Lyapunov methods in robustness—an overview

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Abstract

In this survey we present some basic results and concepts concerning the robust analysis and synthesis of uncertain systems based on Lyapunov methods.

Index terms– Uncertain Systems, Lyapunov Function, Robustness.

1 Introduction

Any model of a real system presents inaccuracies. This is the reason why robustness with respect to system variations is perhaps one of the most important aspects in the analysis and control of dynamical systems. In simple words, a system which has to guarantee certain properties, is said robust if satisfies the requirements not only for its nominal values but also in the presence of perturbations. In this survey we present an overview of a specific approach to system robustness and precisely that based on the Lyapunov theory. Although this approach is a classical one, it is still of great interest in view of the powerful tools it considers. We first introduce some generic concepts related to the theory of Lyapunov Functions and Control Lyapunov Functions. Then we investigate more specific topics such that the stability and stabilization via quadratic Lyapunov functions. We subsequently discuss some classes of non–quadratic functions such as the polyhedral ones. We finally briefly present some successful applications of the theory.

1.1 The concept of robustness

The term robustness is deeply known in control theory since any real system is affected by uncertainties. Uncertainties may be of different nature and they can be essentially divided in the following categories.

- Unpredictable events.
- Unmodelled dynamics.
- Unavailable data.

Unpredictable events are typically those due to factors which perturb the systems and depends on the external environment (i.e. the air turbulence for an aircraft). Unmodeled dynamics is typical in any system modeling in which a simplification is necessary to have a reasonably simple model (for instance if we consider the air pressure inside a plenum in dynamic conditions we often forget about the fact that the pressure is not in general uniform in space). Unavailable data is a very frequent problem in practice since in many cases some quantities are known only when the system operates (how much weight will be carried by a lift?).

Therefore, during the design stage, we cannot consider a single system but a family of systems. Formally the concept of robustness can be stated as follows.

Definition 1.1 *A property $\mathcal{P}$ is said* **robust** *for the family $\mathcal{F}$ of dynamic systems if any member of $\mathcal{F}$ satisfies $\mathcal{P}$*

The family $\mathcal{F}$ and the property $\mathcal{P}$ must be properly specified. For instance if $\mathcal{P}$ is "stability" and $\mathcal{F}$ is a family of systems with uncertain parameters ranging in a set, we have to specify if these parameters are constant or time–varying.

In the context of robustness the family $\mathcal{F}$ represents the uncertainty in the knowledge of the system. There are basically two categories of uncertainties. Precisely we talk about

Parametric uncertainties : when we deal with a class of models depending upon parameters which are unknown; in this case the typical available information is given by bounds on these parameters:

Non–parametric uncertainties : when we deal with a systems in which some of the components are not modeled; the typical available information is provided in terms of the input–output–induced norm of some operator.

In this work we mainly consider parametric uncertainties.

Example 1.1 *Consider the system represented in Fig. 1 having equations*

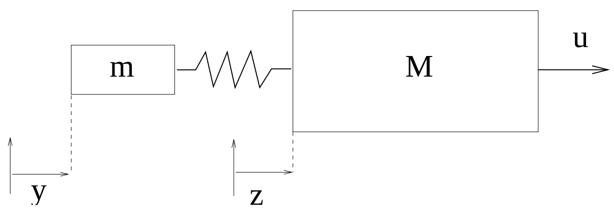


Figure 1: The elastic system

$$
\begin{aligned}
M\ddot{y}(t) &= k(z-y)+\alpha(\dot{z}-\dot{y})+u \\
m\ddot{z}(t) &= k(y-z)+\alpha(\dot{y}-\dot{z})
\end{aligned}
$$

where z and x represent the distance from the equilibrium position, M is the known mass of the main body subject to the force u and where m is the mass of another object elastically connected to the previous. A typical situation is that in which the elastic constant k, the friction α and the mass m are not known. This situation can be modeled in two ways. The first is to take these equations as they are and impose some bound to the parameters as

$$k^- \le k \le k^+, \quad m^- \le m \le m^+, \quad \alpha^- \le \alpha \le \alpha^+.$$

A different possibility is the following. Consider the new variables $\eta = k(z-y)+\alpha(\dot{z}-\dot{y})$ and $\xi = y$. Then we can write (by adopting the Laplace transform)

$$
\begin{aligned}
Ms^2y(s) &= u(s)-\eta(s), \\
\xi(s) &= y(s) \\
\eta(s) &= \Delta(s)\xi(s)
\end{aligned}
$$

where

$$\Delta(s) = \frac{ms^2(\alpha s+k)}{ms^2+\alpha s+k}$$

This corresponds to the situation depicted in Fig. 2. The connected object is represented by the transfer function $\Delta(s)$ (more in general an operator) which is unknown–but–bounded. A typical assumption on this kind of uncertainties is that $\Delta(s)$ is stable and norm–bounded as

$$\|\Delta\| \le \mu$$

where $\|\cdot\|$ any appropriate norm for the transfer function. A quite commonly used norm is

$$\|\Delta\| = \sup_{\omega\ge 0} \ |\Delta(j\omega)|$$

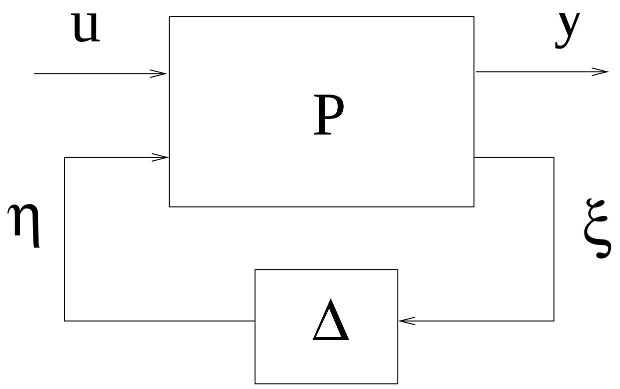


Figure 2: The delta configuration

(in the considered example its value is k although, in principle, the example does not fit in the considered uncertainties since it is not stable). The main advantage of this setup is that one can consider specification of the form $\|\Delta\| \leq \mu$ *even if the equations of the device are not known since such kind of uncertainty specifications do not depend on the physical structure. The shortcoming is that this kind of uncertain specification are quite often a very rough and conservative approximation of the true uncertainties.*

1.2 Time varying parameters

As mentioned above, the family of systems under consideration is represented by a parameterized model in which the parameters are uncertain but they are known to take their values in a prescribed set. Then a fundamental distinction has to be considered.

- The parameters are unknown but constant;
- The parameters are unknown and time–varying;

Even if we consider the same bound for a parameter of a vector of parameters, assuming it constant or time varying may lead to complete different situations. Indeed parameter variation may have a crucial effect on stability. Consider the system

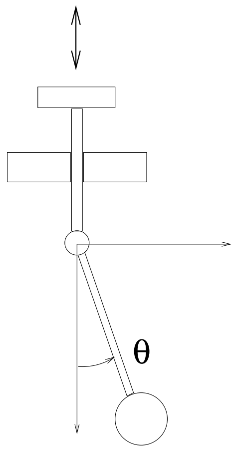
$$\dot{x}(t) = A(w(t))x(t)$$

with

$$A(w) = \begin{bmatrix} 0 & 1 \\ -1+w & -a \end{bmatrix} \qquad |w| \leq \bar{w},$$

where $a > 0$ is a damping parameter and $\bar{w} < 1$ is an uncertainty bound.

For any constant $w < \bar{w}$ and $a > 0$, the corresponding time-invariant system is stable. However, there exists $\bar{w} < 1$ and a (small enough) such that for suitable time-varying $w(t)$, with $|w(t)| \leq \bar{w}$, the system is unstable (precisely $\|x(t)\|$ diverges if $x(0) \neq 0$). This model is not only an academic example. Indeed it can be physically relized as shown in Fig. 1.2 The equation of



this system is

$$J\ddot{\theta}(t) = -(g + b(t))\sin(\theta(t)) - a(t)\dot{\theta}(t)$$

where $b(t)$ is the vertical acceleration of the reference frame. For small variations this equation becomes

$$J\ddot{\theta}(t) = -(g + b(t))\theta(t) - a(t)\dot{\theta}(t)$$

which is the considered type.

A further distinction is important as far as we are considering a stabilization problem. The fact that the parameters are "unknown" may be intended in two ways.

- The parameter are unknown in the design stage but are measurable on–line.
- The parameter are unknown and not measurable on–line.

Obviously (with some exceptions) the possibility of measuring the parameter on–line is an advantage. The compensator which modifies its parameter based on the parameter measurement are often referred to as gain–scheduling or full information. The compensator which do not have this option, are called robust.

2 State space models

In most of the work we consider systems that, in the most general case, are governed by ordinary differential equations of the form

$$\begin{aligned} \dot{x}(t) &= f(x(t), w(t), u(t)) && (1) \\ y(t) &= h(x(t), w(t)) && (2) \end{aligned}$$

or by difference equations of the form

$$\begin{aligned} x(t+1) &= f(x(t), w(t), u(t)) && (3) \\ y(t) &= h(x(t), w(t)) && (4) \end{aligned}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $w(t) \in \mathbb{R}^q$ is an external input (non controllable and whose nature will be specified later), $u(t) \in \mathbb{R}^m$ is a control input and $y(t)$ is the system output. Being the paper mainly devoted to control problems, we introduce the expression of the most general form of dynamic finite–dimensional regulator we are considering and precisely

$$\begin{aligned} \dot{x}_c(t) &= f_c(x_c(t), w(t), y(t)) \quad (5) \\ u(t) &= h_c(x_c(t), w(t), y(t)) \quad (6) \end{aligned}$$

or, in the discrete–time case,

$$\begin{aligned} x_c(t+1) &= f_c(x_c(t), w(t), y(t)) \quad (7) \\ u(t) &= h_c(x_c(t), w(t), y(t)) \quad (8) \end{aligned}$$

Note that no control action has been considered in the output equation (2) and (4) and this assures the well posedness of the feedback connection.

It is well known that the connection of the systems (1)-(2) and (5)-(6) produces a dynamic systems of augmented dimension whose state is the compound vector

$$z(t) = \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} \quad (9)$$

This state augmentation, intrinsic of a feedback loop, may trouble a theory which has the state space as natural environment. This is not a trouble (as long as the dimension of $x_c(t)$ is known) because a dynamic feedback can be always regarded as the static feedback

$$\begin{aligned} v(t) &= f_c(x_c(t), w(t), y(t)) \\ u(t) &= h_c(x_c(t), w(t), y(t)) \end{aligned}$$

for the augmented system

$$\dot{z}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{x}_c(t) \end{bmatrix} = \begin{bmatrix} f(x(t), w(t), u(t)) \\ v(t) \end{bmatrix} \quad (10)$$

with output

$$y(t) = h(x(t), w(t))$$

Therefore, with few exceptions, we will usually refer to static type of control systems. Obviously the same considerations can be done for the discrete-time version of the problem.

In some problems the equations above have to be considered along with constraints which are imposed on both control and output, typically of the form

$$u(t) \in \mathcal{U} \quad (11)$$

and

$$y(t) \in \mathcal{Y} \quad (12)$$

where $\mathcal{U} \subset \mathbb{R}^m$ and $\mathcal{Y} \subset \mathbb{R}^p$ are assigned "admissible" sets.

As far as the input $w(t)$ is concerned, this signal can play several roles depending on the problem, such as that of the reference signal, that of the noise or that of a time–varying parameter vector. A typical specification for such a function is given in the form

$$w(t) \in \mathcal{W} \quad (13)$$

The set $\mathcal{W}$ will be therefore either the set of variation of the admissible reference signals or the set of possible variation of an unknown–but–bounded parameter.

An important comment concerns the compensator nature. Any compensator is assumed to be designed for a specific purpose. Then let us assume that the compensator has to assure a certain property, say property $\mathcal{P}$ to the closed loop system. We will use the following terminology. The compensator is

Gain scheduled if it is of the form (5) (6) (respectively (7) (8)) with no restrictions. Finding a control of this form which assures property $\mathcal{P}$ is a gain–scheduling design problem.

Robust if we assume that the compensator equations are independent on w we will name the controller *robust* and we say that $\mathcal{P}$ is satisfied robustly. Designing a compensator is which independent of w is a robust design problem.

The joint presence of a control $u(t)$ and the signal $w(t)$ can be also interpreted in terms of dynamic game theory in which two individuals u, the "good guy", and w, the "bad guy", play against each other with opposite goals [6].

3 Notations

Throughout the paper we will use the following notations. Given a function $\Psi(x)$ we define the sub-level set

$$\mathcal{N}[\Psi, \beta] = \{x : \ \Psi(x) \leq \beta\}.$$

and the set

$$\mathcal{N}[\Psi, \alpha, \beta] = \{x : \ \alpha \leq \Psi(x) \leq \beta\}.$$

If Ψ is differentiable, we denote by

$$\nabla\Psi(x) = \ [\ \frac{\partial \Psi}{\partial x_1} \frac{\partial \Psi}{\partial x_2} \ \dots \ \frac{\partial \Psi}{\partial x_n}\ ]$$

its gradient. Given a matrix P we use the standard notation

$$P \ > (\geq, <, \leq) \ 0 \quad \Leftrightarrow \quad x^T P x \ > (\geq, <, \leq) \ 0, \ \ \forall x \neq 0.$$

We say that $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a κ–function if it is continuous, strictly increasing and $\phi(0) = 0$.

4 Lyapunov derivative

In this work we will refer to systems which do not have the standard regularity property (in some cases they are not even continuous) and Lyapunov functions which are non differentiable. As a consequence this section which introduces preliminary material is quite involved. Indeed to render the exposition rigorous from a mathematical point of view we need to introduce the derivative in a generalized sense (the Dini derivative). On the other hand, the reader who is not mathematically oriented should not be discouraged from reading the work since the full comprehension of this section is not strictly necessary for the comprehension of the following chapters. Given a differentiable function $\Psi : \mathbb{R}^n \to \mathbb{R}$ and a derivable trajectory $x(t)$ we can always consider the composed function $\Psi(x(t))$ whose derivarive is

$$\dot{\Psi}(x(t)) = \nabla\Psi(x(t))\dot{x}(t)$$

It is well known from the elementary theory of Lyapunov that this is a preliminary step to define the Lyapunov derivative. If the trajectory is generated by the system $\dot{x}(t) = f(x(t))$ and at time t we have $x(t) = x$, then we can determine $\dot{\Psi}$ without the knowledge of the solution $x(t)$

$$\dot{\Psi}(x(t))|_{x(t)=x} = \nabla\Psi(x)\dot{x} = \nabla\Psi(x)f(x)$$

which is a function of x. Unfortunately, writing $\dot{\Psi}$ in this way is not possible if either Ψ is not differentiable or x is not derivable.

Nevertheless, the reader can always consider the material of the section by "taking the picture" instead of entering in the detail of the Dini derivative if this concept turns out to be too nasty.

4.1 Solution of a system of differential equations

Consider a system of the form (possibly resulting from a feedback connection)

$$\dot{x}(t) = f(x(t), w(t)) \tag{14}$$

We will always assume that $w(t)$ is a piecewise continuous function of time. Unfortunately, we cannot rely on continuity assumptions of the function f, since we will sometimes refer to systems with discontinuous controllers. This will cause some mathematical difficulties.

Definition 4.1 *Given a function $x : \mathbb{R}^+ \to \mathbb{R}^n$, which is componentwise absolutely continuous in any compact interval we say that x is a solution if it satisfies (14) for almost all $t \geq 0$.*

The above definition is quite general, but it is necessary to deal with problems in which the solution $x(t)$ is not derivable. It is known that under more stronger assumptions, such as continuity of both w and f, the solution is differentiable everywhere in the regular sense. In most of the paper (but with several exception) we will refer to differential functions admitting regular (i.e. differentiable everywhere) solutions. As far as the existence of global solution (i.e. defined on all the positive axis $\mathbb{R}^+$) we will not enter in this question since we always assume that the system (14) will be globally solvable. In particular, we will not consider equations with finite escape time.

4.2 The upper right Dini derivative

Let us now consider a function $\Psi : \mathbb{R}^n \to \mathbb{R}$ defined and locally Lipschitz on the state space. As long as we are interested in the behavior of this function in terms of its monotonicity along the system trajectories we need to exploit the concept of Lyapunov derivative along the system trajectory. For any solution $x(t)$, we can consider the composed function

$$\psi(t) \doteq \Psi(x(t))$$

This new function $\psi(t)$ is not usually differentiable. However, the composition of a locally Lipschitz function Ψ and an absolutely continuous function is also absolutely continuous, and thus it is differentiable almost everywhere. Therefore we need to introduce the upper right Dini derivative defined as

$$D^+\psi(t) \doteq \lim\sup_{h\to 0^+} \frac{\psi(t+h)-\psi(t)}{h} \tag{15}$$

As long as the function $\psi(t)$ is derivable in the regular sense we have just that

$$D^+\psi(t) = \dot{\psi}(t).$$

There are four Dini derivatives which are said upper or lower if we consider limsup or liminf and they are said right or left if we consider the right or left limit in the quotient ratio. These are denoted as D^+, D_+, D^- and D_-. We limit our attention only to D^+. If an absolutely continuous function $\psi(t)$ defined on and interval $[t_1, t_2]$, has the right Dini derivative $D^+\psi(t)$ nonpositive almost everywhere, than it is non–increasing on such interval as in the case of differentiable functions. The assumption of absolute continuity is fundamental, because there exist examples of continuous functions with zero derivative almost everywhere which are indeed increasing.

