mathrm{He}\begin{bmatrix} PA_i - PB_qT_iH & PB_i \\ 0 & -\frac{I}{2} \end{bmatrix} \le 0 \quad \forall i \in I[1, 2^m].$$

By item 2) of Lemma 1, this ensures that $\dot{V}(x,w) \le w^Tw$ for all x and w for (12). Also, the condition (13) ensures that $s\mathcal{E}(Q^{-1}) \subseteq \mathcal{L}(\bar{U}^{-1}H)$ and hence (12) is valid within $s\mathcal{E}(Q^{-1})$. Therefore, we have $\dot{V}(x,w) \le w^Tw$ for all $x \in s\mathcal{E}(Q^{-1}), w \in \mathbb{R}^r$ for system (1). If $x(0) = 0$ and $\|w\|_2 \le s$, then by integrating both sides of $\dot{V} \le w^Tw$, we have $V(x(t)) \le s^2$, i.e., $x(t) \in s\mathcal{E}(Q^{-1})$ for all $t \ge 0$.

3) We note that (16) implies (15). So by item 2), it is ensured that $x(t) \in s\mathcal{E}(Q^{-1})$ for all $t \ge 0$ if $x(0) = 0$ and $\|w\|_2 \le s$. Hence, the LDI (12) is valid for system (1) for all $\|w\|_2 \le s$ and $x(0) = 0$. If we multiply (16) on the left and the right by $\mathrm{diag}\{P, I, I\}$, we obtain

$$\mathrm{He}\begin{bmatrix} PA_i - PB_qT_iH & PB_i & 0 \\ 0 & -\frac{I}{2} & 0 \\ C_i - D_{zq}T_iH & D_i & -\frac{\gamma^2 I}{2} \end{bmatrix} \le 0$$

for all $i \in I[1, 2^m]$. By Lemma 1, this ensures that $\dot{V}(x,w) + (1/\gamma^2)z^Tz \le w^Tw$ for all $x \in \mathbb{R}^n, w \in \mathbb{R}^r$ for system (12). For system (1), the inequality holds for all $x \in s\mathcal{E}(Q^{-1})$ and $w \in \mathbb{R}^r$. By integrating both sides of the inequality, we have $\|z\|_2 \le \gamma\|w\|_2$ as long as $\|w\|_2 \le s$ and $x(0) = 0$. □

It can be verified that for the special case where $D_{yq} = 0$, $D_{yw} = 0$, $D_{zq} = 0$ and $D_{zw} = 0$, items 1) and 3) reduce to the corresponding results in [25] and [13], respectively. The three parts in Theorem 1 can be respectively used to estimate the domain of attraction, the reachable set and the nonlinear $\mathcal{L}_2$ gain for system (1). For these purposes, we may formulate corresponding optimization problems with linear matrix inequality (LMI) constraints. For the estimation of the nonlinear $\mathcal{L}_2$ gain, we need to minimize γ for a selection of s over $[0, \infty)$.

1) Problem 1: Estimation of the Domain of Attraction: For the purpose of enlarging the estimation of the domain of attraction, we may choose a shape reference set X_R (see, e.g., [22], [25], and [26]) and maximize a scaling $\alpha > 0$ such that $\alpha X_R \subseteq \mathcal{E}(Q^{-1})$, with Q satisfying (13) and (14). The optimizing parameters are Q and Y. When X_R is a polygon or an ellipsoid, the resulting optimization problem has an LMI formulation.

2) Problem 2: Estimation of the Reachable Set: Under the condition (13) and (15), an estimate of the reachable set is given by $s\mathcal{E}(Q^{-1})$. Since smaller (or tighter) estimates are desirable, we may formulate an optimization problem to minimize the size of $s\mathcal{E}(Q^{-1})$. There are different measures of size for ellipsoids, such as the trace of Q and the determinant of Q, among which the trace of Q is a convex measure and is much easier to handle. In a practical situation, we may be interested in knowing the size of a certain state or an output during the operation of the system. For instance, given a row vector $C \in \mathbb{R}^{1\times n}$, we would like to estimate the maximal value of $|Cx(t)|$ for all $t \ge 0$. Since $x(t) \in s\mathcal{E}(Q^{-1})$, the maximal value of $|Cx(t)|$ is less than

$$\bar{\alpha} := (\max\{x^TC^TCx : x^T(s^2Q)^{-1}x \le 1\})^{1/2}.$$

Given $\alpha > 0$. Consider the set $\mathcal{E}(C^TC/\alpha^2) = \{x : x^TC^TCx \le \alpha^2\} = \{x : |Cx| \le \alpha\}$. It is the region between the two hyperplanes $Cx = \alpha$ and $Cx = -\alpha$. It can also be considered as a degenerated ellipsoid corresponding to a positive semidefinite matrix C^TC. Hence we have $\alpha \ge \bar{\alpha}$ if and only if $\mathcal{E}((s^2Q)^{-1}) \subset \mathcal{E}(C^TC/\alpha^2)$, which is equivalent to $C^TC/\alpha^2 \le (s^2Q)^{-1}$. Thus, $\bar{\alpha} = \min\{\alpha : C^TC \le \alpha^2(s^2Q)^{-1}\}$. Note that $C^TC \le \alpha^2(s^2Q)^{-1}$ is equivalent to $Q^{1/2}C^TCQ^{1/2} \le \alpha^2/s^2 I$ and to $CQC^T \le \alpha^2/s^2$, we have

$$\bar{\alpha} = \min\left\{\alpha : CQC^T \le \frac{\alpha^2}{s^2}\right\}.$$

To minimize $\bar{\alpha}$, we can minimize α^2 satisfying the linear (in Q and α^2) constraint $CQC^T \le \alpha^2/s^2$ with Q satisfying (13) and (15). With α determined this way, we have $|Cx(t)| \le \alpha$ for all $t \ge 0$. We may choose different C's, such as $C_i, i = 1, 2, \ldots, N$, and obtain a bound α_i on $|C_ix(t)|$ for each i. The polytope formed as $\{x \in \mathbb{R}^n : |C_ix| \le \alpha_i, i = 1, \ldots, N\}$ will also be an estimate of the reachable set.

3) Problem 3: Estimation of the Nonlinear $\mathcal{L}_2$ Gain: The problem of minimizing a bound on the $\mathcal{L}_2$ gain follows directly from item 3 of Theorem 1 by minimizing γ along with parameters Q and Y satisfying (13) and (16). For each $s > 0$, denote $\gamma^*(s)$ as the minimal γ, then we have

$$\|z\|_2 \le \gamma^*(\|w\|_2)\|w\|_2$$

for all w. In other words, $\gamma^*(s)$ serves as an estimate for the nonlinear $\mathcal{L}_2$ gain.

C. Analysis Based on the Norm-Bounded Differential Inclusion

For easy reference, the NDI description for (1) is repeated as follows. If $|\bar{U}^{-1}Hx|_\infty \le 1$, then

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} \in \left\{\begin{bmatrix} A_\Omega & B_\Omega \\ C_\Omega & D_\Omega \end{bmatrix}\begin{bmatrix} x \\ w \end{bmatrix} : \|\Omega\| \le 1\right\} \tag{17}$$

where

$$\begin{bmatrix} A_\Omega & B_\Omega \\ C_\Omega & D_\Omega \end{bmatrix} = \begin{bmatrix} A & B_w \\ C_z & D_{zw} \end{bmatrix} + \begin{bmatrix} B_q \\ D_{zq} \end{bmatrix} MS^{-1}(I+\Omega)S^{-1}M^{-1}\begin{bmatrix} C_y - H & D_{yw} \end{bmatrix} \tag{18}$$

and $M > 0$ is diagonal, S is symmetric and nonsingular such that $S^2 = 2I - M^{-1}D_{yq}M - MD_{yq}^TM^{-1}$.

The next lemma will be used to handle the norm-bounded differential inclusion (17).

Lemma 2: Given X, Y, Z, S of compatible dimensions, where S is symmetric and nonsingular. If

$$\mathrm{He}\begin{bmatrix} Z & X \\ Y & -\frac{S^2}{2} \end{bmatrix} \le 0 \tag{19}$$

then $\mathrm{He}(Z + XS^{-1}(I+\Omega)S^{-1}Y) \leq 0 \quad \forall \|\Omega\| \leq 1$.

This lemma follows directly using Schur complements and from $M\Omega N + N^T\Omega^T M^T \leq MM^T + N^T N$ for all $\|\Omega\| \leq 1$.

Theorem 2: Given $Q \in \mathbb{R}^{n\times n}$, $Q = Q^T > 0$. Let $V(x) = x^T Q^{-1} x$. Consider system (1).

1) If there exist $Y \in \mathbb{R}^{m\times n}$ and a diagonal $U > 0$ satisfying (13) with $s = 1$ and

$$\mathrm{He}\begin{bmatrix} AQ & B_q U \\ C_y Q - Y & -U + D_{yq}U \end{bmatrix} < 0 \tag{20}$$

then $\mathcal{E}(Q^{-1})$ is a contractively invariant ellipsoid.

2) Given $s > 0$, if there exist $Y \in \mathbb{R}^{m\times n}$ and a diagonal $U > 0$ satisfying (13) and

$$\mathrm{He}\begin{bmatrix} AQ & B_w & B_q U \\ 0 & -\frac{I}{2} & 0 \\ C_y Q - Y & D_{yw} & -U + D_{yq}U \end{bmatrix} \leq 0 \tag{21}$$

then $\dot V(x,w) \leq w^T w$ for all $x \in s\mathcal{E}(Q^{-1}), w \in \mathbb{R}^r$. If $x(0) = 0$ and $\|w\|_2 \leq s$, then $x(t) \in s\mathcal{E}(Q^{-1})$ for all $t \geq 0$.

3) Given $\gamma, s > 0$, if there exist $Y \in \mathbb{R}^{m\times n}$ and a diagonal $U > 0$ satisfying (13) and

$$\mathrm{He}\begin{bmatrix} AQ & B_w & 0 & B_q U \\ 0 & -\frac{I}{2} & 0 & 0 \\ C_z Q & D_{zw} & -\frac{\gamma^2 I}{2} & D_{zq}U \\ C_y Q - Y & D_{yw} & 0 & -U + D_{yq}U \end{bmatrix} \leq 0 \tag{22}$$

then $\dot V(x,w) + (1/\gamma^2) z^T z \leq w^T w$ for all $x \in s\mathcal{E}(Q^{-1}), w \in \mathbb{R}^r$. If $x(0) = 0$ and $\|w\|_2 \leq s$, then $\|z\|_2 \leq \gamma \|w\|_2$.

Proof: The procedure is very similar to the proof of Theorem 1 except we need to establish that the conditions (20)–(22) imply the respective conditions in Lemma 1 for the NDI (17). This is a little more complicated than the counterpart for Theorem 1.

Here, we only show that (22) guarantees (11) when the differential inclusion (10) is specified to (17). The other correspondences in items 1) and 2) are similar and simpler. For system (17), the condition (11) in Lemma 1 can be written as

$$\mathrm{He}\begin{bmatrix} PA_\Omega & PB_\Omega & 0 \\ 0 & -\frac{I}{2} & 0 \\ C_\Omega & D_\Omega & -\frac{\gamma^2 I}{2} \end{bmatrix} \leq 0 \qquad \forall \|\Omega\| \leq 1. \tag{23}$$

From (18), we have

$$\begin{bmatrix} PA_\Omega & PB_\Omega & 0 \\ 0 & -\frac{I}{2} & 0 \\ C_\Omega & D_\Omega & -\frac{\gamma^2 I}{2} \end{bmatrix} = \begin{bmatrix} PA & PB_w & 0 \\ 0 & -\frac{I}{2} & 0 \\ C_z & D_{zw} & -\frac{\gamma^2 I}{2} \end{bmatrix} + \begin{bmatrix} PB_q \\ 0 \\ D_{zq} \end{bmatrix} MS^{-1}(I+\Omega)S^{-1}M^{-1}\begin{bmatrix} C_y - H & D_{yw} & 0 \end{bmatrix}.$$

By Lemma 2, to guarantee (23), it suffices to have

$$\mathrm{He}\begin{bmatrix} PA & PB_w & 0 & PB_q M \\ 0 & -\frac{I}{2} & 0 & 0 \\ C_z & D_{zw} & -\frac{\gamma^2 I}{2} & D_{zq}M \\ M^{-1}(C_y - H) & M^{-1}D_{yw} & 0 & -\frac{S^2}{2} \end{bmatrix} \leq 0. \tag{24}$$

Multiplying on the left and the right by $\mathrm{diag}\{Q, I, I, M\}$, noticing that $\mathrm{He}(-S^2/2) = \mathrm{He}(-I + M^{-1}D_{yq}M)$, $Q = P^{-1}$, $Y = HQ$, (24) is equivalent to

$$\mathrm{He}\begin{bmatrix} AQ & B_w & 0 & B_q M^2 \\ 0 & -\frac{I}{2} & 0 & 0 \\ C_z Q & D_{zw} & -\frac{\gamma^2 I}{2} & D_{zq}M^2 \\ C_y Q - Y & D_{yw} & 0 & -M^2 + D_{yq}M^2 \end{bmatrix} \leq 0$$

which is (22) with $U = M^2$. □

Remark 2: If we take $Y = 0$ in (22), then the inequality reduces to [16, eq. (10a)] (with some permutation). A nonzero parameter Y introduces additional degrees of freedom for regional analysis and makes the results applicable to the case where the system wrapped around the saturation is not globally exponentially stable. ○

As with Theorem 1, different optimization problems with LMI constraints can be formulated for stability and performance analysis of the original system (1) based on the three parts of Theorem 2. Since the NDI is a more conservative description than the PDI and since Theorems 1 and 2 are developed from the same framework, it is easy to see that the analysis results from using Theorem 2 are more conservative than those from using Theorem 1. Actually, even for the special case $m = 1$ for which the NDI and PDI descriptions are the same, Theorem 2 could still be more conservative than Theorem 1 because of using Lemma 2 to derive (24). The advantage of Theorem 2 is that the conditions involve fewer LMIs (but of larger size, i.e., $+m$ more than those in Theorem 1).

We should note that the results in Theorem 2 were established in [28] through the S-procedure. The approach taken in this paper helps us to understand the relationship between the results based on two different types of differential inclusions.

IV. Analysis With Nonquadratic Lyapunov Functions

In this section, we will use a pair of conjugate functions, the convex hull quadratic function and the max quadratic function to perform stability and performance analysis of system (1). For the PDI (6), significant improvement may be achieved with these nonquadratic functions. However, for the NDI (9), there is no advantage in using these nonquadratic functions over quadratic functions. As a matter of fact, this result also applies to any norm-bounded linear differential inclusion (NLDI) (see Remark 5). We first review some results about this pair of conjugate functions.

A. The Max Quadratic Function and the Convex Hull Quadratic Function

Given a family of positive–definite matrices $P_j \in \mathbb{R}^{n\times n}$, $P_j = P_j^T > 0, j \in I[1, J]$, the pointwise maximum quadratic function is defined as

$$V_{\max}(x) := \max\{x^T P_j x : \ j \in I[1,J]\}. \tag{25}$$

Given $Q_j \in \mathbb{R}^{n\times n}$, $Q_j = Q_j^T > 0, j \in I[1,J]$. Let

$$\Gamma := \left\{\gamma \in \mathbb{R}^J : \ \ \gamma_1 + \gamma_2 + \cdots + \gamma_J = 1, \gamma_j \geq 0\right\}$$

the convex hull quadratic function is defined as

$$V_c(x) := \min_{\gamma\in\Gamma} x^T \left(\sum_{j=1}^J \gamma_j Q_j\right)^{-1} x. \tag{26}$$

For simplicity, we say that V_c is composed from Q_j's. It was shown in [15] that $(1/2)V_{\max}$ is conjugate to $(1/2)V_c$ if $Q_j = P_j$ for each $j \in I[1,J]$. It is evident that V_c and $V_{\max}$ are homogeneous of degree 2, i.e., $V_c(\alpha x) = \alpha^2 V_c(x)$, $V_{\max}(\alpha x) = \alpha^2 V_{\max}(x)$. Also established in [15], [23] are that V_c is convex and continuously differentiable and that $V_{\max}$ is strictly convex.

The 1-level set of $V_{\max}$ and that of V_c are, respectively

$$\begin{aligned} L_{V_{\max}} &:= \{x \in \mathbb{R}^n : \ \ V_{\max}(x) \leq 1\} \\ L_{V_c} &:= \{x \in \mathbb{R}^n : \ \ V_c(x) \leq 1\}. \end{aligned}$$

Since $V_{\max}$ and V_c are homogeneous of degree 2, we have

$$\begin{aligned} sL_{V_{\max}} &= \left\{x \in \mathbb{R}^n : \ \ V_{\max}(x) \leq s^2\right\} \\ sL_{V_c} &= \left\{x \in \mathbb{R}^n : \ \ V_c(x) \leq s^2\right\}. \end{aligned}$$

It is easy to see that $L_{V_{\max}}$ is the intersection of the ellipsoids $\mathcal{E}(P_j)$'s. In [23], It was established that L_{V_c} is the convex hull of the ellipsoids $\mathcal{E}(Q_j^{-1})$'s, i.e.,

$$L_{V_c} = \left\{\sum_{j=1}^J \gamma_j x_j : \ x_j \in \mathcal{E}(Q_j^{-1}), \gamma \in \Gamma\right\}.$$

For a compact convex set S, a point $x \in S$ is called an extreme point if it cannot be represented as the convex combination of any other points in S. Clearly an extreme point must belong to the boundary of S (denoted as ∂S). For a strictly convex set, such as $L_{V_{\max}}$, every boundary point is an extreme point. In what follows, we characterize the set of extreme points of L_{V_c}. Since L_{V_c} is the convex hull of $\mathcal{E}(Q_j^{-1})$'s, an extreme point must be on the boundaries of both L_{V_c} and $\mathcal{E}(Q_j^{-1})$ for some $j \in I[1,J]$ (If $x \in \partial L_{V_c} \setminus \cup_{j=1}^J \mathcal{E}(Q_j^{-1})$, then x must be the convex combination of at least two points from $\cup_{j=1}^J \mathcal{E}(Q_j^{-1})$ and thus not an extreme point of L_{V_c}). Denote

$$\begin{aligned} E_j &:= \partial L_{V_c} \cap \partial\mathcal{E}(Q_j^{-1}) \\ &= \left\{x \in \mathbb{R}^n : V_c(x) = x^T Q_j^{-1} x = 1\right\}. \end{aligned}$$

Then, $\bigcup_{j=1}^J E_j$ contains all the extreme points of L_{V_c}. The exact description of E_j is given as follows (see [30] for the proof).

Lemma 3: For each $j \in I[1,J]$, define $F_j = \{x \in \mathbb{R}^n : x^T Q_j^{-1}(Q_k - Q_j)Q_j^{-1} x \leq 0 \quad \forall k \in I[1,J]\}$. Then $E_j = \partial L_{V_c} \cap F_j$.

It is clear that $\alpha F_j = F_j$ for any $\alpha > 0$. Since L_{V_c} is convex and contains the origin in its interior, we have $L_{V_c} = \bigcup_{\delta\in[0,1]} \delta(\partial L_{V_c})$. It follows from Lemma 3 that $\bigcup_{\delta\in[0,1]} \delta E_j = L_{V_c} \cap F_j$.

The following lemma combines some results from [23] and [24].

