Stable LPV realization of parametric transfer functions and its application to gain-scheduling control design

Franco Blanchini, Daniele Casagrande, Stefano Miani, and Umberto Viaro

Abstract

The paper deals with the stabilizability of linear plants whose parameters vary with time in a compact set. First, necessary and sufficient conditions for the existence of a linear gain-scheduled stabilizing compensator are given. Next, it is shown that, if these conditions are satisfied, any compensator transfer function depending on the plant parameters and internally stabilizing the closed-loop control system when the plant parameters are constant, can be realized in such a way that the closed-loop asymptotic stability is guaranteed under arbitrary parameter variations. To this purpose it is shown first that any transfer function that is stable for all constant parameters values admits a realization that is stable under arbitrary parameter variations (LPV stability). Then, the Youla–Kucera parametrization is exploited. Precisely, LPV stability can be ensured by taking an LPV stable realization of the Youla–Kucera parameter. To find one such realization, a reasonably simple and general algorithm based on Lyapunov equations and Cholesky's factorization is provided. The result can profitably be used to achieve both pointwise optimality (or pole placement) and LPV stability. Some potential applications in adaptive control and online tuning are pointed out.

I. Introduction

Linear parameter-varying (LPV) systems are a useful generalization of linear time-invariant (LTI) systems not only because they provide the natural setting for the adaptive (gain-scheduling) control of linear plants whose parameters vary in time, but also because many nonlinear plants can conveniently be embedded into a linear differential inclusion and, therefore, treated as LPV systems (see, for instance, Sect. 4.3 in [8] and [24], [15]). In fact, recent surveys have pointed out the importance given to the LPV framework in the modern control system literature [23], [19].

Nevertheless, the stability analysis of LPV systems is still a challenging subject since most of the classic tools used in the LTI case are no longer valid. Lyapunov theory is a notable exception. In particular, it has been shown that the stability or stabilizability of an LPV system is equivalent to the existence of a Lyapunov norm [18], [10], [9], [3], although not a quadratic one, in general.

It is common practice in the control of LPV systems to determine a family of compensators each of which is suitable for particular parameter values. However, the systematic stability analysis of this type of design solution is rather recent [25]. Since the early 90s, many papers have dealt with gain-scheduling stabilization and performance of LPV systems [2], [1], [21], [12]. Most of them exploit quadratic Lyapunov functions, with some exceptions that consider parameter–dependent and polyhedral Lyapunov functions [11], [4]. A technique based on pole assignment via state feedback has been considered in [22].

The first part of this paper relates the stabilizability of an LPV plant by means of a linear gain-scheduled compensator to the existence of a polyhedral Lyapunov function. This result exploits a duality property and the separation principle presented in [4], [6]. It generalizes significantly the results obtained in [4] for LPV systems, where the input and output matrices were assumed constant and the stabilizability conditions were only sufficient.

The second part of this paper is concerned with the fundamental problem of finding an LPV stabilizing controller realization from a parametric compensator transfer function that ensures internal stability for every constant value of the plant parameters. Indeed, as already suggested in [19], to guarantee stability under parameter variations, the parametric transfer function should be properly realized but, to these authors' knowledge, no general conditions have yet been provided concerning the existence of such a realization. By adapting results from [13] and [6], it is shown that an LPV stabilizing compensator realization of any stabilizing parametric compensator transfer function indeed exists. The main results of the paper are summarized next.

- Necessary and sufficient conditions for LPV stabilizability are provided in terms of bilinear equations that entail the separation of state estimation and feedback (separation principle).
- It is shown that for any parametric proper rational transfer function that is stable for any fixed value of the parameters there exists a realization that is stable under parameter variations.
- This result is applied to the problem of determining a realization of a parametric compensator transfer function for a parametric plant in such a way that closed–loop LPV stability is guaranteed. Moreover, resorting to the Youla–Kucera

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parametrization in its observer–based version [14], [27], [28], [20], it is proved that LPV stability can be achieved by taking an LPV stable realization of the Youla–Kucera parameter, which provides a characterization of all LPV stabilizing compensators.

- Several applications of the suggested technique, such as pointwise optimality with LPV stability, pole placement and online tuning, are pointed out.

II. Problem statement

Given a parametric proper rational transfer function

$$W(s,w) = \frac{N(s,w)}{d(s,w)}, \qquad w \in \mathcal{W}, \tag{1}$$

where w is a parameter and $\mathcal{W}$ a compact set, the state–space representation

$$\begin{array}{rcl} \dot z(t) & = & F(w)\, z(t) + G(w)\, \omega(t), \\ \xi(t) & = & H(w)\, z(t) + K(w)\, \omega(t), \end{array} \tag{2}$$

or, more concisely,

$$\Sigma(w) = \left[\begin{array}{c|c} F(w) & G(w) \\ \hline H(w) & K(w) \end{array}\right], \tag{3}$$

is a realization of $W(s,w)$ if

$$W(s,w) = H(w)[sI - F(w)]^{-1}G(w) + K(w)\,, \qquad \forall w \in \mathcal{W}\,. \tag{4}$$

As long as $w \in \mathcal{W}$ is constant and (3) is minimal, the following two properties are equivalent: (i) $d(s,w)$ is a Hurwitz polynomial; (ii) the realization (3) is asymptotically stable. Instead, if w varies in time, the condition that $d(s,w)$ is a Hurwitz polynomial for any fixed value of w is only necessary for (3) to be asymptotically stable. Therefore, the following definition is opportune.

Definition 2.1: **LPV stable realization**. Assuming that $d(s,w)$ is a Hurwitz polynomial for all fixed $w \in \mathcal{W}$, the realization (3) of (1) is *LPV stable* if

$$\dot z(t) = F(w(t))\, z(t) \tag{5}$$

is asymptotically stable for any function $w(t)$ taking values in $\mathcal{W}$.

As will be shown soon, finding an LPV stable realization, if any, of a transfer function with Hurwitz denominator is very useful in parametric control design. Consider now a strictly–proper *LPV plant* described by

$$\begin{array}{rcl} \dot x(t) & = & A(w)\, x(t) + B(w)\, u(t), \\ y(t) & = & C(w)\, x(t), \end{array} \tag{6}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$, and $w \in \mathcal{W}$, with $A(\cdot), B(\cdot), C(\cdot)$ continuous functions of w and $\mathcal{W}$ compact. Let (1) be the transfer function of a parametric compensator ensuring that the *closed–loop* control system for the plant (6) is internally stable for *any constant value of* w.

The following definition will be adopted throughout this paper.

Definition 2.2: **LPV stabilizing controller realization**. The realization (3) is an *LPV stabilizing* realization of the parametric compensator transfer function (1) if the closed–loop control system with the controller realized as in (3) is asymptotically stable when the parameter w varies with time according to any arbitrary function $w(t)$ taking values in the compact set $\mathcal{W}$.

Taking Definition 2.2 into account, the main problem to be solved can be stated as follows.

Problem 2.1: **LPV synthesis**. Given a plant (6) and a parametric compensator transfer function (1) ensuring internal closed-loop stability for all constant values of $w \in \mathcal{W}$, find an LPV stabilizing controller realization (3).