4.3 Derivative along the solution of a differential equation

Let us consider again a locally Lipschitz function $\Psi : \mathbb{R}^n \to \mathbb{R}$ and a solution $x(t)$ of the differential equation (14). A key point of the theory of Lyapunov is that as long as we wish to consider the derivative of the composed function $\psi(t) = \Psi(x(t))$ we do not need to know $x(\cdot)$ as a function of time but just the current value $x(t)$ and $w(t)$. Let us introduce the upper directional derivative of Ψ with respect to (14) as

$$D^+\Psi(x,w) \doteq \lim\sup_{h\to 0^+} \frac{\Psi(x+hf(x,w))-\Psi(x)}{h} \tag{16}$$

The next fundamental property holds (see [51], Appendix. 1, Th. 4.3)

Theorem 4.1 *If the absolutely continuous function $x(t)$ is a solution of the differential equation (14), $\Psi : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function and if we define $\psi(t) = \Psi(x(t))$, then we have*

$$D^+\psi(t) = D^+\Psi(x(t), w(t)) \tag{17}$$

almost everywhere in t.

The next theorem will be also useful in the sequel.

Theorem 4.2 *If the absolutely continuous function $x(t)$ is a solution of the differential equation (14), $\Psi : \mathbb{R}^n \to \mathbb{R}$ a locally Lipschitz function and we define $\psi(t) = \Psi(x(t))$ then we have for all $0 \le t_1 \le t_2$*

$$\psi(t_2) - \psi(t_1) = \int_{t_1}^{t_2} D^+\psi(\sigma)\, d\sigma = \int_{t_1}^{t_2} D^+\Psi(x(\sigma), w(\sigma))\, d\sigma \tag{18}$$

4.4 Special cases of directional derivatives

There are special but important cases in which the Lyapunov derivative admits an explicit expression, since the directional derivative can be written in a simple way. The most famous and popular case is that in which the function Ψ is continuously differentiable.

Proposition 1 *Assume that Ψ is continuously differentiable on $\mathbb{R}^n$, then*

$$D^+\Psi(x,w) = \nabla\Psi(x) f(x,w), \tag{19}$$

(where we remind that $\nabla\Psi(x) = \; [\partial\Psi(x)/\partial x_1 \; \partial\Psi(x)/\partial x_2 \; \dots \; \partial\Psi(x)/\partial x_n])$

Another important case is that in which the function $\Psi(x)$ is a proper (i.e. locally bounded) convex function. Define the subgradient of Ψ at x the following set (see Fig. 3)

$$\partial\Psi(x) = \{z : \Psi(y) - \Psi(x) \geq z^T(y-x), \quad \text{for all } y \in \mathbb{R}^n\} \tag{20}$$

Note that for a differentiable convex function $\partial\Psi(x)$ is a singleton including the gradient at x

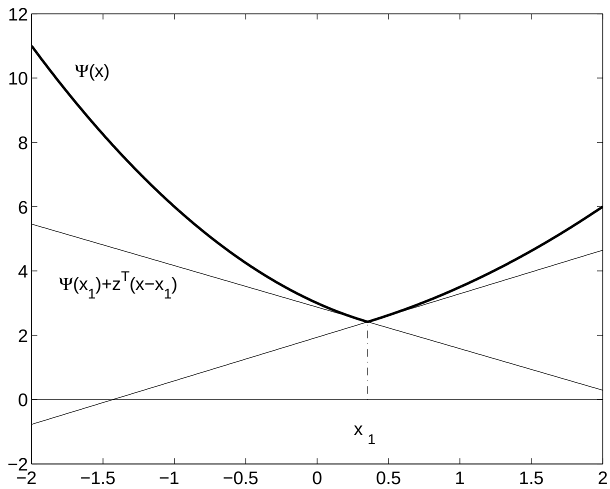


Figure 3: The subgradient

$\partial\Psi(x) = \{\nabla\Psi(x)\}$. Then the following expression holds.

$$D^+\Psi(x,w) = \sup_{z\in\partial\Psi(x)} z^T f(x,w). \tag{21}$$

A further interesting case is that of maximum(minimum)–type functions. Assume that a family of continuously differentiable functions $g_1(x), \; g_2(x), \dots, g_m(x)$ is assigned. The maximum (minimum) function is defined as

$$g(x) = \max_i g_i(x)$$

Define as $\mathcal{I}(x)$ the set of indices where the maximum (minimum) is achieved:

$$\mathcal{I}(x) = \{i : g_i(x) = g(x)\}$$

Then

$$D^+\Psi(x,w) = \max_{i\in\mathcal{I}(x)} \nabla g_i(x) f(x,w) \quad \left(\min_{i\in\mathcal{I}(x)} \nabla g_i(x) f(x,w)\right). \tag{22}$$

5 Lyapunov functions and stability

In this section we remind some basic notions concerning Lyapunov stability for systems of differential equations. The concept of Lyapunov function is widely known in system theory. The main purpose of this section is that of focusing one aspect precisely the relationship between the concept of Lyapunov (and Lyapunov–like) functions and the notion of invariant set.

Generically speaking, a Lyapunov function for a system is a positive definite function monotonically decreasing along the system trajectories. This property can be checked *without knowing the system trajectories* by means of the Lyapunov derivative. If a function Ψ of the state variables is non–increasing along the system trajectory, as an obvious consequence the set

$$\mathcal{N}[\Psi, \nu] = \{x : \quad \Psi(x) \leq \nu\}$$

has the property that if $x(t_1) \in \mathcal{N}[\Psi, \nu]$ then $x(t) \in \mathcal{N}[\Psi, \nu]$ for all $t \geq t_1$. Furthermore, if the trajectory is strictly decreasing and the derivative is bounded away from zero, namely $d\Psi(x(t))/dt < -\gamma$, with $\gamma > 0$ in a set of the form

$$\mathcal{N}[\Psi, \alpha, \beta] = \{x : \quad \alpha \leq \Psi(x) \leq \beta\},$$

then the condition $x(t_1) \in \mathcal{N}[\Psi, \alpha, \beta]$ implies (besides $x(t) \in \mathcal{N}[\Psi, \alpha]$, $t \geq t_1$) that $x(t)$ reaches the smaller set $\mathcal{N}[\Psi, \beta]$ [1]. Properties such as the mentioned one form the core of the section.

5.1 Global stability

Let us introduce the next definition

Definition 5.1 *We say that a locally Lipschitz function $\Psi : \mathbb{R}^n \to \mathbb{R}$ is*

Positive definite *: if $\Psi(0) = 0$ and $\Psi(x) > 0$ for all $x \neq 0$.*

Negative definite *: if $\Psi(0) = 0$ and $\Psi(x) < 0$ for all $x \neq 0$.*

Positive semi-definite *: if $\Psi(0) = 0$ and $\Psi(x) \geq 0$ for all x.*

Negative semi-definite *: if $\Psi(0) = 0$ and $\Psi(x) \leq 0$ for all x.*

Radially unbounded *: if*

$$\lim_{\|x\| \to \infty} |\Psi(x)| = \infty.$$

With the exception of radial unboundedness, the above definition admits a "local" version if we replace the conditions "for all x" by "for all $x \in \mathcal{S}$", where $\mathcal{S}$ is a certain neighborhood of the origin.

Definition 5.2 *We say that a continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a κ–function if it is continuous, strictly increasing and $\phi(0) = 0$.*

[1] it is not difficult to show that this will happen for $t \leq (\beta - \alpha)/\gamma + t_1$

Consider a model of the form

$$\dot{x}(t) = f(x(t), w(t)), \quad w(t) \in \mathcal{W}, \tag{23}$$

and assume that the following condition is satisfied

$$f(0, w) = 0, \quad \text{for all } \; w \in \mathcal{W} \tag{24}$$

which is well known to be equivalent to the fact that $x(t) \equiv 0$ is a trajectory of the system. We assume that system (23) admits a solution for each $x(0) \in \mathbb{R}^n$.

Definition 5.3 *We say that system (23) is Globally Uniformly Asymptotically Stable if, for all functions $w(t) \in \mathcal{W}$, it is*

Locally Stable: *for all $\nu > 0$ there exists $\delta > 0$ such that if $\|x(0)\| \leq \delta$ then*

$$\|x(t)\| \leq \nu, \quad \textit{for all } t \geq 0; \tag{25}$$

Globally Actractive: *for all $\mu > 0$ and $\epsilon > 0$ there exist $T(\mu, \epsilon) > 0$ such that if $\|x(0)\| \leq \mu$ then*

$$\|x(t)\| \leq \epsilon, \quad \textit{for all } t \geq T(\mu, \epsilon); \tag{26}$$

Since we require that the properties of uniform stability and actractivity holds for all functions w, the property above is often referred to as Robust Global Uniform Asymptotic Stability. The meaning of the definition above is that for any neighborhood of the origin the evolution of the system is bounded inside it, provided that we start sufficiently close to 0, and it converges to zero uniformly in the sense that for all the initial states $x(0)$ inside a μ–ball, the ultimate capture of the state inside any ϵ–ball occurs in a time that admits an upper bound not depending on $w(t)$.

Definition 5.4 *We say that a locally Lipschitz function $\Psi : \mathbb{R}^n \to \mathbb{R}$ is a Global Lyapunov Function (GLF) for the system if it is positive definite, radially unbounded and there exists a κ–function ϕ such that*

$$D^+\Psi(x, w) \leq -\phi(\|x(t)\|) \tag{27}$$

The following theorem is a well established result in system theory. Its first formulation is due to Lyapunov [42] and several other versions have been introduced in the literature.

Theorem 5.1 *Assume that the system (23) admits a Global Lyapunov function Ψ. Then it is globally uniformly asymptotically stable.*

Proof From Theorem 4.2 we have that

$$\Psi(x(t)) - \Psi(x(0)) = \int_0^t \; D^+(x(\sigma), w(\sigma)) \; d\sigma \leq -\int_0^t \; \phi(\|x(\sigma)\|) \; d\sigma \tag{28}$$

Therefore $\Psi(x(t))$ is non–increasing. To show stability let $\nu > 0$ be arbitrary and let ξ any positive value such that $\mathcal{N}[\Psi, \xi] \subseteq \mathcal{N}[\| \cdot \|, \nu]$. Since Ψ is radially unbounded and positive definite, we have that such a $\xi > 0$ exists. Since Ψ is positive definite there exists $\delta > 0$

such that $\mathcal{N}[\|\cdot\|,\delta] \subseteq \mathcal{N}[\Psi,\xi]$. Then if $\|x(0)\| \leq \delta$ we have $\Psi(x(0)) \leq \xi$ and, in view of the monotonicity, $\Psi(x(t)) \leq \xi$ thus $x(t) \in \mathcal{N}[\Psi,\xi] \subseteq \mathcal{N}[\|\cdot\|,\nu]$ and therefore $\|x(t)\| \leq \nu$.

To show uniform convergence let $\mu > 0$ be given and $\|x(0)\| \leq \mu$. Take $\rho^* < \infty$ such that

$$\mathcal{N}[\|\cdot\|,\mu] \subseteq \mathcal{N}[\Psi,\rho^*]$$

(for instance we can take $\rho^* = \max_{\|x\|\leq\mu} \Psi(x)$, the smallest value such that the inclusion holds). Now let $\epsilon > 0$ be arbitrary. We have to show that all the solutions originating inside $\mathcal{N}[\|\cdot\|,\mu]$ are ultimately confined in $\mathcal{N}[\|\cdot\|,\epsilon]$, in a finite time that admits an upper bound which does not depend on $w(t)$ and $x(0)$. Take $\rho_* >$ such that

$$\mathcal{N}[\Psi,\rho_*] \subset \mathcal{N}[\|\cdot\|,\epsilon]$$

(again, we can always take the largest of such values which is necessarily positive since Ψ is radially unbounded). We are able to show that for $\|x(0)\| \leq \mu$ condition

$$x(T) \in \mathcal{N}[\Psi,\rho_*] \subseteq \mathcal{N}[\|\cdot\|,\epsilon] \tag{29}$$

occurs in a finite time $T = T(\mu,\epsilon)$. This completes the proof since $\Psi(x(t))$ in non–increasing and therefore $x(t) \in \mathcal{N}[\Psi,\rho_*]$ (thus $\|x(t)\| \leq \epsilon$) for $t \geq T$. Consider the set

$$\mathcal{N}[\Psi,\rho_*,\rho^*] = \{x : \rho_* \leq \Psi(x) \leq \rho^*\}$$

The basic idea is to show that the state $x(t)$ cannot remain in this set indefinitely but it must necessarily reach $\mathcal{N}[\Psi,\rho_*]$ in a time T given by

$$T(\mu,\epsilon) = [\rho^* - \rho_*]/\zeta$$

where we have denoted by ζ

$$\zeta \doteq \min_{x\in\mathcal{N}[\Psi,\rho_*,\rho^*]} \phi(\|x\|) > 0$$

Since $x(0) \in \mathcal{N}[\Psi,\rho^*]$, and therefore $x(t) \in \mathcal{N}[\Psi,\rho^*]$ we can have two cases.

First case: there exists $\bar{t} < T(\mu,\epsilon)$ such that $x(\bar{t}) \in \mathcal{N}[\Psi,\rho_*]$. Again, since $\Psi(x(T)) \leq \Psi(x(\bar{t}))$ we have (29) for $T = T(\mu,\epsilon)$.

Second case: the previous condition does not hold. We show that this is impossible. Since $x(t) \in \mathcal{N}[\Psi,\rho^*]$, for all $t \geq 0$, then $x(t) \in \mathcal{N}[\Psi,\rho_*,\rho^*]$ (at least) for $0 \leq t \leq \bar{t}$ with $\bar{t} > T(\mu,\epsilon) = [\rho^* - \rho_*]/\zeta$. Consider again the integral inequality (28) and write it as

$$\begin{aligned} \Psi(x(\bar{t})) &= \Psi(x(0)) + \int_0^{\bar{t}} D^+(x(\sigma),w(\sigma))\, d\sigma \leq \Psi(x(0)) - \zeta\bar{t} \leq \\ &\leq \rho^* - \zeta\bar{t} < \rho^* - \zeta T(\mu,\epsilon) < \rho_*, \end{aligned}$$

in contradiction with $x(t) \in \mathcal{N}[\Psi,\rho_*,\rho^*]$ for $0 \leq t \leq \bar{t}$.

∎

There is a stronger notion of stability which will be often used in the sequel.

Definition 5.5 *We say that system (23) is Globally Exponentially Stable if there exists $\mu,\gamma > 0$ such that for all $\|x(0)\|$ the condition*

$$\|x(t)\| \leq \mu\|x(0)\|e^{-\gamma t}, \tag{30}$$

holds for all $t \geq 0$ and all functions $w(t) \in \mathcal{W}$.

The factor γ in the definition above will be named the convergence speed while factor μ will be named the transient estimate. Robust exponential stability can be assured by the existence of a Lyapunov function whose decreasing rate along the system trajectories is expressed in terms of the magnitude of the function. Let us assume that the positive definite function $\Psi(x)$ is upper and lower polynomially bounded as

$$\alpha\|x\|^p \leq \Psi(x) \leq \beta\|x\|^p, \quad \text{for all} \;\; x \in \mathbb{R}^n, \tag{31}$$

for some positive reals α and β and some positive integer p. We have the following

Theorem 5.2 *Assume that the system (23) admits a positive definite locally Lipschitz function Ψ, which has polynomial grows as in (31) and*

$$D^+\Psi(x,w) \leq -\gamma\Psi(x) \tag{32}$$

for some positive γ. Then it is Globally Exponentially Stable.