Lemma 4: For a given $x_0 \in \mathbb{R}^n$, let $\gamma^* \in \Gamma$ be an optimal γ such that

$$x_0^T \left(\sum_{j=1}^J \gamma_j^* Q_j\right)^{-1} x_0 = \min_{\gamma\in\Gamma} x_0^T \left(\sum_{j=1}^J \gamma_j Q_j\right)^{-1} x_0 = V_c(x_0).$$

For simplicity and without loss of generality, assume that $\gamma_j^* > 0$ for $j \in I[1,J_0]$ and $\gamma_j^* = 0$ for $j \in I[J_0+1,J]$. Denote

$$Q_0 = \sum_{j=1}^{J_0} \gamma_j^* Q_j \quad x_j = Q_j Q_0^{-1} x_0, \qquad j \in I[1,J_0].$$

Then, $V_c(x_j) = V_c(x_0)$ and $x_j \in V_c(x_0)^{1/2} E_j$, $j \in I[1,J_0]$. Moreover, $x_0 = \sum_{j=1}^{J_0} \gamma_j^* x_j$, and

$$\nabla V_c(x_0) = \nabla V_c(x_j) = 2Q_j^{-1} x_j = 2Q_0^{-1} x_0, \ j \in I[1,J_0]$$

where $\nabla V_c(x)$ denotes the gradient of V_c at x.

The following lemma is adapted from a result of [27] to the slightly different definition of V_c and $V_{\max}$ (the two functions in [27] have the coefficient $1/2$ and the saturation levels in $\bar{U}$ are also included here).

Lemma 5: [27]Let $H \in \mathbb{R}^{m\times n}$, $\bar{U} \in \mathbb{R}^{m\times m}$ be positive–definite diagonal and denote the ℓth row of H by H_ℓ and the ℓ-th diagonal element of $\bar{U}$ by $\bar{u}_\ell$. We have

1) $L_{V_c} \subseteq \mathcal{L}(\bar{U}^{-1}H)$ if and only if $(1/\bar{u}_\ell)H_\ell^T \in L_{V_{\max}}$ for all $\ell \in I[1,m]$;
2) $L_{V_{\max}} \subseteq \mathcal{L}(\bar{U}^{-1}H)$ if and only if $(1/\bar{u}_\ell)H_\ell^T \in L_{V_c}$ for all $\ell \in I[1,m]$.

B. Analysis With Convex Hull Quadratic Functions

In this section, we apply the convex hull quadratic function to the analysis of system (1) through the polytopic differential inclusion (6), which is repeated as follows for easy reference:

$$\begin{bmatrix}\dot{x} \\ z\end{bmatrix} \in \mathrm{co}\left\{\begin{bmatrix} A_i x + B_i w - B_q T_i h(x) \\ C_i x + D_i w - D_{zq} T_i h(x)\end{bmatrix} : \ i \in I[1,2^m]\right\}. \tag{27}$$

This PDI is a valid description for (1) as long as $|\bar{U}^{-1}h(x)|_\infty \leq 1$. We will restrict our attention to a level set sL_{V_c}, where $|\bar{U}^{-1}h(x)|_\infty \leq 1$ for all $x \in sL_{V_c}$. As with the case of using quadratic functions, the crucial point is to guarantee that $x(t) \in sL_{V_c}$ under the class of norm-bounded w and the set of initial states under consideration.

It may appear that choosing $h(x)$ as a linear function Hx within sL_{V_c} should lead to simpler results than choosing it as a nonlinear function. However, it turns out that a nonlinear $h(x)$

not only reduces conservatism but also leads to cleaner and numerically more tractable results. As expected, the derivation of the results is more involved than the former cases in Section III because of the nonquadratic Lyapunov function and the nonlinear function $h(x)$. For this reason, we present the results separately for the estimation of the domain of attraction, the reachable set and the $\mathcal{L}_2$ gain. Based on technical considerations, we first present the result about the reachable set.

Theorem 3: (Reachable set by $\mathcal{L}_2$-norm-bounded inputs) Given $Q_j = Q_j^T > 0$, $j \in I[1,J]$, let V_c be composed from Q_j's as in (26). Given $s > 0$. System (1) with $x(0) = 0$ satisfies $x(t) \in sL_{V_c}$ for all $t \geq 0$ and for all w such that $\|w\|_2 \leq s$ if there exist $Y_j \in \mathbb{R}^{m\times n}$ and $\lambda_{ijk} \geq 0$, $i \in I[1,2^m]$, $j,k \in I[1,J]$ such that

$$\mathrm{He}\begin{bmatrix} A_iQ_j - B_qT_iY_j + \sum_{k=1}^{J}\lambda_{ijk}(Q_j - Q_k) & B_i \\ 0 & -\frac{I}{2}\end{bmatrix} \leq 0 \quad \forall i \in I[1,2^m], j \in I[1,J] \quad (28)$$

$$\begin{bmatrix} \frac{\bar{u}_\ell^2}{s^2} & Y_{j,\ell} \\ Y_{j,\ell}^T & Q_j \end{bmatrix} \geq 0 \quad \forall \ell \in I[1,m], j \in I[1,J] \quad (29)$$

where $Y_{j,\ell}$ is the ℓth row of Y_j.

Proof: We will prove the theorem by showing that for all $x \in sL_{V_c}$ and $w \in \mathbb{R}^r$, we have $\dot{V}_c(x,w) \leq w^Tw$, where $\dot{V}_c(x,w)$ is the time derivative of V_c in the direction of the right-hand side of (1), which depends on x and w.

Let $P_j = Q_j^{-1}$, $H_j = Y_jQ_j^{-1}$. Multiplying (28) on the left and the right by $\mathrm{diag}\{P_j, I\}$, we have

$$\mathrm{He}\begin{bmatrix} P_j(A_i - B_qT_iH_j) + \sum_{k=1}^{J}\lambda_{ijk}P_j(Q_j - Q_k)P_j & P_jB_i \\ 0 & -\frac{I}{2}\end{bmatrix} \leq 0.$$

This implies that for all $i \in I[1,2^m]$, $j \in I[1,J]$

$$\begin{aligned}&2x^TP_j(A_ix + B_iw - B_qT_iH_jx) - w^Tw \\ &\quad \leq 2\sum_{k=1}^{J}\lambda_{ijk}x^TP_j(Q_k - Q_j)P_jx \quad \forall x \in \mathbb{R}^n, w \in \mathbb{R}^r. \quad (30)\end{aligned}$$

Given $j \in I[1,J]$ and any $\delta > 0$. Consider $x \in \delta E_j$. By Lemma 3, we have

$$\sum_{k=1}^{J}\lambda_{ijk}x^TP_j(Q_k - Q_j)P_jx \leq 0.$$

It follows from (30) that

$$\begin{aligned}&2x^TP_j(A_ix + B_iw - B_qT_iH_jx) - w^Tw \leq 0 \\ &\qquad\qquad \forall x \in \delta E_j, w \in \mathbb{R}^r,\ \delta > 0. \quad (31)\end{aligned}$$

(In view of (27) and (29), this actually shows that $\dot{V}_c(x,w) \leq w^Tw$ for all $x \in s(L_{V_c} \cap E_j)$, recalling from Lemma 4 that $\nabla V_c(x) = 2P_jx$ for $x \in E_j$. More explanation can be seen later). We proceed to show that $\dot{V}_c(x,w) \leq w^Tw$ holds for all $x \in sL_{V_c}$ by exploiting the properties of V_c.

Now, consider $x_0 \in sL_{V_c}$. Then $V_c(x_0) = \delta^2$ for some $\delta \in (0,s]$. By Lemma 4, there exist $x_j \in \delta E_j$, $\gamma_j > 0$, $j \in I[1,J_0]$ with $J_0 \leq J$ such that $\sum_{j=1}^{J_0}\gamma_j = 1$ and $x_0 = \sum_{j=1}^{J_0}\gamma_jx_j$ (we note that the indexes j can always be reordered to make this true for each x_0). Let

$$Q_0 = \sum_{j=1}^{J_0}\gamma_jQ_j \quad Y_0 = \sum_{j=1}^{J_0}\gamma_jY_j \quad H_0 = Y_0Q_0^{-1}. \quad (32)$$

Then, we also have $x_0^TQ_0^{-1}x_0 = V_c(x_0) = \delta^2$ and

$$\nabla V_c(x_0) = 2Q_0^{-1}x_0 = 2Q_j^{-1}x_j, \qquad j \in I[1,J_0]. \quad (33)$$

Applying convex combination to the inequalities in (29), we have

$$\begin{bmatrix} \frac{\bar{u}_\ell^2}{s^2} & Y_{0,\ell} \\ Y_{0,\ell}^T & Q_0 \end{bmatrix} \geq 0 \Leftrightarrow \begin{bmatrix} \frac{\bar{u}_\ell^2}{s^2} & H_{0,\ell} \\ H_{0,\ell}^T & Q_0^{-1} \end{bmatrix} \geq 0 \ \forall \ell \in I[1,m].$$

By (2), this implies that $s\mathcal{E}(Q_0^{-1}) \subseteq \mathcal{L}(\bar{U}^{-1}H_0)$. Since $x_0^TQ_0^{-1}x_0 = \delta^2 \leq s^2$, we have $|\bar{U}^{-1}H_0x_0| \leq 1$. Thus, (27) is valid at x_0 with $h(x_0) = H_0x_0$. Hence, we have

$$\dot{x}|_{x=x_0} \in \mathrm{co}\{A_ix_0 + B_iw - B_qT_iH_0x_0 :\ i \in I[1,2^m]\}. \quad (34)$$

and

$$\begin{aligned}&\dot{V}_c(x_0,w) \\ &\in \mathrm{co}\{(\nabla V_c(x_0))^T(A_ix_0 + B_iw - B_qT_iH_0x_0) : i \in I[1,2^m]\}. \quad (35)\end{aligned}$$

Recalling that

$$\begin{aligned}x_0 &= \sum_{j=1}^{J_0}\gamma_jx_j, \quad x_j \in \delta E_j \\ \nabla V_c(x_0) &= 2Q_0^{-1}x_0 = 2Q_j^{-1}x_j = 2P_jx_j. \quad (36)\end{aligned}$$

Applying (31) to x_j and replacing $2x_j^TP_j$ with $(\nabla V_c(x_0))^T$, we obtain

$$(\nabla V_c(x_0))^T(A_ix_j + B_iw - B_qT_iH_jx_j) - w^Tw \leq 0 \ \forall w \in \mathbb{R}^r. \quad (37)$$

By the definition of Q_0, H_0 and Y_0 in (32),

$$H_0x_0 = Y_0Q_0^{-1}x_0 = \left(\sum_{j=1}^{J_0}\gamma_jY_j\right)Q_0^{-1}x_0 \quad (38)$$

and, from (33), we have

$$H_jx_j = Y_jQ_j^{-1}x_j = Y_jQ_0^{-1}x_0, \qquad j \in I[1,J_0]. \quad (39)$$

Combining (36), (38), and (39), and noting that $\gamma_1+\gamma_2+\cdots+\gamma_{J_0}=1$, we have

$$\begin{aligned}&A_ix_0+B_iw-B_qT_iH_0x_0\\&\quad=\sum_{j=1}^{J_0}\gamma_jA_ix_j+\sum_{j=1}^{J_0}\gamma_jB_iw-B_qT_i\sum_{j=1}^{J_0}\gamma_jY_jQ_0^{-1}x_0\\&\quad=\sum_{j=1}^{J_0}\gamma_j(A_ix_j+B_iw-B_qT_iH_jx_j)\qquad\forall\, w\in\mathbb{R}^r\end{aligned}\tag{40}$$

Note that this is satisfied for all $i\in I[1,2^m]$. It follows from (37) that for each $i\in I[1,2^m]$ and $w\in\mathbb{R}^r$,

$$\begin{aligned}&(\nabla V_c(x_0))^T(A_ix_0+B_iw-B_qT_iH_0x_0)-w^Tw\\&\quad=\sum_{j=1}^{J_0}\gamma_j[(\nabla V_c(x_0))^T(A_ix_j+B_iw-B_qT_iH_jx_j)-w^Tw]\\&\quad\le 0.\end{aligned}$$

By (35), we obtain $\dot{V}_c(x_0,w)-w^Tw\le 0$ for all $w\in\mathbb{R}^r$. Note that x_0 is an arbitrary point in sL_{V_c}.

Hence, we have that $\dot{V}_c(x,w)\le w^Tw$ for all $x\in sL_{V_c}$ and $w\in\mathbb{R}^r$. Now, suppose $x(0)=0$ and $\|w\|_2^2\le s^2$. Then, for any $t_0>0$, as long as $x(t)\in sL_{V_c}$ for all $t\in(0,t_0)$, we have $V_c(x(t_0))\le\int_0^{t_0}w^T(\tau)w(\tau)d\tau\le s^2$, i.e., $x(t_0)\in sL_{V_c}$. On the other hand, if there exists $t_0>0$ such that $V_c(x(t))\le s^2$ for all $t\in(0,t_0)$ and $V_c(x(t_0))=s^2$ then we must have $\int_{t_0}^{\infty}w^T(\tau)w(\tau)d\tau=0$ and $\dot{V}_c(x(t),w(t))\le 0$ for almost all $t\ge t_0$. Hence, $V_c(x(t))\le s^2$ for all $t\ge t_0$. Therefore, we conclude that $x(t)\in sL_{V_c}$ for all $t\ge 0$. □

Remark 3: (Optimization issues) With conditions (28) and (29), we may formulate an optimization problem to minimize the estimate of the reachable set as with the quadratic function case. We observe that (28) is a bilinear matrix inequality (BMI) which contains some bilinear terms as the product of a full matrix and a scalar at the (1,1) block of the left-hand side matrix. Similar bilinear terms are contained in the matrix inequalities in [14], [15], and [27] for stability and performance analysis of linear differential/difference inclusions. A direct method to solve BMI problems is to alternatively fix one set of parameters and optimize the other set. In [14], [15], and [27], we adopted the path-following method from [20] and our experience with a set of numerical examples shows that the path-following method is much more effective than the straightforward iterative method. We actually implemented a two-step algorithm which combines the path-following method and the direct iterative method. The first step uses the path-following method to update all the parameters at the same time. The second step fixes λ_{ijk}'s and solves the resulting LMI problem which includes Q_j's and Y_j's as variables. This two-step method proves very effective on the BMI problems in [14], [15], and [27], and also works well on the example in Section V. We also see that if we take $Q_j=Q$ and $Y_j=Y$ for all j, then the bilinear terms vanish and the conditions reduce to the LMIs in (13) and (15). In our computation, we first solve the resulting optimization problem with LMI constraints and then use the optimal Q^* and Y^* to start the two-step algorithm, with $Q_j=Q^*$ and $Y_j=Y^*$ for all j and $\lambda_{ijk}\ge 0$ randomly chosen. This approach also proves effective for the problems of estimating the $\mathcal{L}_2$ gain and the domain of attraction, which will be addressed in Theorems 4 and 5.

Although there is no guarantee that the global optimal solution can be located, the convergence of the algorithms is satisfactory. Furthermore, since the initial value of the optimizing parameters can be inherited from the optimal solution obtained with quadratic functions, the algorithms ensure that the results are at least as good as those from using quadratic functions in Theorem 1. The aforementioned discussion also applies to the optimization problems resulting from Theorems 4 and 5. ∘

Remark 4: (About the nonlinear function $h(x)$): From the proof of Theorem 3, we see that a nonlinear function $h(x_0)=H_0(x_0)x_0$ is constructed from Q_j's and Y_j's so that $|\bar{U}^{-1}H_0(x_0)x_0|\le 1$ for all $x_0\in sL_{V_c}$ [see (32) where H_0 is constructed and the subsequent discussion up to (34)]. This makes the proof more complicated than with a linear function Hx but the result turns out to be cleaner and more easily tractable numerically. If we attempt to use a linear function $h(x)=Hx$ such that $|\bar{U}^{-1}Hx|_\infty\le 1$ for all $x\in sL_{V_c}$, we would have Y_j in (28) replaced with HQ_j and $Y_{j,\ell}$ in (29) replaced with $H_\ell Q_j$. When we formulate an optimization problem to estimate the reachable set by taking H and Q_j's as optimizing parameters, this would result in more complex BMI terms including HQ_j which may cause difficulties in the algorithms, such as slow convergence or getting stuck easily at a local solution. ∘

Remark 5: (Discussion about results based on NDIs): With similar developments as in the proof of Theorem 3, we can obtain a corresponding condition by using the norm-bounded differential inclusion (9) instead of using the PDI (6). The resulting condition involves the existence of Y_j's, $\lambda_{jk}\ge 0$, and a diagonal $U>0$ satisfying (29) and

$$\mathrm{He}\begin{bmatrix}AQ_j+\sum_{k=1}^{J}\lambda_{jk}(Q_j-Q_k) & B_w & B_qU\\ 0 & -\frac{I}{2} & 0\\ C_yQ_j-Y_j & D_{yw} & -U+D_{yq}U\end{bmatrix}\le 0\tag{41}$$

for all $j\in I[1,J]$. The bilinear terms in the first block seem to inject extra degrees of freedom as compared with (21) in Theorem 2 but they actually wouldn't help to reduce the conservatism. In other words, (41) implies the existence of a Q satisfying (21). To see this, we form a matrix

$$\Psi=\begin{bmatrix}-\sum_{k=2}^{J}\lambda_{1k} & \lambda_{21} & \cdots & \lambda_{J1}\\ \lambda_{12} & -\sum_{k=1,k\ne 2}^{J}\lambda_{2k} & \cdots & \lambda_{J2}\\ \vdots & \vdots & \vdots & \vdots\\ \lambda_{1J} & \lambda_{2J} & \cdots & -\sum_{k=1}^{J-1}\lambda_{Jk}\end{bmatrix}.$$

Then Ψ is a Metzler matrix. Since the sum of each column of Ψ is 0, the eigenvalue with the maximal real part is 0. Hence there exists a vector $c\ne 0$ with $c_i\ge 0$ such that $\Psi c=0$ (e.g., see [37]) and in particular we assume $\sum_{j=1}^{J}c_j=1$ (i.e., $c\in\Gamma$). If we let $Q=\sum_{j=1}^{J}c_jQ_j$, and $Y=\sum_{j=1}^{J}c_jY_j$, then Q and Y will satisfy (21) and (13). Furthermore, $s\mathcal{E}(Q)\subseteq sL_{V_c}$ is a smaller estimate of the reachable set. This means that with the NDI description, using the convex hull quadratic Lyapunov function offers no advantage to using the quadratic Lyapunov function. The same situation occurs for the estimation of the $\mathcal{L}_2$ gain or the domain of attraction, or, when applying a max quadratic function to NDIs.

For the special case where $H = 0$, the regional NDI (9) becomes a global NLDI. Thus we can conclude that for any NLDI, the convex hull quadratic function or the max quadratic function offers no advantage over quadratic functions when these stability and performance issues are concerned. $\circ$

We next address the problems of estimating the $\mathcal{L}_2$ gain and the domain of attraction.

Theorem 4: ($\mathcal{L}_2$ gain for norm-bounded w): Given $Q_j = Q_j^T > 0$, $j \in I[1, J]$, let V_c be composed from Q_j's as in (26). Consider system (1). Given $s, \gamma > 0$. If there exist $Y_j \in \mathbb{R}^{m \times n}$ and $\lambda_{ijk} \geq 0$, $i \in I[1, 2^m]$, $j, k \in I[1, J]$ such that

$$\mathrm{He}\begin{bmatrix} A_iQ_j - B_qT_iY_j + \sum_{k=1}^{J}\lambda_{ijk}(Q_j - Q_k) & B_i & 0 \\ 0 & -\frac{I}{2} & 0 \\ C_iQ_j - D_{zq}T_iY_j & D_i & -\frac{\gamma^2 I}{2} \end{bmatrix} \leq 0 \quad \forall i \in I[1, 2^m], j \in I[1, J] \tag{42}$$

$$\begin{bmatrix} \frac{\bar{u}_\ell^2}{s^2} & Y_{j,\ell} \\ Y_{j,\ell}^T & Q_j \end{bmatrix} \geq 0 \qquad \forall \ell \in I[1, m], j \in I[1, J] \tag{43}$$

then for all w such that $\|w\|_2 \leq s$ and $x(0) = 0$, we have $\|z\|_2 \leq \gamma \|w\|_2$.

