A decentralized solution for the constrained minimum norm flow

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Abstract

This paper deals with finding a minimum norm network flow constrained by capacity bounds and demand balance conditions. We derive a distributed feedback control algorithm which assures stability and produces the minimum–norm flow at steady–state in a decentralized way. The control can also be viewed as a recursive distributed algorithm having the form of a continuous–time dynamical system, which converges to the minimum norm flow. We explore the same decentralized control in the case of non–fully connected networks. Here, the control at steady state minimizes the distance from the unconstrained least square solution thus deviating counter-intuitively from the solution of the constrained least square problem. A third and last scenario is considered where the network has an intrinsic linear dynamics. Even in this case our decentralized control drives the network flow to an equilibrium point and the steady state control still enjoys some optimality property in the least-square sense.

I. Introduction and motivation

Minimum norm flows play a central role in flow network optimization as they are not affected by cycles and therefore can be reviewed as some kind of minimum energy solutions. Constraints for the minimum norm flow are typically due to capacity bounds and demand balance conditions. Minimum norm flows have a nice interpretation in several important applications such as communication [8], [14], water distribution control [11], traffic [9], [13], manufacturing [7], [17].

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In this note, we look for a distributed control which stabilizes the network and assures the constrained minimum flow at steady state. In passing, the idea can be used, by “playing around stability", to develop a distributed and *fully decentralized* algorithm for minimum flow. Decentralization is intended in the sense of networks [9], [10] and is a first challenging aspect. A second element, which gives a special flavor to our problem, is that the demand is unknown and therefore the problem is solved *robustly*. Besides this, such a robust setup allows us to deal with potential network failures. Although the literature on flow networks is quite wide, “robustness” has been brought into the picture only recently [1], [2], [5], [12], [15].

A robust decentralized strategy has been proposed in [6], where the authors present a Lyapunov based control that guarantees robustness against uncertain demand in the presence of both buffer and link constraints. However, [6] is only concerned with stability while here we are mainly concerned with flow optimality. Long–term optimality, in flow networks with uncertain demand has been considered in [4], but the provided solutions are *centralized*.

We can recapitulate the main contributions of this work by saying that, under some necessary and sufficient conditions, a simple linear saturated control is proven to

- drive the buffer level arbitrarily close to an assigned reference level under flow capacity constraints;
- require only local information, namely, each link controls its flow based only on the buffer levels of its extreme nodes and the solution is robust with respect to failures.
- assure minimum–norm steady state flow.

We also face the non–full row–rank case in which the describing matrix is a “tall" full column rank matrix and the stabilization problem may be non–solvable since the system state evolves on a affine manifold. In this case, we show how to design a control strategy which solves the minimum–distance solution to the unconstrained least square solution. Finally, we deal with the case of networks with natural flow.

II. Problem definition and preliminary results

To introduce the problem let $B \in \mathbb{R}^{n\times m}$ be the control matrix, $u(t) \in \mathbb{R}^m$ the controlled flow, call it simply control, and $w(t) \in \mathbb{R}^n$ the uncontrolled flow, say it also demand. These are related one to each other through the first order continuous time dynamics

$$\dot{x}(t) = Bu(t) - w(t), \quad \forall t \geq 0. \tag{1}$$

The above dynamics tells us that matrix B combines the controlled flows $u(t)$ in order to counterbalance the demand $w(t)$. A state variable $x(t)$ accumulates up to time t any discrepancies in the counterbalancing action. Such a dynamics often arises in network flow [1], inventory [3], [4], [5], and supply-chain applications [16].

The control is bounded in a hyperbox $\mathscr{U} = \{u \in \mathbb{R}^m, u_i^- \leq u_i \leq u_i^+, \forall i\}$ defined on assigned vectors u^- and u^+ of compatible dimensions, whereas the demand is bounded in a convex and compact set:

$$u(t) \in \mathscr{U}, \qquad w(t) \in \mathscr{W}. \tag{2}$$

The above conditions are referred to as capacity constraints in inventory theory. Now, with respect to the constrained dynamics (1)-(2), a previous result establishes that the set dominance expressed by (3) is a necessary and sufficient condition for the existence of a stabilizing control, namely a control that keeps the state bounded [6]. Let us denote by $B\mathscr{U} = \{y : y = Bu, \ \ u \in \mathscr{U}\}$ and suppose as follows.