Proof Consider the integral inequality (18) and write it as

$$\begin{aligned} \Psi(x(t+T)) &\leq \Psi(x(t)) + \int_t^{t+T} D^+\Psi(x(\sigma),w(\sigma))d\sigma \\ &\leq \Psi(x(t)) - \gamma \int_t^{t+T} \Psi(x(\sigma))d\sigma \leq \Psi(x(t)) - T\gamma\Psi(x(t+T)) \end{aligned}$$

where the last inequality follows by the fact that $\Psi(x(t))$ is non–increasing. This implies

$$\Psi(x(t+T)) \leq \frac{1}{1+T\gamma}\Psi(x(t))$$

Therefore for all k

$$\Psi(x(kT)) \leq \left[\frac{1}{1+T\gamma}\right]^k \Psi(x(0))$$

Let now $t>0$ be arbitrary and $T=t/k$, with k integer. We get

$$\Psi(x(t)) \leq \left\{\left[\frac{1}{1+\gamma t/k}\right]^{-\frac{k}{\gamma t}}\right\}^{-\gamma t} \Psi(x(0)) = \left\{[1+\gamma t/k]^{\frac{k}{\gamma t}}\right\}^{-\gamma t} \Psi(x(0))$$

The number inside the graph brackets converges to e as $k \to \infty$ and since the inequality holds for any k, we have that

$$\Psi(x(t)) \leq e^{-\gamma t}\Psi(x(0))$$

Now we use condition (31) that, after simple mathematics, yields

$$\|x(t)\| \leq \sqrt[p]{\frac{\beta}{\alpha}}\; e^{-\frac{\gamma}{p}t}\; \|x(0)\|$$

which implies exponential convergence with convergence speed γ/p and transient estimate $\sqrt[p]{\frac{\beta}{\alpha}}$. ∎

The previous theorem admits a trivial proof if we assume that $\Psi(x(t))$ is derivable in the regular sense. Indeed the inequality (32) would become the differential inequality $\dot{\Psi}(x(t)) \leq -\gamma\Psi(x)$, that implies $\Psi(x(t)) \leq e^{-\gamma t}\Psi(x(0))$. It is obvious that exponential stability implies robust global asymptotic stability (the proof is very easy and not reported).

5.2 Local stability and ultimate boundedness

Global stability can be somewhat a too ambitious requirement in practical control theory, basically for the next two reasons.

- requiring convergence from arbitrary initial conditions can be too restrictive;
- in practice persistent disturbances can prevent the system from approaching the origin.

For this reason it is very useful to introduce the notion of local stability and uniform ultimate boundedness. Let us denote by $\mathcal{S}$ a neighborhood of the origin.

Definition 5.6 *Let $\mathcal{S}$ be a neighborhood of the origin. We say that system (23) is Uniformly Locally Asymptotically Stable with basin of attraction $\mathcal{S}$ if the next two conditions hold for all functions $w(t) \in \mathcal{W}$.*

Local Stability *: For all $\mu > 0$ there exists $\delta > 0$ such that $\|x(0)\| \leq \delta$ implies $\|x(t)\| \leq \mu$ for all $t \geq 0$.*

Local Uniform Convergence *For all $\epsilon > 0$ there exists $T(\epsilon) > 0$ such that if $x(0) \in \mathcal{S}$, then $\|x(t)\| \leq \epsilon$, for all $t \geq T(\epsilon)$;*

Definition 5.7 *Let $\mathcal{S}$ be a neighborhood of the origin. We say that system (23) is Uniformly Ultimately Bounded in $\mathcal{S}$ if for all $\mu > 0$ there exists $T(\mu) > 0$ such that for $\|x(0)\| \leq \mu$*

$$x(t) \in \mathcal{S}$$

for all $t \geq T(\mu)$ and all functions $w(t) \in \mathcal{W}$.

To assure the conditions of the previous definitions we introduce the following concepts of Lyapunov functions *inside* and *outside* $\mathcal{S}$.

Definition 5.8 *Let $\mathcal{S}$ be a neighborhood of the origin. We say that the locally Lipschitz positive definite function is a Lyapunov function inside $\mathcal{S}$ if there exists $\nu > 0$ such that*

$$\mathcal{S} \subseteq \mathcal{N}[\Psi, \nu]$$

and for all $x \in \mathcal{N}[\Psi, \nu]$ the inequality

$$D^+\Psi(x, w) \leq -\phi(\|x(t)\|)$$

holds for some κ–function ϕ.

Definition 5.9 *Let $\mathcal{S}$ be a neighborhood of the origin. We say that the locally Lipschitz positive definite function is a Lyapunov function outside $\mathcal{S}$ if there exists $\nu > 0$ such that*

$$\mathcal{N}[\Psi, \nu] \subseteq \mathcal{S}$$

and for all $x \notin \mathcal{N}[\Psi, \nu]$ the inequality

$$D^+\Psi(x, w) \leq -\phi(\|x(t)\|)$$

holds for some κ–function ϕ.

The next two theorems hold.

Theorem 5.3 *Assume that the system (23) satisfying condition (24) admits a Lyapunov function* Ψ *inside* $\mathcal{S}$. *Then it is Locally Stable with basin of attraction* $\mathcal{S}$.

Theorem 5.4 *Assume that the system (23) admits a Lyapunov function* Ψ *outside* $\mathcal{S}$. *Then it is uniformly ultimately bounded in* $\mathcal{S}$.

It is intuitive that there are as many possible stability definitions as the possible numbers of permutation of the requirements (Global–Local–Uniform–Exponential–Robust an so on ...). For instance we can define exponential local stability if we require the condition (30) to be satisfied only for $x(0) \in \mathcal{S}$. We can define the exponential ultimate boundedness in the set $\mathcal{S}$ by requiring that $\mathcal{N}[\|\cdot\|, \nu] \subset \mathcal{S}$ and $\|x(t)\| \leq \max\{\mu e^{-\gamma t}\|x(0)\|, \nu\}$. The problem is well known and in the literature classifications of stability concepts have been proposed (see [51] section VI). Clearly further investigation in this sense is beyond the scope of this paper.

6 Control Lyapunov function

In the previous section we have presented the basic results of the Lyapunov theory for a dynamical system with an external input. We now extend these concepts to systems of the form (5) and (6) with a control input. Essentially, we define Control Lyapunov Function a positive definite (locally Lipschitz) function which becomes a Lyapunov functions whenever a proper control action is applied.

As we have observed, any dynamic finite–dimensional feedback controller can be viewed as a static output feedback for a properly augmented system. Therefore, in this section we consider a system of the form

$$\begin{cases} \dot{x}(t) &= f(x(t), w(t), u(t)) \\ y(t) &= h(x(t), w(t)) \end{cases} \tag{33}$$

associated with a static feedback. To introduce the main definition we have to refer to a class $\mathcal{C}$ of controllers. The main classes considered here are

Output feedback : if $u(t) = \Phi(y(t))$;

State feedback : if $u(t) = \Phi(x(t))$;

Output feedback with feedforward : if $u(t) = \Phi(y(t), w(t))$;

State feedback with feedforward : if $u(t) = \Phi(x(t), w(t))$ (full information);

Definition 6.1 *Given a class of controllers* $\mathcal{C}$ *and a locally Lipschitz positive definite function* Ψ *(and possibly a set* $\mathcal{P}$*) we say that* Ψ *is a global control Lyapunov function (a Lyapunov function outside* $\mathcal{P}$ *or a Lyapunov function inside* $\mathcal{P}$*) if there exists a controller in* $\mathcal{C}$ *such that:*

- *for each initial condition* $x(0)$ *there exists a solution* $x(t)$*, for any admissible* $w(t)$*, and each of such solutions is defined for all* $t \geq 0$*;*

- *the function Ψ is a global Lyapunov function (a Lyapunov function outside $\mathcal{P}$ or a Lyapunov function inside $\mathcal{P}$) for the closed–loop system.*

An important generalization of the previous definition concerns the case of control with constraints (11)

$$u(t) \in \mathcal{U}.$$

In this case we say that Ψ is a global control Lyapunov function (a Lyapunov function outside $\mathcal{P}$ or a Lyapunov function inside $\mathcal{P}$) if there exists a controller (in a specified class $\mathcal{C}$) such that, beside the conditions in Definition 6.1, satisfies the constraints. Note also that the problem with state constraints can be easily addressed. If we assume that

$$x(t) \in \mathcal{X}$$

is a hard constraint to be satisfied, we can immediately argue that as long as $\mathcal{N}[\Psi, \mu] \subseteq \mathcal{X}$, for some μ, and Ψ is a control Lyapunov function (either global, inside or outside $\mathcal{P}$), then the constraints can be satisfied satisfied by means of a proper control action as long as $x(0) \in \mathcal{N}[\Psi, \mu]$.

6.1 Associating a feedback control with a Control Lyapunov Function

According to the previous considerations we now take into account a domain of the form

$$\mathcal{N}[\Psi, \alpha, \beta] = \{x : \quad \alpha \leq \Psi(x) \leq \beta\} \tag{34}$$

By possibly assuming $\alpha = 0$ or $\beta = +\infty$, we may include all the "meaningful" cases of Lyapunov function inside a set, outside a set or global. In this section we will basically consider the state feedback and the full information feedback cases. The output feedback case will be briefly discussed at the end. Assume that a locally Lipschitz function Ψ is given and consider the next inequality

$$D^+\Psi[x, u, w] \doteq \lim \sup_{h \to 0^+} \frac{\Psi(x + hf(x, u, w)) - \Psi(x)}{h} \leq -\phi(\|x\|). \tag{35}$$

Consider the next two conditions:

- For all $x \in \mathcal{N}[\Psi, \alpha, \beta]$, there exists u such that (35) is satisfied for all $w \in \mathcal{W}$;
- For all $x \in \mathcal{N}[\Psi, \alpha, \beta]$, and $w \in \mathcal{W}$ there exists u such that 35 is satisfied;

These conditions are clearly necessary for Ψ to be a control Lyapunov function with state or full information feedback, because, by definition, they are satisfied by assuming $u = \Phi(x)$ and $u = \Phi(x, w)$ respectively. A fundamental question is then the following: assume that these condition holds, how can we define the feedback function Φ? The problem can be thought as follows. Let us first analyze the state feedback case. Consider the set

$$\Omega(x) = \{u : \quad \text{(35) is satisfied for all } w \in \mathcal{W}\}$$

Then the question becomes whether does there exist a state feedback control function $u = \Phi(x)$ such that

$$\Phi(x) \in \Omega(x).$$

This question appears a philosophic one because as long as the set $\Omega(x)$ is non–empty for all x, then we can always associate with x a point $u \in \Omega(x)$, and "define" such a function Φ. The matter is different if one requires to the function a certain regularity such as that of being continuous. This fact is important (at least from a mathematical point of view because the resulting closed–loop system must be solvable). A positive answer to this problem can be given for control–affine systems namely systems of the form

$$\dot{x}(t) = a(x(t), w(t)) + b(x(t), w(t))u(t) \tag{36}$$

with a and b continuous terms and with $a(0, w) = 0$ for all $w \in \mathcal{W}$. Assume that continuously differentiable positive definite function Ψ is given and that (35) is satisfied for all x. From the differentiability of Ψ we have that (35) can be written as follows

$$\nabla\Psi(x)[a(x, w) + b(x, w)u] \leq -\phi(\|x\|),$$

Then the set $\Omega(x)$ turns out to be

$$\Omega(x) = \{u : \quad \nabla\Psi(x)b(x, w)u \leq -\nabla\Psi(x)a(x, w) - \phi(\|x\|), \quad \text{for all} \;\; w \in \mathcal{W}\} \tag{37}$$

This nonempty set is convex for each x, being the intersection of hyperplanes, and together with the continuity of a and b and the continuity of $\nabla\Psi(x)$, this property is sufficient to state the next theorem.

Theorem 6.1 *Assume that the set $\Omega(x)$ as in (37) is nonempty. Then there always exists a function $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ continuous everywhere, possibly with the exception of the origin, such that*

$$\Phi(x) \in \Omega(x) \tag{38}$$

Proof See [25]

∎

The previous theorem considers the fundamental concept of *selection* of a set–valued map. A set–valued map f from $\mathcal{X}$ to $\mathcal{Y}$ is a multivalued function which associates to any element x of $\mathcal{X}$ a subset Y of $\mathcal{Y}$. A selection is a single–valued function which maps x one of the element in $Y = f(x)$. In our case, $\Omega(x)$ is the set–valued map of all feasible control values which assure a certain decreasing rate to the Lyapunov derivative.

In the case of full–information control, the appropriate set–valued map must be defined in the state–disturbance product space:

$$\Omega(x, w) = \{u : \quad \nabla\Psi(x)b(x, w)u \leq -\nabla\Psi(x)a(x, w) - \phi(\|x\|)\} \tag{39}$$

If this set is not empty for all x and w then we may seek for a function

$$\Phi(x, w) \in \Omega(x, w) \tag{40}$$

which is a stabilizing full–information control. In view of the convexity of the set $\Omega(x, w)$ and the continuity, it can be shown that a continuous selection always exists, namely, that Theorem 6.1 can be stated by replacing (38) by (40).

The next question is: how can we determine this function in an analytic form? To this aim, let us think first to the full information case. Let us also consider, for the moment being that the region of interest is of the form $\mathcal{N}[\Psi, \epsilon, \kappa]$ with k finite and ϵ small. Assume that there exists a continuous function $\hat{\Psi}(x, w)$ which satisfies (40). Then consider the next minimum effort control [48]

$$\Phi_{ME}(x, w) = \arg \min_{u \in \Omega(x,w)} \|u\|_2 \tag{41}$$

($\| \cdot \|_2$ is the euclidean norm). Such a control function always exists and it has the obvious property that

$$\|\Phi_{ME}(x, w)\| \leq \|\hat{\Phi}(x, w)\|$$

for any admissible controller $\hat{\Phi}(x, w)$, therefore it is bounded in $\mathcal{N}[\Psi, \epsilon, \kappa]$. The minimum effort control admits an analytic expression which can be easily derived as follows. For fixed x and w $\Omega(x, w)$ is defined by a linear inequality for u

$$\nabla\Psi(x)b(x, w)u \leq -\nabla\Psi(x)a(x, w) - \phi(\|x\|) \doteq -c(x, w) \tag{42}$$

The vector u of minimum norm which satisfies (42) can be determined analytically as follows

$$\Phi_{ME}(x, w) = \begin{cases} -\frac{b(x,w)^T \nabla\Psi(x)^T}{\|\nabla\Psi(x)b(x,w)\|^2} \; c(x, w), & \text{if } \; c(x, w) > 0 \\ 0, & \text{if } \; c(x, w) \leq 0 \end{cases} \tag{43}$$

The singularity due to the condition $\nabla\Psi(x)b(x, w) = 0$, for some x and w is not a problem since this automatically implies that $c(x, w) \leq \phi(\|x\|) < 0$ (this is basically the reasons of working inside the set $\mathcal{N}[\Phi, \epsilon, \kappa]$ with small but positive ϵ, so excluding $x = 0$). This expression admits an immediate extension to the state feedback case, if we assume that the term b does not depend on w and precisely

$$\dot{x}(t) = a(x(t), w(t)) + b(x(t))u(t)$$

in this case the condition becomes

$$\nabla\Psi(x)b(x)u \leq -\nabla\Psi(x)a(x, w) - \phi(\|x\|) \doteq -c(x, w) \tag{44}$$

which has to be satisfied for all w by an appropriate choice of u. Define now the value

$$\hat{c}(x) = \max_{w \in \mathcal{W}} c(x, w)$$

which is a continuous function of x [2]. The condition to be considered is then

$$\nabla\Psi(x)b(x)u \leq -\hat{c}(x) \tag{45}$$

which yields the following expression for the control:

$$\Phi_{ME}(x) = \begin{cases} -\frac{b(x)^T \nabla\Psi(x)^T}{\|\nabla\Psi(x)b(x)\|^2} \; \hat{c}(x), & \text{if } \; \hat{c}(x) > 0 \\ 0, & \text{if } \; \hat{c}(x) \leq 0 \end{cases} \tag{46}$$

The minimum effort control (46) belongs to the class of gradient–based controllers of the form

$$u(t) = -\gamma(x)b(x)^T \nabla\Psi(x)^T \tag{47}$$

This type of controllers is well known and includes other types of control functions. For instance if the control effort is not of concern, one can just consider (47) with $\gamma(x) > 0$ "sufficiently large function" [3]

[2] $\hat{c}(x, w)$ is referred to as Marginal Function see [24] for further details

It can be shown that as long as (46) is a suitable controller, any function having the property

$$\gamma(x) \geq \max\left\{\frac{\hat{c}(x)}{\|\nabla\Psi(x)b(x)\|^2}, 0\right\}$$

is also a suitable controller. This fact reveals an important property of the proposed control and precisely

- If Ψ is a control Lyapunov function, then the controllers of the form $\gamma(x)\nabla\Psi(x)b(x)$ have infinite positive gain margin in the sense that if $-\gamma(x)b(x)^T\nabla\Psi(x)^T$ is a stabilizing control, then $-\gamma'(x)b(x)^T\nabla\Psi(x)^T$ is also stabilizing for all $\gamma'(x)$ such that $\gamma'(x) \geq \gamma(x)$.

The problem becomes more involved if one admits that also the term b depends on the uncertain parameters. In this case finding a state feedback controller is related to the following min–max problem

$$\min_{u\in\mathcal{U}}\max_{w\in\mathcal{W}}\{\nabla\Psi(x)[a(x,w)+b(x,w)u]\} \leq -\phi(\|x\|), \tag{48}$$

where, for the sake of generality, we assumed $u \in \mathcal{U}$. If this condition is pointwise–satisfied, then there exists a robustly stabilizing feedback control. However, determining the minimizer function $u = \Phi(x)$ can be very hard.