A. Motivation

Problem 2.1 is motivated by the fact that the closed-loop LPV system may be unstable if the parametric compensator transfer function is not realized properly. To show this, consider the simple fluid plant represented in Fig. 1. Its (stable) first-order model is

$$\dot y(t) = -\alpha\, y(t) + w(t)\, u(t), \;\; \alpha > 0 \tag{7}$$

where the state/output $y(t) = h(t) - \bar h$ is the deviation of the actual level $h(t)$ in the reservoir from the equilibrium level $\bar h$, the plant input $u(t)$ is the valve opening, $p(t)$ is the (measured) pressure in the tank, $h_1(t)$ is the level of the fluid in the tank and $w(t) = \rho\sqrt{p(t)/\gamma + h_1(t)}$, with γ denoting the fluid density and with ρ constant. Assume that $0 < w^- \le w(t) \le w^+$, $\forall t$, and that the feedback controller transfer function is

$$-\frac{\kappa(w)}{s+\beta}, \;\; \beta > 0, \tag{8}$$

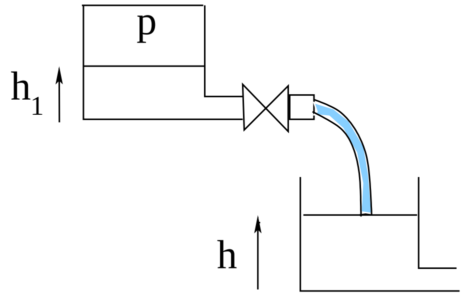

Fig. 1. Simple fluid–flow plant.

with

$$\kappa(w) = \frac{\kappa_0}{w}, \;\; \kappa_0 > 0\,, \tag{9}$$

to compensate for pressure variations.
When w is constant, the feedback control system is internally stable with characteristic polynomial:

$$d(s) = s^2 + (\beta+\alpha)s + \alpha\beta + \kappa_0. \tag{10}$$

The controller (8) admits the two realizations:

$$\Sigma_1(w) = \left[\begin{array}{c|c} -\beta & 1 \\ \hline -\kappa_0/w & 0 \end{array}\right], \;\; \Sigma_2(w) = \left[\begin{array}{c|c} -\beta & -\kappa_0/w \\ \hline 1 & 0 \end{array}\right], \tag{11}$$

yielding, respectively, the closed–loop system matrices:

$$A_1(w) = \begin{bmatrix} -\alpha & -\kappa_0 \\ 1 & -\beta \end{bmatrix}, \;\; A_2(w) = \begin{bmatrix} -\alpha & w \\ -\kappa_0/w & -\beta \end{bmatrix}. \tag{12}$$

$\Sigma_1(w)$ leads to an asymptotically stable feedback system independently of how w varies with time, whereas $\Sigma_2(w)$ may not. Indeed, if α and β are small enough and w varies over a sufficiently large range, the system may be unstable [16]. In [15] this situation has been attributed to the existence of "hidden coupling terms". Conditions ensuring that such terms do not exist have been derived for the case in which the LPV system arises from the linearization of a nonlinear plant (see Theorem 10 in [19]). It will be shown later that framing the design problem as an LPV gain-scheduling problem avoids this issue.

III. Preliminary results

This section extends to LPV systems the results obtained in [6] and [13] for linear switching systems. To this purpose, some stabilizability and quadratic stabilizability conditions will be derived. Either of the following assumptions will be made.

Assumption 1: There exist two positive–definite matrices P and Q, and two matrices $U(w)$ and $Y(w)$ dependent on parameter w such that

$$PA(w)^T + A(w)P + B(w)U(w) + U(w)^T B(w)^T < 0\,, \tag{13}$$

$$A(w)^T Q + QA(w) + Y(w)C(w) + C^T(w)Y(w)^T < 0\,, \tag{14}$$

where $A(w)$, $B(w)$ and $C(w)$ are defined by (6).

Assumption 1 is standard in quadratic stabilizability studies (see, e.g., [2]). When B and C are constant, conditions (13) and (14) correspond to linear matrix inequalities (LMI's). Unfortunately, quadratic stabilizability is restrictive; conditions that remove conservatism will be given soon.
To state the second assumption, the following definition is needed.

Definition 3.1: A matrix $M(w)$, continuous function of $w \in \mathcal{W}$, $\mathcal{W}$ compact, is of class $\mathcal{H}_1$ if there exists $\tau > 0$ such that $\|I + \tau M(w)\|_1 < 1$ for all $w \in \mathcal{W}$. Similarly, $M(w)$ is of class $\mathcal{H}_\infty$ if there exists $\tau > 0$ such that $\|I + \tau M(w)\|_\infty < 1$ for all $w \in \mathcal{W}$.

Remark 3.1: The condition $M \in \mathcal{H}_1$, respectively, $M \in \mathcal{H}_\infty$, is equivalent to the fact that the measure μ_1, respectively, μ_∞, reported in [5] satisfies the constraint $\mu_1(M) < 1$, respectively, $\mu_\infty(M) < 1$ (see Exercise 16, p. 147, in [5]), which means that a systems with such an M as the state matrix is LPV stable.

Assumption 2: There exist a full row rank $n \times \mu$ matrix X, a full column rank $\nu \times n$ matrix R, as well as an $m \times \mu$ matrix $U(w)$, a $\nu \times p$ matrix $L(w)$ and matrices $P(w) \in \mathcal{H}_1$ and $Q(w) \in \mathcal{H}_\infty$ dependent on w such that

$$A(w)X + B(w)U(w) = XP(w)\,, \tag{15}$$

$$RA(w) + L(w)C(w) = Q(w)R\,. \tag{16}$$

The importance of the previous assumptions is illustrated by the following theorem.

Theorem 3.1: The following two conditions are equivalent:

i) the inequalities (13) and (14) of Assumption 1 are satisfied;

ii) the LPV plant (6) is *quadratically* stabilizable by means of a compensator of the form (3).

The following two conditions are also equivalent:

iii) the equations (15) and (16) of Assumption 2 are satisfied;

iv) the LPV plant (6) is stabilizable by means of a compensator of the form (3) and the closed–loop system admits a *polyhedral* Lyapunov function.

Proof: The equivalence of i) and ii) can be proven as in [2] to which the reader is referred. Note only that, if these conditions are satisfied, the observer–based controller described by

$$\begin{aligned}\frac{d}{dt}\hat{x}(t) &= [A(w)+L(w)C(w)+B(w)J(w)]\hat{x}(t)-L(w)y(t), &(17)\\ u(t) &= J(w)\hat{x}(t)+v(t), &(18)\\ v(t) &= 0, &(19)\end{aligned}$$

with $J(w)=-B^T(w)P$ and $L(w)=-QC^T(w)$, is a quadratically stabilizing compensator [1].