Assumption 1 *Matrix B is full row rank and set $\mathscr{W}$ is in the interior of the set $B\mathscr{U}$, that is,*

$$\mathscr{W} \subset int\{B\mathscr{U}\}. \tag{3}$$

The following assumption elaborates on the nature of the demand as unknown but constant parameter.

Assumption 2 *The demand w is constant and not available for the synthesis of control $u(t)$.*

This assumption is not necessary as far as the stability of the proposed algorithm is concerned. It is clearly necessary to prove flow optimality at steady state.

In the case of networks [6], [9], [10], a control is decentralized if the flow on each single link is a function of the inventories at the extremes nodes of the link itself as formalized next.

Definition II.1 Decentralized control. *The control strategy $u = \Phi(x)$ is* decentralized *with respect to the network structure induced by B if each component u_j is function only of the components of x corresponding to the non–zero entries of the jth column of B, denoted by $B_{\cdot,j}$.*

We are finally in the position of formally stating the problem. Assume that a reference is given for the buffer level and, without restriction we assume that such a level is equal to 0.

Problem 1 *Given a tolerance $\varepsilon > 0$, find a decentralized control law $u(t) = \Phi(x(t))$ such that*

- *the state converges to an equilibrium point, say it $\bar{x} \doteq \lim_{t\to\infty} x(t)$, which is bounded in norm, i.e., $\|\bar{x}\| \leq \varepsilon$;*
- *the control at the equilibrium, denoted by $\bar{u} \doteq \lim_{t\to\infty} u(t)$ is of minimum norm, i.e.:*

$$\bar{u} = \arg\min_{u\in\mathscr{U}} \frac{1}{2}\; u^T u : \quad Bu = w. \tag{4}$$

Remark II.1 *Note that we can easily include weights in the cost by considering*

$$J_{weighted} = \frac{1}{2}\; u^T \Sigma^2 u \tag{5}$$

where Σ is a diagonal positive definite weighting matrix. Actually, we can turn cost (5) into (4) by rearranging the model in the form $\dot{x} = (B\Sigma^{-1})(\Sigma u) - w$.

III. Towards the optimum of problem (4)

In this section, we give a detailed analysis of the optimization problem (4). In this context, a key role is played by the saturation function $sat[\cdot]$ which takes on the componentwise form (see Fig. 1)

$$sat[u]_i = \begin{cases} u_i^- & if \quad u_i < u_i^- \\ u & if \quad u_i^- \leq u_i \leq u_i^+ \\ u_i^+ & if \quad u_i > u_i^+ \end{cases}.$$

The main result of this section is to show that the optimal solution of (4) has the structure of a saturation function.

Theorem III.1 *The optimal solution of problem (4) is of the form $u = sat[B^T\xi]$.*

Proof: We show that $u = sat[B^T\xi]$ and $\mu = B^T\xi - sat[B^T\xi]$ satisfy the Karush-Khun-Tuker (KKT) condition. Let u^* be the optimal solution of Problem 1. The stationary KKT condition imposes that

$$u^* = B^T(BB^T)^{-1}w + (B^T(BB^T)^{-1}B - I)\mu,$$

where $\mu \in \mathbb{R}^n$ must satisfy the complementary slackness KKT conditions, that is, $\mu_i \neq 0$ only if u_i^* is either equal to u_i^+ or u_i^-, for $i = 1,\ldots,n$. The above argument implies that, if we define $\xi = (BB^T)^{-1}w + B^T(BB^T)^{-1}B\mu$, and take for μ the expression below

$$\mu = B^T\xi - sat[B^T\xi],$$

then it turns out that,

$$u^* = B^T\xi - I\mu = B^T\xi - B^T\xi + sat[B^T\xi] = sat[B^T\xi].$$

■

The next theorem is some kind of converse result.