An important fact worth mentioning is the relation between the above min–max problem an the corresponding full information problem

$$\max_{w\in\mathcal{W}}\min_{u\in\mathcal{U}}\{\nabla\Psi(x)[a(x,w)+b(x,w)u]\} \leq -\phi(\|x\|), \tag{49}$$

in which the "min" and the "max" are reversed. If condition (49) is satisfied, then there exists a full information control. In fact condition (48) always implies (49). There are important classes of systems for which the two conditions are equivalent. For instance, in the case in which b does depend on x only, then the two problems are equivalent. This means that the existence of a full–information stabilizing controller implies the existence of a pure state feedback controller [45]. A further class of control affine uncertain systems for which the same property holds is the so called convex processes [13]

There are other forms of controllers which can be associated with a control Lyapunov function. In particular an interesting class, strongly related to the minimum effort control is the limited–effort control. Assume that the control is constrained as

$$\|u(t)\| \leq 1. \tag{50}$$

Assuming the magnitude equal to 1 is not a restriction since the actual magnitude or weighting factors can be discharged on $b(x)$. Here essentially two norms are worth of consideration and precisely the 2–norm $\|u\|_2 = \sqrt[2]{u^Tu}$ and the ∞–norm $\|u\|_\infty = \max_j |u_j|$. We consider the case of local stability and we assume that a control Lyapunov function inside $\mathcal{N}[\Psi,\kappa]$ exists which stabilizes with a control which does not violate the constraint (50).

A reasonable approach is that of considering the control which minimizes the Lyapunov derivative while preserving this bound. The problem to be solved is

$$\min_{\|u\|\leq 1} \nabla\Psi(x)[a(x,w)+b(x)u]$$

For instance if we consider the two norm, this control is

$$u = \begin{cases} -\frac{b(x)^T \nabla\Psi(x)^T}{\|b(x)^T \nabla\Psi(x)^T\|} & \text{if} \quad \nabla\Psi(x) b(x) \neq 0, \\ 0 & \text{if} \quad \nabla\Psi(x) b(x) = 0 \end{cases} \tag{51}$$

in the case of infinity norm we have

$$u = -sgn[b(x)^T \nabla\Psi(x)^T] \tag{52}$$

where the $sgn[v]$ $v \in \mathbb{R}^n$ is the component–wise sign function, (its ith component is such that $(sgn[v])_i = sgn[v_i]$). Both the controls (51) and (52) are discontinuous, but can be approximated by the continuous controllers:

$$u = -\frac{b(x)^T \nabla\Psi(x)^T}{\|b(x)^T \nabla\Psi(x)^T\| + \delta}$$

for δ sufficiently small and

$$u = -sat[\kappa b(x)^T \nabla\Psi(x)^T]$$

for κ sufficiently large, respectively, (sat is the vector saturation function).

The literature about control Lyapunov functions is huge. This concept is of fundamental importance in the control of nonlinear uncertain systems. For further details, the reader is referred to specialized work such as [24] and [34].

6.2 Output feedback case

As far as the output feedback case is concerned, the problem of determining a control to associate with a Control Lyapunov Function is much harder. We will briefly consider the problem to explain the reasons of these difficulties. Consider the system

$$\begin{aligned} \dot{x}(t) &= f(x(t), w(t), u(t)) \\ y(t) &= h(x(t), w(t)) \end{aligned}$$

and a candidate Control Lyapunov function Ψ defined on $\mathbb{R}^n$. Consider a static output feedback of the form

$$u(t) = \Phi(y(t))$$

Since only output information are available, the control value u that renders the Lyapunov derivative negative must assure this property for a given output value y for all possible values of x and w that produce that output. Let $\mathcal{Y}$ be the domain of g

$$\{y = g(x, w), \quad x \in \mathbb{R}^n, w \in \mathcal{W}\}$$

Given $y \in \mathcal{Y}$ define the preimage set as

$$g^{-1}(y) = \{(x, w), \quad x \in \mathbb{R}^n, \; w \in \mathcal{W} : \quad y = g(x, w) \; \}$$

Then a condition for Ψ to be a control Lyapunov function under output feedback is then the following. Consider the following set

$$\Omega(y) = \{u : \quad D^+(x, w, u) \leq -\Phi(\|x\|), \quad \text{for all} \quad (x, w) \in g^{-1}(y)\}$$

A necessary and sufficient condition for $\Phi(y)$ to be a proper control function is that

$$\Phi(y) \in \Omega(y), \quad \text{for all} \;\; y \in \mathcal{Y} \tag{53}$$

This theoretical condition is simple to state, but useless in most cases, since the set $g^{-1}(y)$ can be hard (not to say impossible) to determine. It is not hard to find a similar theoretical conditions for a control of the form $\Phi(y, w)$ which are again hard to apply.

So far we have considered the problem of associating a control to a Control Lyapunov Function. Obviously, a major problem is how to find the function. The are lucky cases in which the function can be determined, but it is well known that in general, the problem is very difficult especially in the output feedback case. Special classes of systems for which the problem is solvable will be considered later. The reader is referred to specialized literature for further details [24] [34].

6.3 Fake Control Lyapunov functions

In this section we sketch a simple concept which is related with notions in other fields such as the greedy or myopic strategy in Dynamic optimization. Let us introduce a very heuristic approach to control a system. Given a plant

$$\dot{x}(t) = f(x(t), u(t))$$

(we do not introduce uncertainties for brevity) and a positive definite functions $\omega(x)$, let us just adopt the following strategy that in some sense can be heuristically justified. Let us just consider the controller (possibly among those of a certain class) that in some sense renders maximum the decreasing rate of $\Psi(x(t))$, regardless of the fact that the basic condition (35) is satisfied. Assuming a constraint of the form $u(t) \in \mathcal{U}$ and assuming, for the sake of simplicity, ω to be a differentiable function, this reduces to

$$u = \Phi(x) = \arg\min_{u \in \mathcal{U}} \nabla\omega(x) f(x, u)$$

If we assume that an integral cost of the form

$$\int_0^\infty \omega(x(t))dt$$

is assigned, this type of strategy is known as greedy or myopic strategy. Indeed, it minimizes the derivative at each time in order to achieve the "best" instantaneous results. It is well known that this strategy is far from achieving the optimum of the integral cost (with the exception of special cases [44].

What we show here is that it may also produce instability. To prove this fact, we can even consider the simple case of a linear time–invariant system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with a scalar input u and the function

$$\omega(x) = x^T P x$$

Consider the case in which the system $(A, B, B^T P)$ is strictly non–minimum phase, precisely $F(s) = B^T P(sI - A)^{-1}B$ admits zeros with positive real parts. Now let us first consider a gradient–based controller which tries to render the derivative $\dot{\omega}(x(t)) = 2x^T P(Ax + Bu)$ as negative as possible. According to the previous considerations the gradient–based control is in this case

$$u(t) = -\gamma B^T P x(t)$$

with γ a large positive. However, due to the non–minimum phase nature of the system this control can lead the system to instability. If one considers a limitation for the input such as $|u| \leq 1$, the pointwise minimizing control is

$$u = \arg \min_{|u|\leq 1} 2x^T P(Ax + Bu) = -sgn[x^T PB]$$

As it is known this system becomes locally unstable at the origin if, as we have assumed, $A, B, B^T P$ is strictly non–minimum phase.

Example 6.1 *Consider the (open–loop stable!) linear system with*

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad P = I$$

It is immediate that the gradient based controller is

$$u = -\gamma B^T P = -\gamma x_2$$

and destabilizes the systems to instability for $\gamma \geq 1$. The discontinuous control

$$u = -sgn[x_2]$$

clearly produces similar destabilizing effects as well as its "approximation"

$$u = -sat[\gamma x_2].$$

7 Discrete–time systems

We have presented the main concepts of this section in the context of continuous–time systems. However the same concepts hold in the case of discrete–time systems although there are several technical differences.

Let us now consider the case of a system (possibly resulting from a feedback connection) of the form

$$x(t+1) = f(x(t), w(t)) \tag{54}$$

where now functions $x(t)$ and $w(t) \in \mathcal{W}$ are indeed sequences, although they will be referred to "functions" as well. Let us now consider a function $\Psi : \mathbb{R}^n \to \mathbb{R}$ defined and continuous on the state space. It is known that, as a counterpart of the Lyapunov derivative we have to consider the Lyapunov difference. Again, for any solution $x(t)$, we can consider the composed function

$$\psi(t) \doteq \Psi(x(t))$$

and let us consider the increment

$$\Delta\Psi(t) = \Psi(t+1) - \Psi(t) = \Psi(x(t+1)) - \Psi(x(t))$$

Then the Lyapunov difference is defined

$$\Delta\Psi(t) = \Psi(f(x(t), w(t))) - \Psi(x(t)) \doteq \Delta\Psi(x(t), w(t)) \tag{55}$$

Therefore the behavior of the function Ψ along the system trajectories can be studied by considering the function $\Delta\Psi(x, w)$, thought as a function of x and w.

Consider a model of the form (54) and again assume that the following condition is satisfied

$$f(0, w) = 0, \quad \text{for all} \;\; w \in \mathcal{W} \tag{56}$$

namely, $x(t) \equiv 0$ is a trajectory of the system.

For this discrete–time model the same definitions of uniform asymptotic stability (Definition 5.3) holds unchanged. Definition 5.4 of global Lyapunov function remains unchanged up to the fact that we replace the Lyapunov derivative with the Lyapunov difference.

Definition 7.1 *We say that a locally Lipschitz function $\Psi : \mathbb{R}^n \to \mathbb{R}$ is a Global Lyapunov Function (GLF) for the system if it is positive definite, radially unbounded and there exists a κ–function ϕ such that*

$$\Delta\Psi(x, w) \leq -\phi(\|x(t)\|) \tag{57}$$

The next theorem is the discrete–time counterpart of Theorem 5.1.

Theorem 7.1 *Assume that the system (54) admits a Lyapunov function Ψ. Then it is globally uniformly stable.*

Also in the discrete–time case we can introduce a stronger notion namely the exponential stability.

Definition 7.2 *We say that system (23) is Globally Exponentially Robustly Stable if there exists a positive $\lambda < 1$ such that for all $\|x(0)\|$ we have the condition*

$$\|x(t)\| \leq \mu \|x(0)\| \lambda^t \tag{58}$$

for all $t \geq 0$ and all sequences $w(t) \in \mathcal{W}$.

The coefficient λ is the discrete–time convergence speed. As in the discrete–time case exponential stability can be assured by the existence of a Lyapunov function whose decreasing rate along the system trajectories is expressed in terms of the magnitude of the function. Let us assume that the positive definite function $\Psi(x)$ is upper and lower polynomially bounded, as in (31). We have the following

Theorem 7.2 *Assume that the system (54) admits a positive definite locally Lipschitz function Ψ, which has polynomial grows as in (31) and*

$$\Delta\Psi(x, w) \leq \beta\Psi(x) \tag{59}$$

for some positive $\beta < 1$. Then it is globally exponentially uniformly stable with speed of convergence $\lambda = (1 - \beta) < 1$.

Proof It is left as an exercise to the reader. ∎

Note that the condition (59) of the theorem may be equivalently stated as

$$\Psi(f(x,w)) \leq \lambda\Psi(x)$$

for all x and $w \in \mathcal{W}$.

The concept of Local Stability and Uniform Ultimate Boundedness for discrete–time system are introduced by Definitions 5.6 and 5.7 which hold without modifications. The concepts of Lyapunov functions *inside* and *outside* $\mathcal{S}$ sounds now as follows.

Definition 7.3 *We say that the locally Lipschitz positive definite function is a Lyapunov function inside $\mathcal{S}$ if there exists $\nu > 0$ such that $\mathcal{S} \subseteq \mathcal{N}[\Psi, \nu]$ and for all $x \in \mathcal{N}[\Psi, \nu]$ the inequality*

$$\Delta\Psi(x,w) \leq -\phi(\|x(t)\|)$$

holds for some κ–function ϕ and all $w \in \mathcal{W}$.

Definition 7.4 *We say that the locally Lipschitz positive definite function is a Lyapunov function outside $\mathcal{S}$ if there exists $\nu > 0$ such that $\mathcal{N}[\Psi, \nu] \subseteq \mathcal{S}$ and for all $x \notin \mathcal{N}[\Psi, \nu]$ the inequality*

$$\Delta\Psi(x,w) \leq -\phi(\|x(t)\|)$$

holds for some κ–function ϕ and

$$\Psi(f(x,w)) \leq \nu$$

for all $x \in \mathcal{N}[\Psi, \nu]$ and all $w \in \mathcal{W}$.

Note that the last condition in the previous definition has no analogous statement in Definition 5.9 and its meaning is that once the set $\mathcal{N}[\Psi, \nu]$ is reached by the state, it cannot be escaped. This condition is automatically satisfied in the continuous time. The next two theorems hold.

Theorem 7.3 *Assume that the system (54) admits a positive definite locally Lipschitz function Ψ inside $\mathcal{S}$. Then it is Locally Stable with basin of attraction $\mathcal{S}$.*

Theorem 7.4 *Assume that the system (54) admits a positive definite locally Lipschitz function Ψ outside $\mathcal{S}$. Then it is uniformly ultimately bounded in $\mathcal{S}$.*

We consider now the case of a controlled discrete–time system (7) and (8). As we have observed, any dynamic finite–dimensional feedback controller can be viewed as a static output feedback for a properly augmented system. Therefore, we consider a system of the form

$$\begin{cases} \dot{x}(t) &= f(x(t), w(t), u(t)) \\ y(t) &= h(x(t), w(t)) \end{cases} \tag{60}$$

associated with a static feedback. Again we should refer to one of the classes of controllers specified above: output feedback, state feedback output feedback with feedforward, state feedback with feedforward.

The definition of control Lyapunov function is identical to Definition 6.1. In practice a Control Lyapunov function (Global, Inside, Outside) is a Lyapunov function once a proper control is applied. If we need to take into account control constraints of the form (11) we just include in the definition that the control constraints have to be satisfied, and if we have constraints on the state $x(t) \in \mathcal{X}$ then the key condition is that $x(0) \in \mathcal{N}[\Psi, \mu] \subseteq \mathcal{X}$, for some μ. It is understood that, in the difference equation case, there is no need to specify that the closed loop system has to admit a solution which, in the discrete–time case, exists as long as the control function is well defined.

So far we have seen that there is no conceptual difference between discrete and continuous time systems in the definition of Control Lyapunov Function. However technical differences are present which are particularly evident when we associate a control with a Control Lyapunov Function.

Again take into account a domain of the form

$$\mathcal{N}[\Psi, \alpha, \beta] = \{x : \quad \alpha \leq \Psi(x) \leq \beta\} \tag{61}$$

where by possibly assuming $\alpha = 0$ or $\beta = +\infty$, we include all the "meaningful" cases. Assume that a positive definite continuous function Ψ is given and consider the next inequality

$$\Delta\Psi(x, u, w) \doteq \Psi(x + hf(x, u, w)) - \Psi(x) \leq -\phi(\|x\|). \tag{62}$$

The problem can be thought in either of the following way. If for all x there must exist u such that (62) is satisfied for all $w \in \mathcal{W}$, then this condition implies that Ψ is a control Lyapunov function with state feedback. Conversely, if we allow for u to be a function also of w, then the condition becomes: for all x and $w \in \mathcal{W}$ there exists u such that (62) is satisfied. In this second case we are dealing with a control Lyapunov function for the full information feedback.

To characterize the control function for the state feedback, consider the set

$$\Omega(x) = \{u : \quad \text{(62) is satisfied for all } w \in \mathcal{W}\}$$

Then any proper state feedback control function $u = \Phi(x)$ has to be such that

$$\Phi(x) \in \Omega(x) \tag{63}$$

In the case of full–information control, consider the set

$$\Omega(x, w) = \{u : \quad \text{(62) is satisfied}\}$$

Then the control function have to be such that

$$\Phi(x, w) \in \Omega(x, w) \tag{64}$$

Now the question of regularity of function $\Phi(x)$ is not essential from the mathematical point of view. It may be nevertheless important since discontinuous controllers may have practical troubles such as actuator over–exploitation.

The problem of determining a feedback control in an analytic form does not admit general solutions as in the continuous case. The main reasons is that even in the case of a smooth Control Lyapunov functions the gradient does not play any role. Once the gradient is known, in the continuous–time case, basically the control is chosen in order to push the system in the opposite direction as much as it can. In the discrete–time case this property does not hold. Let us consider a very simple example.