To prove that iv) implies iii), assume that a stabilizing compensator exists. The related stable closed–loop system matrix is

$$\begin{bmatrix} A(w)+B(w)K(w)C(w) & B(w)H(w) \\ G(w)C(w) & F(w) \end{bmatrix}. \tag{20}$$

Since the system stability implies the existence of a polyhedral Lyapunov function, the following equation holds for some $P(w)\in\mathcal{H}_1$ (see [5] for details):

$$\begin{bmatrix} A(w)+B(w)K(w)C(w) & B(w)H(w) \\ G(w)C(w) & F(w) \end{bmatrix}\begin{bmatrix} X \\ Z \end{bmatrix}=\begin{bmatrix} X \\ Z \end{bmatrix}P(w) \tag{21}$$

whose upper block row yields

$$A(w)X+B(w)K(w)C(w)X+B(w)H(w)Z=A(w)X+B(w)U(w)=XP(w), \tag{22}$$

where $U(w)=K(w)C(w)X+H(w)Z$, which proves (15). The necessity of (16) can be proven in a similar way by duality [4].

To prove that iii) implies iv), assume that (15) and (16) are satisfied, and consider the compensator of order $\nu+\mu-n$ described by

$$\begin{aligned}\dot{r}(t) &= Q(w)r(t)-L(w)y(t)+RB(w)u(t), &(23)\\ \hat{x}(t) &= Mr(t), &(24)\\ \dot{z}(t) &= F_{SF}(w)z(t)+G_{SF}(w)\hat{x}(t), &(25)\\ u(t) &= H_{SF}(w)z(t)+K_{SF}(w)\hat{x}(t)+v(t), &(26)\\ v(t) &= 0, &(27)\end{aligned}$$

where v will be used later [1], matrix M is any left inverse of R, i.e.,

$$MR=I, \tag{28}$$

and $F_{SF}(w), G_{SF}(w), H_{SF}(w), K_{SF}(w)$ can be computed from

$$\begin{bmatrix} K_{SF}(w) & H_{SF}(w) \\ G_{SF}(w) & F_{SF}(w) \end{bmatrix}=\begin{bmatrix} U(w) \\ V(w) \end{bmatrix}\begin{bmatrix} X \\ Z \end{bmatrix}^{-1}, \tag{29}$$

where Z is any complement of X that makes the square matrix $\begin{bmatrix} X \\ Z \end{bmatrix}$ invertible and

$$V(w)\doteq ZP(w). \tag{30}$$

Letting

$$s(t)\doteq R\;x(t)-r(t) \tag{31}$$

[1]The reason for introducing the dummy signal $v(t)=0$ will become clear later.

and choosing $[x^T \ z^T \ s^T]^T$ as the state vector, after simple manipulations the following closed–loop matrix is obtained:

$$\left[\begin{array}{cc|c} A(w)+B(w)K_{SF}(w) & B(w)H_{SF}(w) & -B(w)K_{SF}(w)M \\ G_{SF}(w) & F_{SF}(w) & -G_{SF}(w)M \\ \hline 0 & 0 & Q(w) \end{array}\right]. \tag{32}$$

The LPV system is stable if and only if the diagonal blocks of the block–triangular state matrix (32) are LPV stable (see Part I of [18]). Now, the upper left diagonal block is LPV stable because it satisfies equation (21), while the lower right block $Q(w(t)) \in \mathcal{H}_\infty$ is the state matrix of an LPV stable system, according to Remark 3.1. ■

The conditions of Theorem 3.1 are numerically hard, in general. However, if $A(w)$ has the polytopic structure:

$$A(w) = \sum_{i=1}^{s} A_i w_i, \quad w_i \geq 0, \quad \sum_{i=1}^{s} w_i = 1, \tag{33}$$

for some integer s, and B and C are constant matrices, the algorithms suggested in [7], [5] can profitably be used to compute X and R.
The following result, taken from [6], will be used later.

Proposition 3.1: Assume that either pair of stabilizability conditions of Theorem 3.1 is satisfied and let $W(s,w)$ be the transfer function of a compensator ensuring that the closed loop is internally stable *for fixed* w. Then, $W(s,w)$ can be realized in the form (17) – (19) or, respectively, in the form (23) – (27) with (19) (respectively, (27)) replaced by

$$v(s) = T(s,w)[C(w)\hat{x}(s) - y(s)] = T(s,w)\, o(s), \tag{34}$$

where $T(s,w)$ is a stable transfer function (Youla–Kucera parameter [14], [27]) and $o(s) = C(w)\hat{x}(s) - y(s)$.

The structure of the resulting compensator is shown in Fig.2.

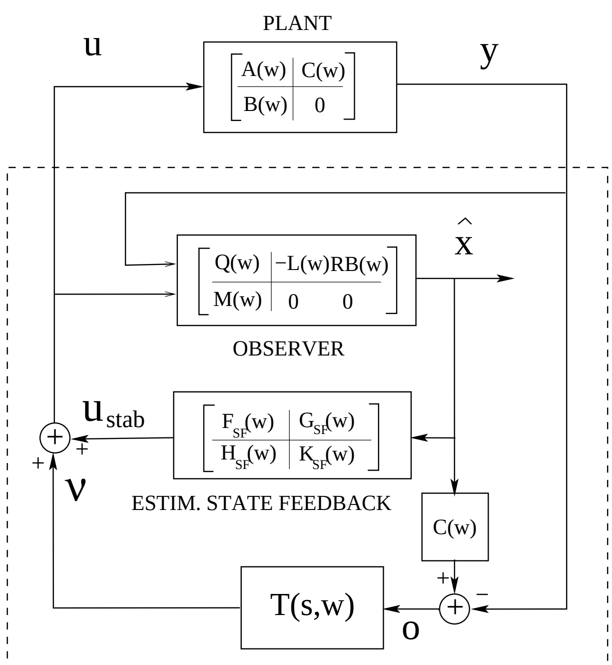

Fig. 2. Observer-based controller parametrization.

Remark 3.2: The compensator structure represented in Fig. 2 contains a generalized observer as defined in [17], [26]. A special compensator with this structure is obtained under the assumption of quadratic stabilizability for $Q(w) = A(w) + B(w)K(w) + L(w)C(w)$, and $R = M = I$ with a merely static estimated–state compensator $K(w)$.

IV. LPV STABLE REALIZATION OF A PARAMETRIC TRANSFER FUNCTION

In the sequel, we will exploit the fact that the closed-loop system in Fig. 2 is LPV stable if we take an LPV stable realization of the block denoted by $T(s,w)$. To this aim the following definition is useful.

Definition 4.1: **Stable regular parametric (SRP) transfer function.** A proper rational transfer function $N(s,w)/d(s,w)$, where N is a $p \times m$ polynomial matrix in s, $d(s,w)$ a monic polynomial of degree ν in s, with N and d continuous functions of $w \in \mathcal{W} \subset \mathbb{R}^s$ with $\mathcal{W}$ an assigned compact set, is called a *stable regular parametric* (SRP) transfer function if $d(s,w)$ is Hurwitz for all $w \in \mathcal{W}$.

It will be proved presently that the next procedure provides an LPV stable realization of an SRP transfer function.

Procedure 4.1:

1) Take any realization $\{\tilde{F}(w), \tilde{G}(w), \tilde{H}(w), \tilde{K}(w)\}$ of the SRP transfer function $N(s,w)/d(s,w)$, so that

$$N(s,w)/d(s,w) = \tilde{H}(w)(I - \tilde{F}(w))^{-1}\tilde{G}(w) + \tilde{K}(w). \tag{35}$$

where $F(w)$ is a Hurwitz matrix continuous in w for all $w \in \mathcal{W}$.