Theorem III.2 *Any vector of the form $u = sat[B^T\xi]$, such that $Bsat(B^T\xi) = w$, is an optimal solution of (4).*

Proof: Denoting by $y = B^T\xi - sat[B^T\xi]$, we rewrite $u = sat[B^T\xi]$ as

$$u = B^T\xi - Iy,$$

Then, we observe that the pair (u,y) satisfies the KKT conditions for problem (4). In particular, by definition, vector y satisfies the complementary slackness KKT conditions. To prove that (u,y) satisfies also the stationary KKT condition, we show that

$$u = B^T\xi - Iy = B^T(BB^T)^{-1}w + (B^T(BB^T)^{-1}B - I)y$$

holds. Indeed, the above equation is a consequence of the following identity

$$B^T\xi = B^T(BB^T)^{-1}w + B^T\xi - B^T(BB^T)^{-1}w = \tag{6}$$

$$= B^T(BB^T)^{-1}w + B^T\xi - B^T(BB^T)^{-1}Bsat[B^T\xi] = \tag{7}$$

$$= B^T(BB^T)^{-1}w + B^T(BB^T)^{-1}B(B^T\xi - sat[B^T\xi]) = \tag{8}$$

$$= B^T(BB^T)^{-1}w + B^T(BB^T)^{-1}By. \tag{9}$$

The rhs of (6) implies (7) as $Bsat[B^T\xi] = w$. In addition, (8) implies (9), as $y = B^T\xi - sat[B^T\xi]$.

■

IV. Decentralized linear saturated control

So far, we have discussed the structure of the optimum of problem (4). Strong of the result in Theorem III.2, we now inspect some converging properties of decentralized linear saturated control strategies of the form

$$u(t) = sat\left[-\frac{B^T}{\gamma}x(t)\right], \quad \gamma \text{ positive scalar.} \tag{10}$$

As a prelude to the convergence analysis, we need to define the set of states that, once passed on to the strategy (10), return a control $u(t)$ able to counterbalance a given demand. Specifically, we look for states x satisfying

$$Bsat\left[-\frac{B^T}{\gamma}x\right] = w. \tag{11}$$

We can collect all the above states in a new set $\mathscr{L}_\gamma$ as defined below, which we will make use of in the statement of the next lemma:

$$\mathscr{L}_\gamma = \{x : \quad (11) \quad \text{holds}\}.$$

Before giving the lemma, notice that under Assumption 1 we have that, for some positive β

$$\min_{u\in\mathscr{U}} \; x^T Bu - x^T w \leq \beta\|x\|. \tag{12}$$

Using this property, it can be easily seen that, taking $\|x\|^2$ as Lyapunov function (see [6]), the decentralized control (13) assures convergence of the buffer level to zero:

$$\Phi_{BB} = \arg\min_{u\in\mathscr{U}} \; x^T Bu. \tag{13}$$

It must be noted that control (10) is a regularized version of (13). More precisely we have that (13) is the limit of (10) for $\gamma \to 0$. The price we pay for the regularization is that we cannot achieve convergence to the origin, but to the set $\mathscr{L}_\gamma$ which is bounded and arbitrarily small for decreasing γ as shown in the following lemma.

Lemma IV.1 *Under Assumption 1, $\mathscr{L}_\gamma$ is bounded and convex for any $\gamma > 0$. Moreover, if we denote by $\mathscr{L}_1$ the set of solutions of (11) with $\gamma = 1$, the set $\mathscr{L}_\gamma$ is achieved by scaling $\mathscr{L}_1$ as*

$$\mathscr{L}_\gamma = \gamma\mathscr{L}_1.$$

Proof: We first prove the second part. If $y \in \mathscr{L}_1$ then $x = \gamma y \in \mathscr{L}_\gamma$. Indeed, as we can write $y = \frac{x}{\gamma}$, then $Bsat\left[-B^T y\right] = w$ implies $Bsat\left[-\frac{B^T}{\gamma}x\right] = w$. Symmetrically, If $x \in \mathscr{L}_\gamma$, then $y = \frac{x}{\gamma} \in \mathscr{L}_1$. As we can write $x = \gamma y$, then $Bsat\left[-\frac{B^T}{\gamma}x\right] = w$ implies $Bsat\left[-B^T y\right] = w$.