Proof: We will prove the theorem by showing that for all $x \in sL_{V_c}$ and $w \in \mathbb{R}^r$, $\dot{V}_c(x, w) + (1/\gamma^2)z^Tz \leq w^Tw$. Since (42) implies (28), by Theorem 3, we have $x(t) \in sL_{V_c}$ for all t and for all $\|w\|_2 \leq s$, $x(0) = 0$. Also, all the relationships established in the proof of Theorem 3 are true under the conditions of the current theorem.

Let $P_j = Q_j^{-1}$, $H_j = Y_jQ_j^{-1}$ and

$$W_{ij} = P_jA_i - P_jB_qT_iH_j + \sum_{k=1}^{J}\lambda_{ijk}P_j(Q_j - Q_k)P_j.$$

Multiplying (42) on the left and the right by $\mathrm{diag}\{P_j, I, I\}$, we have

$$\mathrm{He}\begin{bmatrix} W_{ij} & P_jB_i & 0 \\ 0 & -\frac{I}{2} & 0 \\ C_i - D_{zq}T_iH_j & D_i & -\frac{\gamma^2 I}{2} \end{bmatrix} \leq 0.$$

By Schur complements, this is equivalent to

$$\mathrm{He}\begin{bmatrix} W_{ij} & P_jB_i \\ 0 & -\frac{I}{2} \end{bmatrix} + \frac{1}{\gamma^2}\begin{bmatrix} (C_i - D_{zq}T_iH_j)^T \\ D_i^T \end{bmatrix}[C_i - D_{zq}T_iH_j \;\; D_i] \leq 0. \tag{44}$$

Denote

$$f_{ij}(x, w) = A_ix + B_iw - B_qT_iH_jx$$
$$g_{ij}(x, w) = C_ix + D_iw - D_{zq}T_iH_jx.$$

Then (44) implies that for all $x \in \mathbb{R}^n$, $w \in \mathbb{R}^r$,

$$2x^TP_jf_{ij}(x, w) + \frac{1}{\gamma^2}g_{ij}^T(x, w)g_{ij}(x, w) - w^Tw \leq 2\sum_{k=1}^{J}\lambda_{ijk}x^TP_j(Q_k - Q_j)P_jx. \tag{45}$$

Consider $x \in \delta E_j$ for $\delta > 0$. Like in the proof of Theorem 3, we have

$$\sum_{k=1}^{J}\lambda_{ijk}x^TP_j(Q_k - Q_j)P_jx \leq 0 \;\forall x \in \delta E_j, w \in \mathbb{R}^r, \delta > 0.$$

It follows from (45) that

$$2x^TP_jf_{ij}(x, w) + \frac{1}{\gamma^2}g_{ij}^T(x, w)g_{ij}(x, w) - w^Tw \leq 0, \quad \forall x \in \delta E_j, w \in \mathbb{R}^r, \delta > 0. \tag{46}$$

We note that this is true for all $i \in I[1, 2^m]$ and $j \in I[1, J]$.

Now, consider $x_0 \in sL_{V_c}$. Then $V_c(x_0) = \delta^2$ for some $\delta \in (0, s]$. Like in the proof of Theorem 3, there exist $x_j \in \delta E_j$, $\gamma_j > 0, j \in I[1, J_0]$ such that $\sum_{j=1}^{J_0}\gamma_j = 1$ and $x_0 = \sum_{j=1}^{J_0}\gamma_jx_j$. Let H_0, Q_0, Y_0 be defined as in (32). Then, we also have $|\bar{U}^{-1}H_0x_0| \leq 1$. Applying Proposition 1 at x_0, we have

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} \in \mathrm{co}\left\{\begin{bmatrix} A_ix_0 + B_iw - B_qT_iH_0x_0 \\ C_ix_0 + D_iw - D_{zq}T_iH_0x_0 \end{bmatrix} : i \in I[1, 2^m]\right\}.$$

Let

$$f_{i0}(x_0, w) = A_ix_0 + B_iw - B_qT_iH_0x_0$$
$$g_{i0}(x_0, w) = C_ix_0 + D_iw - D_{zq}T_iH_0x_0.$$

Then, see (47), as shown at the bottom of the page.

Since $2x_j^TP_j = 2x_0^TQ_0^{-1} = (\nabla V_c(x_0))^T$ [see (33)], applying (46) at x_j, we obtain

$$(\nabla V_c(x_0))^Tf_{ij}(x_j, w) + \frac{1}{\gamma^2}|g_{ij}(x_j, w)|^2 - w^Tw \leq 0 \quad \forall w \in \mathbb{R}^r, \; i \in I[1, 2^m].$$

$$\dot{V}_c(x_0, w) + \frac{1}{\gamma^2}z^Tz - w^Tw \leq \max\{(\nabla V_c(x_0))^Tf_{i0}(x_0, w) + \frac{1}{\gamma^2}|g_{i0}(x_0, w)|^2 - w^Tw : i \in I[1, 2^m]\} \tag{47}$$

Like in (40), we have

$$f_{i0}(x_0,w)=\sum_{j=1}^{J_0}\gamma_j f_{ij}(x_j,w)$$
$$g_{i0}(x_0,w)=\sum_{j=1}^{J_0}\gamma_j g_{ij}(x_j,w).$$

It follows that

$$(\nabla V_c(x_0))^T f_{i0}(x_0,w)+\frac{1}{\gamma^2}|g_{i0}(x_0,w)|^2-w^Tw\le 0$$

and from (47)

$$\dot V_c(x_0,w)+\frac{1}{\gamma^2}z^Tz-w^Tw\le 0 \tag{48}$$

which is satisfied for all $x_0\in sL_{V_c}$ and $w\in\mathbb{R}^r$. Since $x(0)=0$, $x(t)\in sL_{V_c}$ for all t and for all $\|w\|_2\le s$, integrating both sides of (48), we have $\|z\|_2^2\le\gamma^2\|w\|_2^2$. This completes the proof. □

Theorem 5: (Estimation of the domain of attraction): Given $Q_j=Q_j^T>0, j\in I[1,J]$, let V_c be composed from Q_j's as in (26). Consider system (1) with $w\equiv 0$. We have $\dot V_c(x)<0$ for all $x\in L_{V_c}\setminus\{0\}$ if there exist $\lambda_{ijk}\ge 0, Y_j\in\mathbb{R}^{m\times n}, i\in I[1,2^m], j,k\in I[1,J]$ such that

$$\mathrm{He}(A_iQ_j-B_qT_iY_j+\sum_{k=1}^{J}\lambda_{ijk}(Q_j-Q_k))<0 \quad \forall i\in I[1,2^m], j\in I[1,J]$$
$$\begin{bmatrix}1 & Y_{j,\ell}\\ Y_{j,\ell}^T & Q_j\end{bmatrix}\ge 0 \quad \forall \ell\in I[1,m], j\in I[1,J].$$

Proof: The proof can be adapted from the proof of Theorem 3 by assuming that $B_i=0$. Then, with the same procedure, it can be shown that $\dot V_c(x)<0$ for all $x\in L_{V_c}\setminus\{0\}$. □

Remark 6: Note that the condition in Theorem 5 is similar to (but less conservative than) that of [27, Th. 4], which is developed for a special case without algebraic loops. Similar numerical complexity can be expected. ∘

C. Analysis With Max Quadratic Functions

The max quadratic function is not differentiable everywhere. Following the definition of [43, p. 215], a subgradient of a convex function $f:\mathbb{R}^n\to\mathbb{R}$ at x_0 is a vector $v\in\mathbb{R}^n$ such that

$$f(x)-f(x_0)\ge v^T(x-x_0)\ \ \forall x\in\mathbb{R}^n \tag{49}$$

and the subdifferential, denoted as $\partial f(x_0)$ (not to be confused as the boundary of a set) , is the set of all subgradient at x_0. The function $f(x)$ is differentiable at x_0 if and only if $\partial f(x_0)$ is single valued. We use $\partial V_{\max}(x)$ to denote the sub-differential of $V_{\max}$ at x.

Lemma 6: Consider $x_0\in\mathbb{R}^n$. Suppose that there exists $J_0\in I[1,J]$ such that $V_{\max}(x_0)=x_0^TP_jx_0$ for $j\in I[1,J_0]$ and $V_{\max}(x_0)>x_0^TP_jx_0$ for $j>J_0$. Then

1) $\partial V_{\max}(x_0)=\mathrm{co}\{2P_jx_0: j\in I[1,J_0]\}$;
2) for a vector $\zeta\in\mathbb{R}^n$, the directional derivative of $V_{\max}$ at x_0 along ζ is

$$\lim_{t\to 0^+}\frac{V_{\max}(x_0+t\zeta)-V_{\max}(x_0)}{t}=\max_{\xi\in\partial V_{\max}(x_0)}\{\xi^T\zeta\}.$$

Proof: See [30]. □

For simplicity and with some abuse of notation, for $\dot x$ given by (1), denote

$$\begin{aligned}\dot V_{\max}(x,w)&:=\max_{\xi\in\partial V_{\max}(x)}\{\xi^T\dot x\}\\ &=\max_{\xi\in\partial V_{\max}(x)}\{\xi^T(Ax+B_qq+B_ww)\}.\end{aligned}$$

Then, by Lemma 6, with $\zeta=\dot x$, $V_{\max}$ is decreasing along $\dot x$ if and only if $\dot V_{\max}(x,w)<0$.

Theorem 6: (Reachable set by bounded inputs) Given $P_j=P_j^T>0, j\in I[1,J]$, let $V_{\max}$ be the max quadratic function formed by P_j's as in (25). Given $s>0$. System (1) with $x(0)=0$ satisfies $x(t)\in sL_{V_{\max}}$ for all $t\ge 0$ and for all w such that $\|w\|_2\le s$ if there exist $H\in\mathbb{R}^{m\times n}$, $\lambda_{ijk}\ge 0, \alpha_{\ell j}\ge 0, j,k\in I[1,J], i\in I[1,2^m], \ell\in I[1,m]$, such that $\sum_{j=1}^{J}\alpha_{\ell j}=1$, and

$$\mathrm{He}\begin{bmatrix}P_jA_i-P_jB_qT_iH+\sum_{k=1}^{J}\lambda_{ijk}(P_j-P_k) & P_jB_i\\ 0 & -\frac{I}{2}\end{bmatrix}\le 0 \quad \forall i\in I[1,2^m], j\in I[1,J] \tag{50}$$

$$\begin{bmatrix}\frac{\bar u_\ell^2}{s^2} & H_\ell\\ H_\ell^T & \sum_{j=1}^{J}\alpha_{\ell j}P_j\end{bmatrix}\ge 0\ \forall \ell\in I[1,m]. \tag{51}$$

Proof: By the definition of V_c, condition (51) implies that $V_c((s/\bar u_\ell)H_\ell)\le 1$ for all $\ell\in I[1,m]$. By Lemma 5, this implies that $L_{V_{\max}}\subseteq\mathcal{L}(s\bar U^{-1}H)=(1/s)\mathcal{L}(\bar U^{-1}H)$, i.e., $sL_{V_{\max}}\subseteq\mathcal{L}(\bar U^{-1}H)$. Hence, $|\bar U^{-1}Hx|_\infty\le 1$ for all $x\in sL_{V_{\max}}$. By Proposition 1, we have

$$\dot x\in\mathrm{co}\{A_ix+B_iw-B_qT_iHx: i\in I[1,2^m]\}\ \forall x\in sL_{V_{\max}}.$$

On the other hand, it can be verified that (50) implies that

$$2x^TP_j(A_ix+B_iw-B_qT_iHx)-w^Tw \le 2\sum_{k=1}^{J}\lambda_{ijk}x^T(P_k-P_j)x \tag{52}$$

for all $j\in I[1,J], i\in I[1,2^m]$. The state–space of x can be partitioned as the following subsets:

$$S_j=\{x\in\mathbb{R}^n: x^T(P_k-P_j)x\le 0, k\in I[1,J]\},\ j\in I[1,J].$$

If $x \in S_j \setminus \cup_{k \neq j} S_k$, then $V_{\max}(x) = x^T P_j x$ and $\partial V_{\max}(x) = 2P_j x$. If $x \in \cap_{j=1}^{J_0} S_j \setminus \cup_{j=J_0+1}^{J} S_j$, then $V_{\max}(x) = x^T P_j x, j \in I[1, J_0]$ and $\partial V_{\max}(x) = \mathrm{co}\{2P_j x : j \in I[1, J_0]\}$.

We first consider $x \in S_j \setminus \cup_{k \neq j} S_k$. Then

$$\sum_{k=1}^{J} \lambda_{ijk} x^T (P_k - P_j) x \leq 0 \tag{53}$$

and

$$\dot{V}_{\max}(x, w) - w^T w \leq \max_{i \in I[1, 2^m]} (2x^T P_j (A_i x + B_i w - B_q T_i H x) - w^T w).$$

If $x \in \cap_{j=1}^{J_0} S_j \setminus \cup_{j=J_0+1}^{J} S_j$, then (53) is satisfied for all $j \in I[1, J_0]$ and we have

$$\dot{V}_{\max}(x, w) - w^T w \leq \max_{j \in I[1, J_0]} \max_{i \in I[1, 2^m]} (2x^T P_j (A_i x + B_i w - B_q T_i H x) - w^T w).$$

It follows from (52) and (53) that $\dot{V}_{\max}(x, w) - w^T w \leq 0$. The remaining part of the proof is similar to the proof of Theorem 3. $\square$

The following results can be proven similarly to Theorem 6 (see [30] for proofs).

Theorem 7: ($\mathcal{L}_2$ gain for norm-bounded w) Given $P_j = P_j^T > 0, j \in I[1, J]$. Consider system (1) and $s, \gamma > 0$. If there exist $H \in \mathbb{R}^{m \times n}$, $\lambda_{ijk} \geq 0$, $\alpha_{\ell j} \geq 0$, $j, k \in I[1, J]$, $i \in I[1, 2^m]$, $\ell \in I[1, m]$, such that $\sum_{j=1}^{J} \alpha_{\ell j} = 1$ and

$$\mathrm{He} \begin{bmatrix} P_j (A_i - B_q T_i H) + \sum_{k=1}^{J} \lambda_{ijk} (P_j - P_k) & P_j B_i & 0 \\ 0 & -\frac{I}{2} & 0 \\ C_i - D_{zq} T_i H & D_i & -\frac{\gamma^2 I}{2} \end{bmatrix} \leq 0 \quad \forall i \in I[1, 2^m], j \in I[1, J] \tag{54}$$

$$\begin{bmatrix} \frac{\bar{u}_\ell^2}{s^2} & H_\ell \\ H_\ell^T & \sum_{j=1}^{J} \alpha_{\ell j} P_j \end{bmatrix} \geq 0 \ \ \forall \ell \in I[1, m] \tag{55}$$

then for all w such that $\|w\|_2 \leq s$ and $x(0) = 0$, we have $\|z\|_2 \leq \gamma \|w\|_2$.

Theorem 8: (Estimation of the domain of attraction): Given $P_j = P_j^T > 0, j \in I[1, J]$. Consider system (1) with $w \equiv 0$. We have $\dot{V}_{\max}(x, 0) < 0$ for all $x \in L_{V_{\max}} \setminus \{0\}$ if there exist $H \in \mathbb{R}^{m \times n}$, $\lambda_{ijk} \geq 0$, $\alpha_{\ell j} \geq 0$, $j, k \in I[1, J]$, $i \in I[1, 2^m]$, $\ell \in I[1, m]$, such that $\sum_{j=1}^{J} \alpha_{\ell j} = 1$ and

$$\mathrm{He} \left(P_j A_i - P_j B_q T_i H + \sum_{k=1}^{J} \lambda_{ijk} (P_j - P_k) \right) < 0 \quad \forall i \in I[1, 2^m], j \in, I[1, J] \tag{56}$$

$$\begin{bmatrix} \bar{u}_\ell^2 & H_\ell \\ H_\ell^T & \sum_{j=1}^{J} \alpha_{\ell j} P_j \end{bmatrix} \geq 0 \quad \forall \ell \in I[1, m]. \tag{57}$$

As compared with the counterpart results from using convex hull quadratic functions, the conditions (50), (54), and (56) in Theorems 6–8 appear to be less tractable because of the bilinear term $P_j B_q T_i H$ in the first blocks of the matrices. Also, the same H for all P_j's seems to offer fewer degrees of freedom as compared with different Y_j's for different Q_j's in Theorems 3–5. However, numerical examples show that Theorems 6–8 may produce better results in some cases.

V. Examples

1) Example 1: Consider system (1) with the following parameters:

$$\begin{bmatrix} A & B_q & B_w \\ C_y & D_{yq} & D_{yw} \\ C_z & D_{zq} & D_{zw} \end{bmatrix} = \left[\begin{array}{ccc|cc|cc} 0 & 0 & -1 & 1 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 & 1 & 1 & 0 \\ 0 & 1 & -3 & 1 & -1 & 1 & 1 \\ \hline 1 & 0 & 1 & -3 & -1 & 1 & -1 \\ 0 & 1 & 0 & -2 & -4 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 \end{array}\right].$$

The well-posedness of the system is easily verified through Claim 2. We use the four methods in Theorems 1, 2, 4, and 7 to estimate the nonlinear $\mathcal{L}_2$ gain. The resulting estimates are plotted in Fig. 2, where the dotted curve is from applying quadratics via NDI (Theorem 2), the dashed–dotted one is from applying quadratics via PDI (Theorem 1), the dashed one is from applying max quadratics (with $J = 2$) via PDI (Theorem 7) and the solid one is from applying convex hull quadratics ($J = 2$) via PDI (Theorem 4). Each of the four curves tends to a constant value as $\|w\|_2$ goes to infinity. This constant value will be an estimate of the global $\mathcal{L}_2$ gain. As expected, applying quadratics via PDI always leads to better results than applying quadratics via NDI, and applying one of the two nonquadratics always leads to better results than applying quadratics. However, the relationship between the results from applying the two nonquadratic functions is not definite. The situation exhibited in Fig. 2 can be reversed if we change the parameters of the system. In what follows, we present several scenarios through some adjustments of the parameters.

2) Case 2: If we change D_{yq} to $D_{yq} = \begin{bmatrix} -3 & -1.3 \\ -2.3 & -4 \end{bmatrix}$ (well-posedness ensured), then the global $\mathcal{L}_2$ gain by using quadratics via NDI is unbounded (or, global stability is not confirmed), while that by using quadratics via PDI is 170.1473. By using max quadratics and convex hull quadratics, the global $\mathcal{L}_2$ gains are, respectively, 20.7833 and 19.3307.

3) Case 3: If we change D_{yq} to $D_{yq} = \begin{bmatrix} -3 & -2 \\ -2 & -4 \end{bmatrix}$ (well-posedness ensured), then the global $\mathcal{L}_2$ gain by using quadratics via either NDI or PDI is unbounded. By using max quadratics and convex hull quadratics, the global $\mathcal{L}_2$ gains are, respectively, 42.3354 and 31.6731.