2) Compute the positive–definite solution $P(w)$ of the Lyapunov equation:

$$\tilde{F}^T(w)P(w) + P(w)\tilde{F}(w) = -I. \tag{36}$$

3) Factorize $P(w)$ as

$$P(w) = R^T(w)R(w), \tag{37}$$

where $R(w)$ is an upper triangular matrix (Cholesky's decomposition).

4) Realize the given SRP function according to

$$\begin{array}{rcl} \dot{z}(t) &=& F(w)z(t) + G(w)u(t), \\ y(t) &=& H(w)z(t) + K(w)u(t), \end{array} \tag{38}$$

where $F(w) = R(w)\tilde{F}(w)R^{-1}(w)$, $G(w) = R(w)\tilde{G}(w)$ $H(w) = \tilde{H}(w)R^{-1}(w)$ and $K(w) = \tilde{K}(w)$.

The following result holds.

Theorem 4.1: Any SRP transfer function admits an LPV stable realization computable by means of Procedure 4.1.

Proof: Consider an arbitrary SRP transfer function $N(s,w)/d(s,w)$ and, for any fixed $w \in \mathcal{W}$, realize it according to Procedure 4.1. To prove that this realization remains stable under arbitrary variations $w(t)$ of w, note that

$$R^{-T}(w)\tilde{F}^T(w)R^T(w)R(w)R^{-1}(w) + R^{-T}(w)R^T(w)R(w)\tilde{F}(w)R^{-1}(w) = -R^{-T}(w)R^{-1}(w). \tag{39}$$

Therefore

$$F^T(w)I + IF(w) = -\hat{P}^{-1}(w), \tag{40}$$

where $\hat{P}(w) = R(w)R^T(w) = R(w)[R^T(w)R(w)]R(w)^{-1}$ is similar to $P(w)$.
By continuity, $\hat{P}^{-1}(w)$ is positive definite and lower bounded, that is,

$$\hat{P}^{-1}(w) \geq \beta I \tag{41}$$

for some positive β, which means that $\dot{z}(t) = F(w)\,z(t)$ is quadratically stable since it admits the identity matrix as a Lyapunov matrix. ■

Remark 4.1: To avoid confusion, observe that the matrix $R(w)$ cannot the thought of as a state transformation matrix when w varies with time.

A. Implementation

To make Procedure 4.1 amenable to online implementation, Step 3 uses Cholesky's factorization rather than the more natural product of square roots $P^{1/2}$ adopted in [6]. In this way, the number of flops required to find the LPV stable realization can be estimated. Precisely, given the realization (35) of order ν, the Lyapunov equation (36) entails solving a set of linear equations in $(\nu+1)\nu/2$ unknowns, which requires about $((\nu+1)\nu/2)^3/3$ flops. Cholesky's algorithm to find R in (37) requires $\nu^3/3$ flops. The determination of F from $FR = R\tilde{F}$ requires about ν^3 flops. Finally, the computation of $G = R\tilde{G}$ requires $p \times \nu^2/2$ flops and that of H from $RH = \tilde{H}$ requires $m \times \nu^2/2$ flops. All of these operations have to be performed within the sampling time, which can be accomplished using current technology.

Clearly, simpler solutions are possible in particular cases. For instance, if the Hurwitz denominator $d(s,w)$ of the transfer function is expressed in the factored form:

$$d(s,w) = \prod_i [s + \lambda_i(w)] \quad \prod_j [s^2 + 2\sigma_j(w)s + \sigma_j^2(w) + \omega_j^2(w)], \tag{42}$$

where $\lambda_i(w)$ and $\sigma_j(w)$ are positive, the transfer function can be expanded into partial fractions as

$$\frac{N(s,w)}{d(s,w)} = D + \sum_i \frac{\hat{\alpha}_i}{s + \lambda_i(w)} \; + \sum_j \frac{\hat{\beta}_j s + \hat{\gamma}_j}{s^2 + 2\sigma_j(w)s + \sigma_j^2(w) + \omega_j^2(w)}, \tag{43}$$

whose numerators coefficients $\hat{\alpha}_i$, $\hat{\beta}_j$, $\hat{\gamma}_j$ can be computed by solving linear equations. In this case, an LPV stable realization is provided by

$$\begin{aligned} F(w) &= \text{block–diag}\left\{-\lambda_i(w), \begin{bmatrix} -\sigma_j(w) & \omega_j(w) \\ -\omega_j(w) & -\sigma_j(w) \end{bmatrix}\right\}, && (44) \\ G(w) &= [\ 1 \quad 1 \quad \ldots\ [\ 0 \qquad 1\ ]\ [\ 0 \quad 1\ ]\ \ldots\ ]^T, && (45) \\ H(w) &= [\ \alpha_1\ \alpha_2\ \ldots\ [\ \beta_1 \quad \gamma_1\ ]\ [\ \beta_2 \quad \gamma_2\ ]\ \ldots\ ], && (46) \\ K(w) &= D, && (47) \end{aligned}$$

where α_i, β_j and γ_j are simply related to $\hat{\alpha}_i$, $\hat{\beta}_j$ and $\hat{\gamma}_j$.

V. Parametrization of all LPV stabilizing compensator realizations

A. Controller parametrization: the case of LPV stable plants

If the plant to be controlled is LPV stable as in the example illustrated at the end of Section II, the required controller realization can be structured as in Fig. 3, where $T(s, w)$ is a stable transfer function (Youla–Kucera parameter).

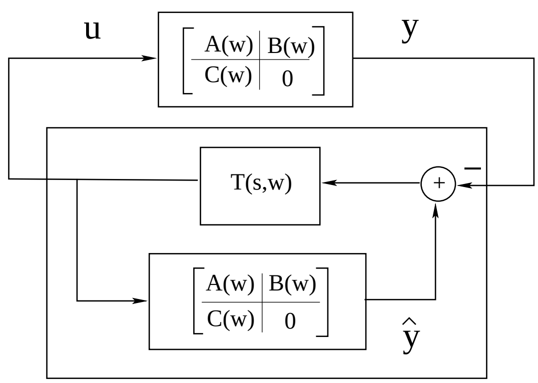

Fig. 3. Structure of an LPV stable plant.

By realizing $T(s, w)$ according to (2), the state–space representation of the closed–loop system becomes:

$$\begin{aligned} \frac{d}{dt}x(t) &= A(w)x(t) + B(w)u(t), && (48) \\ \frac{d}{dt}\hat{x}(t) &= A(w)\hat{x}(t) + B(w)u(t), && (49) \\ \frac{d}{dt}z(t) &= F(w)z(t) + G(w)(C(w)\hat{x}(t) - y(t)), && (50) \\ u(t) &= H(w)z(t) + K(w)(C(w)\hat{x}(t) - y(t)). && (51) \end{aligned}$$

Letting $e(t) \doteq \hat{x}(t) - x(t)$ and choosing $[x^T(t)\ z^T(t)\ e^T(t)]^T$ as the state vector, the state matrix of the closed–loop system takes the form:

$$\begin{bmatrix} A(w) & B(w)H(w) & B(w)K(w)C(w) \\ 0 & F(w) & G(w)C(w) \\ 0 & 0 & A(w) \end{bmatrix}. \tag{52}$$

Therefore the closed loop is LPV stable as long as $F(w(t))$ is stable.