Now we prove that $\mathscr{L}_1$ is convex (which proves that $\mathscr{L}_\gamma$ is such). Since (4) has a unique optimal solution, and recalling (11), then if both x^1 and x^2 are included in $\mathscr{L}_\gamma$, we have that the following relation holds componentwise

$$sat\left[-B^T x^1\right] = sat\left[-B^T x^2\right].$$

As a consequence, for all unsaturated components i for which it holds $u_i^- < -B_{i\bullet}^T x^1 = B_{i\bullet}^T x^2 < u_i^+$, it also holds $u_i^- < -B_{i\bullet}^T(\lambda x^1 + (1-\lambda)x^2) < u_i^+$. On the other hand, saturated components i, which are characterized by $-B_{i\bullet}^T x^k \geq u_i^+$, respectively $-B_{i\bullet}^T x^k \leq u_i^-$, for $k = 1,2$, satisfy the condition $-B_{i\bullet}^T(\lambda x^1 + (1-\lambda)x^2) \geq u_i^+$, respectively $-B_{i\bullet}^T(\lambda x^1 + (1-\lambda)x^2) \leq u_i^-$. Therefore, $x = \lambda x^1 + (1-\lambda)x^2$ is in $\mathscr{L}_\gamma$ for any $0 \leq \lambda \leq 1$.

To prove boundedness of $\mathscr{L}_1$ (and then of $\mathscr{L}_\gamma$) we start from expression (13) and note that it is satisfied by any vector Φ_{BB} such that, componentwise,

$$\Phi_{BBj} = \Phi_{BBj}(-B_{j\bullet}^T x) = \begin{cases} u_j^+ & \text{if } -B_{j\bullet}^T x > 0 \\ u_j^- & \text{if } -B_{j\bullet}^T x < 0 \\ u \in [u^-, u^+] & \text{otherwise} \end{cases}.$$

Take arbitrary $\hat{x}$ and $x \neq 0$ in $\mathscr{L}_\gamma$, and consider

$$\sigma = \sup \quad \{\lambda \geq 0 \quad \text{such that} \quad \hat{x} + \lambda x \in \mathscr{L}_\gamma\}.$$

Note that $\sigma \geq 1$, since $\mathscr{L}_\gamma$ is convex, and that boundedness is assured if σ is finite. Consider the expression

$$\begin{aligned} 0 &= x^T B\ sat[-B^T(\hat{x} + \lambda x)] - x^T w = x^T B\ \Phi_{BB}(-B_{j\bullet}^T x) - x^T w + \\ &+ x^T B\ sat[-B^T(\hat{x} + \lambda x)] - x^T B\ \Phi_{BB}(-B_{j\bullet}^T x) \leq -\beta \|x\| + \phi(\hat{x}, x, \lambda) \end{aligned} \tag{14}$$

with

$$\phi(\hat{x}, x, \lambda) = x^T B\ sat[-B^T(\hat{x} + \lambda x)] - x^T B\ \Phi_{BB}(-B_{j\bullet}^T x).$$

The first equality in (14) must hold for all $\lambda \geq 0$. In addition, denoting by $v = -Bx$ and by $\hat{v} = -B\hat{x}$, we get

$$\phi(\hat{x}, x, \lambda) = \sum_{j=1}^{m} v_j(sat[\hat{v}_j + \lambda v_j] - \Phi_{BB}(v_j)),$$

where each of the terms in the sum is null for λ large. Therefore, the sequence of (in)equalities (14) returns the contradiction $0 \leq -\beta \|x\|$ for $x \neq 0$ for λ large enough. ■

We build upon the above lemma to accommodate the next convergence result.

Theorem IV.1 *Under Assumption 1, the control (10), with arbitrary $\gamma > 0$, is such that:*

- $x(t)$ *is bounded,*
- $x(t) \to \mathscr{L}_\gamma$,

- $u(t) \to \bar{u} = sat\left[-\frac{B^T}{\gamma}\bar{x}\right] = u^*$ *the optimal solution of (4).*