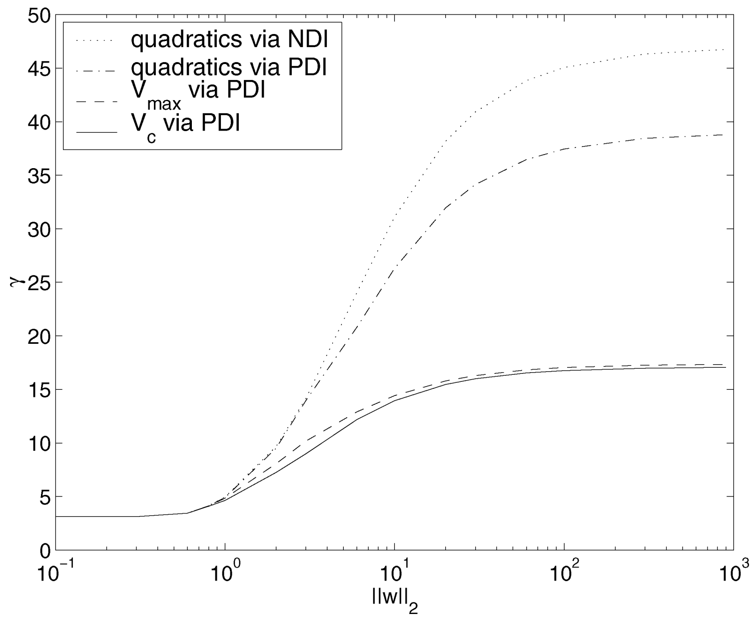

Fig. 2. Different estimates of the nonlinear $\mathcal{L}_2$ gain: Case 1.

The above two situations also show how the stability and performance results by the same method can be affected by the parameter D_{yq} which describes the algebraic loop. As discussed in [40], this parameter is one of the two key design parameters in static anti-windup synthesis and can have a dramatic impact on antiwindup performance.

Due to space limitation, we will not present computational results about the estimation of the domain of attraction or the estimation of the reachable set. Interested readers are referred to [27] for some numerical results. From the different situations exhibited through the $\mathcal{L}_2$ gain, it is not hard to infer that the difference among the estimations by using quadratics/non-quadratics via NDI/PDI can be made arbitrarily large through adjusting the four elements of D_{yq}. For instance, Case 2 suggests that the estimate of the domain of attraction by using quadratics via NDI is bounded while that by using quadratics via PDI is the whole state space. Case 3 suggests that the domain of attraction estimated by nonquadratic functions is the whole state space while that by quadratics (via PDI or NDI) is bounded. On the other hand, the estimate of the reachable set by nonquadratics can be bounded while that by quadratics is not.

We should remark that for this particular example, the algorithm for applying convex hull quadratics converges very well for all the values of s that we considered in our numerical computation, even under different parameter changes. The algorithm for applying max quadratics generally converges well but for some values of s it showed some difficulties where we needed to stop the algorithm and restart it from different initial values of λ_{ijk} which are randomly generated. In any case, improvement is expected from the nonquadratic functions.

4) Example 2: We adopt Example 2 from [16]. The plant is a cart-spring-pendulum system with one control input, one disturbance input, four states and one measurement output. The plant and controller parameters can be found in [16]. For this example, the closed-loop system without antiwindup compensation is not globally stable. Also, there exists no static antiwindup compensation to make the global $\mathcal{L}_2$ gain bounded. With dynamic anti-windup augmentation, an upper bound for the achievable global $\mathcal{L}_2$ gain is found to be 181.1424 (by using quadratic Lyapunov functions). When this achievable gain is approached, some parameters of the antiwindup compensator will approach infinity. To make the parameters within a reasonable range, we have to allow a slightly larger global $\mathcal{L}_2$ gain. A particular dynamic antiwindup compensator is given as follows, with notation adopted from [16]:

$$\left[\begin{array}{c|c}\Lambda_1 & \Lambda_2 \\ \hline \Lambda_3 & \Lambda_4\end{array}\right] = \left[\begin{array}{cccc|c} -10.04 & -8.67 & 5.95 & -34.81 & 625.3 \\ 16.78 & -0.0077 & -50.52 & 33.4 & 214.6 \\ 27.26 & 12.9 & -176.84 & -20.2 & 1534 \\ 6.8 & 9.56 & -54.1 & -35 & 410 \\ \hline 157 & -10 & -148 & 105 & 1467 \\ 3209 & -1315 & 1458 & 6281 & 3452 \\ 972 & -763 & 1102 & -196 & -6949 \\ 74719 & -50878 & 27569 & -10528 & 24840 \\ -1152 & -367 & 5992 & 387 & -54618 \end{array}\right].$$

When quadratic Lyapunov functions are used via the PDI, the estimated global $\mathcal{L}_2$ gain is 182.3080. When V_c (with $J = 2$) is used via the PDI, a slightly smaller estimate is given as 181.2326. For other values of bound on $\|w\|_2$, the improvement by using V_c is also small. However, if we change some parameters of the system, the difference between estimates by quadratics and nonquadratics can be arbitrarily large.

For this particular system, we have $D_{yq} = \Lambda_4(5)$. Hence, the algebraic loop is directly affected by $\Lambda_4(5)$. Suppose that we change $\Lambda_4(5)$ from -54618 to -52618. Two estimates of the nonlinear $\mathcal{L}_2$ gain are plotted in Fig. 3, where the dashed curve corresponds to the estimate obtained by applying quadratic functions and the solid one to that obtained by applying V_c (with $J = 2$), both via PDI description. Also plotted as a

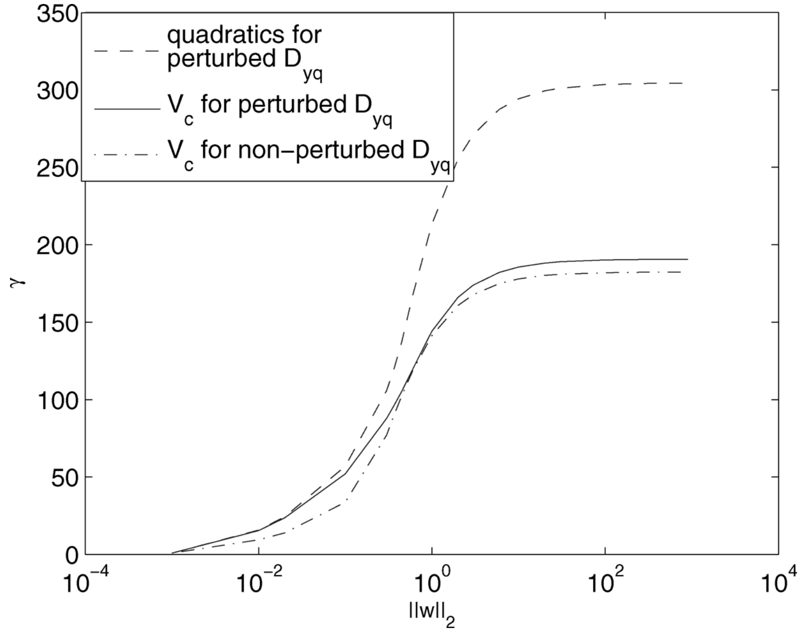

Fig. 3. Different estimates of $\mathcal{L}_2$ gain under parameter perturbation.

dashed–dotted curve is the estimate obtained by using V_c when $\Lambda_4(5) = -54618$. The above computational results suggest that nonquadratic functions may also have advantage for analyzing robust performance under parameter perturbations. This will motivate further research problems.

The order of the closed-loop system for this example is 12, including the state of the plant, the controller and the dynamic antiwindup compensator. The BMI problem for V_c with $J = 2$ involves 189 variables (the two matrices Q_1 and Q_2 for V_c contain 156 variables). It takes about 2 h to generate the solid curve (a connection of 18 points). The smoothness of the curve suggests the uniformity of the convergence to some optimal or suboptimal solutions, considering that the algorithm was run only once for each value of $\|w\|_2$ and the initial values of λ_{ijk}'s were chosen randomly.

VI. Conclusion

For a general system with saturation or deadzone components, regional stability and performance analysis relies on an effective regional treatment of the algebraic loop and the deadzone function. This paper provides such a treatment which yields two forms of parameterized differential inclusions. Applying available tools based on quadratic Lyapunov functions to these differential inclusions, we obtained conditions for stability and performance in the form of LMIs. These conditions are easily tractable but could be conservative in view of the quadratic Lyapunov functions applied. Further improvement relies on using non-quadratic Lyapunov functions. We explored a pair of conjugate Lyapunov functions in this paper and reduced the conservatism of the conditions with a series of BMI conditions. Numerical experience shows that these BMI conditions can be effectively solved with the path following method. Although there is no guarantee that the global optimal solutions will be obtained, the great potential of these nonquadratic Lyapunov functions has been revealed by numerical examples. The effectiveness demonstrated through these examples motivates further investigation on these nonquadratic Lyapunov functions and the development of more efficient algorithms to handle them for more complicated situations. This paper's results lay foundations for the design of saturated controllers and for the design of anti-windup compensators. Preliminary results have been obtained in [29] for regional dynamic anti-windup design which is based on the analysis result by applying quadratic functions via NDI. The analysis results based on PDI and nonquadratic functions can be applied for design purposes by incorporating controller design parameters into the existing optimization problem. In this regard, main efforts will be devoted to making the optimization problems more tractable through careful algebraic manipulation and appropriate parameter transformations.

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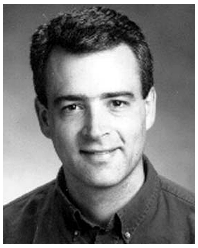

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Piecewise-quadratic Lyapunov functions for systems with deadzones or saturations

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ABSTRACT

A piecewise quadratic Lyapunov function is developed for the analysis of the global and regional performances for systems with saturation/deadzone in a general feedback configuration with an algebraic loop. This piecewise quadratic Lyapunov function effectively incorporates the structure of the saturation/deadzone nonlinearity. Several sector-like conditions are derived to describe the complex nonlinear algebraic loop. These conditions transform several performance analysis problems into optimization problems with linear (or bilinear) matrix inequalities. The effectiveness of the results is demonstrated with numerical examples.

1. Introduction

Saturation is a common nonlinear phenomenon in engineering systems. It exists in different parts of control systems, such as the actuators and/or the sensors, due to the capacity limits of those physical components. Among saturation occurring at different locations, of special interest is actuator saturation, which has been widely studied since the 1950s, with significant advancement over the last decade.

Among the recent literature within this context, one important approach used for analysis and synthesis is the Lyapunov approach. Generally, as indicated in [1], the Lyapunov approach consists of two steps. In the first step, the saturation or the deadzone functions are bounded with proper global or regional sectors. In the second step, the system satisfying the sector conditions is analyzed with various Lyapunov functions. The Lyapunov approach has been used to establish quantitative measures of stability and performances, such as the size of the domain of attraction, the convergence rate, and the nonlinear $\mathcal{L}_2$ gain. In light of the available numerical tools to solve convex optimization problems, these analysis and synthesis problems are usually cast into linear matrix inequalities (LMIs), which can be solved numerically with little difficulty (see e.g., [3,2,4–11]).

With the global or regional sector conditions, there is a great potential to be explored in the second step for choosing effective and tractable Lyapunov functions. In most existing literature for systems with saturation/deadzone, the quadratic Lyapunov function has been used to cast various analysis and design problems into the LMI framework. Due to the conservatism that may arise from quadratic functions, some recent efforts have been made toward the construction and application of nonquadratic Lyapunov functions that may lead to LMIs or bilinear matrix inequalities (BMIs) (see, e.g., [4,1,6,12,13].)

In [4,6] (for earlier works, see e.g., [14,15]), a Lure–Postnikov type Lyapunov function is used. In [6], robust stability and performance are addressed for the direct design of a static full-state feedback and a dynamic output feedback controller, while [4] focuses on enlarging the domain of attraction by using a static anti-windup compensator. The type of piecewise quadratic functions introduced in [16] was applied for the design of anti-windup controllers in [13]. Piecewise quadratic Lyapunov functions also appear in model predictive control (MPC) for constrained linear systems (see [17]), the analysis of which has been linked to saturated systems with algebraic loops in [8]. Some recent publications, e.g., [18,1,19,10], provided novel ideas in constructing Lyapunov functions for saturated systems. By

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relying on a regional sector condition, [1] considers two different differential inclusions, the norm-bounded differential inclusion (NDI) and the polytopic differential inclusion (PDI), to describe systems with saturation/deadzone. In [1], both quadratic and non-quadratic Lyapunov functions are used to analyze the system stability and performances.

In this paper, we consider systems with saturation and/or deadzone in a general configuration which allows for algebraic loops including saturation/deadzone and exogenous input/disturbances. The problems to be considered include the estimation of the domain of attraction, the reachable set under a class of disturbances with bounded energy and the nonlinear $\mathcal{L}_2$ gain.

Motivated by the Lure–Postnikov type Lyapunov function, we develop a novel piecewise quadratic function which effectively incorporates the structure of the saturation/deadzone nonlinearity. This piecewise quadratic function generalizes the traditional Lure–Postnikov type with additional design parameters but is as easily tractable. All the analysis problems will be converted into optimization problems with LMI/BMI constraints. This is achieved by capturing the properties of saturation/deadzone and the algebraic loop with several sector-like conditions. As compared to existing papers that use the Lure–Postnikov type function, we address a general system configuration which allows for algebraic loops including saturation/deadzone and exogenous inputs/disturbances. As compared to results based on the type of piecewise quadratic function introduced in [16], we do not require the explicit solution of the algebraic loop, which could be very involved, if possible, for systems with two or more saturated components, or with exogenous disturbances entering the saturated components. As compared to the non-quadratic Lyapunov functions in [1], the piecewise quadratic function directly incorporates the structure of saturation/deadzone. In a nutshell, what we address here is the problem of stability and performance analysis for linear systems involving (multiple) saturations and deadzones, which can all be reduced to the general block diagram in [1, Figure 1], where the block $\mathcal{H}$ is linear. What we provide in this paper is a Lyapuonv-based tool which reduces the conservativeness as compared to the classical absolute stability results and also as compared to the results arising from the use of the nonquadratic Lyapunov functions in [1]. We will use a numerical example to compare the results from this paper's method and those from [1] in Section 5.

This paper is organized as follows. In Section 2, we provide a general description of the system and define the piecewise quadratic Lyapunov function. Some sector or sector-like conditions are presented in this section to describe the properties of the deadzone and the algebraic-loop. In Section 3, the piecewise quadratic Lyapunov function is used to establish matrix conditions for global stability and performances. Section 4 performs respective regional analysis. In Section 5, two numerical examples are used to show the effectiveness of this paper's approach and the great potential of the piecewise quadratic Lyapunov function.

Notation. For $u \in \mathbb{R}^r$, we denote $|u| := \sqrt{u^{\mathrm{T}}u}$, $|u|_\infty := \max_i |u_i|$; for $u \in \mathcal{L}_2$, $\|u\|_2 := (\int_0^\infty u^{\mathrm{T}}(t)u(t)\mathrm{d}t)^{1/2}$. For a matrix $A \in \mathbb{R}^{c\times r}$, $|A| := \max_{|x|=1} |Ax|$ for $x \in \mathbb{R}^r$. For a square matrix X, $\mathrm{He}\,X := X + X^{\mathrm{T}}$. For $Q = Q^{\mathrm{T}} > 0$, denote $\mathcal{E}(Q) := \{x : x^{\mathrm{T}}Qx \le 1\}$.

2. System description and the Lyapunov function

2.1. System description

Generally, a system with saturation or deadzone can be described with the following compact form:

$$\mathcal{H} \quad \begin{cases} \dot{x} = Ax + B_q q + B_w w \\ y = C_y x + D_{yq} q + D_{yw} w \\ z = C_z x + D_{zq} q + D_{zw} w \\ q = \mathrm{dz}(y) \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}^m$ contains all the variables affected by saturation/deadzone, $w \in \mathbb{R}^r$ is the exogenous input such as the references and disturbances, and $z \in \mathbb{R}^p$ is the performance output. $\mathrm{dz}(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is the deadzone function defined as $\mathrm{dz}(y) := y - \mathrm{sat}(y)$ where $\mathrm{sat}(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is the symmetric saturation function having saturation levels $\bar{u}_1, \ldots, \bar{u}_m > 0$. The i-th component of $\mathrm{sat}(y)$, i.e., $\mathrm{sat}_{\bar{u}_i}(\cdot)$ depends on the i-th input component y_i via $\mathrm{sat}_{\bar{u}_i}(y_i) := \frac{y_i}{\max\{1, \frac{|y_i|}{\bar{u}_i}\}}$. Denote $U = \mathrm{diag}[\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_m]$.

When $D_{yq} \neq 0$, a nonlinear algebraic loop is imposed by the second equation in (1) as

$$y = C_y x + D_{yq}\mathrm{dz}(y) + D_{yw} w. \tag{2}$$

This algebraic loop is said to be well-posed if y is uniquely determined from x and w from this equation. A necessary and sufficient condition for the well-posedness is given in [1]. A sufficient condition, which may be more convenient for deriving matrix conditions in some cases, is that there exists a diagonal matrix $X > 0$ satisfying $2X - D_{yq}X - XD_{yq}^{\mathrm{T}} > 0$ (see, e.g., [7,11]).

This general system description has been used to study anti-windup systems in [2,5,13,7]. Many linear systems with saturation/deadzone components can be transformed into (1) through a loop transformation. In most literature, various assumptions are made on the general configuration (1), such as, the absence of the algebraic loop ($D_{yq} = 0$) and the disturbance not entering the algebraic loop ($D_{yw} = 0$). These assumptions simplify the analysis and design of the system but can be restrictive under certain situations. For example, it was shown in some works, such as [7,1], that the algebraic loop can be purposely introduced into the anti-windup configuration to reduce the global $\mathcal{L}_2$ gain. We will further address this point in Section 5.

2.2. Some sector-like conditions for the deadzone and algebraic-loop

In this section, we review two sector conditions that have been used/derived by previous papers (e.g., [20,5]), and introduce three sector-like conditions that will be useful in this paper.

Fact 1. *For every diagonal matrix $\Delta \in \mathbb{R}^{s\times s}$, $\Delta > 0$, $\mathrm{dz}(\cdot)$ satisfies*

$$\mathrm{dz}(v)^{\mathrm{T}}\Delta\{v - \mathrm{dz}(v)\} \ge 0, \quad \forall v \in \mathbb{R}^s. \tag{3}$$

The above inequality is referred to as the global sector condition for the deadzone function, or, simply, $\mathrm{dz}(\cdot) \in [0, \ I]$. This global sector condition might be conservative for examining the performance over a local region. The following regional sector condition was used in earlier papers, e.g., [20,5].

Fact 2. *Given $r \in \mathbb{R}^m$ such that $-\bar{u}_i \le r_i \le \bar{u}_i$, , $\forall i = 1, \ldots, m$, the following inequality holds for any diagonal matrix $\Delta \in \mathbb{R}^{m\times m}$, $\Delta > 0$:*

$$\mathrm{dz}(v)^{\mathrm{T}}\Delta\{v - \mathrm{dz}(v) - r\} \ge 0, \quad \forall v \in \mathbb{R}^m. \tag{4}$$

Next we introduce several sector-like conditions that describe the properties of the algebraic-loop with deadzone. As mentioned in [1], if the algebraic loop $y = C_y x + D_{yq}\mathrm{dz}(y) + D_{yw} w$ is well-posed, then y is a piecewise linear function of x and w. This piecewise linear function can be very involved for $m \ge 2$ and may not be usable in the presence of w. Instead of solving the algebraic loop, we examine some properties that will be useful for the analysis.