Since the structure in Fig. 3 parametrizes all stabilizing compensators (see p. 67 of [20]), when the plant is LPV stable, Problem 2.1 admits a solution for any stabilizing controller transfer function $W(s, w)$. According to Fig. 3, $W(s, w)$ can be written as

$$W(s, w) = -[I - T(s, w)C(w)(sI - A(w))^{-1}B(w)]^{-1}T(s, w), \tag{53}$$

so that the Youla–Kucera parameter is

$$T(s, w) = W(s, w)[C(w)(sI - A(w))^{-1}B(w)W(s, w) + I]^{-1} \tag{54}$$

which must be realized properly to ensure LPV stability.

B. Controller parametrization: the general case

Problem 2.1 can be solved, provided the plant satisfies the stabilizability conditions of Section III, as stated by the following theorem that draws on [6] with some minor technical adjustment.

Theorem 5.1: If the plant (6) satisfies the stabilizability conditions (15) – (16) of Assumption 2, Problem 2.1 (LPV synthesis) is solvable for an arbitrary stabilizing controller transfer function $W(s,w)$. Moreover, if the plant satisfies the quadratic stabilizability conditions (13) – (14) of Assumption 1, Problem 2.1 is solvable with a compensator of order $2n+\nu$, where n is the plant state dimension and ν the degree of the denominator $d(s,w)$ of $W(s,w)$.

Proof: In view of Proposition 3.1, any compensator transfer function $W(s,w)$ can be given the structure of Fig. 2. To prove LPV stability, it is enough to observe that, according to Theorem 4.1, an LPV stable realization of the Youla–Kucera parameter $T(s,w)$ can always be found starting from any asymptotically stable realization of $T(s,w)$ using Procedure 4.1. ■

The practical implementation of the LPV stabilizing compensator requires the online realization of the Youla–Kucera parameter from the compensator transfer function. In the case of quadratic stabilizability, the following result simplifies this task.

Theorem 5.2: Assume that the plant $\{A(w),B(w),C(w)\}$ satisfies the quadratic stabilizability conditions of Assumption 1 and that a state–space realization $\{F(w),G(w),H(w),K(w)\}$ of the compensator is given. Then, an LPV stabilizing compensator realization solving Problem 2.1 is

$$\left[\begin{array}{cc|c} A(w)+B(w)J(w)+L(w)C(w)-B(w)K^{(T)}(w)C(w) & B(w)H^{(T)} & B(w)K^{(T)}(w)-L(w) \\ -G^{(T)}(w)C(w) & F^{(T)}(w) & G^{(T)}(w) \\ \hline J(w)-K^{(T)}(w)C(w) & H^{(T)}(w) & K^{(T)}(w) \end{array}\right] \tag{55}$$

with $J(w)=-B^T(w)P$ and $L(w)=-QC^T(w)$, as in Theorem 3.1, and the Youla–Kucera parameter realized as

$$\left[\begin{array}{c|c} F^{(T)}(w) & G^{(T)}(w) \\ \hline H^{(T)}(w) & K^{(T)}(w) \end{array}\right] = \\ \left[\begin{array}{cc} R(w) & 0 \\ 0 & I \end{array}\right]\left[\begin{array}{cc|c} A(w)+B(w)K(w)C(w) & B(w)H(w) & B(w)K(w)-L(w) \\ G(w)C(w) & F(w) & G(w) \\ \hline -J(w)+K(w)C(w) & H(w) & K(w) \end{array}\right]\left[\begin{array}{cc} R^{-1}(w) & 0 \\ 0 & I \end{array}\right], \tag{56}$$

where $R(w)$ is computed according to Procedure 4.1.

Proof: By combining (55) and (56) with $R(w)=I$ and dropping the argument w to simplify notation, the compensator turns out to be

$$\left[\begin{array}{ccc|c} A+LC+BJ-BKC & -BJ+BKC & BH & -L+BK \\ LC-BKC & A+BKC & BH & -L+BK \\ -GC & GC & F & G \\ \hline J-KC & -J+KC & H & K \end{array}\right]. \tag{57}$$

Denote the state vector for (57) by $[\hat{x}^T(t) \;\; \xi^T(t) \;\; \eta^T(t)]^T$. Consider the equivalent compensator realization by taking the new state vector $[(\hat{x}(t)-\xi(t))^T \;\; \xi^T(t) \;\; \eta^T(t)]^T$. The compensator realization becomes

$$\left[\begin{array}{ccc|c} A+BJ & 0 & 0 & 0 \\ LC-BKC & A+LC & BH & -L+BK \\ -GC & 0 & F & G \\ \hline J-KC & 0 & H & K \end{array}\right] \tag{58}$$

leading, for any fixed $w \in \mathcal{W}$, to the same transfer function as $\{F,G,H,K\}$ since the variables $\hat{x}(t)-\xi(t)$ and $\xi(t)$ are unreachable and unobservable, respectively.

Owing to the particular structure of (56), the same compensator transfer function is obtained for any arbitrary $R(w)$. Then we can prove stability by exploiting Theorem 4.1 and computing $R(w)$ as in Procedure 4.1. Essentially, the LPV stability of the overall closed–loop is assured if we take an LPV stable realization of the Youla–Kucera parameter,$\{F^{(T)},G^{(T)},H^{(T)},K^{(T)}\}$. Indeed, the closed–loop system is LPV stable if $F^{(T)}$ is LPV stable. In fact, with reference to the state vector $[(\hat{x}(t)-\xi(t))^T \;\; \xi^T(t) \;\; x^T(t) \;\; \eta^T(t)]^T$, the overall system matrix is

$$\left[\begin{array}{cc} I_{2n} & 0 \\ 0 & R \end{array}\right]\left[\begin{array}{cccc} A+BJ & 0 & 0 & 0 \\ LC-BKC & A+LC & (-L+BK)C & BH \\ B(J-KC) & 0 & A+BKC & BH \\ -GC & 0 & GC & F \end{array}\right]\left[\begin{array}{cc} I_{2n} & 0 \\ 0 & R^{-1} \end{array}\right]. \tag{59}$$

It follows that the system is LPV stable if and only if the three blocks:

$$A+BJ, \qquad A+LC, \qquad R\left[\begin{array}{cc} A+BKC & BH \\ GC & F \end{array}\right]R^{-1} \tag{60}$$

are LPV stable [18]. Now, the first two blocks are stable by construction, while the third block is stable since, in view of (40), all the matrices share the Lyapunov function $\|\cdot\|_2^2$. ■

Remark 5.1: Often the plant parameters are constant most of the time and are subject to variations only occasionally. In these cases, it is reasonable to design the controller in such a way that it is optimal for the prevailing parameter values but ensures LPV stability too. Optimality can be achieved, e.g., by determining the Youla–Kucera parameter according to the Wiener–Hopf design [27]. The only additional requirement is that the Youla-Kucera parameter be properly realized.