Proof: Denote by $\bar{x} \in \mathscr{L}_\gamma$ any solution of (11). Consider the Lyapunov–like function

$$\Psi(x-\bar{x}) = (x-\bar{x})^T(x-\bar{x})/2 = \|x-\bar{x}\|^2/2.$$

If we take for $\bar{x}$ the barycenter of $\mathscr{L}_\gamma$, then $\Psi(\cdot)$ can be interpreted as the distance of x from $\mathscr{L}_\gamma$. Denoting by $v = -B^T x/\gamma$ and $\bar{v} = -B^T\bar{x}/\gamma$, the derivative of $\Psi(\cdot)$ takes on the form

$$\begin{aligned}\dot{\Psi}(x-\bar{x}) &= (x-\bar{x})^T B\left[sat\left[-\frac{B^T}{\gamma}x\right]-w\right] = -\gamma(x-\bar{x})^T\frac{B^T}{\gamma}\left[sat\left[-\frac{B^T}{\gamma}x\right]-sat\left[-\frac{B^T}{\gamma}\bar{x}\right]\right] \\ &= -\gamma(v-\bar{v})^T\left[sat[v]-sat[\bar{v}]\right] = \sum_{i=1}^{m}\ (v_i-\bar{v}_i)\left[sat[v_i]-sat[\bar{v}_i]\right].\end{aligned}$$

It is easy to see that each function $(v_i-\bar{v}_i)\left[sat[v_i]-sat[\bar{v}_i]\right]$ is non negative, and it is positive for $v_i \neq \bar{v}_i$ if $\bar{v}_i$ is an interior point of the interval $[u_i^-, u_i^+]$(see Fig. 1). Then $\|x(t)-\bar{x}\| \leq \|x(0)-\bar{x}\|$ so that $x(t)$ is bounded. Furthermore, $\dot{\Psi}(x-\bar{x})$ is a Lipschitz, in fact piecewise affine, function

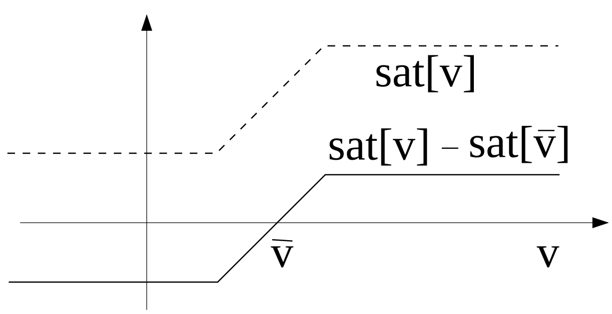

Fig. 1. The sat–function and its translation

of v and it is strictly negative unless the condition below holds for all i:

$$(v_i-\bar{v}_i)\left[sat[v_i]-sat[\bar{v}_i]\right] = 0.$$

This proves that $sat[v_i(t)]-sat[\bar{v}_i] \to 0$ (possibly $v_i(t)-\bar{v}_i(t) \to 0$) which means in turn that

$$sat\left[-\frac{B^T}{\gamma}x(t)\right] \to sat\left[-\frac{B^T}{\gamma}\bar{x}\right].$$

From Lemma 1, we can immediately conclude that $u(t) \to u^*$. ■

Corollary IV.1 *By taking $\gamma > 0$ small enough we can assure convergence of the state x to any arbitrary small neighborhood of the origin, thus the control solves Problem 1.*

Under the uniqueness of $\bar{x}$ we have the following.

Corollary IV.2 *If the equilibrium state $\bar{x}$ is the unique solution of (11) it is globally uniformly asymptotically stable*

Remark IV.1 (Unbounded control) *In the case $\mathscr{U} = \mathbb{R}^m$, the control (10) collapses to the linear feedback control $u(t) = -\frac{B^T}{\gamma}x(t)$. The above consideration is somehow not surprising as the resulting system is stable (since BB^T/γ is positive definite) and trivial computations show that the dynamics converges to the equilibrium point $\bar{x} = -\gamma(BB^T)^{-1}w$. Thus, the corresponding control $\bar{u}$ at the equilibrium is $\bar{u} = B^T(BB^T)^{-1}$ and does not depend on γ. Note that this latter expression for $\bar{u}$ is the well–known expression of the minimum 2-norm solution of $Bu = w$.*

Remark IV.2 (Failures) *The control is robust again failures as long as the necessary and sufficient condition (3) remains satisfied. Indeed, assume that a link has a failure and changes its bounds from u_j^-, u_j^+ to $\tilde{u}_j^- \geq u_j^-$ or $\tilde{u}_j^+ \leq u_j^+$ (possibly including total failure $\tilde{u}_j^- = \tilde{u}_j^+ = 0$) so that it must work at a reduced capacity. Then the new minimum–norm flow will be (distributedly) resumed after a transient.*