Since the algebraic loop allows for possibly unknown exogenous inputs, we introduce an estimate of the solution of the algebraic loop through the equation $u = C_y x + D_{yq}\mathrm{dz}(u)$ and examines the relationship between u and y.

Fact 3. *According to the non-decreasing properties of saturation and deadzone functions, the following inequality holds for every diagonal matrix $\Delta \in \mathbb{R}^{m\times m}$, $\Delta > 0$:*

$$\{\mathrm{dz}(v_1) - \mathrm{dz}(v_2)\}^{\mathrm{T}}\Delta\{\mathrm{sat}(v_1) - \mathrm{sat}(v_2)\} \geq 0, \quad \forall v_1, v_2 \in \mathbb{R}^m. \tag{5}$$

Substituting v_1, v_2 with u and y, respectively, and using $\mathrm{sat}(u) = u - \mathrm{dz}(u)$, we obtain

$$\{\mathrm{dz}(u) - \mathrm{dz}(y)\}^{\mathrm{T}}\Delta\{(D_{yq} - I)\mathrm{dz}(u) + (I - D_{yq})\mathrm{dz}(y) + D_{yw}w\} \geq 0 \tag{6}$$

for all u, y satisfying the respective algebraic-loop.

The last conditions are derived to describe the properties of the time derivatives of u and $\mathrm{dz}(u)$, wherever they exist. For convenience, denote $\dot{u} = \mathrm{d}u/\mathrm{d}t$ and $\phi(x, w) = \mathrm{d}(\mathrm{dz}(u))/\mathrm{d}t$. It is easy to see that

$$\phi_i(x, w) = \begin{cases} 0, & \text{if } |u_i| < \bar{u}_i, \\ \dot{u}_i, & \text{if } |u_i| > \bar{u}_i. \end{cases} \tag{7}$$

Note that $\phi_i(x, w)$ may not exist where $u_i = \pm\bar{u}_i$. Consequently, we have

Fact 4. *For every diagonal matrix $\Delta \in \mathbb{R}^{m\times m}$, the following equalities hold almost everywhere:*

$$\phi(x, w)^{\mathrm{T}}\Delta\{\dot{u} - \phi(x, w)\} \equiv 0, \tag{8}$$

$$\mathrm{dz}(u)^{\mathrm{T}}\Delta\{\dot{u} - \phi(x, w)\} \equiv 0, \tag{9}$$

where, by definition of u, $\phi(x, w)$ and (1), $\dot{u} = C_yAx + C_yB_q\mathrm{dz}(y) + C_yB_w w + D_{yq}\phi(x, w)$.

2.3. Piecewise quadratic Lyapunov functions

Define $\xi(x) := \begin{bmatrix} x^{\mathrm{T}} & \mathrm{dz}(u(x))^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}$. Consider a novel Lyapunov function candidate as follows:

$$V(x) = \xi(x)^{\mathrm{T}}P\xi(x) \tag{10}$$

where $P = P^{\mathrm{T}} > 0$. It is clear that V is positive definite and radially unbounded. If the algebraic loop $u = D_{yq}\mathrm{dz}(u) + v$ is well-posed, then the solution u is a piecewise affine function of v (see [1], Remark 1 for details). Hence V is piecewise quadratic in x and is differentiable almost everywhere. The idea of defining the Lyapunov function as a quadratic function involving the (nonlinear) vector ξ is new and the resulting Lyapunov function directly inherits in an implicit way, the piecewise affine structure of the nonlinearity acting on the system with saturation/deadzone. This nonlinearity is the solution of the algebraic loop in Eq. (2). As compared to the piecewise quadratic function in [1], which is the pointwise maximum of several quadratics, the function in this paper utilizes the structure of the nonlinear algebraic loop. Instead of relying on the explicit solution to the algebraic loop, as done in [13], we will use the sector-like conditions to derive matrix conditions for stability and performances.

The Lyapunov function in (10) is inspired by the Lure–Postnikov type Lyapunov function in [6,4], where the Lure–Postnikov type Lyapunov function is defined as

$$V_L(x) = x^{\mathrm{T}}Qx + 2\sum_{i=1}^{m}\int_0^{\eta_i}\mathrm{dz}(\sigma)W_{ii}\mathrm{d}\sigma \tag{11}$$

where $Q = Q^{\mathrm{T}} > 0$, $W > 0$ is a diagonal matrix and $\eta = Kx$ is a certain control input. In this paper, instead of choosing η explicitly as $\eta = Kx$ (as in [6,4]), we allow it to be dependant on x in a more complicated way, in particular, we let $\eta = u$, where u satisfies $u = C_yx + D_{yq}\mathrm{dz}(u)$.

In what follows, we establish a connection between the function V_L in (11) and the function V in (10) and see how V_L is generalized to V. First we observe that

$$2\int_0^{u_i}\mathrm{dz}(\sigma)\mathrm{d}\sigma = \mathrm{dz}(u_i)^2. \tag{12}$$

The above equality can be easily checked by considering two cases (Assume $u_i > 0$ for simplicity): $u_i \leq \bar{u}_i$ and $u_i > \bar{u}_i$, where $\bar{u}_i$ is the saturation level for the ith component of $\mathrm{sat}(\cdot)$. In the first case, clearly both sides are zero. In the second case, the integral should be $2\int_{\bar{u}_i}^{u}(\sigma - \bar{u}_i)\mathrm{d}\sigma = (u_i - \bar{u}_i)^2 = \mathrm{dz}(u_i)^2$.

By (12) and letting $\eta = u$, we have $V_L(x) = x^{\mathrm{T}}Qx + \mathrm{dz}(u)^{\mathrm{T}}W\mathrm{dz}(u)$, which is a special case of V when P is a block diagonal matrix. In other words, V is generalized from V_L by adding cross terms in the form of $x^{\mathrm{T}}N\mathrm{dz}(u) + \mathrm{dz}(u)^{\mathrm{T}}N^{\mathrm{T}}x$ and allows W to be non-diagonal. This clearly gives extra degree of freedom in the resulting optimization problems and will lead to less conservative results.

Before proceeding to analyzing the system with the piecewise quadratic function V, we need to address one more concern. Here we note that V_L is differentiable everywhere but V is generally not. In what follows, we will see that we only need to examine the time derivative of V where it is differentiable, which is almost everywhere in the state space.

Denote the time-derivative of V as $\dot{V}$, wherever it exists. Suppose V is differentiable at x, then $\dot{V} =< \nabla V(x), f(x, w) >$, where $\nabla V(x)$ is the gradient of V at x and f is the right-hand side of the first equation in (1). Since V is piecewise quadratic, $\dot{V}$ is well-defined for almost all $x \in \mathbb{R}^n$. If $\dot{V} \leq \alpha(x, w)$ for almost all $x \in \mathbb{R}^n$ and all $w \in \mathbb{R}^r$, and α is continuous, then the time derivative of the function $t \mapsto V(x(t))$, which is defined for almost all t, is upper bounded by $\alpha(x(t), w(t))$ for almost all t (see e.g., [9, p. 99–100]).

For a well-posed algebraic loop $u = C_yx + D_{yq}\mathrm{dz}(u)$, the signal $u(x)$ can be represented as $u(x) = (I - D_{yq})^{-1}C_yx + b(x)$, where $b(x) = (I - D_{yq})^{-1}D_{yq}\ \mathrm{sat}(u)$ is a globally bounded function, i.e., there exists $\theta > 0$ such that $|b(x)| \leq \theta$ for all x. It can be verified that there always exists some $k > 0$ such that $|u| \leq k|x|$. Consider the following two cases: In the case $|\overline{U}^{-1}C_yx|_\infty < 1$, the algebraic loop gives $u = C - yz$ and $|u| \leq |C_y||x|$; and in the case $|\overline{U}^{-1}C_yx|_\infty \geq 1$, $|u| \leq |(I - D_{yq})^{-1}C_y||x| + \bar{\theta}|x|$ where $\bar{\theta} = \theta|\overline{U}^{-1}C_y|$. Thus $|\mathrm{dz}(u)| \leq \bar{k}|x|$ for some $\bar{k} > 0$, and it follows that $\alpha_1|x| \leq |\xi(x)| \leq \alpha_2|x|$ for some $\alpha_1, \alpha_2 > 0$. Therefore, there exist $\beta_1 > 0$ and $\beta_2 > 0$ such that $\beta_1|x|^2 \leq V(x) \leq \beta_2|x|^2$. If there exists $\varepsilon > 0$ such that $\dot{V} < -\varepsilon|x|^2$ for almost all x and for $w = 0$ then the origin is globally exponentially stable when $w = 0$, i.e., there exists $k > 0$ and $\lambda > 0$ such that, for all $x(0) \in \mathbb{R}^n$, the solutions of (1) with $w = 0$ satisfy

$$|x(t)| \leq k|x(0)|\exp(-\lambda t) \quad \forall t \geq 0. \tag{13}$$

If the bound $\dot{V} < -\varepsilon|x|^2$ only holds on a forward invariant set $\mathcal{E}$ containing a neighborhood of the origin, then the origin is locally exponentially stable and the bound (13) can be asserted for all solutions starting in $\mathcal{E}$. In such a situation, we will say that the origin is locally exponentially stable from $\mathcal{E}$.

3. Global analysis

To facilitate the presentation of the main results, we define a list of matrices that will be used to form matrix inequalities as conditions for stability and performances. These matrices will be used for both global analysis in this section and regional analysis in the next section.

Denote I_m as the identity matrix of dimension m, and $0_{m\times n}$ as the $m\times n$ zero matrix. For $P \in \mathbb{R}^{(n+m)\times(n+m)}$, and diagonal matrices $\Delta_i \in \mathbb{R}^{m\times m}$, $i = 1, \dots, 5$, denote

$$\begin{aligned}
\Psi &:= \begin{bmatrix} I_{n+m} & 0_{(n+m)\times(2m+r)} \end{bmatrix}^{\mathrm{T}} P \begin{bmatrix} A & 0 & 0 & B_q & B_w \\ 0 & 0 & I_m & 0 & 0 \end{bmatrix};\\
\bar{\Psi}_1 &:= \begin{bmatrix} 0_{m\times(n+2m)} & I_m & 0_{m\times r} \end{bmatrix}^{\mathrm{T}} \Delta_1 \\
&\quad \times \begin{bmatrix} C_y & 0 & 0 & D_{yq} - I_m & D_{yw} \end{bmatrix};\\
\bar{\Psi}_2 &:= \begin{bmatrix} 0_{m\times n} & I_m & 0_{m\times(2m+r)} \end{bmatrix}^{\mathrm{T}} \Delta_2 \begin{bmatrix} C_y & D_{yq} - I_m & 0 & 0 & 0 \end{bmatrix};\\
\bar{\Psi}_3 &:= \begin{bmatrix} 0_{m\times n} & -I_m & 0_{m\times m} & I_m & 0_{m\times r} \end{bmatrix}^{\mathrm{T}} \Delta_3 \\
&\quad \times \begin{bmatrix} 0 & I - D_{yq} & 0 & D_{yq} - I_m & D_{yw} \end{bmatrix};\\
\bar{\Psi}_4 &:= \begin{bmatrix} 0_{m\times(n+m)} & I_m & 0_{m\times(m+r)} \end{bmatrix}^{\mathrm{T}} \Delta_4 \\
&\quad \times \begin{bmatrix} C_yA & 0 & D_{yq} - I_m & C_yB_q & C_yB_w \end{bmatrix};\\
\bar{\Psi}_5 &:= \begin{bmatrix} 0_{m\times n} & I_m & 0_{m\times(2m+r)} \end{bmatrix}^{\mathrm{T}} \Delta_5 \\
&\quad \times \begin{bmatrix} C_yA & 0 & D_{yq} - I_m & C_yB_q & C_yB_w \end{bmatrix};\\
\bar{\Psi}_6 &:= \begin{bmatrix} 0_{r\times(n+3m)} & I_r \end{bmatrix}^{\mathrm{T}},
\end{aligned} \tag{14}$$

and

$$\bar{\Psi} = \Psi + \Psi^{\mathrm{T}} + \sum_{i=1}^{5} (\bar{\Psi}_i + \bar{\Psi}_i^{\mathrm{T}}). \tag{15}$$

We see that the matrix $\bar{\Psi}$ is linear in P and Δ_i's. From the structure of the matrices, we also see that Ψ, $\bar{\Psi}_i$, $i = 1, 2, 3, 4, 5$ and $\bar{\Psi}$ can be partitioned into 5 by 5 blocks, where the row/column partition is $[n, m, m, m, r]$. It should be noted that the second diagonal block of $\bar{\Psi}$ is $(\Delta_2 + \Delta_3)(D_{yq} - I_m) + (D_{yq} - I_m)^{\mathrm{T}}(\Delta_2 + \Delta_3)$. If this block is negative definite for certain $\Delta_2, \Delta_3 > 0$, then the algebraic-loop is well-posed.

Theorem 1. *Consider the system (1).*

1. *(Exponential stability) If there exist a matrix $P \in \mathbb{R}^{(n+m)\times(n+m)}$, $P = P^{\mathrm{T}} > 0$ and diagonal matrices $\Delta_i \in \mathbb{R}^{m\times m}$, $i = 1, \dots, 5$, $\Delta_{i=\{1,2,3\}} > 0$, satisfying the LMI,*

$$\begin{bmatrix} I_{n+3m} & 0_{(n+3m)\times r} \end{bmatrix} \bar{\Psi} \begin{bmatrix} I_{n+3m} \\ 0_{r\times(n+3m)} \end{bmatrix} < 0, \tag{16}$$

then for the Lyapunov function $V(x) = \xi(x)^{\mathrm{T}} P \xi(x)$, there exists $\varepsilon > 0$ such that $\dot{V} < -\varepsilon|x|^2$ for almost all $x \in \mathbb{R}^n$ and $w = 0$. Thus, the origin of the system (1) is globally exponentially stable.

2. *(Reachable region) If there exist a matrix $P \in \mathbb{R}^{(n+m)\times(n+m)}$, $P = P^{\mathrm{T}} > 0$ and diagonal matrices $\Delta_i \in \mathbb{R}^{m\times m}$, $i = 1, \dots, 5$, $\Delta_{i=\{1,2,3\}} > 0$ satisfying the LMI,*

$$\bar{\Psi} - \bar{\Psi}_6 \bar{\Psi}_6^{\mathrm{T}} < 0 \tag{17}$$

then $\dot{V} < w^{\mathrm{T}} w$ for almost all $x \in \mathbb{R}^n$ and all $w \in \mathbb{R}^r$. If $x(0) = 0$ and $\|w\|_2 \le s$, then $\xi(x(t)) \in \mathcal{E}(P/s^2)$ for all $t \ge 0$.

3. *(Global $\mathcal{L}_2$ gain) If there exist a matrix $P \in \mathbb{R}^{(n+m)\times(n+m)}$, $P = P^{\mathrm{T}} > 0$ and diagonal matrices $\Delta_i \in \mathbb{R}^{m\times m}$, $i = 1, \dots, 5$, $\Delta_{i=\{1,2,3\}} > 0$ satisfying the LMI,*

$$\begin{bmatrix} \bar{\Psi} - \bar{\Psi}_6 \gamma \bar{\Psi}_6^{\mathrm{T}} & \star \\ \begin{bmatrix} C_z & 0 & 0 & D_{zq} & D_{zw} \end{bmatrix} & -\gamma I \end{bmatrix} < 0 \tag{18}$$

then $\dot{V} + \frac{1}{\gamma} z^{\mathrm{T}} z < \gamma w^{\mathrm{T}} w$ for almost all $x \in \mathbb{R}^n$ and all $w \in \mathbb{R}^r$. If $x(0) = 0$, then $\|z\|_2 \le \gamma\|w\|_2$, i.e., the global $\mathcal{L}_2$ gain of (1) is bounded by γ.

Proof. First, for each problem, the well-posedness of the algebraic is guaranteed since the second diagonal block in the matrix $\bar{\Psi}$ is negative definite. Let $\zeta = \begin{bmatrix} x^{\mathrm{T}} & \mathrm{dz}(u)^{\mathrm{T}} & \phi(x, w)^{\mathrm{T}} & \mathrm{dz}(y)^{\mathrm{T}} & w \end{bmatrix}^{\mathrm{T}}$. Recall that $\phi(x, w)$ is defined as the time derivative of $\mathrm{dz}(u)$. Thus,

$$\dot{\xi} = \begin{bmatrix} \dot{x} \\ \phi(x, w) \end{bmatrix} = \begin{bmatrix} A & 0 & 0 & B_q & B_w \\ 0 & 0 & I & 0 & 0 \end{bmatrix} \zeta. \tag{19}$$

For x such that the time-derivative of $\mathrm{dz}(u)$ exists, we have

$$\dot{V}(x) = \xi^{\mathrm{T}} P \dot{\xi} + \dot{\xi}^{\mathrm{T}} P \xi = \zeta^{\mathrm{T}} \Psi \zeta + \zeta^{\mathrm{T}} \Psi^{\mathrm{T}} \zeta \tag{20}$$

where Ψ is defined in (14).

Next, we interpret the sector or sector-like conditions in Facts 1–4 with the following inequalities:

$$\zeta^{\mathrm{T}} (\bar{\Psi}_i + \bar{\Psi}_i^{\mathrm{T}}) \zeta \ge 0; \quad i = 1, \dots, 5. \tag{21}$$

where $\Delta_{i=1,2,3} \ge 0$. This is explained as follows. By substituting v with y (or u) in (3), and Δ with Δ_1 (or Δ_2), we obtain the inequalities for $i = 1$ (or $i = 2$). Similarly, (21) with $i = 3$ is obtained by substituting v_1 and v_2 with u and y, and Δ with Δ_3 (see (6)). The inequalities for $i = 4, 5$ can be obtained by replacing Δ in (8) and (9) with Δ_4 and Δ_5 respectively.

1. To examine the exponential stability, we set $w = 0$. The inequality (16) implies that for all $\zeta \ne 0$, $w = 0$,

$$\zeta^{\mathrm{T}} \left(\Psi + \Psi^{\mathrm{T}} + \sum_{i=1}^{5} (\bar{\Psi}_i + \bar{\Psi}_i^{\mathrm{T}}) \right) \zeta < 0.$$

It follows from (20) and (21) that $\dot{V}(x) < 0$ for almost all $x \ne 0$. Thus exponential stability is guaranteed.

2. To examine the reachable region, we note that $\dot{V} - w^{\mathrm{T}} w = \zeta^{\mathrm{T}} (\Psi + \Psi^{\mathrm{T}} - \bar{\Psi}_6 \bar{\Psi}_6^{\mathrm{T}}) \zeta$. Following the same procedure as for exponential stability, the LMI in (17) implies that $\dot{V} \le w^{\mathrm{T}} w$ for almost all x, w. Integrating both sides with $V(0) = 0$, we have $V(x(t)) \le \|w\|_2^2 \le s^2$ for all $t \ge 0$, i.e., $\xi(x(t)) \in \mathcal{E}(P/s^2)$ for all $t \ge 0$.