VI. Applications and examples

A. Fixed pole assignment

Consider the case in which the compensator transfer function is designed so as to locate the closed–loop poles in the same place for all (constant) values of the parameters, which means that

$$\Sigma_{CL}(w) = \left[\begin{array}{c|c} A(w) + B(w)K(w)C(w) & B(w)H(w) \\ \hline G(w)C(w) & F(w) \end{array}\right] \tag{61}$$

must exhibit fixed poles. In view of equation (56), this specification is equivalent to the requirement that the Youla–Kucera parameter has fixed poles. To this purpose, use can be made of the procedures illustrated in Section IV. For simplicity, assume that the desired closed–loop eigenvalues are distinct. Then, the transformation matrix $R(w)$ in equation (56) can be chosen in such a way that $F^{(T)}(w) = F^{(T)} = const$. The following proposition formalizes this result.

Proposition 6.1: Assume that the plant is LPV stable and that the assigned compensator transfer function $N(s,w)/d(s)$ allocates the poles of the closed–loop transfer function in fixed positions. Then, an LPV stabilizing realization of the form (3) with F constant can be obtained for this compensator.

If the plant is LPV stable, the compensator can be structured as in Fig. 3 and the sensitivity function can be written as

$$S(s,w) = 1 - [C(w)(sI - A(w))^{-1}B(w)]T(s,w). \tag{62}$$

Pole assignment can be achieved by making $T(s,w)$ cancel all of the plant eigenvalues and replace them by its own poles. In this way, stability is obviously guaranteed for every constant value of w. To achieve LPV stability, $T(s,w)$ must be realized properly. Since the poles of $T(s,w)$ are fixed, we can just realize it by taking a constant state matrix F so that no transformation $R(w)$ is needed.

Example 6.1: **Flow control revisited**

Consider the hydraulic plant in Fig. 1. The compensator (8) locates the close–loop poles at the roots of (10). An *ad-hoc* LPV stabilizing realization was already derived at the end of Subsection II-A. However, the controller transfer function (8) can also be realized from (53) and (54) with

$$T(s,w) = \frac{\kappa(w)(s+\alpha)}{(s+\alpha)(s+\beta) + \kappa(w)w} = \frac{(\kappa_0/w)(s+\alpha)}{(s+\alpha)(s+\beta) + \kappa_0} \tag{63}$$

leading to a compensator that ensures closed–loop LPV stability for the desired poles.

B. Online tuning for LTI plants

The procedures conceived for LPV plants can profitably be used to synthesize parametric compensators for LTI plants whose parameters are known accurately. Still, to improve the system performances, some controller parameter $w \in \mathcal{W}$ can be tuned online according to the scheme of Fig. 4.

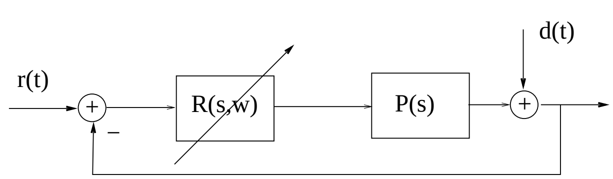

Fig. 4. LTI plant with LPV compensator.

Assume that internal stability is ensured for each *constant value of the parameter* $w \in \mathcal{W}$. The main question is whether system stability can be ensured *even under tuning*. An answer is provided by the following result.

Corollary 6.1: Assume that $P(s)$ is a stabilizable (in the LTI sense) and $R(s,w)$ is a family of compensator transfer functions, parametrized with respect to w, that ensure the internal stability of the closed-loop system for all constant $w \in \mathcal{W}$. Then, there always exists a realization of $R(s,w)$ that guarantees LPV stability too.

Proof: It is enough to observe that (13) and (14) are always satisfied in the case of LTI plants. ■

Remark 6.1: The previous result is valid no matter how the parameter tuning is carried out. It may be performed by a human operator or by an adaptive device.

Example 6.2: Let $P(s)$ in Fig. 4 be

$$P(s) = \frac{1}{s+\mu}. \tag{64}$$

To (partly) suppress a disturbance $d(t)$ with a (dominant) sinusoidal component of frequency ω_0, the controller transfer function can be chosen as

$$R(s) = \frac{\kappa\omega^2(s+\alpha)}{s^2+2\xi\omega s+\omega^2} \tag{65}$$

with $\omega = \omega_0$ and ξ small.
If the disturbance frequency varies in a range $\omega^- \le \omega_0 \le \omega^+$ (as is often the case in practice), the parameter ω in (65) must be adjusted accordingly (which, of course, requires measuring the disturbance frequency) and the control system becomes an LPV system whose stability depends on the controller realization. By realizing the controller in companion form, the state, input and output matrices of the resulting closed–loop system turn out to be:

$$\left[\begin{array}{c|c} A_{CL} & B_{CL} \\ \hline C_{CL} & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -\mu & \kappa\omega^2\alpha & \kappa\omega^2 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -\omega^2 & -2\xi\omega & 1 \\ \hline 1 & 0 & 0 & 0 \end{array}\right]. \tag{66}$$

Assume for simplicity $\alpha = \mu$ in (65) to cancel the pole of (64). Then the realization is nonminimal and, indeed, equivalent to

$$\left[\begin{array}{c|c} A_{CL} & B_{CL} \\ \hline C_{CL} & 0 \end{array}\right] = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -(\kappa+1)\omega^2 & -2\xi\omega & 1 \\ \hline \kappa\omega^2 & 0 & 0 \end{array}\right]. \tag{67}$$

It is known that this system is unstable for ξ small and w varying with time. However, the compensator can be realized in such a way that, for any fixed w, its transfer function matches the desired one and, at the same time, LPV stability is guaranteed. Since the plant is LPV stable, the theory of Subsection V-A applies. In particular, the compensator can be given the form:

$$R(s,w) = -\frac{T(s,w)}{1-T(s,w)P(s)}, \tag{68}$$

from which:

$$T(s,w) = \frac{R}{1+RP} = \frac{\kappa\omega^2(s+\mu)}{s^2+2\xi\omega s+(1+\kappa)\omega^2}. \tag{69}$$

Therefore the problem can be solved by finding an LPV stable realization of $T(s,w)$. According to Subsection IV-A and assuming complex eigenvalues, one such realization is

$$\Sigma^{(T)} = \left[\begin{array}{c|c} F^{(T)} & G^{(T)} \\ \hline H^{(T)} & 0 \end{array}\right] = \left[\begin{array}{cc|c} -\xi\omega & \omega\sqrt{1+\kappa-\xi^2} & 0 \\ -\omega\sqrt{1+\kappa-\xi^2} & -\xi\omega & 1 \\ \hline h_1(w) & h_2(w) & 0 \end{array}\right], \tag{70}$$

where $h_1(w)$, $h_2(w)$ are given by

$$h_1(w) = \kappa\omega(\mu-\xi\omega)/\sqrt{1+\kappa-\xi^2}, \qquad h_2(w) = \kappa\omega^2. \tag{71}$$

The resulting controller realization as in (68)is thus

$$\begin{array}{rcl} \dot{x}_c & = & \left[\begin{array}{ccc} -\xi\omega & \omega\sqrt{1+\kappa-\xi^2} & 0 \\ -\omega\sqrt{1+\kappa-\xi^2} & -\xi\omega & 1 \\ h_1(w) & h_2(w) & -\mu \end{array}\right] x_c + \left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right] e \\ u & = & \left[\begin{array}{ccc} h_1(w) & h_2(w) & 0 \end{array}\right] e \end{array} \tag{72}$$

In figure 5 the evolution of the output y_{lpv} with the so realized controller and with the same controller realized in minimal form (y_{min}) are depicted for $\mu = 1$, $\kappa = 10$ and $\xi = 0.01$. It is apparent how the latter leads to instability (note that the size scale for y_{lpv} and y_{min} are quite different). The tuning parameter time evolution ω is also depicted.