Example 7.1 *Let us seek for a state feedback for the scalar system*

$$\dot{x}(t)\,(x(t+1)) = f(x(t), w(t)) + u(t), \quad |w| \leq 1. \tag{65}$$

Assume that $|f(x(t), w(t))| \leq \xi|x|$. *Consider the control Lyapunov function* $x^2/2$. *The continuous time problem is straight forward. Since the "gradient" is* x *take* u *pushing against for instance* $u = -\kappa x$ *with* κ *large enough, in any case* $\kappa > \xi$. *The Lyapunov derivative is*

$$x[f(x, w) + u] \leq -(\kappa - \xi)x^2$$

so the closed–loop system is globally asymptotically stable.

The discrete–time version of the problem is completely different. To have the Lyapunov function $x^2/2$ *decreasing the basic condition is:*

$$[f(x, w) + u(x)]^2 - x^2 \leq -\phi(|x|), \quad \textit{for all} \;\; |w| \leq 1$$

The only information we can derive is that $u(x)$ *has to be such that*

$$u(x) \in \Omega(x) = \{x: \;\; [f(x, w) + u(x)]^2 \leq x^2 - \phi(|x|), \;\; \textit{for all} \;\; |w| \leq 1 \;\}$$

this condition heavily involves function f. *Furthermore, it is not difficult to show that the bound* $|f(x(t), w(t))| \leq \xi|x|$ *does not assure a non–empty* $\Omega(x)$ *for all* x *(hence the system stabilizability). For instance the system*

$$x(t+1) = [a + bw(t)]x + u(t), \quad |w| \leq 1. \tag{66}$$

is not stabilizable by state feedback for all values of the constant a *and* b. *A necessary and sufficient stabilizability condition via state feedback is* $|b| < 1$.

The previous example shows that there are no analogous controllers of that proposed for uncertain systems in [3] for continuous–time systems. By the way, one can see that the discrete time system (66) is always stabilizable by means of the full information feedback $u = -f(x, w)$. This implies that the equivalence between state and full information feedback shown in [45] does not hold in the discrete–time case.

7.1 Literature Review

Needless to say, the literature on the Lyapunov theory is so huge that it is not possible to provide but a limited review on the subject. Nevertheless we would like to remind some seminal works as well as some fundamental textbooks as specific references. Beside the already mentioned work of Lyapunov [42]. It is fundamental to quote the work of La Salle, [38] , Krasowski [35], Hann [30] and Hale [31] as pioneering work concerning the stability of motion. An important work on Lyapunov theory is the book [51]. The reader is referred to this book for further details on the theory of stability and a complete list of references.

The Lyapunov direct method provides sufficient conditions to establish the stability of dynamic systems. A major problem in the theory is that it is non–constructive in many cases. How to construct Lyapunov and control Lyapunov functions, will be one of the main deal of this paper. It is however fundamental to note that Lyapunov–type theorems admit several converse theorems which basically state that if a system is asymptotically stable (under appropriate assumptions) then it admits a Lyapunov function. Famous results in this sense are due to Persisdki and Kurzweil and Massera. The reader is again referred to the book [51]. Lyapunov Theory has also played an important role in robustness analysis and robust synthesis of control systems. In connection with the robust stabilization problem, pioneer papers for the construction of quadratic functions are [29] [3] [28] [39]. Converse Lyapunov Theorems for uncertain systems are provided in [45] and [41].

The problem of associating a feedback control function with a Control Lyapunov Function has been considered by Artstein [1] and a universal formula can be found in [52]. This problem has been considered in the context of systems with uncertainties in [24]. It is worth mentioning that the concept of Lyapunov–Like function is in some sense related with the concept of partial stability. Basically, a system is partially stable with respect to part of its state variables if these remain bounded and converge, regardless what the remaining do. For further details on this matter the reader is referred to [54].

8 Quadratic stability and stabilization

It is well known that an important class of candidate Lyapunov function is that of the quadratic ones. A quadratic candidate Lyapunov function is a function of the form

$$\Psi(x) = x^T P x \tag{67}$$

where P is a symmetric positive definite matrix. The gradient of such function is

$$\nabla \Psi(x) = 2x^T P \tag{68}$$

So that the Lyapunov derivative for the controlled system

$$\dot{x}(t) = f(x(t), u(t), w(t))$$

is

$$\dot{\Psi}(x, u, w) = 2x^T P f(x, u, w)$$

This expression is not particularly useful since for a generic f it can be hard to analyze. The situation is different if we consider linear uncertain systems:

$$\dot{x}(t) = A(w(t))x(t) + B(w(t))u(t) \tag{69}$$

where $A(w)$ and $B(w)$ are matrices whose entries are continuous functions of the parameter $w \in \mathcal{W}$ and $\mathcal{W}$ is a compact set. The Lyapunov derivative in this case is

$$\dot{\Psi}(x, u, w) = x^T(A(w)^T P + PA(w))x + x^T(u^T B(w)^T P + PB(w)u)x$$

Let us first consider a stability analysis problem and let us set $u = 0$. We get

$$\dot{\Psi}(x, w) = x^T(A(w)^T P + PA(w))x \doteq -x^T(Q(w))x$$

Then the Lyapunov derivative is negative if and only if the matrix $Q(w)$ is positive definite for all $w \in \mathcal{W}$. Note that the fact that $\mathcal{W}$ is compact plays a fundamental role. If we consider that system

$$\dot{x}(t) = -wx(t), \quad 0 < w \leq 1$$

and $P = 1$ then

$$\dot{\Psi} = -2wx^2 < 0$$

However for $w(t) = e^{-t} \in (0, 1]$, $x(t)$ does not converge to 0 indeed

$$x(t) = x(0)e^{e^{-t}-1}$$

(thus $x(\infty) = x(0)/e$) because, although negative, $\dot{\Psi}$ gets arbitrarily close to 0.

Let us now consider the stabilization problem and let us first consider the case in which B is certain $B(w) = B$

$$\dot{\Psi}(x, u, w) = x^T(A(w)^T P + PA(w))x + 2x^T(PBu)x \tag{70}$$

According to the considerations of the previous section we can seek a control of the form

$$u = -\gamma B^T P x \tag{71}$$

Precisely, it can be shown that if there exists a continuous control such that $\dot{\Psi}(x, u, w) \leq -\alpha(\|x\|)$ then there exists a control of the form (71) which assures the same property and, in fact, exponential stability.

Definition 8.1 *A (system (controlled system) is said to be quadratically stable (stabilizable) if it admits a quadratic Lyapunov Function (Control Lyapunov Function).*

To state some conditions about the solvability of the robust stability and robust stabilization problem we introduce two special (although quite general) cases of uncertainty structure.

An interesting case, is that of *non–parametric uncertainty* in which the matrices A and B are affected by an uncertain norm–bounded term Δ as follows

$$\begin{aligned} A(\Delta) &= A_0 + D\Delta E, \\ B(\Delta) &= B_0 + D\Delta F, \quad \|\Delta\| \leq 1, \end{aligned}$$

where D, E and F are known matrices. A further structure of interest is that of *polytopic systems* whose matrices $A(w), B(w)$ are the elements of the convex hull of a finite set of known matrices A_i, B_i, $i = 1, 2, \dots, r$

$$\begin{aligned} A(w) &= \sum_{i=1}^{s} w_i A_i, \qquad \sum_{i=1}^{s} w_i = 1, \quad w_i \geq 0, \\ B(w) &= \sum_{i=1}^{s} w_i B_i, \qquad \sum_{i=1}^{s} w_i = 1, \quad w_i \geq 0, \end{aligned}$$

This case includes, as special case that of interval matrices, namely matrices in which some of the entries belongs to independent intervals. Also affine combinations of matrices with uncertain parameters can be considered as special case. For instance all the matrices of the form

$$A = A_0 + A_1 p_1 + A_2 p_2, \quad p_1^- \leq p_1 \leq p_1^+, \quad p_2^- \leq p_2 \leq p_2^+,$$

can be expressed as convex combination of the four vertex matrices

$$\begin{aligned} \hat{A}_1 &= A_0 + A_1 p_1^+ + A_2 p_2^+, \quad \hat{A}_2 = A_0 + A_1 p_1^+ + A_2 p_2^-, \\ \hat{A}_3 &= A_0 + A_1 p_1^- + A_2 p_2^+, \quad \hat{A}_4 = A_0 + A_1 p_1^- + A_2 p_2^- \end{aligned}$$

8.1 The relation with $\mathcal{H}_\infty$

A fundamental connection can be established between the quadratic stability of an uncertain system with non–parametric uncertainties and the $\mathcal{H}_\infty$ norm of an associated transfer function. Given a stable strictly proper rational transfer function $W(s)$ we define as $\mathcal{H}_\infty$ norm the value

$$\|W(s)\|_\infty = \sup_{Re(s)\geq 0} \sqrt{\sigma[W^T(s^*)W(s)]} \tag{72}$$

where $\sigma[M]$ is the maximum modulus of the eigenvalues.

The following extremely important property holds.

Theorem 8.1 *Given the system*

$$A(\Delta) = A_0 + D\Delta E \qquad \|\Delta\| \leq \rho,$$

Then there exists a positive definite matrix P such that

$$x^T PA(\Delta)x < 0, \quad \textit{for all complex} \quad \|\Delta\| \leq \rho$$

iff A_0 is stable and

$$\|E(sI - A_0)^{-1}D\|_\infty < \frac{1}{\rho}$$

This theorem admits an extension to the stabilization problem. Precisely we have the following.

Theorem 8.2 *Consider the system*

$$\begin{aligned} \dot{x}(t) &= [A_0 + D\Delta E]x(t) + [B_0 + D\Delta F]u(t) \\ y(t) &= C_0 x(t), \qquad ||\Delta(t)|| \leq 1. \end{aligned}$$

Then the control $u(s) = K(s)y(s)$ is quadratically stabilizing iff the d-to-z transfer function of the loop

$$\begin{aligned} sx(s) &= A_0 x(s) + Dd(s) + B_0 u(s) \\ z(s) &= Ex(s) + Fu(s) \\ y(s) &= C_0 x(s) \\ u(s) &= K(s)y(s) \end{aligned}$$

satisfies the conditions

$$\|W_{zd}(s)\| \leq 1$$

The proof of both theorems can be found in [36].

The previous theorems, have been a break–through in the development of the robust control, since they show that the available efficient methods based on $\mathcal{H}_\infty$ theory, are indeed useful to analyze robustness and to design robust compensators.

8.2 LMI conditions for polytopic systems

Let us now consider the case of a polytopic system.

$$\dot{x}(t) = A(w(t))x(t) + B(w(t))u(t),$$

$$\begin{array}{lrcll} e.g. & A(w) & = & \sum_{i=1}^{s} w_i A_i, & \sum_{i=1}^{s} w_i = 1, \quad w_i \geq 0, \\ & B(w) & = & \sum_{i=1}^{s} w_i B_i, & \sum_{i=1}^{s} w_i = 1, \quad w_i \geq 0, \end{array}$$

and let us consider the problem of the existence of a (control) Lyapunov function. Let us consider first the case $B = 0$. The question is if there exists a positive definite matrix P such that

$$2x^T PA(w)x = x^T (PA(w) + A(w)^T P)x = -x^T Q(w)x < 0$$

for all $x \neq 0$. This condition is easily shown to be equivalent to

$$PA_i + A_i^T P = -Q_i < 0,$$

This implication means that there exists a Lyapunov function if and only if the matrices A_i share a common Lyapunov matrix. This lead to the condition

$$\begin{array}{rcl} PA_i + A_i^T P & < & 0, \quad i = 1, 2, \ldots, s \\ P & > & 0 \end{array} \tag{73}$$

This type of conditions are know as Linear Matrix Inequality (LMI) [20] and are very convenient to be handled numerically. This is due to the following strong property.

- the set of all matrices P satisfying (73) is convex.

Let us now consider the problem of determining a quadratic control Lyapunov function. We first consider the special problem of determining a quadratic CLF associated with a linear controller $u = Kx$. The condition now becomes

$$(A_i + B_i K)^T P + P(A_i + B_i K) < 0, \quad i = 1, 2, \ldots, s$$

unfortunately this condition is nonlinear. However, if we set

$$Q = P^{-1}, \quad KQ = R$$

then we get

$$QA_i^T + A_i Q + R^T B_i^T + B_i R < 0, \quad i = 1, 2, \ldots, s, \quad Q > 0$$

which is still a LMI condition in Q and R. Once Q and R are found we can derive $P = Q^{-1}$ and $K = RP$.

It is to say that a control Lyapunov function cannot always be associated with a linear control. There are examples of system which are quadratically stabilizable but not quadratically stabilizable via linear compensators [47]. As already mentioned, in the case the a known matrix B if there exists a quadratic control Lyapunov function, then there always exists a linear controller associated with such a function.

8.3 Limits of the quadratic functions

Quadratic functions are well known in the control theory and they are known to be fundamental as practical tools in the stability analysis and synthesis of systems. Nevertheless, although commonly accepted, quadratic stability and stabilization is a conservative concepts since there are stable systems which are not quadratically stable and stabilizable systems which are not quadratically stabilizable. For instance consider the system

$$A(w) = \begin{bmatrix} 0 & 1 \\ -1 + w(t) & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad |w| \leq \rho$$

the system is stable iff

$$\rho < \rho_{ST} = 1, \quad \text{(robust stability radius)}$$

However the system is quadratically stable iff

$$\rho < \rho_Q = \frac{\sqrt{3}}{2}, \quad \text{(quadratic stability radius)}$$

This can be immediately checked by computing the $\mathcal{H}_\infty$ norm $\|F(s) = D(sI - A_0)^{-1}E\|_\infty = 2/\sqrt{3}$ where

$$A_0 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad E = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Therefore requiring the existence of a quadratic function is usually restrictive. As far as stability is concerned, the next counterexample shows that the conservativity of the methods based on quadratic Lyapunov functions can be arbitrarily high. For instance, consider again the system

$$\dot{x}(t) = A(w(t))x(t) + B(w(t))u(t)$$

where

$$w \in \rho\mathcal{W}$$

where $\rho \geq 0$ is an uncertain measure, and define the following stabilizability margins

$$\begin{aligned} \rho_{ST} &= \sup\{\rho : \text{(S) is stabilizable}\} \\ \rho_Q &= \sup\{\rho : \text{(S) is quadratically stabilizable}\} \end{aligned}$$

There are systems for which

$$\frac{\rho_{ST}}{\rho_Q} = \infty$$

For instance

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} w(t) \\ 1 \end{bmatrix}.$$

If we take $\mathcal{W} = [-1, 1]$, then

$$\rho_{ST} = \infty$$

and

$$\rho_Q = 1$$

As far as stabilization is concerned, we have said that this system is stabilizable for arbitrary ρ. But it can be also shown that it is not stabilizable by means of a linear static state feedback of the form

$$u = k_1 x_1 + k_2 x_2$$

(in which k_1 and k_2 do not depend on w). There are examples of stabilizable systems which cannot be stabilized via linear (even dynamic) compensators. [8]. Therefore seeking quadratic Lyapunov functions and/or linear compensators may conservative.

9 Non quadratic stability and stabilizability

Since quadratic functions are conservative, a natural question is whether exit classes of functions which are universal as candidate Lyapunov function. We say that a class of functions $\mathcal{C}$ is universal for the stability analysis (stabilization) problem if stability (stabilizability) is equivalent to the existence of a Lyapunov function (control Lyapunov function) in this class. Such classes exist and have a more recent history than the quadratic functions. The polytopic functions are in particular interesting because they have the property of being universal and that of being computable by means of algorithms based on linear programming.

9.1 Polyhedral Lyapunov functions

Let us introduce the class of symmetric polyhedral Lyapunov function. A symmetric polyhedral Lyapunov function is any function that can be written in the form

$$\Psi(x) = \|Fx\|_\infty$$

where F is a full column rank matrix (see Fig. 4).

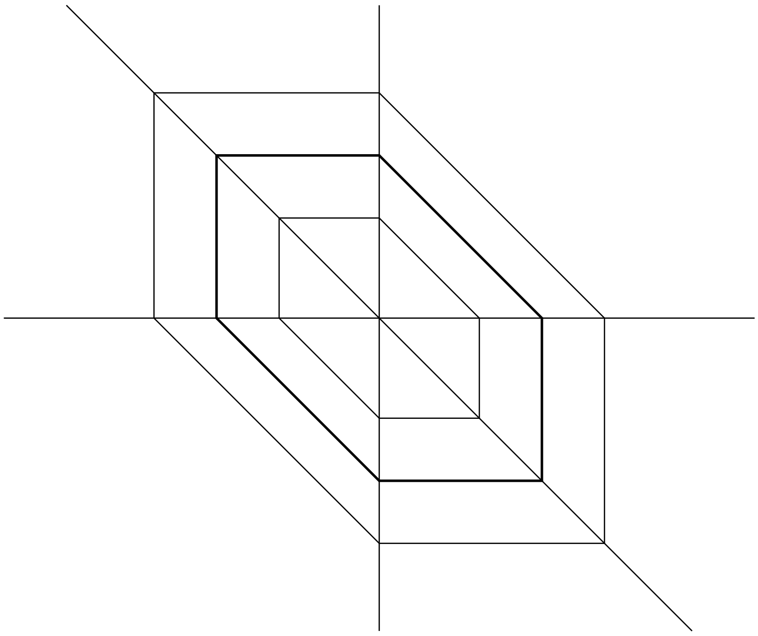


Figure 4: A polyhedral function

The main result concerning this class is the following [21] [46].