3. For the global $\mathcal{L}_2$ gain, we note that

$$\dot{V} + \frac{1}{\gamma} z^{\mathrm{T}} z - \gamma w^{\mathrm{T}} w = \zeta^{\mathrm{T}} (\Psi + \Psi^{\mathrm{T}} - \gamma \bar{\Psi}_6 \bar{\Psi}_6^{\mathrm{T}}) \zeta + \frac{\begin{bmatrix} C_z & 0 & 0 & D_{zq} & D_{zw} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} C_z & 0 & 0 & D_{zq} & D_{zw} \end{bmatrix}}{\gamma} < 0.$$

By Schur complement and using (21), the LMI in (18) ensures that $\dot{V} + \frac{1}{\gamma} z^{\mathrm{T}} z - \gamma w^{\mathrm{T}} w \le 0$ for almost all x, w. Integrating both sides, we obtain $\|z\|_2 \le \gamma\|w\|_2$ under initial condition $x(0) = 0$. ○

4. Regional analysis

For systems that are not globally exponentially stable, we need to perform regional analysis to evaluate the domain of attraction or a set of initial conditions that would not be driven unbounded by the disturbance. Even for a globally stable system, regional analysis can be used to reduce the conservatism when the system operates within a bounded region, or the energy of the disturbance is bounded by a certain value. Furthermore, we would like to examine the nonlinear relationship between the output energy and the input energy. This will be studied under the term "Nonlinear $\mathcal{L}_2$ gain".

When a system operates in a bounded region, in particular, when $\xi(x)$ is bounded, we can find a matrix H such that $|\bar{U}^{-1} H\xi(x)|_\infty \le 1|$. Let $r = H\xi(x)$. Then $|r_i| \le \bar{u}_i$ and we can use this r in the regional sector condition (4). Because there are infinitely many H satisfying the condition, we can take it as an additional variable for optimization, so that the conservatism is reduced to the maximal extent. For a given matrix H, define $\mathcal{L}(H) := \{\xi \in \mathbb{R}^{n+m} : |\overline{U}^{-1} H\xi(x)|_\infty \le 1\}$.

The main idea in regional analysis is to ensure an invariant set, possibly in the presence of disturbance, such as $\mathcal{E}(P/s^2) := \{\xi \in \mathbb{R}^{n+m} : \xi(x)^{\mathrm{T}} \frac{P}{s^2} \xi(x) \le 1\}$, so that $\xi(x(t)) \in \mathcal{E}(P/s^2)$ for all $t \ge 0$. Then use a matrix H satisfying the set inclusion condition $\mathcal{E}(P/s^2) \subset \mathcal{L}(H)$. Under these conditions, we can incorporate the term $r = H\xi(x)$ into (4) to reduce the conservatism. Since all these conditions (set invariance, set inclusion, and the sector condition) are related, they will be imposed together. As a result, an optimization problem can be formulated to reduce the conservatism with P, H and other matrices as joint optimizing variables.

In what follows, we interpret the condition $\mathcal{E}(P/s^2) \subset \mathcal{L}(H)$ with matrix inequalities.

Lemma 1 (*See, e.g., [5]*). *Let $s > 0$. For $H_1, H_2 \in \mathbb{R}^{m\times(n+m)}$ and $P \in \mathbb{R}^{(n+m)\times(n+m)}$, $\mathcal{E}(P/s^2) \subseteq \mathcal{L}(H_1) \cap \mathcal{L}(H_2)$ if and only if*

$$\begin{bmatrix} \bar{u}_l^2/s^2 & H_{1l} \\ \star & P \end{bmatrix} \ge 0, \quad \forall l = 1, 2, \ldots, m \tag{22a}$$

$$\begin{bmatrix} \bar{u}_l^2/s^2 & H_{2l} \\ \star & P \end{bmatrix} \ge 0, \quad \forall l = 1, 2, \ldots, m \tag{22b}$$

where H_{1l}, H_{2l} are the l-th row of H_1, H_2, respectively.

To facilitate the presentation of Theorem 2, for $H_1, H_2 \in \mathbb{R}^{m\times(n+m)}$, $\Delta_1, \Delta_2 \in \mathbb{R}^{m\times m}$ we denote

$$\bar{\Omega}_1 := \begin{bmatrix} 0_{m\times(n+2m)} & I_m & 0_{m\times r} \end{bmatrix}^{\mathrm{T}} \Delta_1 \begin{bmatrix} H_1 & 0 & 0 & 0 & 0 \end{bmatrix};$$

$$\bar{\Omega}_2 := \begin{bmatrix} 0_{m\times n} & I_m & 0_{m\times(2m+r)} \end{bmatrix}^{\mathrm{T}} \Delta_2 \begin{bmatrix} H_2 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{\Omega} = \bar{\Omega}_1 + \bar{\Omega}_2.$$

These matrices will be used to impose the regional sector condition (4).

Theorem 2. *Consider system (1).*

1. *(Exponential stability) If there exist matrices $P \in \mathbb{R}^{(n+m)\times(n+m)}$, $P = P^{\mathrm{T}} > 0$, $H_1, H_2 \in \mathbb{R}^{m\times(n+m)}$ satisfying (22) with $s = 1$, and diagonal matrices $\Delta_i \in \mathbb{R}^{m\times m}$, $i = 1, \ldots, 5$, $\Delta_{i=\{1,2,3\}} > 0$ satisfying*
$$\begin{bmatrix} I_{n+3m} & 0_{(n+3m)\times r} \end{bmatrix} (\bar{\Psi} - \bar{\Omega} - \bar{\Omega}^{\mathrm{T}}) \begin{bmatrix} I_{n+3m} \\ 0_{r\times(n+3m)} \end{bmatrix} < 0, \tag{23}$$
then for $V(x) = \xi(x)^{\mathrm{T}} P \xi(x)$, there exists $\varepsilon > 0$ such that $\dot{V} < -\varepsilon|x|^2$ for almost all $\xi(x) \in \mathcal{E}(P)\backslash\{0\}$ and $w = 0$. Thus, the origin of the system (1) is locally exponentially stable. If $\xi(x(0)) \in \mathcal{E}(P)$, then $\xi(x(t)) \in \mathcal{E}(P)$ for all $t > 0$ and $\lim_{t\to\infty} x(t) = 0$.
2. *(Reachable region) Let $s > 0$. If there exist matrices $P \in \mathbb{R}^{(n+m)\times(n+m)}$, $P = P^{\mathrm{T}} > 0$, $H_1, H_2 \in \mathbb{R}^{m\times(n+m)}$ satisfying (22), and diagonal matrices $\Delta_i \in \mathbb{R}^{m\times m}$, $i = 1, \ldots, 5$, $\Delta_{i=\{1,2,3\}} > 0$ satisfying*
$$\bar{\Psi} - \bar{\Omega} - \bar{\Omega}^{\mathrm{T}} - \bar{\Psi}_6 \bar{\Psi}_6^{\mathrm{T}} < 0 \tag{24}$$
then $\dot{V} < w^{\mathrm{T}} w$ for almost all $\xi(x) \in \mathcal{E}(P/s^2)$ and all $w \in \mathbb{R}^r$. If $\xi(x(0)) = 0$ and $\|w\|_2 \le s$, then $\xi(x(t)) \in \mathcal{E}(P/s^2)$ for all $t \ge 0$.
3. *(Regional $\mathcal{L}_2$ gain) Let $s > 0$. If there exist matrices $P \in \mathbb{R}^{(n+m)\times(n+m)}$, $P = P^{\mathrm{T}} > 0$, H_1 and H_2 satisfying (22), diagonal matrices $\Delta_i \in \mathbb{R}^{m\times m}$, $i = 1, \ldots, 5$, $\Delta_{i=\{1,2,3\}} > 0$ satisfying*
$$\begin{bmatrix} \bar{\Psi} - \bar{\Omega} - \bar{\Omega}^{\mathrm{T}} - \bar{\Psi}_6 \gamma \bar{\Psi}_6^{\mathrm{T}} & \star \\ \begin{bmatrix} C_z & 0 & 0 & D_{zq} & D_{zw} \end{bmatrix} & -\gamma I \end{bmatrix} < 0 \tag{25}$$
then $\dot{V} + \frac{1}{\gamma} z^{\mathrm{T}} z < \gamma w^{\mathrm{T}} w$ for almost all $\xi(x) \in \mathcal{E}(P/s^2)$ and all $w \in \mathbb{R}^r$. If $\xi(x(0)) = 0$ and $\|w\|_2 \le s$, then $\|z\|_2 \le \gamma \|w\|_2$.

Proof. From Lemma 1, if the LMI (22) is satisfied, then for all $\xi(x) \in \mathcal{E}(P/s^2)$, $\xi(x) \in \mathcal{L}(H_1) \cap \mathcal{L}(H_2)$. Let $r = H_1\xi(x)$, then $|r_i| \le \bar{u}_i$ for each i. The sector condition (4) with $\Delta = \Delta_1$ and $v = u$ can be written as $\zeta^{\mathrm{T}}[\bar{\Psi}_1 - \bar{\Omega}_1]\zeta \ge 0$. Similarly, let $r = H_2\xi(x)$, and apply the sector condition (4) with $\Delta = \Delta_2$ and $v = y$, we have $\zeta^{\mathrm{T}}[\bar{\Psi}_2 - \bar{\Omega}_2]\zeta \ge 0$.

The rest part of the proof is similar to that for Theorem 1 by replacing $\bar{\Psi}_1, \bar{\Psi}_2$ with $\bar{\Psi}_1 - \bar{\Omega}_1$ and $\bar{\Psi}_2 - \bar{\Omega}_2$, respectively. Here we note that the regional sector condition is satisfied for all $t \ge 0$ under respective condition for each problem. This is because that the matrix inequalities ensure that $\xi(x(t)) \in \mathcal{E}(P/s^2)$ for all $t \ge 0$. ○

Remark 1. When $s \to \infty$, the LMI (22) enforces $H_1, H_2 \to 0$, and the results in (23)–(25) will converge to the global results in Theorem 1.

Remark 2. In Theorem 2, the inequalities (23)–(25) are not convex due to the bilinear terms $\Delta_1 H_1$, $\Delta_2 H_2$ in $\bar{\Omega}$. Each of them cannot be combined into one matrix variable because the constraint (22) also involves H_1, H_2.

From Theorem 2, different optimization problems can be formulated to maximize the estimation of the domain of attraction, to minimize the reachable region, or to minimize a bound on the regional $\mathcal{L}_2$ gain. Combining the bound on the regional $\mathcal{L}_2$ gain for all $s > 0$, where s is the bound on the input energy, we can obtain a curve as the bound for the nonlinear $\mathcal{L}_2$ gain. Here we would like focus our discussion on the nonlinear $\mathcal{L}_2$ gain since the other problems are similar or relatively simpler.

The nonlinear (or regional) $\mathcal{L}_2$ gain has also been addressed in our earlier papers [5,1]. In [5], the problem is to design an anti-windup compensator so that the regional $\mathcal{L}_2$ gain is minimized for a given $s > 0$. Since the quadratic Lyapunov function is used, the optimization problem turns out to be convex. The paper [1] uses quadratic functions and two types of nonquadratic Lyapunov functions to estimate the regional $\mathcal{L}_2$ gain. When nonquadratic Lyapunov functions are used, the optimization problems involve some bilinear matrix inequalities.

There are different algorithms to deal with optimization problems with BMI constraints, such as direct iteration, the path-following method and a combination of these as used in [1]. For the optimization problem in this paper, the direct iteration method works very well. The detailed procedure is given next.

Procedure 1. We choose a sequence of $s_1 < s_2 < \cdots < s_{N-1} < s_N$ where N is some positive integer.

Step 1. *Initial step*. Select $i = 1$ and $s = s_i$. An initial value of the optimizing parameters can be inherited from the optimal solution obtained with quadratic functions (see the convex results from the regional analysis in [1]). For example, we let $H_1 = H_2 = [H_a\ 0]$ where H_a is the optimal solution from the quadratic approach in [1]). Go to step 2.

Step 2. *Optimization with fixed H_1, H_2*. With $s = s_i$ and fixed H_1, H_2 from the previous step, we minimize γ under constraints (22) and (25), which are LMIs in P and Δ_i's.

Step 3. *Optimization with fixed Δ_1, Δ_2*. With $s = s_i$ and fixed Δ_1, Δ_2 from the previous step, we minimize γ under constraints (22) and (25), which are LMIs in P, H_1, H_2 and Δ_i, $i = 3, 4, 5$. If the difference between γ obtained in this step and that from the previous iteration is greater than the desired accuracy, return to step 2. Otherwise go to step 4.

Step 4. *Initial estimation for s_{i+1}*. If $i = N$, then finish. Otherwise set $i = i + 1$ and select the optimal values of H_1, H_2 from step 3 and go to step 2.

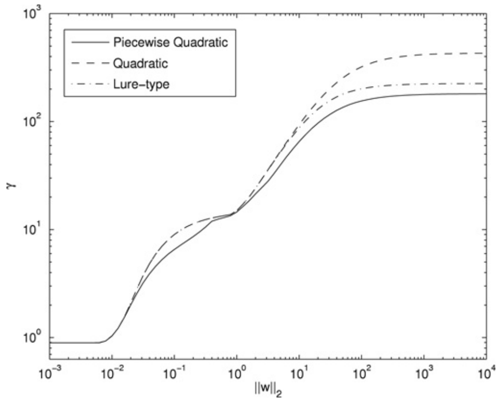

Fig. 1. Different estimates of the nonlinear $\mathcal{L}_2$ gain for Example 1.

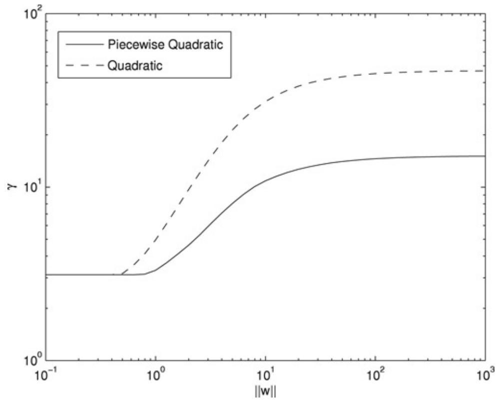

Fig. 2. Different estimates of the nonlinear $\mathcal{L}_2$ gain for Example 2.

Although there is no guarantee that a global optimal solution will be obtained, the algorithm in Procedure 1 has shown to work well in many example studies. Perhaps one important property of the algorithm is that since the initial value of the optimizing parameters can be inherited from the optimal solution obtained with quadratic functions, the algorithm ensures that the results are at least as good as those from using quadratic functions in [1].

5. Examples

Example 1. Consider the cart-spring-pendulum system used in [2]. In [2], the synthesis leads to a dynamic anti-windup compensator based on an unconstrained LQG controller, which generates desirable responses under a certain saturation condition. We adopted the parameters of the closed-loop system from [2] for the following analysis.

When quadratic Lyapunov function is used, the estimated global $\mathcal{L}_2$ gain is 431.01. If Lure-type Lyapunov function in (11) is used, the global $\mathcal{L}_2$ gain is found to be 203.24. When the piecewise quadratic function in (10) is used to solve the optimization problem based on Theorem 1, the global $\mathcal{L}_2$ gain is 181.142. The bounds for the nonlinear $\mathcal{L}_2$ gain are plotted in Fig. 1, where the dashed curve is from applying quadratics, the dash-dotted is from applying Lure-type Lyapunov function, and the solid one is from this paper's piecewise quadratic function.

Example 2. This example is adopted from Example 1 in [1]. The system parameters can be found in [1]. We use the result in Theorem 2 to estimate the nonlinear $\mathcal{L}_2$ gain. The bounds on the nonlinear $\mathcal{L}_2$ gain are plotted in Fig. 2, where the dashed curve is from applying quadratics and the solid one is from this paper's piecewise quadratics. Each of the two curves tends to a constant value as $\|w\|_2$ goes to infinity. This constant value is an estimate of the global $\mathcal{L}_2$ gain, which coincides with the value as shown in Table 1.

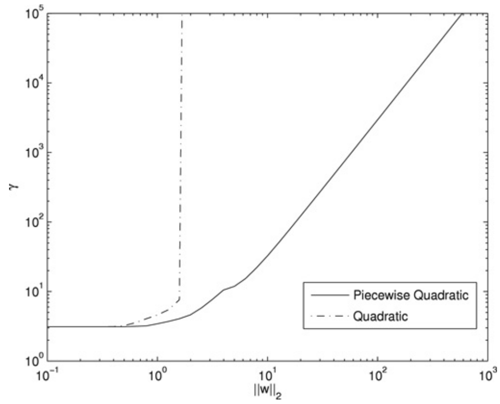

Fig. 3. Different estimates of the nonlinear $\mathcal{L}_2$ gain.

In this table, we present several scenarios through adjustments of the algebraic loop parameter D_{yq}, and compare the estimates of global $\mathcal{L}_2$ gains obtained from Theorem 1 with those obtained from other methods in the literature. Note that the convex hull quadratic function and the max quadratic function in [1] are composed of two quadratic functions. Moreover, the quadratic estimates in Table 1 are computed by using two different differential inclusions in [1]: PDI and NDI, respectively. For the readers' convenience, we report the explicit functions from [1] at the bottom of Table 1. From the comparison, we observe that the piecewise quadratic Lyapunov approach in Theorem 1 always gives a better estimate than other approaches. Even in the case that quadratic Lyapunov approach and Lure-type Lyapunov approach cannot give a finite estimate of the global $\mathcal{L}_2$ gain, our approach gives a satisfactory finite estimate.

If we change D_{yq} to $D_{yq} = \begin{bmatrix} -3 & -3 \\ -3 & -4 \end{bmatrix}$, the piecewise quadratic Lyapunov approach in this paper and all other Lyapunov approaches mentioned in the table cannot give a finite global $\mathcal{L}_2$ gain (or, global exponential stability is not confirmed). Fig. 3 shows the estimated nonlinear $\mathcal{L}_2$ gain, where the solid curve is from applying the piecewise quadratic Lyapunov approach based on Theorem 2, and the dash-dotted one is from quadratic functions. Both curves diverge to infinity as the bound s on $\|w\|_2$ goes to infinity. Nevertheless, we observe that the solid curve diverges at a much slower rate than the dash-dotted one, and for large enough value of $\|w\|_2$, the piecewise Lyapunov approach still gives finite estimates of $\mathcal{L}_2$ gain while the quadratic ones cannot.

The above example also shows how the stability and performance results by the same method can be affected by the parameter D_{yq}, which describes the algebraic loop. This is also discussed in [7,1]. From the different situations illustrated by the different $\mathcal{L}_2$ gains, it is clear that the piecewise quadratic approach never does worse than the quadratic ones (such as [2,5,1]) and the Lure-type Lyapunov approach. In some cases, the $\mathcal{L}_2$ gain obtained from the piecewise Lyapunov approach is smaller than that obtained from the quadratic and non-quadratic approaches in [1]. These two numerical examples show the great potential of the piecewise quadratic Lyapunov approach in the analysis and synthesis of saturated systems.

6. Conclusions

In this paper, a Lyapunov approach based on piecewise quadratic functions is developed to analyze the global and regional stability and performances for a general linear system with saturation or deadzone functions. This analysis relies on an effective treatment of the algebraic loop with exogenous

Table 1
Global $\mathcal{L}_2$ gain estimates for different Lyapunov functions.

Lyapunov function	$D_{yq} = \begin{bmatrix} -3 & -1 \\ -2 & -4 \end{bmatrix}$	$\begin{bmatrix} -3 & -1.3 \\ -2.3 & -4 \end{bmatrix}$	$\begin{bmatrix} -3 & -2 \\ -2 & -4 \end{bmatrix}$
Piecewise quadratic in (10)	15.13	17.19	25.86
Convex hull quadratic[a] in [1]	17.06	19.33	31.67
Max quadratic[b] in [1]	17.37	20.78	42.34
Quadratic[c] via PDI in [1]	38.96	170.15	∞
Lure–Postnikov type in (11)	46.96	∞	∞
Quadratic[c] via NDI in [2,1]	46.96	∞	∞

[a] Corresponds to $V_c(x) := \min_{\{\gamma_1+\gamma_2=1,\gamma_{1,2}>0\}} x^{\mathrm{T}}(\gamma_1 Q_1 + \gamma_2 Q_2)^{-1}x$, with $Q_{1,2} = Q_{1,2}^{\mathrm{T}} > 0$.

