**System and Control Theory - Exercises - January 07, 2019**
Some of the proposed problems may have no solution. If so, explain why.

1. **Mechanical device.** Consider the system

$$\dot{x}_1(t) = x_2(t) \quad (1)$$
$$\dot{x}_2(t) = \sin(x_1(t)) + u(t) \quad (2)$$

with $y(t) = x_1(t)$. The matrices $A$, $B$, $C$ of the linearized system at the equilibrium $\bar{x} = [\bar{x}_1, \bar{x}_2]^\top$, $\bar{u}$ are

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad (3)$$

find the values $\bar{x}_1$, $\bar{x}_2$ and $\bar{u}$ (assume $-\pi \leq \bar{x}_1 \leq \pi$).

2. **Assign poles.** Compute the state feedback control $u(t) = Kx(t)$ for the linear system (3) that assigns the closed-loop eigenvalues $\{-1, -1\}$.

3. **Assign zeros.** Compute a state feedback control $u(t) = Kx(t)$ for the linear system (3) so that the transfer function $W(s) = C(sI - (A + BK))^{-1}B$ has a zero in $-1$: $W(-1) = 0$.

4. **Optimal control.** For the linear system (3), compute the optimal control $u(t) = Kx(t)$ with $Q = 3I$ and $R = 1$.

5. **Global convergence.** For the nonlinear system (1)–(2), compute a possibly nonlinear state feedback control $u(t) = \phi(x(t))$, such that $x(t)$ converges to the equilibrium for any initial condition $x(0)$ (globally).

6. **Kalman.** Consider the linear system (3) with the additional presence of disturbances:

$$\dot{x}(t) = Ax(t) + Bu(t) + Ed, \qquad y = Cx(t) + w(t)$$

with $E = \sqrt{3}I$. Find the Kalman filter.

7. **Identification.** Consider the model

$$W(s) = \frac{b}{s + a},$$

where $a$ and $b$ are real numbers, and the measurements $W(0) = 1$, $W(j) = 1/2 - j/2$, $W(j2) = 1/5 - j2/5$. Write the over-determined (to be solved via least-square) system to find $a$ and $b$.

8. **Convexity.** Consider an asymptotically stable system $\dot{x}(t) = f(x(t))$, $f(0) = 0$, such that the corresponding linearized system is asymptotically stable. Consider a quadratic Lyapunov function $V(x) = 2x^\top Pf(x)$, with $P$ positive definite. Show that the set $\mathcal{P}$ of all positive definite matrices $P$ such that $x^\top Pf(x) < 0$ for $x \neq 0$,

$$\mathcal{P} = \{P \ \text{ positive definite}: \ \ x^\top Pf(x) < 0, \quad x \neq 0\},$$

is a convex set.

9. **Reachability with minimum effort.** Consider the discrete-time system

$$x(k+1) = Ax(k) + Bu(k), \quad y(k) = Cx(k)$$

with constraints $a \leq y(k) \leq b$ and $c \leq u(k) \leq d$. Write the constrained optimization problem to determine the control sequence $u(0)$, $u(1)$, $\ldots u(T-1)$, which drives the state from $\bar{x}_0$ to $\bar{x}_T$ (both assigned) in $T$ steps, namely $x(T) = \bar{x}_T$, and minimizes the effort

$$J = \sum_{h=0}^{T-1} \|u(h)\|^2.$$