Theorem 9.1

$$\dot{x}(t) = A(w(t))x(t), \quad w \in \mathcal{W}$$

is stable if and only if it admits a polyhedral Lyapunov function.

The polyhedral functions are a universal class even for the stabilization problem. Indeed the system

$$\dot{x}(t) = A(w(t))x(t) + B(w(t))u(t), \qquad w \in \mathcal{W}$$

is stabilizable if and only if it admits a polyhedral control Lyapunov function [10].

Although the polyhedral Lyapunov functions have such nice properties, their computation can be a non–trivial task. We can explain this fact by considering necessary and sufficient condition for a polyhedral function to be a Lyapunov function. Consider the polytopic system

$$\dot{x}(t) = A(w(t))x(t) + B(w(t))u(t),$$

with

$$\begin{aligned} A(w) &= \sum_{i=1}^{s} w_i A_i, \qquad \sum_{i=1}^{s} w_i = 1, \quad w_i \geq 0, \\ B(w) &= \sum_{i=1}^{s} w_i B_i, \qquad \sum_{i=1}^{s} w_i = 1, \quad w_i \geq 0, \end{aligned}$$

We first note that the most general representation for a polyhedral (possibly non–zero symmetric) function is the expression

$$\Psi(x) = \max_i \quad F_i x = \max(Fx) \tag{74}$$

where $\max(x)$ denotes the maximum component of vector x and where F is the generating $(r \times n$ matrix. Expression (74) provides a positive definite function if and only if the polytope

$$\mathcal{P} = \{x : \ Fx \leq \bar{1}\} = \{x : \ F_i x \leq 1, \quad i = 1, 2, \ldots, r\},$$

where $\bar{1}$ denotes the vector

$$\bar{1} = \ [1\ 1\ \ldots\ 1\ ]^T,$$

includes the origin as an interior point Note that the symmetric case $\Psi(x) = \|Gx\|_\infty$ can be always reduced to the more general expression (74) by taking F as

$$F = \begin{bmatrix} G \\ -G \end{bmatrix}$$

To provide necessary and sufficient conditions for a polyhedral function to be a Lyapunov function, we need the next definition.

Definition 9.1 *A square matrix M is said a $\mathcal{M}$–matrix if $M_{ij} \geq 0$ for $i \neq j$.*

Theorem 9.2 *A positive definite polyhedral function of the form (74) is a Lyapunov function for the system $\dot{x}(t) = A(w(t))x(t)$ if and only if the following condition holds: there exist r $\mathcal{M}$–matrices H_1, H_2, $\ldots H_s$ such that*

$$\begin{aligned} FA_k &= H_k F \qquad (75) \\ H_k \bar{1} &\leq -\beta \bar{1} \qquad (76) \end{aligned}$$

for some $\beta > 0$.

The coefficient β is important because it measures the convergence rate since we have

$$\|x(t)\| \leq C\|x(0)\|e^{-\beta t}, \quad C > 0.$$

The main problem is the following. As long as the matrix F is assigned this condition is linear with respect to the unknowns H_k. Therefore checking if (74) is a Lyapunov function can be performed via linear programming. Conversely, if F is not known but has to be determined, the situation is quite different because the condition becomes bilinear due to the products $H_k F$.

The previous theorem admits a dual version. A polyhedral function can be also expressed in a dual form. Let X be the matrix whose columns are the vertices of the unit ball of Ψ. Then

$$\Psi(x) = \min\{\sum_{j=1}^{v} \alpha_j : \quad \sum_{j=1}^{v} \alpha_j x_j = x\} = \min\{\bar{1}^T \alpha : \quad X\alpha = x\} \tag{77}$$

Theorem 9.3 *A positive definite polyhedral function of the form (77) is a Lyapunov function for the system $\dot{x}(t) = A(w(t))x(t)$ if and only if the following condition holds: There exist r $\mathcal{M}$–matrices P_1, P_2, $\dots P_s$ such that*

$$A_k X = X P_k \tag{78}$$

$$\bar{1}^T P_k \leq -\beta \bar{1}^T \tag{79}$$

for some positive constant β.

Clearly the dual version suffers of the same problem as far as the determination of the function is considered, in view of the product XP_k.

The next theorem concerns the stabilization problem.

Theorem 9.4 *A positive definite polyhedral function of the form (77) is a control Lyapunov function for the system $\dot{x}(t) = A(w(t))x(t) + B(w(t))u(t)$ if and only if the following condition holds: There exists r $\mathcal{M}$–matrices P_1, P_2, $\dots P_s$ and a matrix U such that*

$$\begin{array}{rcl} A_k X + B_k U & = & X P_k \\ \bar{1}^T P_k & \leq & -\beta \bar{1}^T \end{array} \tag{80}$$

for some positive constant β.

To the author's knowledge, there does not exist a dual "plane–type" formulation of the theorem.

The bilinearity of the equation renders this expression hard to use. An iterative method, based on linear programming, to produce the matrix F has been proposed in [10]. Such a procedure is shown to be convergent, but the resulting number of rows of F or the number of columns of X can be arbitrarily high. Such a procedure is based on the reduction to a suitable discrete–time problem it will be described later.

9.2 Other types of non–quadratic Lyapunov functions

It is worth mentioning that other types of non–quadratic Lyapunov functions have been considered in the literature beside the polyhedral ones. Polynomial functions for stability analysis have been considered in [49] and [23]. In [50] piecewise quadratic functions have been considered as candidate Lyapunov functions for hybrid systems. A certain class of smooth control Lyapunov functions for uncertain systems have been proposed in [11].

10 The discrete–time case

10.1 Quadratic stabilization for unstructured uncertainty

Let us not consider the discrete–time version of the problem and precisely a system of the form

$$\begin{aligned} x(t+1) &= [A_0 + D\Delta E]x(t) + [B_0 + D\Delta F]u(t) \\ y(t) &= C_0 x(t), \quad ||\Delta(t)|| \leq 1. \end{aligned}$$

Then the stabilizing control $u(z) = K(z)y(z)$ is robustly quadratically stabilizing iff the d-to-z transfer function of the loop

$$\begin{aligned} zx(z) &= A_0 x(z) + Dd(z) + B_0 u(z) \\ z(z) &= Ex(z) + Fu(z) \\ y(z) &= C_0 x(z) \\ u(z) &= K(z)y(z) \end{aligned}$$

satisfies the condition

$$\|W_{zd}(z)\|_\infty \doteq \sup_{|z|\geq 1} \sqrt{\sigma[W_{zd}(z)^T W_{zd}(z)]} \leq 1$$

Therefore also in the discrete–time case the quadratic (stability) stabilizability reduces to an $\mathcal{H}_\infty$ type problem.

Let us consider the case of parametric uncertainties.

$$x(t+1) = A(w(t))x(t) + B(w(t))u(t), \tag{81}$$

$$\begin{aligned} e.g. \quad A(w) &= \sum_{i=1}^{s} w_i A_i, \quad \sum_{i=1}^{s} w_i = 1, \quad w_i \geq 0, \\ B(w) &= \sum_{i=1}^{s} w_i B_i, \quad \sum_{i=1}^{s} w_i = 1, \quad w_i \geq 0. \end{aligned}$$

The conditions for quadratic stability become

$$A^T(w)PA(w) - P < 0, \quad \text{for all } w \in \mathcal{W} \tag{82}$$

for some $P > 0$. Again, this condition is true if and only if

$$A_i^T P A_i - P < 0, \quad \text{for all } i, \quad P > 0 \tag{83}$$

The proof is simple. Define the vector norm

$$\|x\|_P \doteq \sqrt[2]{x^T P x}$$

Then $x^T P x$ is a discrete–time Lyapunov function if and only if the corresponding induced matrix norm of $A(w)$ is less than 1. This means that for all x

$$\|A(w)x\|_P = \|\sum_{i=1}^{s} w_i A_i x\|_P \leq \|x\|_P$$

But for any fixed x, the term $\| \sum_{i=1}^{s} w_i A_i x \|_P$ reaches it maximum on the vertices therefore the above condition is equivalent to

$$\|A_i x\|_P \leq \|x\|_P, \quad \text{for all } i$$

which implies (83). As far as the quadratic synthesis is concerned, assuming a linear controller of the form $u = Kx$, we have to find a positive definite matrix P such that

$$(A_i + B_i K)^T P (A_i + B_i K) - P < 0$$

Let us now pre and post multiply by $Q = P^{-1}$ and let $KQ = R$. We get

$$(Q A_i^T + R^T B_i^T) Q^{-1} (A_i Q + B_i R) - Q < 0$$

which is know to be equivalent to

$$\begin{bmatrix} Q & Q A_i^T + R B_i \\ A_i Q + B_i R & Q \end{bmatrix} > 0, \qquad Q > 0, \quad i = 1, 2, \ldots, s,$$

which turns out to be a set of linear matrix inequalities.

10.2 Polyhedral functions for discrete–time systems

Discrete–time uncertain systems can be also faced by means of polytopic functions. Consider a system of the form (81) and consider a positive definite polyhedral function of the form (74). Then such a function is a Lyapunov function for the system $x(t+1) = A(w(t))x(t)$ if and only if the following condition holds: there exists r nonnegative matrices H_1, H_2, $\ldots H_r$ such that

$$F A_k = H_k F \tag{84}$$

$$H_k \bar{1} \leq \lambda \bar{1} \tag{85}$$

for some positive constant $\lambda < 1$. The coefficient λ replaces β of the continuous–time case and assures the convergence rate

$$\|x(t)\| \leq C \|x(0)\| \lambda^t, \quad C > 0.$$

The next theorem admits a dual version if we consider the dual expression of Ψ (77).

A positive definite polyhedral function of the form (77) is a Lyapunov function for the system $\dot{x}(t) = A(w(t))x(t)$ if and only if the following condition holds: there exists r nonnegative matrices P_1, P_2, $\ldots P_r$ such that

$$A_k X = X P_k \tag{86}$$

$$\bar{1}^T P_k \leq \lambda \bar{1}^T \tag{87}$$

for some positive constant $\lambda < 1$.

The next property concerns the stabilization problem.

Theorem 10.1 *A positive definite polyhedral function of the form (77) is a control Lyapunov function for the system $x(t+1) = A(w(t))x(t) + B(w(t))u(t)$ if and only if the following condition holds: there exists r nonnegative matrices P_1, P_2, $\ldots P_s$ and a matrix U such that*

$$\begin{array}{rcl} A_k X + B_k U & = & X P_k \\ \bar{1}^T P_k & \leq & \lambda \bar{1}^T \end{array} \tag{88}$$

for some positive constant $\lambda < 1$.

The bilinearity of the equation renders this expression hard to use exactly as in the continuous–time case. An iterative method to produce the matrix F has been proposed in [9] and will be briefly presented next.

Assume that an arbitrary polytope P_0, including the origin as an interior point is given and represented in the form

$$\mathcal{P} = \left\{ x : \quad F^{(0)} x \leq g^{(0)} \right\}.$$

Fix a positive contractivity parameter $\bar{\lambda} < 1$ and a tolerance $\epsilon > 0$ (small enough in such a way that $\lambda(1 + \epsilon) < 1$).

Procedure 10.1 *Set $i = 0$ and $\mathcal{P}^{(0)} = \mathcal{P}$.*

1. *Form the polytope*

$$\mathcal{S}^{(i+1)} = \left\{ (x, u) : \quad F^{(k)} [A_k x + B_k u] \leq \bar{\lambda} g^{(k)}, \quad k = 1, 2, \ldots, s \right\}$$

   in the extended state space (x, u).

2. *Compute the projection of $\mathcal{S}^{(k)}$ on the space associated with the state component*

$$\tilde{\mathcal{P}}^{(i+1)} = \{ x : \quad \exists\ u : \quad (x, u) \in \mathcal{S}^{(i+1)} \}$$

3. *Set*

$$\mathcal{P}^{(i+1)} = \tilde{\mathcal{P}}^{(i+1)} \bigcap \mathcal{P},$$

4. *If $\tilde{\mathcal{P}}^{(i)} \subseteq \tilde{\mathcal{P}}^{(i+1)}(1 + \epsilon)$, then stop. Equation (88) is satisfied with $\lambda = \bar{\lambda}(1 + \epsilon)$ and X being the set of vertices of $\mathcal{P}^{(i)}$. Else go to step 1.*

In [9] it has been shown that if the system admit a polyhedral control Lyapunov function with a speed of convergence $\bar{\lambda}$ then the procedure stops in *a finite number of steps*, providing a polyhedral control Lyapunov function with a speed of convergence $\lambda = \bar{\lambda}(1 + \epsilon)$, thus the small relaxation ϵ is what we need pay to assure convergence in a finite number of steps.

This very procedure can be applied to continuous–time systems

$$\dot{x}(t) = A(w(t)) x(t) + B(w(t)) u(t), \tag{89}$$

if one considers the following discrete–time system

$$x(t + 1) = [I + \tau A(w)] x(t) + \tau B(w) u(t). \tag{90}$$

It can be shown that equation (80) is satisfied for the continuous–time system (89) if and only if there exists $\tau > 0$ (sufficiently small) such that (88) is satisfied for (90). Furthermore a speed of convergence λ for (90) implies a convergence $\beta = (1 - \lambda)/\tau$ for (89).

This type of iterative procedures are well–known in the literature and trace back to the work [7] and [27] for the construction of the largest controlled–invariant set within a region. Indeed for $\lambda = 1$ the sequence of set $\mathcal{P}^{(i)}$ converges to$\mathcal{P}^{\infty}$ the largest λ–contractive set inside $\mathcal{P}$ [9]. It is worth mentioning that this type of procedures can include control constrains without conceptual difficulties. Let us consider the next simple example (without uncertainties)

Example 10.1 *Consider the system* [3] *with*

$$\mathcal{P} = \{x : \quad \|x\|_\infty \leq 2\}$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and the constraints

$$|u| \leq 1,$$

then the sequence of sets, computed for $\lambda = 1$, is reported in figure 5. In this case we have that $\mathcal{P}^{(2)} = \mathcal{P}^{(2)} = \mathcal{P}^{\infty}$) is the largest controlled–invariant set inside $\mathcal{P}$. This very simple example can be carried out without a computer. Unfortunately, the instances in computation of polyhedral

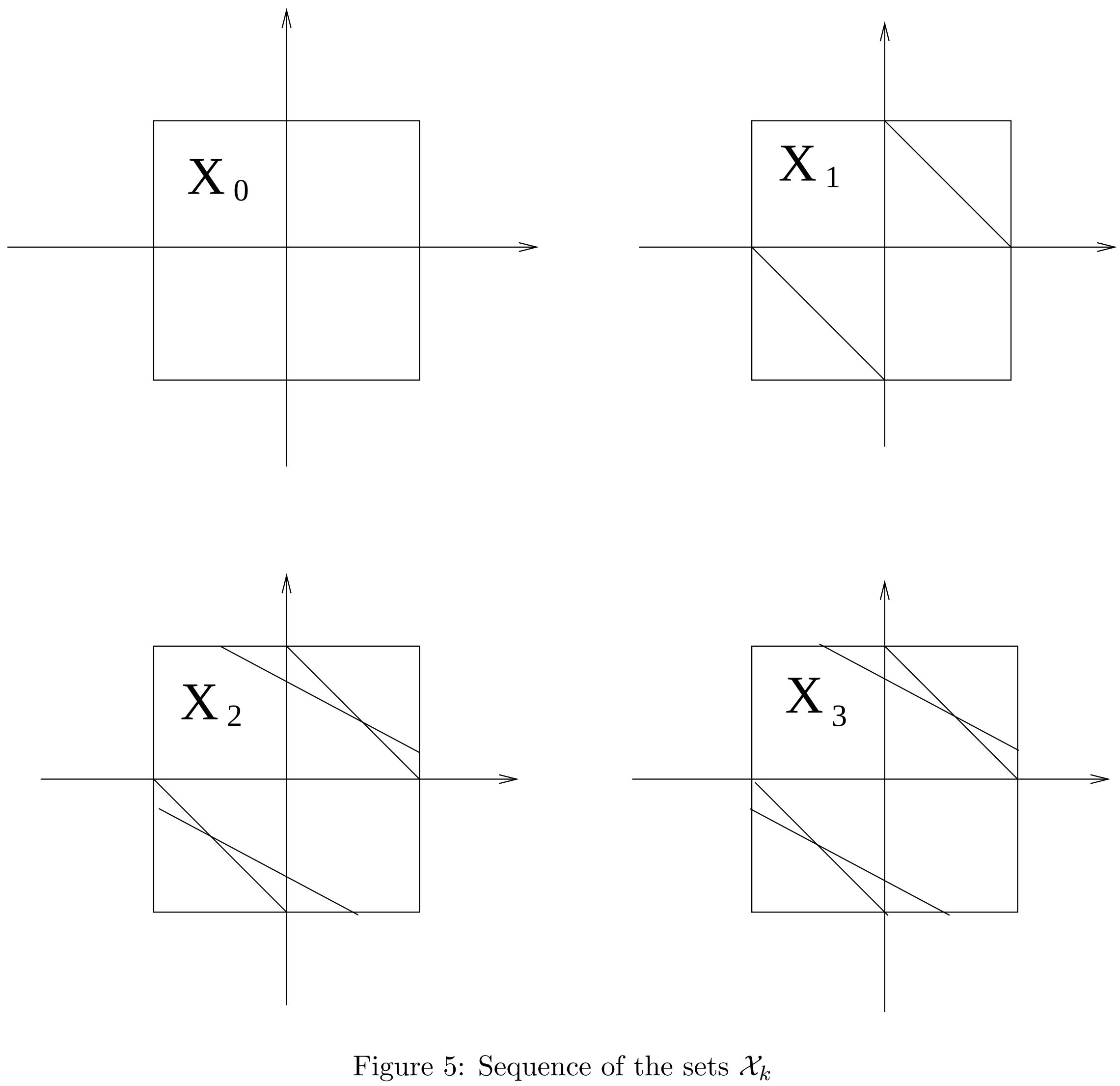


Figure 5: Sequence of the sets $\mathcal{X}_k$

Lyapunov or control Lyapunov functions are not always so lucky. There are examples in which the largest controlled invariant set is delimited by thousands of planes. This is undoubtedly a limit of the technique.