[b] Corresponds to $V_{max}(x) := \max\{x^{\mathrm{T}}P_1x, x^{\mathrm{T}}P_2x\}$, with $P_{1,2} = P_{1,2}^{\mathrm{T}} > 0$.

[c] Corresponds to $V_q(x) = x^{\mathrm{T}}Px$, with $P = P^{\mathrm{T}} > 0$.

inputs. Applying some existing sector conditions and some newly derived sector-like conditions, we obtain global conditions for stability and performance in the form of LMIs. Corresponding regional conditions are converted into BMIs. Numerical experience shows that the BMI conditions can be effectively solved with an iterative algorithm provided in Procedure 1. The great potential of the proposed piecewise quadratic Lyapunov function has been revealed by numerical examples.

Acknowledgment

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Control of saturated linear plants via output feedback containing an internal deadzone loop

Dan Dai, Tingshu Hu, Andrew R. Teel and Luca Zaccarian

Abstract— **In this paper we address a LMI-based optimization method for designing output feedback control laws to achieve regional performance and stability of linear control systems with input saturation. Algorithms are developed for minimizing the upper bound on the regional $\mathcal{L}_2$ gain for exogenous inputs with $\mathcal{L}_2$ norm bounded by a given value, and for minimizing this upper bound with a guaranteed reachable set or domain of attraction. Based on the structure of the optimization problems, using the projection lemma, the output feedback controller synthesis is cast as a convex optimization over linear matrix inequalities. The problems are studied in a general setting where the only requirement on the linear plant is detectability and stabilizability.**

keywords: output feedback control, input saturation, $\mathcal{L}_2$ gain, reachable set, domain of attraction, LMIs

I. Introduction

The behavior of linear, time-invariant (LTI) systems subject to actuator saturation has been extensively studied over the past several decades. More recently, some systematic design procedures based on rigorous theoretical analysis have been proposed through various framework. Most of the research efforts geared toward the constructive linear or nonlinear control for saturated plants can be divided into two main strands.

In the first one, called anti-windup design, a pre-designed controller is given, so that its closed-loop with the plant without input saturation is well behaved (at least asymptotically stable but possibly inducing desirable unconstrained closed-loop performance). Given the predesigned controller, anti-windup design addresses the controller augmentation problem aimed at maintaining the predesigned controller behavior before saturation and introducing suitable modifications after saturation so that global (or regional) asymptotic stability is guaranteed (local asymptotic stability already holds by the properties of the saturation nonlinearity). Anti-windup research has been largely discussed and many constructive design algorithms have been formally proved to induce suitable stability and performance properties. Many of these constructive approaches (see, e.g., [4], [5], [8], [9], [10], [11], [12], [19], [23], [30]) rely on convex optimization techniques and provide Linear Matrix Inequalities (LMIs) [2] for the anti-windup compensator design.

The second research strand, can be called "direct design", to resemble the fact that saturation is directly accounted for in the controller design and that no specification or constraint is imposed on the behavior of the closed-loop for small signals. Direct designs for saturated systems range from the well-known Model Predictive Control (MPC) techniques [20], especially suitable for discrete-time systems) to sophisticated nonlinear control laws which are able to guarantee global asymptotic stability for all linear saturated and globally stabilizable plants (see, e.g., the scheduled Riccati approach in [21] and the nested saturations of [25], [27]). Several LMI-based methods for direct controller design for linear plants with input saturation have also been proposed (see, e.g., [6], [18], [22], [24]). It is not our scope to mention here all the extensive literature on direct design for saturated systems, but it is worth mentioning that several constructive methods are available that differ in simplicity, effectiveness and formality.

Compared to anti-windup control, direct design is a simpler task to accomplish, because there's no constraint on the closed-loop behavior for small signals, therefore, the designer has full freedom in the selection of the controller dynamics. Anti-windup design, on the other hand, allows to guarantee that a certain prescribed unconstrained performance is met by the closed-loop as long as the saturation limits are not exceeded (this performance often consists in some linear performance measure when a linear plant+controller pair is under consideration) and that this performance is gradually deteriorated as signals grow larger and larger outside the saturation limits.

In this paper, we will propose a synthesis method for the construction of output feedback controllers with an internal deadzone loop. This type of structure corresponds to the typical framework used since the 1980's for the design of control systems for saturated plants. See for example the work in [28], [3], [29], [13], [26], [6] and other references in [1]. In our approach we will use the same tools used in our recent papers [17] for static anti-windup design, and we will recast the underlying optimization problem for the selection of all the controller matrices (whereas in [17] only the static anti-windup gain was selected and the underlying linear controller matrices were fixed). This approach parallels the approach proposed in [22] where classical sector conditions were used and extra assumptions on the direct input-output link of the plant were enforced. A similar assumption was also made in the recent paper [7] which uses similar tools to ours to address both magnitude and rate saturation problems in a compensation scheme with lesser degrees of freedom than ours. Here we use the regional analysis tool adopted in [17], and we extend the general output feedback synthesis to characterize the regional $\mathcal{L}_2$ gain and reachable set for a class of norm bounded disturbance inputs, as well as the estimate of domain of attraction. The overall synthesis is cast as an optimization over LMIs, and under a detectability and stabilizability condition on the plant, the proposed design procedure will always lead to regionally stabilizing controllers if the plant is exponentially unstable, to semi-global

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results if the plant is non-exponentially unstable, and to global results if the plant is already exponentially stable. An interesting advantage of the approach proposed here is that due to the type of transformation that we use, it is possible to derive system theoretic interpretation of the feasibility conditions for the controller design (such as stabilizability and detectability of the pant). This result is novel and was not previously observed in [22].

The paper is organized as follows: In Section II we formulate three problems that will be addressed in the paper; in Section III we state the LMI-based main conditions for output feedback controller synthesis and the procedure for the controller construction; in Section IV we give the feasible solutions for the problems we presented in Section II; in Section V we illustrate the proposed constructions on a simulation example.

Notation For compact presentation of matrices, given a square matrix X we denote $\mathrm{He}X := X + X^T$. For $P = P^T > 0$, we denote $\mathcal{E}(P) := \{x : \ x^T P x \leq 1\}$.

II. Problem statement

Consider a linear saturated plant,

$$\mathcal{P} \quad \begin{cases} \dot{x}_p = A_p x_p + B_{p,u}\mathrm{sat}(y_c) + B_{p,w} w \\ y = C_{p,y} x_p + D_{p,yu}\mathrm{sat}(y_c) + D_{p,yw} w \\ z = C_{p,z} x_p + D_{p,zu}\mathrm{sat}(y_c) + D_{p,zw} w \end{cases} \tag{1}$$

where $x_p \in \mathbb{R}^{n_p}$ is the plant state, $y_c \in \mathbb{R}^{n_u}$ is the control input subject to saturation, $w \in \mathbb{R}^{n_w}$ is the exogenous input (possibly containing disturbance, reference and measurement noise), $y \in \mathbb{R}^{n_y}$ is the measurement output and $z \in \mathbb{R}^{n_z}$ is the performance output.

The goal of this paper is the synthesis of a plant-order linear output feedback controller with internal deadzone loops:

$$\mathcal{C} \begin{cases} \dot{x}_c = A_c x_c + B_c y + E_1 \mathrm{dz}(y_c) \\ y_c = C_c x_c + D_c y + E_2 \mathrm{dz}(y_c), \end{cases} \tag{2}$$

where $x_c \in \mathbb{R}^{n_c}$ (with $n_c = n_p$) is the controller state, $y_c \in \mathbb{R}^{n_u}$ is the controller output and $\mathrm{dz}(\cdot) : \mathbb{R}^{n_u} \to \mathbb{R}^{n_u}$ is the deadzone function defined as $\mathrm{dz}(y_c) := y_c - \mathrm{sat}(y_c)$ for all $y_c \in \mathbb{R}^{n_u}$ with $\mathrm{sat}(\cdot) : \mathbb{R}^{n_u} \to \mathbb{R}^{n_u}$ being the symmetric saturation function having saturation levels $\bar{u}_1, \ldots, \bar{u}_{n_u}$ with its i-th component $\mathrm{sat}_{\bar{u}_i}(\cdot)$ depending on the i-th input component y_{ci} as follows:

$$\mathrm{sat}_{\bar{u}_i}(y_{ci}) = \begin{cases} \bar{u}_i, & \text{if } y_{ci} \geq \bar{u}_i, \\ y_{ci}, & \text{if } -\bar{u}_i \leq y_{ci} \leq \bar{u}_i, \\ -\bar{u}_i, & \text{if } y_{ci} \leq -\bar{u}_i. \end{cases} \tag{3}$$

The resulting nonlinear closed-loop (1), (2), is depicted in Figure 1.

The same output feedback controller structure was considered in [22], where convex synthesis methods for global (rather than regional, as we consider here) stability and performance were developed. In [22], it was assumed for simplicity that $D_{p,yu} = 0$ (we will remove this assumption here).

It is well known that linear saturated plants are characterized by weak stabilizability conditions. In particular, since by linearity the controller authority becomes almost zero for arbitrarily large signals, then global asymptotic stability can only be guaranteed for plants that are not exponentially unstable, while global exponential stability can only be guaranteed if the plant is already exponentially stable. Due to this fact, global results are never achievable (not even with a nonlinear controller) when wanting to exponentially stabilize plants that are not already exponentially stable. On the other hand, local and regional results are always achievable and semiglobal ones are achievable with non-exponentially unstable plants. The following three regional properties are then relevant for the controller design addressed here:

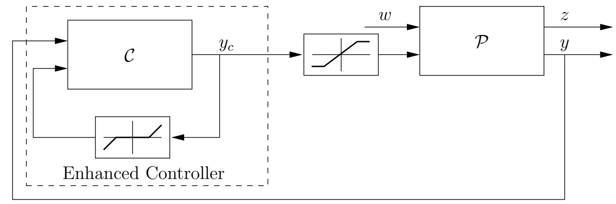

Fig. 1. The linear output feedback control system with deadzone loops.

Property 1: Given a set $\mathcal{S}_p \subset \mathbb{R}^{n_p}$, the plant (1) is $\mathcal{S}_p$-regionally exponentially stabilized by controller (2) if the origin of the closed-loop system (1), (2) is exponentially stable with domain of attraction including $\mathcal{S}_p \times \mathcal{S}_c$ (where $\mathcal{S}_c \subset \mathbb{R}^{n_c}$ is a suitable set including the origin).

Property 2: Given a set $\mathcal{R}_p \subset \mathbb{R}^{n_p}$ and a number $s > 0$, controller (2) guarantees $(s, \mathcal{R}_p)$-reachability for the plant (1) if the response $(x_p(t), x_c(t))$, $t \geq 0$ of the closed-loop system (1), (2) starting from the equilibrium point $(x_p(0), x_c(0)) = (0, 0)$ and with $\|w\|_2 < s$, satisfies $x_p(t) \in \mathcal{R}_p$ for all $t \geq 0$.

Property 3: Given two numbers $s, \gamma > 0$, controller (2) guarantees (s, γ)-regional finite $\mathcal{L}_2$ gain for the plant (1) if the performance output response $z(t)$, $t \geq 0$ of the closed-loop system (1), (2) starting from the equilibrium point $(x_p(0), x_c(0)) = (0, 0)$ and with $\|w\|_2 < s$, satisfies $\|z\|_2 < \gamma \|w\|_2$.

Based on the three properties introduced above, in this paper we are interested in providing LMI-based design tools for the synthesis of an output feedback controller of the form (2) guaranteeing suitable stability, reachability and $\mathcal{L}_2$ gain properties on the corresponding closed-loop. In particular, we will address the following problems:

Problem 1: Consider the linear plant (1), a bound s on $\|w\|_2$, a desired reachability region $\mathcal{R}_p$ and a bound γ on the desired regional $\mathcal{L}_2$ gain. Design a linear output feedback controller (2) guaranteeing $(s, \mathcal{R}_p)$ reachability, (s, γ)-regional finite $\mathcal{L}_2$ gain and which maximizes the exponential stability region $\mathcal{S}_p$ of the closed-loop (1), (2).

Problem 2: Consider the linear plant (1), a bound s on $\|w\|_2$, a desired stability region $\mathcal{S}_p$ and a bound γ on the desired regional $\mathcal{L}_2$ gain. Design a linear output feedback controller (2) guaranteeing $\mathcal{S}_p$ regional exponential stability, (s, γ)-regional finite $\mathcal{L}_2$ gain and which minimizes the $(s, \mathcal{R}_p)$ reachability region of the closed-loop (1), (2).

Problem 3: Consider the linear plant (1), a bound s on $\|w\|_2$, a desired stability region $\mathcal{S}_p$ and a desired reachability region $\mathcal{R}_p$. Design a linear output feedback controller (2) guaranteeing $\mathcal{S}_p$ regional exponential stability, $(s, \mathcal{R}_p)$ reachability and which minimizes the (s, γ)-regional finite $\mathcal{L}_2$ gain of the closed-loop (1), (2).

III. LMI-based design

In this section, a set of main feasibility conditions for solving Problems 1 to 3 will be presented in addition to giving a constructive procedure to design a state space representation of the linear output feedback controller (2).

A. Main feasibility theorem

The results that we will derive are based on the sector description of the deadzone originally developed in [15], [14]. The main idea of the description is as follows. For a scalar saturation function $\mathrm{sat}_{\bar{u}}(\cdot)$, if $|v| \le \bar{u}$ then $\mathrm{sat}_{\bar{u}}(u)$ is between u and v for all $u \in \mathbb{R}$. Applying this description to deal with the saturating actuator in Figure 1, we will have $\mathrm{sat}_{\bar{u}_i}(y_{ci})$ between y_{ci} and $H_i x$ as long as $|H_i x| \le \bar{u}_i$. Here x is the combined state in Figure 1 and H_i can be any row vector of appropriate dimensions. It turns out that the choice of H_i can be incorporated into LMI optimization problems. It should be noted that [6] and [9] exploit the same idea to deal with saturations and deadzones. In this paper, y_c and x are related to each other in a more general way as mentioned above, rather than $y_c = Fx$, as in [6], [9].

The following theorem will be used in the following sections to provide solutions to the problem statements given in Section II.

Theorem 1: Consider the linear plant (1). If the following LMI conditions in the unknowns $Q_{11} = Q_{11}^T > 0$, $P_{11} = P_{11}^T > 0$, $\gamma^2 > 0$, $Y_p \in \mathbb{R}^{n_u \times n_p}$, $K_1 \in \mathbb{R}^{n_p \times n_y}$, $K_2 \in \mathbb{R}^{n_y \times n_y}$, $K_3 \in \mathbb{R}^{n_z \times n_y}$ are feasible:

$$\mathrm{He}\begin{bmatrix} A_pQ_{11} + B_{pu}Y_p & 0 & B_{pw} \\ C_{pz}Q_{11} + D_{p,zu}Y_p & -\frac{\gamma^2 I}{2} & D_{p,zw} \\ 0 & 0 & -\frac{I}{2} \end{bmatrix} < 0 \tag{4a}$$

$$\mathrm{He}\begin{bmatrix} P_{11}A_p + K_1C_{py} & P_{11}B_{pw} + K_1D_{p,yw} & 0 \\ K_2C_{py} & K_2D_{p,yw} - I/2 & 0 \\ C_{pz} + K_3C_{py} & D_{p,zw} + K_3D_{p,yw} & -\frac{\gamma^2 I}{2} \end{bmatrix} < 0 \tag{4b}$$

$$\begin{bmatrix} Q_{11} & I \\ I & P_{11} \end{bmatrix} > 0. \tag{4c}$$

$$\begin{bmatrix} \bar{u}_i^2/s^2 & Y_{pi} \\ Y_{pi}^T & Q_{11} \end{bmatrix} \ge 0, \quad i = 1, \ldots, n_u, \tag{4d}$$

(where Y_{pi} denotes the ith row of Y_p), then there exists an output feedback controller of the form (2) and of oder n_p which guarantees the following three properties for the closed-loop:

1) (s, γ) regional finite $\mathcal{L}_2$ gain;
2) $\mathcal{S}_p$ regional exponential stability with $\mathcal{S}_p = \mathcal{E}((s^2Q_{11})^{-1})$;
3) $(s, \mathcal{R}_p)$ reachability region with $\mathcal{R}_p = \mathcal{E}((s^2Q_{11})^{-1})$.

Moreover, given any feasible solution to the LMI constraints (4), a state-space representation of a controller guaranteeing these properties can be determined based on the matrices (Q_{11}, P_{11}) by way of the Procedure 1 reported next.

Remark 1: Each of the LMI conditions in (4) has a system theoretic interpretation:

1) The fist condition (4a) corresponds to a strengthened stabilizability condition for the plant (1). Indeed, substituting $Y_p = K_pQ_{11}$, (4a) corresponds to the LMI formulation of the bounded real lemma (see, e.g., [10]) characterizing the $\mathcal{L}_2$ gain from w to z for the plant controlled by a state feedback law $u = K_px_p$. Therefore, (4a) constrains γ to be not smaller than the $\mathcal{L}_2$ gain of the plant stabilized by static state feedback. Note however that the corresponding state feedback gain K_p is further constrained by (4d), so that the open-loop plant $\mathcal{L}_2$ gain may be only reduced to a certain extent when larger values of s make the constraint (4d) tighter. As s approaches $+\infty$, Y_p will approach 0 and the constraint on γ enforced by (4a) will approach the global constraint given by the $\mathcal{L}_2$ gain of the open-loop plant.

2) The second condition (4b) corresponds to a strengthened detectability condition for the plant. Indeed, if we had $K_2 = 0$ in the condition above, then the corresponding equation would mean that (if we take $L = P_{11}^{-1}K_1$) there exist L and K_3 such that the observer with unknown input

$$\begin{aligned} \dot{\tilde{x}} &= A_p\tilde{x} + L(y - \tilde{y}) \\ \tilde{y} &= C_{yp}\tilde{x} \\ \tilde{z} &= C_{pz}\tilde{x} - K_3(y - \tilde{y}), \end{aligned}$$

for the system

$$\begin{aligned} \dot{x} &= A_px + B_{pw}w \\ y &= C_{yp}x + D_{p,yw}w \\ z &= C_{pz}x + D_{p,zw}w \end{aligned}$$

guarantees gain γ from w to the output observation error $(z - \tilde{z})$. The fact above can be checked writing down the error dynamics $e = x - \tilde{x}$ and imposing that there exists a disturbance attenuation Lyapunov function $V = e^TP_{11}e$. In particular, the LMI (4b) corresponds to imposing that:

$$2e^TP_{11}(A_p + LC_{py})e + 2e^TP_{11}(B_{p,w} + LD_{p,yw})w + \frac{1}{\gamma^2}z_e^Tz_e - w^Tw < 0,$$

where $z_e = z - \tilde{z}$. $\circ$

B. Controller construction

We provide next a constructive algorithm for determining the matrices of the linear controller whose existence is established in Theorem 1.

Procedure 1: (Output feedback construction)

Step 1. *Solve the feasibility LMIs.* Find a solution $(Q_{11}, P_{11}, K_1, K_2, K_3, Y_p, \gamma)$ to the feasibility LMIs conditions (4).