V. Tall matrices

Consider now the case of "non-full" row–rank B matrices in which $x(t)$ evolves on the affine manifold below, where we have denoted by $Ra[B]$ the range of B:

$$x(t) \in x(0) + Ra[B].$$

Suppose that system (1) has a full column rank matrix B. Then, we introduce a new variable $z(t) = B^T x(t)$ whose dynamics appears as

$$\dot{z}(t) = B^T \dot{x}(t) = B^T B u(t) - B^T w(t).$$

Now, in absence of constraints on flows, we can stabilize the system using the control $u = -z = -B^T x$, so that the system converges to the steady state expressed by (15), and the steady state control takes on the form in (16):

$$B^T B \bar{z} = B^T w, \tag{15}$$

$$\bar{u} = -\bar{z} = (B^T B)^{-1} B^T w. \tag{16}$$

As a matter of fact, the vector $\bar{x} = B\bar{u}$ is the best approximation of w. Indeed, it turns out that $\bar{u}$ in (16) is the least square solution of

$$\min \frac{1}{2}\|Bu - w\|_2^2.$$

Also, note that the original system represented by x may diverge while z does not and this is known as partial stability [18].

Turning back to the constrained case, consider the following saturated control:

$$u = sat[-\frac{z}{\gamma}] = sat[-\frac{B^T}{\gamma}x]. \tag{17}$$

In this case we can see that the steady state flow is not the solution of the constrained least square problem (18), but it is indeed the solution of the constrained minimum distance from the unconstrained least square solution as expressed in (19):

$$\min_{u \in \mathscr{U}} \frac{1}{2}\|Bu - w\|^2 \tag{18}$$

$$\min_{u \in \mathscr{U}} \frac{1}{2}\|u - (B^T B)^{-1} B^T w\|^2. \tag{19}$$

Obviously, the two problems have, in general, different solutions unless $\mathscr{U} = \mathbb{R}^m$, in which case we should come up with a same optimal solution $u^* = (B^T B)^{-1} B^T w$. The next example sheds light on the above topic.

Example V.1 *With reference to problems (18) and (19), let us take*

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad w = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix},$$

and $\mathscr{U} = \{u \in \mathbb{R}^2 : -3 \leq u_i \leq 3, i = 1, 2\}$. *Suppose the objective functions for the two problems (18) and (19) are, respectively,*

$$\|Bu - w\|^2 = \left\| \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} \right\|, \qquad \|u - (B^T B)^{-1} B^T w\|^2 = \left\| \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \end{bmatrix} \right\|.$$

The latter equation comes out after computing $(B^T B)^{-1} B^T w = \begin{bmatrix} 0 & 4 \end{bmatrix}^T$. *After we have done this, it is immediate to verify that the optimal solution for problem (19), call it* u^*, *differs from* *the one of problem (18), denoted by $\hat{u}$, as evidenced by the two expressions below:*

$$u^* = sat[(B^T B)^{-1} B^T w] = \begin{bmatrix} 0 & 3 \end{bmatrix}, \qquad \hat{u} = \begin{bmatrix} 1 & 3 \end{bmatrix}.$$

As usual, we can prove that $u^* = sat[(B^T B)^{-1} B^T w]$ is the optimal solution of (19) verifying that u^* and the vector $\mu \in \mathbb{R}^n$, with $\mu = (B^T B)^{-1} B^T w - sat[(B^T B)^{-1} B^T w]$ satisfy the KKT conditions. Here, note that the stationary KKT condition problem (19) implies $u^* = (B^T B)^{-1} B^T w - \mu$ and, by definition μ satisfies the complementary slackness KKT conditions.

Remark V.1 *We mentioned that $x(t)$ may diverge with the component orthogonal to B while $z = B^T x$ does not. Clearly this scheme may be applied to problems in which divergence of some variables is acceptable, typically when these represent information rather than physical quantities. Furthermore, is not difficult to see that under Assumption 1 state x remains bounded. Indeed if we consider the additional variable $y = B^\perp x$ where $B^\perp B = 0$ the overall system turns out to be represented by z and by y which satisfies*

$$\dot{y}(t) = 0$$

since $w \in B\mathscr{U}$ is equivalent to $B^\perp w = 0$.