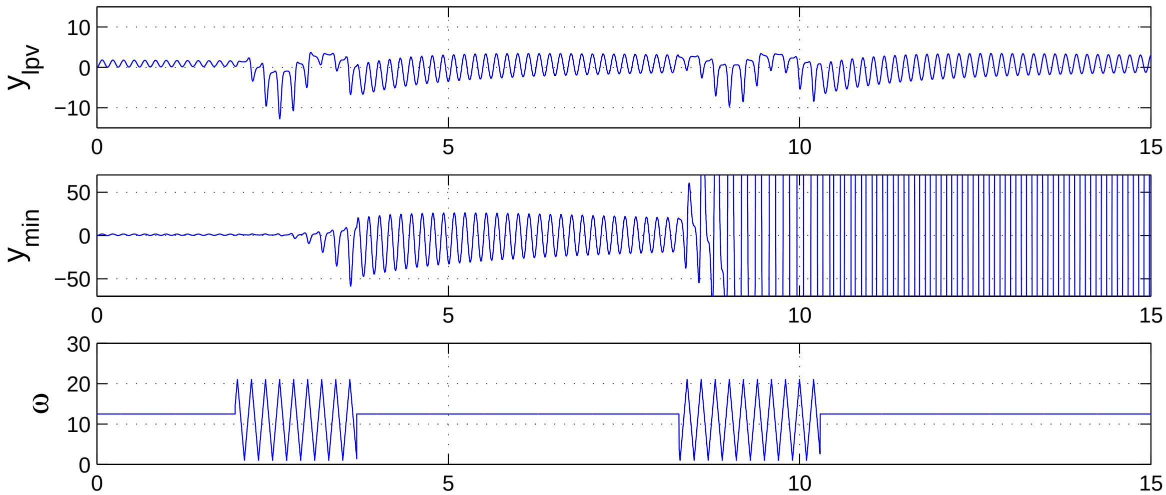

Fig. 5. y_{lpv}: ouput with the proposed realization; y_{min}: ouput corresponding to the unstable closed–loop produced by the minimal realization of the compensator; ω: tuning parameter subject to a piecewise sinusoidal variation.

C. LPV stability within the Hurwitz region

In most practical cases, given a parametric plant and a parametric compensator, we can establish the region of stability in the parameter space by using standard tools such as the Routh-Hurwitz table. Obviously, the resulting analysis is valid as long as the parameters are constant in time. The present approach allows us to ensure LPV stability for all possible variations within the region of stability.

Consider the following simple plant:

$$\left[\begin{array}{c|c} A(w) & B(w) \\ \hline C(w) & 0 \end{array}\right] = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -(1+\rho w) & -\xi & 1 \\ \hline 1 & 0 & 0 \end{array}\right] \tag{73}$$

with ξ small and $\rho \approx 1$ but $\rho < 1$. The controller must satisfy the following specifications:

- for any fixed value of $w \in [0,1]$ the controller transfer function has the PI structure:

$$W_{comp}(s) = k + \frac{h}{s}; \tag{74}$$

- the closed–loop system is LPV stable when $w = w(t)$ varies with time in the interval $[0,1]$;
- the proportional and integral gains k and h can be changed online within the region for which stability is guaranteed for any fixed w, and the closed–loop system is stable regardless of their variation.

Trivial computations based on the Routh–Hurwitz array ensure that the closed–loop system is stable for every fixed w, $0 \le w \le 1$, if and only if $h > 0$ and $h \le \xi(1+k)$. Therefore, for fixed w the stability region in the (k,h)–plane is:

$$\mathcal{R} = \{h \ge \epsilon \;\text{ and }\; h \le \xi(1+k) - \epsilon\} \tag{75}$$

with ϵ positive and arbitrarily small.
This system is quadratically stabilizable. Possible gains $J(w)$ and $L(w)$ of the observer–based controller (see (17) – (18)) are

$$J(w) = [\rho w \quad -1], \quad L(w) = [0 \quad \rho w]^T. \tag{76}$$

Correspondingly the pre–compensator equations are

$$\frac{d}{dt}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(1+\rho w + k) & -\xi \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \rho w \end{bmatrix} y(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(t), \tag{77}$$

$$u(t) = [\,\rho w \quad -1\,]\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + v(t), \tag{78}$$

$$o(t) = -[\,1 \quad 0\,]\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + y(t), \tag{79}$$

$$v(s) = T(s,w)\,o(s). \tag{80}$$

The Youla–Kucera parameter $T(s,w)$ can be realized as in (56) starting from any realization of the compensator. A simple realization of (74) is given by: $F=0$, $G=1$, $H=h$ and $K=k$. In this case:

$$\left[\begin{array}{c|c} F^{(T)}(w) & G^{(T)}(w) \\ \hline H^{(T)}(w) & K^{(T)}(w) \end{array}\right] = \left[\begin{array}{cc} R(w) & 0 \\ 0 & I \end{array}\right] \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ -(1+\rho w+k) & -\xi & h & -(\rho w-k) \\ 1 & 0 & 0 & 1 \\ \hline -(\rho w-k) & -1 & h & k \end{array}\right] \left[\begin{array}{cc} R^{-1}(w) & 0 \\ 0 & I \end{array}\right], \tag{81}$$

where $R(w)$ is the upper–triangular Cholesky factor of the solution $P=R^TR=[p_{ij}]$ of the Lyapunov equation:

$$(F^{(T)}(w))^TP+PF^{(T)}(w)=I. \tag{82}$$

Let $\alpha=\xi$, $\beta=(1+\rho w+k)$ and $\gamma=h$. Then, (82) entails the solution of the linear set:

$$\left[\begin{array}{cccccc} 0 & 0 & 0 & -2\beta & 0 & -2 \\ 0 & -2\alpha & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2\gamma & 0 \\ 1 & -\beta & -\alpha & 0 & 0 & 1 \\ 0 & 0 & -1 & -\gamma & -\beta & 0 \end{array}\right] \left[\begin{array}{c} p_{11} \\ p_{22} \\ p_{33} \\ p_{12} \\ p_{23} \\ p_{31} \end{array}\right] = \left[\begin{array}{c} -1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{array}\right]. \tag{83}$$

The nonzero entries of $R=[r_{ij}]$ turn out to be

$$r_{11}=\sqrt{p_{11}},\quad r_{12}=p_{12}/r_{11},\quad r_{13}=p_{13}/r_{11},\quad r_{22}=\sqrt{p_{22}-r_{12}^2},\quad r_{23}=(p_{23}-r_{12}r_{13})/r_{22},\quad r_{33}=\sqrt{p_{33}-r_{13}^2-r_{23}^2}.$$

$R^{-1}=[s_{ij}]$ is also upper–triangular. Its nonzero entries given by

$$s_{11}=1/r_{11},\quad s_{22}=1/r_{22},\quad s_{33}=1/r_{33},\quad s_{12}=-r_{12}/(r_{11}r_{22}),\quad s_{13}=-(r_{22}s_{12}+r_{13}/r_{33}),\quad s_{23}=-r_{23}/(r_{22}r_{33}).$$

The preceding operations can easily be implemented online because their computational burden is small.

