System and Control Theory - Exercitation - January 22, 2016
Some of the proposed problems could be unsolvable. If so, explain why.

1. **Stabilization.** Consider the chemical system

$$\begin{aligned}\dot{x} &= -xy + u\\ \dot{y} &= -xy + 1\\ \dot{z} &= +xy - z\end{aligned}$$

   Find a state feedback control which asymptotically stabilizes the system at the equilibrium state corresponding to $\bar{y} = 1$.

2. **Observer design.** Consider the previous system. You can put *only one sensor on one of the variables*: a) x; b) y; c) z. Your goal is to design an observer to use along with the feedback computed in Exercise 1, to stabilize the system. Which are good choices among a), b) and c) ?

3. **Bucy–Kalman.** Consider the system $\dot{x}(t) = Ax(t) + Ev(t)$, and $y(t) = Cx(t) + w(t)$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

   where v and w are disturbances. Write the equations of the Bucy–Kalman filter and compute the gain. You have freedom in the choice of the weighting matrices.

4. **Minimum norm flow**. Consider the equations

$$\begin{aligned}u_1 - u_3 &= d_1\\ u_3 + u_2 &= d_2\end{aligned}$$

   where d_1 and d_2 are positive water demands and u_1, u_2 and u_3 are flows to be determined. Find the expression of the flow with mimimum Euclideal norm. If we impose that the flows must be nonnegative, $u_1 \geq 0$, $u_2 \geq 0$ and $u_3 \geq 0$, for which values of $d_1 > 0$ and $d_2 > 0$ this condition is satisfied?

5. **Dual Lyapunov equation.** A linear systems $\dot{x} = Ax$ is asymptotically stable if and only if for any choice of Q symmetric and positive definite, the symmetric solution P of

$$A^\top P + PA = -Q$$

   is positive definite. Prove that the same result ($\tilde{P}$ positive definite for any positive definite Q) is true for the *dual* Lyapunov equation

$$\tilde{P}A^\top + A\tilde{P} = -Q$$

   (note that $\tilde{P}$ and P are different). Please no questions on this exercise.

6. **Robustness** Find the largest value of $\mu \geq 0$ for which $\dot{x} = [A + E]x$ is asymptotically stable for all $\|E\| \leq \mu$ where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

   Please no questions on this exercise.

7. **Constrained stabilization.** Consider the (one dimensional) system

$$\dot{x}(t) = ax(t) + bu(t)$$

   and assume $|u(t)| \leq 1$ is a hard constraint. Assume $a > 0$ (the system is unstable) and $b > 0$. Can we stabilize the system from any arbitrary initial condition $x(0)$? If not, find the interval $[-\mu, \mu]$ of all initial conditions from which we can assure $x(t) \to 0$.