VI. Networks with natural flow

The results of the previous sections can be extended to more general systems. In particular, consider linear systems evolving according to the equation

$$\dot{x} = -Lx + Bu - w \tag{20}$$

where L is a $n \times n$ symmetric positive semidefinite distributed matrix (e.g., the Laplacian one of the underlying network) and B is , again, the $n \times m$ control matrix. The term $-Lx$ describes “the natural system flow", whereas Bu describes the variation of the state induced by external control. We christen the first variation as “natural flow” and the second one as “forced”. Due to the presence of L, we can relax Assumption 1 as follows.

Assumption 3 *$W \in int\{Ra[L] + B\mathscr{U}\}$ where $Ra[L]$ is the range of matrix L.*

Theorem VI.1 *Assumption 3 is necessary and sufficient for the existence of a strategy which keeps the state bounded.*

Proof: We prove necessity, since sufficiency will be proved constructively later on. If the assumption fails there exists a vector w not included in $Ra[L]+B\mathscr{U}$. Consider a new basis matrix $[T^L \; T^{\perp}]$ in which T^L represents the components along $Ra[L]$ and $T^{\perp}$ the orthogonal. Let $[RS]^T$ be its inverse and rewrite

$$\begin{bmatrix} x_L \\ x_{\perp} \end{bmatrix} = \begin{bmatrix} R^T \\ S^T \end{bmatrix} x$$

so that we get

$$\dot{x}_L = -R^T Lx + R^T Bu - R^T w, \qquad \dot{x}_{\perp} = S^T Bu - S^T w.$$

Now, note that Assumption 3 is equivalent to Assumption 1 applied to the second subsystem, namely $S^T \mathscr{W} \subset S^T B\mathscr{U}$ which is necessary to keep $x_{\perp}$ bounded. ■

We seek a stabilizing control law $u \in \mathscr{U}$ which is optimal at steady–state, precisely which minimizes

$$J_L = \frac{1}{2}\, x^T Lx + \frac{1}{2} u^T u, \quad s.t. \quad -Lx + Bu = w, \tag{21}$$

where the term $x^T Lx$ represents the energy of the natural flow and the term $u^T u$ is the energy of the forced flow. For instance if L is the Laplacian of a “natural flow graph", then $x^T Lx$ represents the sum of the square of the difference of potentials at the nodes.

Theorem VI.2 *The decentralized control $u = sat[-B^T x]$ minimizes asymptotically (21).*

Proof: Given any factorization of L, $L = CC^T$ consider the auxiliary system

$$\dot{x} = Cv + Bu - w \tag{22}$$

whose control components are u and v. Despite the usual assumption on u being bounded in $\mathscr{U}$, we let v be unbounded. Now, reformulate Problem 1 in terms of the controls u and v

$$\tilde{J}_L = \frac{1}{2}\, v^T v + \frac{1}{2} u^T u, \qquad s.t. \quad Cv + Bu = w. \tag{23}$$

According to the derivations obtained in Section IV, and specifically reported in Theorem IV.1, the optimum for problem (23) takes on the form

$$\begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} sat[-C^T x] \\ sat[-B^T x] \end{bmatrix} = \begin{bmatrix} -C^T x \\ sat[-B^T x] \end{bmatrix}, \tag{24}$$

where the second equality holds because v is unbounded. The minimum $v = -C^T x$ is a feasible solution, and hence an optimal one, for the original problem (21). Indeed, if we replace control

(24) in (22), we get the stable closed–loop system achieved by (20) when $u = sat[-B^T x]$. This last system asymptotically reaches the equilibrium condition $-Lx + Bu - w = 0$ with x and u optimal in the sense of (21). ■

Consider again system (20), but this time assume that we are given a target state x_0 different form zero. If we define $y = x - \bar{x}$ as the "complaint" vector, we can translate the system into the new form

$$\dot{y} = -Ly + Bu - w - Lx_0 = -Ly + Bu - \hat{w}, \tag{25}$$

where $\hat{w} = w + Lx_0$. Noticing that $\hat{w}$ and w differ by components in the range of L, Assumption 3 remains valid. Therefore we can incorporate a target level with no further modifications in the approach.