Step 2. *Construct the matrix Q.* (see also [11], [16].) Define the matrices $Q_{11} \in \mathbb{R}^{n_p \times n_p}$ and $Q_{12} \in \mathbb{R}^{n_p \times n_c}$, with $n_c = n_p$ as a solution of the following equation:

$$Q_{11}P_{11}Q_{11} - Q_{11} = Q_{12}Q_{12}^T. \tag{5}$$

Since Q_{11} and P_{11} are invertible and $Q_{11}^{-1} < P_{11}$ by the feasibility conditions, $Q_{11}P_{11}Q_{11} - Q_{11}$ is positive definite. Hence there always exists [1] a matrix Q_{12} satisfying equation (5). Define the matrix $Q_{22} \in \mathbb{R}^{n_c \times n_c}$ as

$$Q_{22} := I + Q_{12}^TQ_{11}^{-1}Q_{12}. \tag{6}$$

Finally, define the matrix $Q \in \mathbb{R}^{n \times n}$ $(n = n_p + n_c)$ as

$$Q := \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}. \tag{7}$$

Step 3. *Controller synthesis LMI.* Construct the matrices $\Psi \in \mathbb{R}^{\bar{n} \times \bar{n}}$, $H \in \mathbb{R}^{(n_c+n_u) \times \bar{n}}$, $G \in \mathbb{R}^{(n_c+n_u+n_y) \times \bar{n}}$, and $T \in \mathbb{R}^{\bar{n} \times \bar{n}}$ $(\bar{n} = n_p + n_c + n_u + n_w + n_z)$ as follows:

$$\Psi = \begin{bmatrix} A_pQ_{11} & A_pQ_{12} & -B_{pu}U & B_{pw} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -Y_p & -Y_c & -U & 0 & 0 \\ 0 & 0 & 0 & -\frac{I}{2} & 0 \\ C_{pz}Q_{11} & C_{pz}Q_{12} & -D_{p,zu}U & D_{p,zw} & -\frac{\gamma^2}{2}I \end{bmatrix}$$

[1] Note that equation (5) always admits infinite solutions, parametrizing an infinite compensators inducing the same performance on the plant. Understanding how to exploit this degree of freedom for the selection of a most desirable compensator is subject of future research.

$$H = \begin{bmatrix} 0 & B_{pu} \\ I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & D_{p,zu} \end{bmatrix}^T \quad G = \begin{bmatrix} 0 & I & 0 & 0 & 0 \\ C_{py} & 0 & 0 & D_{p,yw} & 0 \\ 0 & 0 & I & 0 & 0 \end{bmatrix}$$

$$T = \mathrm{diag}\{Q, I, I, I\}$$

where $Y_c \in \mathbb{R}^{n_u \times n_c}$ is defined as $Y_c := Y_p(Q_{11})^{-1}Q_{12}$. Then, we define the unknown variable $\bar{\mathbf{\Omega}}_U \in \mathbb{R}^{(n_c+n_u)\times(n_c+n_u+n_y)}$ as:

$$\bar{\mathbf{\Omega}}_U := \begin{bmatrix} \bar{A}_c & \bar{B}_c & \bar{E}_1 U \\ \bar{C}_c & \bar{D}_c & \bar{E}_2 U \end{bmatrix} \tag{8}$$

Finally, solve the output feedback controller LMI:

$$M = \mathrm{He}(\Psi + \Psi_1) = \mathrm{He}(\Psi + H^T \bar{\mathbf{\Omega}}_U G T) < 0 \tag{9}$$

in the unknowns $\bar{\mathbf{\Omega}}_U$, and $U \in \mathbb{R}^{n_u \times n_u}, U > 0$ diagonal.

Step 4. *Computation of the controller matrices*. From the matrices U and $\bar{\mathbf{\Omega}}_U$ in Step 3, compute the matrix $\bar{\mathbf{\Omega}}$ as

$$\begin{aligned} \bar{\mathbf{\Omega}} &:= \begin{bmatrix} \bar{A}_c & \bar{B}_c & \bar{E}_1 \\ \bar{C}_c & \bar{D}_c & \bar{E}_2 \end{bmatrix} \\ &:= \bar{\mathbf{\Omega}}_U \,\mathrm{diag}(I, I, U^{-1}). \end{aligned}$$

Finally, the controller parameters in $\mathbf{\Omega}$ can be determined applying the following transformation:

$$\begin{aligned} \mathbf{\Omega} &:= \begin{bmatrix} A_c & B_c & E_1 \\ C_c & D_c & E_2 \end{bmatrix} \\ &= \bar{\mathbf{\Omega}} \begin{bmatrix} I & 0 & 0 \\ -\bar{X} D_{p,yu} \bar{C}_c & \bar{X} & \bar{X} D_{p,yu}(I - \bar{E}_2) \\ 0 & 0 & I \end{bmatrix} \end{aligned}$$

where $\bar{X} := (I + D_{p,yu}\bar{D}_c)^{-1}$. $\star$

IV. Feasibility of the Synthesis Problems

In this section, the feasibility conditions established in Theorem 1 will be used to provide conditions for the solvability of Problems 1 to 3.

A. Feasibility of and solution to Problem 1

We first use the result of Theorem 1 to give a solution to Problem 1. To this aim, we use the guaranteed $\mathcal{L}_2$ performance and reachability region of items 1 and 3 of Theorem 1 and maximize the size of the guaranteed stability region by maximizing the size of the ellipsoid $\mathcal{E}((s^2 Q_{11})^{-1})$ which, according to item 2 of Theorem 1 is an estimate of the domain of attraction. We state the corollary below for a generic measure $\alpha_R(\cdot)$ of the size of the ellipsoid $\mathcal{E}((s^2 Q_{11})^{-1})$. This is typically done with respect to some shape reference of the desired stability region $\mathcal{S}_p$.

Corollary 1: Given s, $\mathcal{R}_p$ and γ, a solution to Problem 1 is given (whenever feasible) by applying Procedure 1 to the optimal solution of the following maximization problem:

$$\sup_{Q_{11}, P_{11}, K_1, K_2, K_3, Y_p, \gamma^2} \alpha_R(\mathcal{E}((s^2 Q_{11})^{-1})), \text{ subject to}$$

$$(4a), (4b), (4c), (4d), \tag{10a}$$

$$\mathcal{E}(Q_{11}^{-1}/s^2) \subset \mathcal{R}_p \tag{10b}$$

The formulation in Corollary 1 can be easily particularized to the problem of maximizing the volume of $\mathcal{E}(Q_{11}^{-1}/s^2)$ by selecting in (10) as $\alpha_R(\mathcal{E}((s^2 Q_{11})^{-1})) = \det(s^2 Q_{11})$. Alternative easier selections of α_R can correspond to maximizing the size of a region which has a predefined shape. For example, when focusing on ellipsoids, one can seek for stability regions of the type

$$\mathcal{S}_p = \mathcal{E}(S_p^{-1}) = \{x_p : \; x_p^T (\alpha S_p)^{-1} x_p \le 1\}, \tag{11}$$

where α is a positive scalar such that larger values of α correspond to larger sets $\mathcal{S}_p$. Then the optimization problem (10) can be cast as

$$\sup_{Q_{11}, P_{11}, K_1, K_2, K_3, Y_p, \gamma^2, \alpha} \alpha, \text{ subject to}$$

$$(4a), (4b), (4c), (4d), (10b) \tag{12a}$$

$$\alpha S_p \le s^2 Q_{11}. \tag{12b}$$

Similarly, if one takes a polyhedral reference region: $\mathcal{S}_p = \alpha co\{x_1, x_2, ..., x_{n_p}\}$, then constraint (12b) can be replaced by ${x_{pi}}^T Q_{11}^{-1} x_{pi} \le \alpha s^2$, or equivalently

$$\begin{bmatrix} \alpha s^2 & x_{pi}^T \\ x_{pi} & Q_{11} \end{bmatrix} \ge 0, \quad i = 1, \ldots, n_p. \tag{13}$$

We finally note that the constraint (10b) on the guaranteed reachability region can be expressed by way of different convex (possibly LMI) conditions depending on the shape of the set $\mathcal{R}_p$. Guidelines in this direction are given in the following section.

Remark 2: Based on Corollary 1, reduced LMI conditions can be written to only maximize the estimate of the domain of attraction without any constraint on the other performance measures:

$$\begin{aligned} &\sup_{Q_{11}, P_{11}, K_1, Y_p} \alpha_R(\mathcal{E}((s^2 Q_{11})^{-1})), \text{ subject to} \\ &\mathrm{He}[\; A_p Q_{11} + B_{pu} Y_p \;] < 0 \\ &\mathrm{He}[\; P_{11} A_p + K_1 C_{py} \;] < 0 \\ &(4c), (4d) \end{aligned} \tag{14}$$

From (14) it is straightforward to conclude that if the plant is exponentially stable, then global exponential stability and finite $\mathcal{L}_2$ gain can be achieved by the proposed output feedback controller ($Y_p = 0$ and $K_1 = 0$ are sufficient). If the plant is not exponentially unstable, semiglobal results are obtainable. [2] Regional results can always be obtained in the general case and the size of the maximal feasible domain of attraction depends on the particular problem. $\circ$

B. Feasibility of and solution to Problem 2

We now use the result of Theorem 1 to give a solution to Problem 2 following similar steps as the ones in the previous section. When focusing on reachable sets, smaller estimates are desirable, so that there's a guaranteed bound on the size of the state when the system is disturbed by external inputs. Since by Theorem 1 the reachability region estimate coincides with the estimate of the domain of attraction, the goal addressed in Problem 2 is in contrast with the goal addressed in the previous section. The corresponding equivalent to Corollary 1 is the following (where $\alpha_R(\cdot)$ is a measure of the size of the ellipsoid $\mathcal{E}((s^2 Q_{11})^{-1})$):

Corollary 2: Given s, $\mathcal{S}_p$ and γ, a solution to Problem 2 is given (whenever feasible) by applying Procedure 1 to the optimal solution to the following minimization problem:

$$\min_{Q_{11}, P_{11}, K_1, K_2, K_3, Y_p, \gamma^2} \alpha_R(\mathcal{E}((s^2 Q_{11})^{-1}), \text{ subject to}$$

$$(4a), (4b), (4c), (4d), \tag{15a}$$

[2] The proof of this fact follows the same steps as the proof of [17, Propositions 1 and 2].

$$\mathcal{S}_p \subset \mathcal{E}((s^2 Q_{11})^{-1}) \tag{15b}$$

Similar to the previous section, the volume of the reachability set can be minimized by selecting $\alpha_R(\mathcal{E}((s^2Q_{11})^{-1})) = \det(s^2 Q_{11})$ in (15).

Alternative easier selections of α_R correspond to focusing on ellipsoids when choosing

$$\mathcal{R}_p = \mathcal{E}(\alpha R_p^{-1}) = \{x_p : \; x_p^T(\alpha R_p)^{-1} x_p \le 1\}, \tag{16}$$

where $R_p = R_p^T > 0$, then the optimization problem (15) becomes

$$\min_{Q_{11},P_{11},K_1,K_2,K_3,Y_p,\gamma^2,\alpha} \alpha, \text{ subject to}$$

$$(4a),(4b),(4c),(4d),(15b) \tag{17a}$$

$$s^2 Q_{11} \le \alpha R_p \tag{17b}$$

Similarly, $\mathcal{R}_p$ can be selected as the following unbounded set:

$$\mathcal{R}_p(\alpha) = \{x_p : \; |Cx_p| \le \alpha\},$$

where $C \in \mathbb{R}^{1\times n_p}$ is a given row vector. Then $\mathcal{E}((s^2Q_{11})^{-1}) \subset \mathcal{R}_p(\alpha)$ if and only if $CQ_{11}C^T \le \alpha^2/s^2$. If both (17a) and $CQ_{11}C^T \le \alpha^2/s^2$ are enforced to hold in the LMI optimization, then it follows that $|Cx_p(t)| \le \alpha$ for all t if $\|w\|_2 \le s$. Therefore, if our objective is to minimize the size of a particular output Cx_p, we may formulate the following optimization problem:

$$\min_{Q_{11},P_{11},K_1,K_2,K_3,Y_p,\gamma^2,\alpha^2} \alpha^2, \text{ subject to}$$

$$(4a),(4b),(4c),(4d),(15b) \tag{18a}$$

$$CQ_{11}C^T < \alpha^2/s^2, \tag{18b}$$

where the guaranteed $\mathcal{L}_2$ gain is be incorporated in (4a) and (4b) and the guaranteed stability set is incorporated in (15b).

Remark 3: If there is no interest for a guaranteed $\mathcal{L}_2$ gain, then the problem to minimize the desirable reachable set $\mathcal{R}_p$ can be simplified to the feasibility of the LMIs in (4) and (15b), by removing the second block row and the second block column of the matrices in (4a) and the third block ones in (4b). ∘

C. Feasibility of and solution to Problem 3

Similar to what has been done in the previous two sections with reference to Problems 1 and 2, we use here Theorem 1 to give a solution to Problem 3.

Corollary 3: Given s, $\mathcal{R}_p$ and $\mathcal{S}_p$, a solution to Problem 3 is given (whenever feasible) by applying Procedure 1 to the optimal solution to the following minimization problem:

$$\min_{Q_{11},P_{11},K_1,K_2,K_3,Y_p,\gamma^2} \gamma^2, \text{ subject to}$$

$$(4a),(4b),(4c),(4d), \tag{19a}$$

$$\mathcal{S}_p \subset \mathcal{E}((s^2Q_{11})^{-1}) \subset \mathcal{R}_p \tag{19b}$$

If we only focus on ellipsoidal reachability and stability sets so that for two given matrices $S_p = S_p^T > 0$ and $R_p = R_p^T > 0$ $\mathcal{S}_p := \mathcal{E}(S_p^{-1})$ and $\mathcal{R}_p := \mathcal{E}(R_p^{-1})$, then the optimization problem (19) can be cast as the following convex formulation:

$$\min_{Q_{11},P_{11},K_1,K_2,K_3,Y_p,\gamma^2} \gamma^2, \text{ subject to}$$

$$(4a),(4b),(4c),(4d), \tag{20a}$$

$$S_p \le s^2 Q_{11} \le R_p \tag{20b}$$

Alternative shapes for the guaranteed reachability set $\mathcal{R}_p$ and for the guaranteed stability region $\mathcal{S}_p$ can be selected by following the indications given in the previous two sections.

V. Simulation Example

We consider the system used in [10], which has one control input, one disturbance input, four states and one measurement output. The plant state is $x_p = \begin{bmatrix} p & \dot{p} & \theta & \dot{\theta} \end{bmatrix}^T$, where p is the horizontal displacement of the cart and θ is the angle of the pendulum. The plant parameters are given in [10]. For each $s > 0$, the achievable $\mathcal{L}_2$ gain by a plant order output feedback can be determined with the algorithm based on Theorem 1. If choosing different s over $(0, \infty)$, the achievable performance can be obtained as a function of s.

The solid line in Figure 2 reports the achievable $\mathcal{L}_2$ gain by a suitable plant order output feedback controller, as a function of s. For comparison purposes, we report in the same figure (dashed line) the $\mathcal{L}_2$ gain achievable by a dynamic anti-windup compensation when using a specific unconstrained controller (see [16] for details).

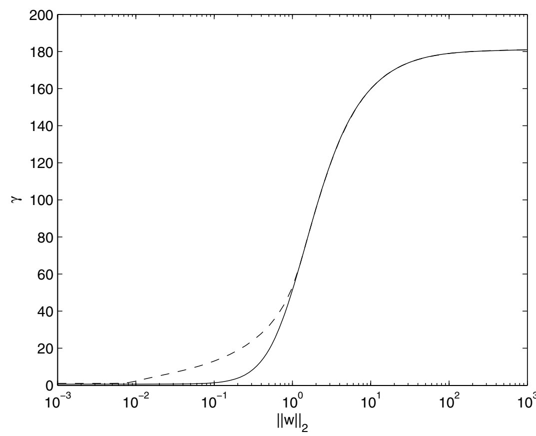

Fig. 2. Achievable nonlinear $\mathcal{L}_2$ gains. Proposed output feedback (thin solid); Dynamic anti-windup (dashed)

In [10], when the cart-spring-pendulum system is subject to the larger pendulum tap, which is modeled as a constant force of $7.94N$ with duration $0.01s$, the closed-loop response with an LQG controller exhibits undesirable oscillations if the control input is constrained in the range of the D/A converter: [-5, +5] Volts. In [10] a dynamic anti-windup compensator is used to preserve the local LQG behavior and improve the response after saturation. We compare that result to our direct design. In particular, we use Procedure 1 to construct an output feedback controller by fixing $s = 0.14$. The corresponding optimal $\mathcal{L}_2$ gain is $\gamma = 2.26$ and the controller matrices are

$$\begin{bmatrix} A_c & B_c & E_1 \\ C_c & D_c & E_2 \end{bmatrix} =$$

$$\left[\begin{array}{cccc|cc|c} -215.1 & 5.7 & 137.7 & -6 & 3558.8 & -3150 & -0.6334 \\ -1781 & -1154 & -62.5 & 360.2 & -4123 & -3202 & -48.76 \\ -93.8 & -53.9 & -159.9 & 21.2 & -1048 & 3788 & -4.44 \\ -4914 & -2441 & -416.8 & 735.1 & 1492 & 1290 & -131.6 \\ \hline -74.44 & -38.27 & -5.93 & 11.77 & 120.6 & 3.75 & -0.999 \end{array}\right]$$

The thin solid curve in Fig 3 represents the response of the closed-loop system with our output feedback controller to the same disturbance that generates the undesirable response arising from the saturated LQG controller (taken from [10]), which is represented by the dashed curve in the same figure.

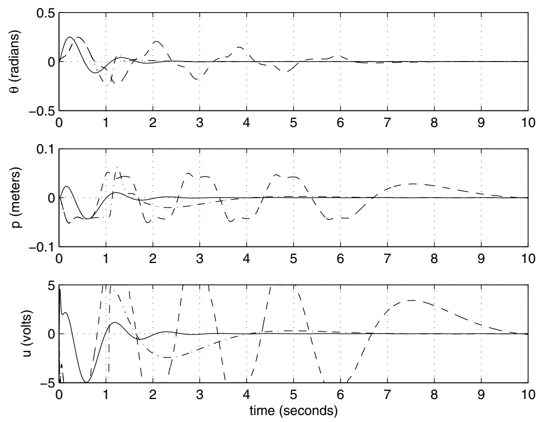

Fig. 3. Simulated response to the large pendulum tap. Constrained response with LQG controller (dashed); Response with the linear output feedback controller (thin solid); Response with the LQG controller and dynamic anti-windup (dash-dotted).

Moreover, for comparison purposes, the dash-dotted curve represents the response when using the dynamic compensator proposed in [10] on top of the LQG controller.

Note that the proposed controller guarantees more desirable large signal responses as compared to the anti-windup closed-loop of [10], indeed our controller is not constrained to satisfy the small signal specification as in the anti-windup approach of [10]. On the other hand, it must be recognized that synthesizing the controller by direct design reduces the small signal performance (before saturation) of the overall closed-loop, which can be imposed as an arbitrary linear performance when using the anti-windup tools as in [10] (or the more advanced techniques recently proposed in [17]).

VI. Conclusions

In this paper we proposed a synthesis method for the construction of a linear output feedback controller with an internal deadzone loop. By using the regional analysis tools also employed in [16], an LMI-based method for the controller synthesis is derived. Different optimization goals have been considered to optimize the $\mathcal{L}_2$ performance level, the domain of attraction or the reachability region of the closed-loop. Each optimization goal corresponds to a different optimization problem to be solved. A simulation example shows the effectiveness of the proposed controller.

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