[3]Further examples can be found in [9] and [10]

11 Construction of Lyapunov functions for nonlinear uncertain systems

We have seen how the construction of a Lyapunov or a control Lyapunov function can be carried out in the case of linear uncertain systems. In the case of uncertain nonlinear systems, the problem is much harder. Therefore solutions are available only for specific cases.

11.1 Matching conditions

One famous case is that of systems with matched conditions, namely systems of the form

$$\dot{x}(t) = f(x(t), w(t), u(t)) = F(x(t)) + B(u(t) + g(x(t), w(t)))$$

where the uncertain term is bounded as

$$\|g(x, w)\| \leq \alpha\|x\| + \beta$$

The basic assumption is that the nominal systems $F(x)$ is stable, where stability is possibly achieved by a pre–compensator (whose only scope is nominal stability). We actually assume that there exists a Lyapunov function $\Psi(x)$ for the nominal system such that

$$\dot{\Psi}_{nom}(x(t)) \doteq \nabla\Psi(x)F(x) \leq -\phi(\|x\|)$$

for some κ–function ϕ. We show how this very function can be used as a Lyapunov function for the perturbed system. Consider the control

$$u = -\gamma(x)\frac{B^T\nabla\Psi(x)^T}{\|B^T\nabla\Psi(x)^T\|}$$

where we assume that γ is any function such that

$$\gamma(x) \geq \alpha\|x\| + \beta$$

The derivative with respect the perturbed dynamics is

$$\begin{aligned}
\dot{\Psi}_\Delta(x(t)) &= \nabla\Psi(x)[F(x) + B(u + g(x, w))] \\
&= \Psi_{nom}(x(t)) - \gamma(x)\|\nabla\Psi(x)B\| + \nabla\Psi(x)Bg(x, w) \leq \\
&\leq -\phi(\|x\|) - \gamma(x)\|\nabla\Psi(x)B\| + \|\nabla\Psi(x)B\|\|g(x, w)\| \leq \\
&\leq -\phi(\|x\|) - \|\nabla\Psi(x)B\|(\gamma(x) - \alpha\|x\| - \beta) \leq -\phi(\|x\|)
\end{aligned}$$

Therefore the control is stabilizing. The problem with this control is that it is not continuous. We can actually find a continuous control which assures practical stability, namely ultimate boundedness within an arbitrary small set. One of such controls is the following

$$u = -\frac{B^T\nabla\Psi(x)^T\gamma^2(x)}{\|B^T\nabla\Psi(x)^T\|\gamma(x) + \delta} \tag{91}$$

The corresponding derivative is

$$\dot{\Psi}_\Delta(x(t)) = \nabla\Psi(x)[F(x) + Bg(x, w) + Bu] \leq$$

$$
\begin{aligned}
&\leq \nabla\Psi(x)F(x) + \|\nabla\Psi(x)\|\|g(x,w)\| + \nabla\Psi(x)Bu \leq \\
&\leq \nabla\Psi(x)F(x) + \|\nabla\Psi(x)B\|\gamma(x) - \frac{\|\nabla\Psi(x)B\|^2\gamma^2(x)}{\|\nabla\Psi(x)B\|\gamma(x)+\delta} \\
&\leq -\phi(\|x\|) + \delta\frac{\|\nabla\Psi(x)B\|\gamma(x)}{\|\nabla\Psi(x)B\|\gamma(x)+\delta} \\
&\leq -\phi(\|x\|) + \delta
\end{aligned}
$$

Take

$$\kappa(\delta): \quad \min_{\phi(\|x\|)\geq\delta} \Psi(x).$$

Then the state is confined in the set

$$\mathcal{S} = \{x: \quad \Psi(x) \leq \kappa\},$$

and since $\kappa(\delta) \to 0$ as $\delta \to 0$, this assures practical stability.

The above control is not unique. For instance (91) can be written as

$$u = -\frac{\gamma^2(x)}{\|B^T\nabla\Psi(x)^T\gamma(x)\| + \delta}\ B^T\nabla\Psi(x)^T = \hat{\gamma}(x) \quad B^T\nabla\Psi(x)^T \tag{92}$$

which is a gradient–based control of the form (47). Any function $\gamma(x) \geq \hat{\gamma}(x)$ corresponds to a suitable. Under appropriate assumptions, such as the linearity of the uncertain model, γ can even chosen as a sufficiently large constant.

11.2 Beyond the matching conditions: Backstepping

Clearly the systems with matched uncertainty form a very special class. Let us now consider a class which is more general namely that of systems in the so called Strict Feedback Form. A system of the form

$$\dot{x}(t) = F(x(t), w(t)) + G(x(t), w(t))u(t)$$

Is said to be in the strict feedback form if it can be represented as

$$
\begin{aligned}
F(x,w) &= \begin{bmatrix} f_{11} & f_{12} & 0 & \dots & 0 \\ f_{21} & f_{22} & f_{23} & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ f_{n-1,1} & f_{n-1,2} & f_{n-1,3} & \dots & f_{n-1,n} \\ f_{n,1} & f_{n,2} & f_{n,3} & \dots & f_{n,n} \end{bmatrix} x + F(0,w) \\
f_{i,j} &= f_{i,j}(x_1, x_2, \dots, x_i, w) \\
G(x,w) &= \begin{bmatrix} 0 \\ 0 \\ : \\ 0 \\ f_{n,n+1} \end{bmatrix} \\
f_{i,i+1} &\neq 0.
\end{aligned}
$$

This structure admits a block version in which $f_{i,i+1}$ are full row rank matrices. The basic idea here is the following. For a system in the form above, any variable x_{i+1} can be seen as a "virtual" control signal for the subsystem associated with the variables x_i. Under appropriate assumptions (for instance that the functions $f_{i,j}$ are all bounded) we can stabilize this system. We present the idea by means of an example.

Example 11.1 *Consider the system*

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)F(x_1(t))w(t) + x_2(t)\\ \dot{x}_2(t) &= u(t)\end{aligned}$$

$$|w| \leq 1, \qquad |F(x_1)| \leq m$$

Consider the first equation with the "virtual" control x_2

$$\begin{aligned}x_2 &= S(x_1)x_1\\ \dot{x}_1 &= x_1[F(x_1)w(t) + S(x_1)]\end{aligned}$$

where $S(x_1)$ *is smooth and bounded with bounded derivative. If*

$$[F(x_1)w(t) + S(x_1)] < 0$$

this system is stable. Unfortunately, x_2 *is* ***not*** *a control variable !! Then we control the first equation*

$$\dot{x}_2(t) = u(t)$$

in such a way that x_2 *is "close" to* $S(x_1)x_1$

$$\dot{x}_2 = u = -k[x_2 - S(x_1)x_1]$$

This is basically the idea of backstepping. Consider the change of variables

$$\begin{cases} z_1 = x_1 \\ z_2 = x_2 - S(x_1)x_1 \end{cases} \qquad \begin{cases} x_1 = z_1 \\ x_2 = z_2 - S(z_1)z_1 \end{cases}$$

and the candidate Lyapunov function

$$\Psi(z_1, z_2) = z_1^2 + z_2^2$$

we have

$$\begin{aligned}\dot{z}_1(t) &= z_1F(z_1)w + z_2 + S(z_1)z_1\\ \dot{z}_2(t) &= S(z_1)[z_1F(z_1)w + z_2 + S(z_1)z_1] + S'(z_1) + u\end{aligned}$$

the control considered above becomes

$$u = -kz_2$$

$$\dot{\Psi} = [F(x_1)w + S(z_1)]z_1^2 + [1 + S'(z_1) + S(z_1)F(x_1)w + S(z_1)]z_1z_2 + [S(z_1) - k]z_2^2$$

If k *is sufficiently large then, in any bounded domain (for instance delimited by* $z_1^2 + z_2^2 \leq \mu^2$*) where we can assume* $[1 + S'(z_1) + S(z_1)F(x_1)w + S(z_1)]| \leq b$*,* $|[F(x_1)w + S(z_1)]| \leq -a$ *and* $|S(z-1)| \leq c$*, we obtain*

$$\dot{\Psi} \leq -az_1^2 + b|z1||z2| - [k - c]z_2^2 < 0, \quad \text{for} \quad (z_1, z_2) \neq 0.$$

Thus the system is stable. The idea can be generalized by reasoning recursively for each subsystem. This idea described in [25] was introduced for the first time in the context of linear uncertain systems in [2].

Let us show how the idea can be extended to the n–dimensional case. For brevity, we work under special assumption that can be easily removed. We assume that the system has a scalar input. Furthermore we assume that the functions $f_{i,j}$ are all bounded as $|f_{i,j}| \leq M_{i,j}$ and that the $f_{i,i+1}$ are bounded away from zero (without restriction we assume all of them positive) $f_{i,i+1} \geq N_i$. We assume also that $F(0, w) = 0$. Then we can proceed as follows. Consider the following "immersion"

$$\dot{x} = F(x,w)x + G(x,w)u \in \{Ax + Bu, \quad A \in \mathcal{A}, \quad \text{and} \quad B \in \mathcal{B}\}$$

where $\mathcal{A}$ and $\mathcal{B}$ are the set of interval matrices of the form

$$A = \begin{bmatrix} a_{11} & \bar{a}_{12} & 0 & \dots & 0 \\ a_{21} & a_{22} & \bar{a}_{23} & \dots & : \\ \dots & \dots & \dots & \dots & 0 \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,n-1} & \dots & \bar{a}_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \dots & a_{n,n} \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 0 \\ : \\ 0 \\ \bar{a}_{n,n+1} \end{bmatrix}$$

such that

$$|a_{i,j}| \leq M_{i,j}, \quad \text{and} \quad b_{i,i+1} \geq N_i$$

The elements of the form $\bar{a}_{i,i+1}$ have a bar to remind that they are non–zero. This kind of immersions will be discussed later. Consider the following change of variables.

$$z = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ k_1 & 1 & 0 & \dots & 0 \\ k_1 k_2 & k_2 & 1 & \dots & 0 \\ : & : & : & : & : \\ k_1 \dots k_{n-1} & \dots & \dots & k_{n-1} & 1 \end{bmatrix} x, \quad x = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ -k_1 & 1 & 0 & \dots & 0 \\ 0 & -k_2 & \dots & \dots & 0 \\ : & : & : & 1 & : \\ 0 & \dots & \dots & -k_{n-1} & 1 \end{bmatrix} z$$

where k_i are design parameters. It is not difficult that with such a change of variables, and the feedback

$$u = -k_n z_n(t)$$

B remains unchanged while the the closed loops state matrix becomes

$$A_{CL}(k) =$$

$$\begin{bmatrix} \tilde{a}_{11} - \bar{a}_{12}k_1 & \bar{a}_{12} & 0 & \cdot & 0 \\ \tilde{a}_{21}(k_1) & \tilde{a}_{22}(k_1) - \bar{a}_{23}k_2 & \bar{a}_{23} & \cdot & 0 \\ \tilde{a}_{3,1}(k_1,k_2) & \tilde{a}_{3,2}(k_1,k_2) & \tilde{a}_{3,3}(k_1,k_2) - \bar{a}_{3,3}k_3 & \cdot & : \\ \dots & \dots & \dots & \cdot & \bar{a}_{n-1,n} \\ \tilde{a}_{n,1}(k_1,\dots,k_{n-1}) & \tilde{a}_{n,2}(k_1,\dots,k_{n-1}) & \dots & \cdot & \tilde{a}_{n,n}(k_1,\dots,k_{n-1}) - \bar{a}_{n,n}k_n \end{bmatrix}$$

where all entries $\tilde{a}_{ij}$ are functions of the original parameter and the coefficients k_i. Now it is fundamental to note that

$$\tilde{a}_{i,j} = \tilde{a}_{i,j}(k_1, k_2, \dots, k_{i-1})$$

and then in the ith row the elements are functions *only* of the previous parameters, with the exception of the diagonal elements which contain the terms $\bar{a}_{i,i+1}k_i$. The proof now proceeds by showing that by an appropriate choice of the parameters k_1, k_2, $\dots k_n$, this system admits the quadratic Lyapunov function $\Psi(z) = z^T z$ which is equivalent to show that $Q(K) = A_{CL}(K)^T + A_{CL}(K)$ is a negative definite matrix. Note that such a a matrix is of the form

$$Q(K) =$$

$$\begin{bmatrix} \tilde{q}_{11} - 2\bar{a}_{12}k_1 & q_{12} & q_{13}(k_1, k_2) & \cdot & q_{1n}(k_1, , \dots, k_{n-1}) \\ \tilde{q}_{21}(k_1) & \tilde{q}_{22}(k_1) - 2\bar{a}_{23}k_2 & \bar{q}_{23} & \cdot & q_{1n}(k_1, \dots, k_{n-1}) \\ \tilde{a}_{3,1}(k_1, k_2) & \tilde{a}_{3,2}(k_1, k_2) & \tilde{q}_{3,3}(k_1, k_2) - 2\bar{a}_{3,4}k_3 & \cdot & \vdots \\ \vdots & \vdots & \dots & \cdot & \vdots \\ \tilde{q}_{n,1}(k_1, \dots, k_{n-1}) & \tilde{q}_{n,2}(k_1, \dots, k_{n-1}) & \dots & \cdot & \tilde{q}_{n,n}(k_1, \dots, k_{n-1}) - 2\bar{a}_{n-1,n}k_n \end{bmatrix}$$

Rendering it positive definite is easy. Indeed one can take k_1 in such a way that $\tilde{q}_{11} - 2\bar{a}_{12}k_1$ is positive. Then one can take k_2 in such a way that the first (2×2) principal matrix has a positive determinant, take k_2 in such a way that the first (3×3) matrix has a positive determinant, and so on.

There are obviously other classes of systems for which the Lyapunov theory turns out to be useful. However, the theory fails to provide general solutions which are not tailored for special categories of plants. On the other hand, it is well known that class of non–linear systems has often to be considered on its own since a general constructive theory is not available.

12 The output feedback problem

The previous sections basically consider the problem of state feedback stabilization. It is well known that this is a practical limitation since quite often the state variable cannot be measured and the feedback must be based on output measurements. There are, of course, noteworthy cases in which the state variables, if not measured, can be estimated (in the robotics systems velocity can be often estimated by means of virtual (filtered) derivators of the positions.

However it can be generically stated that the output feedback problem of uncertain (even linear systems) lacks of sound theoretical results with the exception of very special classes of systems (e.g. minimum–phase systems). Basically the difficulties can be explained as follows.

- The approach based on control Lyapunov functions basically requires the knowledge (or the estimation with an error) of the state.
- Basically, any "observer" must replicate the system dynamics, and thus state estimation for an uncertain system is a hard problem.

We can explain the difficulties by considering the problem of estimating the state for a linear uncertain system. Consider the following plant

$$\begin{aligned} \dot{x}(t) &= A(w(t))x(t) + B(w(t))u(t) \\ y(t) &= Cx(t) \end{aligned}$$

And consider an observer of the form

$$\begin{aligned} \dot{z}(t) &= (A_0 - LC)z(t) + B_0 u(t) + Ly(t) \\ y(t) &= Cx(t) \end{aligned}$$

If we define the error $e = z - x$ we obtain

$$\dot{e}(t) = (A_0 + LC)e(t) + (A_0 - A(w(t)))x(t) + (B_0 - B(w(t)))u(t)$$

Since the dynamics is not known exactly, error convergence depends upon u, w and x. This is related with the problem known as the fragility of the observer principle. Precisely, if we project an observer for an unstable system based on a nominal model, arbitrarily small perturbations can produce a divergent error e.