VII. Example

Consider system (1) with matrix B, upper and lower bounds u^+, u^- and demand w specified below:

$$B = \begin{bmatrix} 1.0 & -1.0 & 0 & 0 & -0.5 & 0 \\ 0 & 1.0 & -1.0 & 0 & 0 & -1.0 \\ 0 & 0 & 1.0 & -1.0 & 0 & 1.0 \\ 0 & 0 & 0 & 1.0 & 0.5 & 0 \end{bmatrix}; \quad \begin{array}{lcl} u^- & = & [1\ 1\ 1\ 0\ 0\ 1] \\ u^+ & = & [7\ 5\ 5\ 3\ 2\ 2] \\ w & = & [1\ 2\ 2\ 1] \end{array}.$$

Then, the minimum–norm solution of $Bu = w$, computed via standard convex optimization software, turns out to be

$$u_{min} = \begin{bmatrix} 4.2 & 3.4 & 1.0 & 0.6 & 0.2 & 1.0 \end{bmatrix}.$$

With a look at Fig. 2, left, we show that using $\gamma = 0.01$ the proposed algorithm drives the buffers virtually to zero:

$$x(\infty) = \begin{bmatrix} -0.0420 & -0.0760 & -0.0400 & -0.0460 \end{bmatrix}.$$

The flow converges to the optimal u_{min} with a numerical tolerance of 10^{-13} as shown in Fig. 2, right.

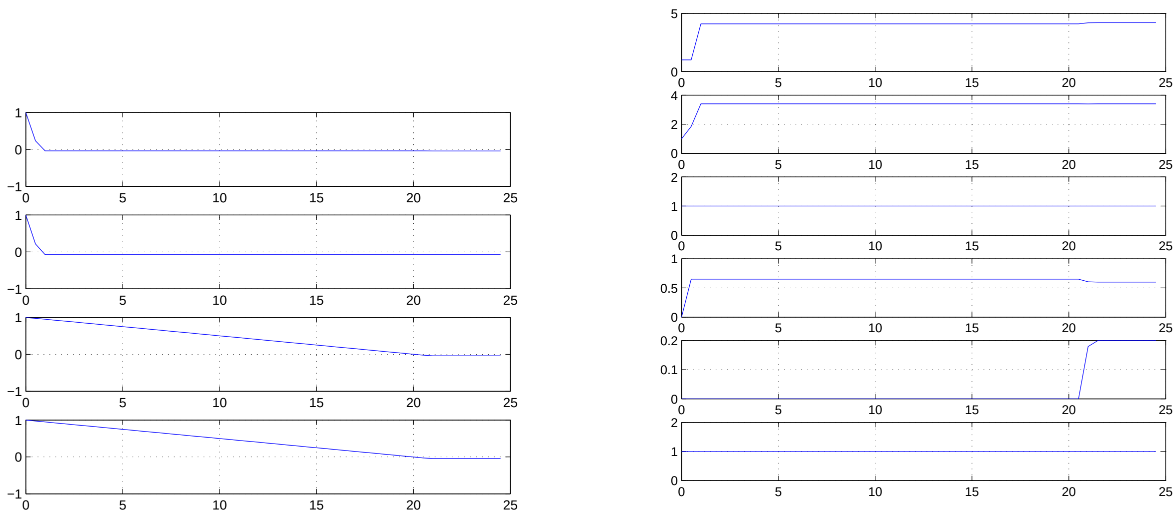

Fig. 2. Time plot of the buffer state $x(t)$ (left) and controlled flow $u(t)$ (right).

VIII. Conclusions

We have considered a decentralized linear–saturated control for constrained network flow. The main result of the paper shows that such a control is optimal at steady state, in the sense that it returns the minimum–norm flow which satisfies the demand. We have considered some sort of “dual" under–determined problem, where exact flow compensation is not possible in general, and we have shown that the same strategy returns the flow that is at the minimum–distance from the unconstrained least–square solution. Note that this last flow is different from the one obtained as solution of the least square problem. Finally we have proven optimality at steady state even in those cases in which the network has a natural flow induced from node potentials.

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