VII. Conclusions

Necessary and sufficient conditions for the existence of an LPV stabilizing compensator have been provided. It has been shown that, if these conditions are satisfied, any given parametric gain–scheduled transfer function ensuring internal stability for constant values of the parameters admits an LPV stabilizing realization (which is nonminimal, in general). A procedure to construct a realization of this kind has been described. Under the assumption of quadratic stabilizability, it turns out that the dimension of an LPV stabilizing compensator realization is twice the dimension of the controlled plant plus the order of the compensator transfer function to be realized.

The results lend themselves to interesting extensions. For instance, pointwise optimality can be ensured along with LPV stability. However, nothing can yet be said about time–varying performance. In this respect, the present paper follows a path opposite to that followed in [1][2], where LPV performance has been considered without pointwise optimality.

Since the suggested realization procedure is independent of the design criterion, it can be combined with any synthesis technique. It seems particularly suited to the online tuning of standard controllers.

References

[1] P. Apkarian and P. Gahinet, “A convex characterization of gain–scheduled $\mathcal{H}_\infty$ controllers,” *IEEE Trans. Automat. Contr.*, vol. 40, no. 5, 1995.
[2] G. Becker and A. Packard, “Robust performance of linear parametrically varying systems using parametrically–dependent linear feedback,” *Systems Control Lett.*, vol. 23, pp. 205–215, 1994.
[3] F. Blanchini, “Nonquadratic Lyapunov functions for robust control,” *Automatica J. IFAC*, vol. 31, no. 3, pp. 451–461, 1995.
[4] F. Blanchini and S. Miani, “Stabilization of LPV systems: state feedback, state estimation and duality,” *SIAM J. Control Optim.*, vol. 42, no. 1, pp. 76–97, 2003.
[5] F. Blanchini and S. Miani, *Set-Theoretic Methods in Control*, Birkhauser, Boston, 2008.
[6] F. Blanchini, S. Miani, and F. Mesquine, “A separation principle for linear switching systems and parametrization of all stabilizing controllers,” *IEEE Trans. Automat. Contr.*, vol. 54, no. 2, pp. 279–292, 2009.
[7] F. Blanchini, S. Miani, and C. Savorgnan, “Stability results for linear parameter varying and switching systems,” *Automatica J. IFAC*, vol. 43, no. 10, pp. 1817–1823, 2007.
[8] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia, PA, 2004.
[9] R. K. Brayton and C. H. Tong, “Stability of dynamical systems: a constructive approach,” *IEEE Trans. Circuits Syst.*, vol. 26, no. 4, pp. 224–234, 1979.
[10] ——, “Constructive stability and asymptotic stability of dynamical systems,” *IEEE Trans. Circuits Syst.*, vol. 27, no. 11, pp. 1121–1130, 1980.
[11] E. Feron, P. Apkarian, and P. Gahinet, “Analysis and synthesis of robust control systems via parameter–dependent Lyapunov functions,” *IEEE Trans. Automat. Contr.*, vol. 41, no. 7, pp. 1041–1046, 1996.
[12] A. Helmerson, “μ synthesis and LFT gain scheduling with real uncertainties,” *Int. J. Robust Nonlinear Contr.*, Vol. 8, pp. 631–642, 1998.
[13] J. Hespanha and A. S. Morse, “Switching between stabilizing controllers,” *Automatica J. IFAC*, vol. 38, no. 11, pp. 1905–1917, 2002.
[14] V. Kucera, *Analysis and Design of Discrete Linear Control Systems*, Prentice Hall, Englewood Cliffs, NJ, 1991.
[15] D. A. Lawrence and W. J. Rugh, “Gain scheduling dynamic controllers for a nonlinear plant,” *Automatica*, Vol. 31, no. 3, pp. 381–390, 1995.
[16] D. Liberzon, *Switching in Systems and Control*, Systems and Control: Foundations & Applications, Birkhauser, Boston, 2003.
[17] D. Luenberger, “An introduction to observers,” *IEEE Trans. Automat. Contr.*, vol. 16, no. 6, pp. 596–602, 1971.

[18] A. P. Molchanov and E. S. Pyatnitskii, "Lyapunov functions that define necessary and sufficient conditions for absolute stability of nonlinear nonstationary control systems. I-II-III," *Autom. Remote Control*, vol. 47, pp. I:344–354, II: 443–451, III:620–630, 1986.

[19] W. J. Rugh and J. S. Shamma, "Research on gain scheduling," *Automatica*, vol. 36, no. 10, pp. 1401–1425, 2000.

[20] R. Sanchez-Pena and M. Sznaier, *Robust Systems, Theory and Applications*, Wiley, New York, 1998.

[21] G. Scorletti and L. El Ghaoui, "Improved LMI conditions for gain scheduling and related control problems," *Int. J. Robust Nonlinear Contr.*, vol. 8, pp. 845–877, 1992.

[22] S. M. Shahruz and S. Behtash, "Designing controllers for linear parameter–varying systems by the gain scheduling technique," *J. Math. Anal. Appl.*, Vol. 168, pp. 195–217, 1992.

[23] J. S. Shamma, "Linearization and gain scheduling," in *The Control Handbook*, W. Levine, Ed.,CRC Press, 1996, pp. 388–396.

[24] J. S. Shamma and M. Athans, "Analysis of gain scheduled control for nonlinear plants," *IEEE Trans. Automat. Contr.*, Vol. 35, no. 8, pp. 898–907, 1990.

[25] J. S. Shamma and M. Athans, "Guaranteed properties of gain scheduled control for linear parameter–varying plants," *Automatica*, Vol. 27, no. 3, pp. 559–564, 1991.

[26] C. Tsui, "A new design approach to unknown input observers," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 464–468, 1996.

[27] D. Youla, H. Jabr, and J. Bongiorno, "Modern Wiener–Hopf design of optimal controllers, ii. the multivariable case," *IEEE Trans. Automat. Contr.*, vol. 21, no. 3, pp. 319–338, 1976.

[28] K. Zhou, J. Doyle, and K. Glower, *Robust and Optimal Control*, Prentice Hall, Englewood, Cliffs, NJ, 1996.