It is also well known that if the observer is part of a feedback loop then infinitesimal perturbations are indeed tolerated, and in particular stability is preserved. But this implies that it is virtually impossible to design an observer without a simultaneous consideration of a feedback of the estimated state, so that it is not clear how to extend the known separation principle to uncertain systems (although some results are available for classes of systems [5]

The situation is quite different for the gain–scheduling problem. It the term $w(t)$ is unknown only in the design stage but is available on–line then one can consider the observer

$$\begin{aligned} \dot{z}(t) &= (A(w(t)) - L(w(t))C)z(t) + B(w(t))u(t) + L(w(t))y(t) \\ y(t) &= Cx(t) \end{aligned}$$

which produces the equation error

$$\dot{e}(t) = (A(w(t)) + L(w(t))C)e(t)$$

It immediately shown that, for polytopic systems, the problem of designing a quadratic Lyapunov function along with a linear gain L reduces to the LMI problem

$$PA_k + PL_kC + A_k^TP + C^TL_k^TP = PA_k + A_k^TP + PS_k + C^TS_k^T < 0, \quad P > 0$$

where we have denoted by $S_k \doteq PL_k$. There is a well–known duality in the synthesis of gain scheduled observer and gain–scheduled stated feedback in the quadratic framework [20]. The aspect of duality in a more general context has been investigated in [12, 14].

13 Application of Lyapunov techniques

The Lyapunov approach is undoubtedly one of the most successful in the field of control of dynamical systems, especially from point of view of the applications. In this section we presents some known problems which can be very effectively faced by means of the described techniques.

13.1 Controlling nonlinear systems via robust control methods

This stabilization method is based on the concept of absorbing system. Given any system of the form

$$\dot{x}(t) = f(x(t), u(t))$$

we say that, this system is absorbed in the system

$$\dot{x}(t) = F(x(t), u(t), w(t))$$

within domains $\mathcal{X}$ and $\mathcal{U}$, if for all $x \in \mathcal{X}$ and $u \in \mathcal{U}$

$$f(x,u) = F(x,u,w), \quad \text{for some } \ w \in \mathcal{W}, \quad (i.e. f(x,u) \in \{F(x,u,w), \ \ w \in \mathcal{W}\})$$

This means that all the trajectories of the first system are included among the trajectories of the second one. Therefore stabilizing the latter implies stabilizing the former. Clearly the second system should have "nice" properties. In particular, the usual trick is basically a *bargain of nonlinearity for uncertainty.* Consider for instance the system

$$\dot{x}(t) = f(x(t)) + Bu(t) \tag{93}$$

and assume that

$$f(x) = A(w)x, \quad w = w(x), \qquad \text{with} \quad A(w) \in \mathcal{A}$$

then, if the control $u = \Phi(x)$ stabilizes

$$\dot{x}(t) = A(w(t))x(t) + Bu(t)$$

for all $A(w(t)) \in \mathcal{A}$, then it stabilizes (93). Clearly the technique is useful if $\mathcal{A}$ is a set of simple representation, for instance when

$$f(x) \in conv\{A_i\}\ x,$$

so that

$$\mathcal{A} = \left\{ A(w) = \sum_{k=1}^{r} w_i A_i, \quad \sum_{k=1}^{r} w_i = 1, \quad w_i \geq 0 \right\}$$

One way to create the set $\mathcal{A}$ is the following. Consider any component of f and assume that it is continuously differentiable inside a convex domain $\mathcal{X}$. Then we can write the following

$$f_i(x_1, x_2, \ldots, x_n) = f_i(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) + \int_{\bar{x}}^{x} \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(x_1, x_2, \ldots, x_n) dx_j$$

If we assume that $\bar{x}$ is an equilibrium point we get $f_i(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) = 0$. This can be written as

$$f_i(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} a_{ij}(x_1, x_2, \ldots, x_n)(x_j - \bar{x}_j)$$

where a_{ij} is the "average" value. Now if we are able to provide bounds to the components $a_{ij}(x_1, x_2, \ldots, x_n)$,

$$a_{ij}^- \leq a_{ij}(x_1, x_2, \ldots, x_n) \leq a_{ij}^+$$

we can merge the original systems in an interval time–varying system

$$\dot{x}(t) = A(t)x(t) + Bu(t), \qquad \text{with} \quad a_{ij}^- \leq a_{ij}(t) \leq a_{ij}^+$$

The robust stabilizability of this system implies the stabilizability of the original system (but the opposite is not true in general).

This technique can be extended also to the case in which the control does not enter linearly in the system. It is also immediately noticed that bounded uncertainties in the original model can be also faced without special difficulties.

Example. Consider the following simplified equation of a magnetic levitator

$$\ddot{y}(t) = -k\frac{i(t)^2}{y(t)^2} + g = f(y(t), i(t))$$

shown in Fig. 6. Where y is the distance of a steel ball from a controlled magnet and i is the current impressed by an amplifier. Let $(\bar{y}, \bar{i})$ the positive values such that $f(\bar{y}, \bar{i}) = 0$ It can be written as follows

$$f(y, u) = \int_{(\bar{y}, \bar{i})}^{(y,i)} \left[\frac{2ki^2}{y^3} dy - \frac{2ki}{y^2} di \right]$$

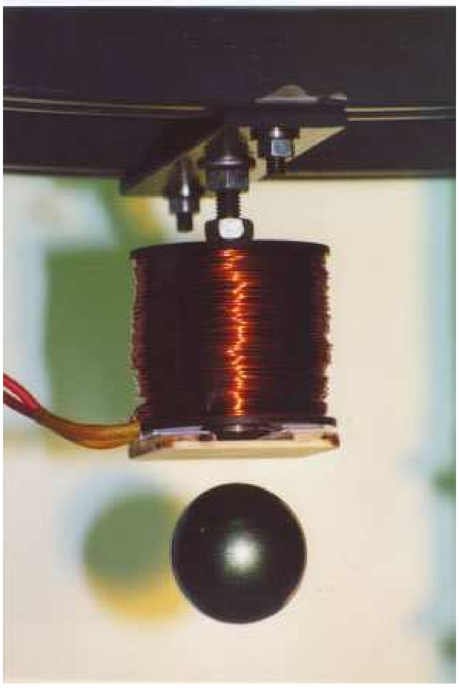


Figure 6: The magnetic levitator

Now it is reasonable (in practice necessary for this system) to assume bounds on the variables such as

$$0 < y^- \leq y \leq y^+, \quad 0 < i^- \leq i \leq i^+, \quad 0 < k^- \leq k \leq k^+.$$

Then we get the absorbing model

$$\ddot{y}(t) = a(t)(y(t) - \bar{y}) - b(t)(i(t) - \bar{i}) \tag{94}$$

with

$$\frac{2k^- {i^-}^2}{{y^+}^3} \leq a(t) \leq \frac{2k^+ {i^+}^2}{{y^-}^3}, \quad \frac{2k^- i^-}{{y^+}^2} \leq b(t) \leq \frac{2k^+ i^+}{{y^-}^2}$$

13.2 Observer design for nonlinear systems by means of robust control algorithms

Let us now consider a nonlinear system of the form

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

and let us consider an observer of the form

$$\dot{z} = f(z(t)) + L(y - Cz) + Bu \tag{95}$$

and the corresponding error equation

$$\dot{e} = [f(e + x) - f(x)] + LCe \tag{96}$$

where $e(t) = z(t) - x(t)$. Under appropriate assumptions, such as the smoothness of f we can write, for any x the following equation

$$[f(e+x) - f(x)] = Ae, \quad A \in \mathcal{A},$$

where $\mathcal{A}$ is a bounding set in the $n \times n$ matrices. By means of the same consideration of the previous subsection, set $\mathcal{A}$ can be any set with the property that

$$\frac{\partial f}{\partial x} \in \mathcal{A}$$

Then, the error equation can be absorbed in the following equation.

$$\dot{e} = (A(t) - LC)e, \qquad A(t) \in \mathcal{A} \tag{97}$$

If this system is is robustly stable, then also the previous error equation (96) is such that $e(t) \to 0$.

A fundamental point is the following. The procedure for the observer design is quite similar to that of the state feedback synthesis. Actually one can see that (97) is the dual equation of

$$\dot{x}(t) = (A(t) - BK)x(t)$$

that can be faced for instance by means of LMI techniques. The substantial difference is that no uncertainties are tolerated in the original plant because we must construct the observer (95), tha could not be done , unless the uncertain parameters are measured on–line by the controller.

Another interesting case is that of systems of the form

$$\begin{aligned} \dot{x}_1(t) &= f_{11}(x_1(t)) + f_{12}(x_1(t))x_2(t) + g_1(x_1(t))u(t) \\ \dot{x}_2(t) &= f_{21}(x_1(t)) + f_{22}(x_1(t))x_2(t) + g_2(x_1(t))u(t) \\ y(t) &= x_1(t) \end{aligned}$$

where $x_1 \in \mathbb{R}^{n_1}$ $x_2 \in \mathbb{R}^{n_2}$, $x_1 + x_2 = n$. The peculiarity of this system is that the unmeasured vector x_2 enters in an affine way in the equations. For this system it is possible to determine a reduced–order observer as follows. Define the variable

$$w(t) = x_2(t) - Lx_1(t)$$

where matrix L is a design parameter. Then we have

$$\begin{aligned} \dot{w} &= (f_{22}(x_1) - Lf_{12})x_2 + f_{21}(x_1) - Lf_{11}(x_1) + (g_2(x_1) - Lg_1(x_1))u \\ &\pm (Lf_{12}(x_1)L - f_{22}(x_1)L) \\ &= (f_{22}(x_1) - Lf_{12}(x_1))w \\ &+ \underbrace{[f_{21}(x_1) - Lf_{11}(x_1) + f_{22}(x_1)L - Lf_{12}(x_1)L] + [g_2(x_1) - Lg_1(x_1)]u}_{\doteq \rho(x_1,u)} \end{aligned}$$

For the resulting system

$$\dot{w} = (f_{22}(x_1) - Lf_{12}(x_1))w + \rho(x_1, u)$$

we consider the reduced observer

$$\dot{z}_w = (f_{22}(x_1) - Lf_{12}(x_1))z_w + \rho(x_1, u)$$

If we consider the error $e = z_w - w = z_2 - x - 2$, we have that its evolution is governed by

$$\dot{e} = (f_{22}(x_1) - Lf_{12}(x_1))e$$

Therefore, if under appropriate assumptions we can find L for which this system is stable, we can asymptotically estimate x_2.

This kind of tricks can be used even with other classes of systems for instance not necessarily u-affine as in the next example.

Example 13.1 *Consider a system of the form*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x_1, u) \end{aligned}$$

where $y = x_1$ is the measured variable. This is the the typical equation of a mechanical system including the levitator. Define $w = x_2 - Lx - 1$ to derive

$$\dot{w} = -Lw + f(x_1, u) - L^2 x_1$$

for which we can construct the observer

$$\dot{z}_w = -Lz_w + f(x_1, u) - L^2 x_1$$

Note that this observer is tolerant against small variations of the model f. Indeed, if the model is not accurate so that the true system is $\dot{x}_2 = f(x_1, u) + \Delta$ the error equation becomes

$$\dot{e} = -Le + \Delta$$

which is small as long as $\|\Delta\|$ is small.

13.3 Domain of attraction

A classical application of the Lyapunov theory is the determination of a domain of attraction [26]. Let us consider the basic idea beyond the technique. Assume that the system

$$\dot{x}(t) = f(x(t))$$

with $f(0) = 0$ is locally stable and its linerization admits a local Lyapunov function $\Psi(x)$. Then as already mentioned, there exists a positive value κ such that $\mathcal{N}[\Psi, \kappa]$ is a domain of attraction. The problem is how to determine one of such κ. Such a number κ must be such that

$$\max\{\kappa : \mathcal{N}[\Psi(x), \kappa] \subset \mathcal{N}[\dot{\Psi}(x), 0]\}$$

This can be achieved by considering the following problem

$$\kappa = \min\{\Psi(x) \geq \delta : \ \dot{\Psi}(x) = 0\} - \epsilon$$

where the small numbes $\delta > 0$ and $\epsilon > 0$ are necessary to eliminate the trivial solution at the origin and to avoid points with zero derivative. It is obvious that if the system is uncertain $\dot{x} = f(x, w)$, then the problem can be modified as follows. Define (with an abuse of notation)

$$\dot{\Psi}_{max}(x) = \max_{w \in \mathcal{W}} \dot{\Psi}(x, w).$$

Then

$$\kappa = \min\{\Psi(x) \geq \delta : \ \dot{\Psi}_{max}(x) = 0\} - \epsilon$$

The choice of the function Ψ basically fixes the "shape" of the domain. Unfortunately this choice is critical as shown in the next example.

Example 13.2 *Consider the following system*

$$\begin{array}{rcl} \dot{x}_1(t) & = & -[x_1(t) - x_1^3(t)] - x_2(t) \\ \dot{x}_2(t) & = & x_1(t) - x_2(t) \end{array}$$

and the function

$$\Psi(x) = x_1^2 + x_2^2$$

So that

$$\dot{\Psi}(x) = -2x_1^2 - 2x_2^2 + 2x_1^4$$

it is easy to see that as $\epsilon \to 0$ $\kappa \to 1$. So the corresponding domain is the unit circle (see Fig. 7). A different possibility is the following. Write the nonlinearity between square brackets as

$$-[x_1(t) - x_1^3(t)] = -[1 - w]x_1, \quad \text{with} \ \ w = x_1^2$$

Now if we impose bounds of the form

$$|x_1| \leq \bar{x}_1 \tag{98}$$

we have

$$|w| \leq \bar{w} = \sqrt{\bar{x}_1} \tag{99}$$

Then we can consider the uncertain linear systems with matrix

$$A(w) = \left[\begin{array}{cc} -[1 - w] & -1 \\ 1 & -1 \end{array} \right]$$

and compute the largest domain of attraction subject to the constraints (98) and (99). In Figure 7, this domain is computed for $\bar{x}_1 = 3/2$ and $\bar{w} = \sqrt{3/2}$ by means of a polyhedral Lyapunov functions. The vertices of the unit ball are denoted by "". The "true" domain of attraction has been also represented by means of several trajectories by backward integration starting from several initial conditions inside the previous domain of attractions. Clearly this example show the limits of the determination of the domain of attraction achieved by guessing the shape of the function or by the merging in an uncertain system.*

13.4 High gain adaptive control

An interesting application associated with a control Lyapunov function is the so called high–gain adaptive control. Consider, for instance, an uncertain system which is control–affine and with constant and certain input matrix.

$$\dot{x}(t) = f(x(t), w(t)) + Bu(t), \quad w(t) \in \mathcal{W}$$

and assume that a control Lyapunov function Ψ is given. We have seen that, if Ψ is smooth, a controller can be given in the form

$$u(t) = -\gamma B^T \nabla \Psi(x) \tag{100}$$

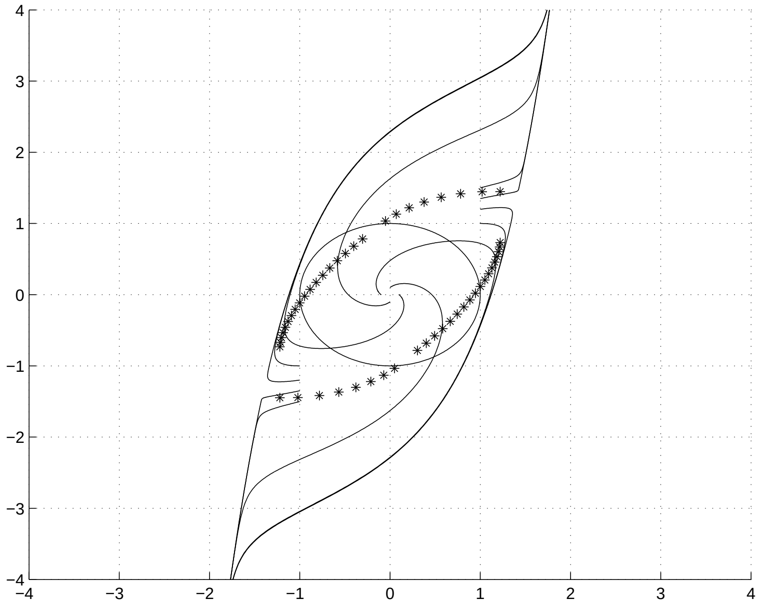


Figure 7: The true and the estimated domains of attraction

with $\gamma > 0$ "large". What does large mean? We have seen that, given any stabilizing control $\hat{\Phi}(x)$, associated with Ψ, if the following condition holds

$$\|\gamma B^T \nabla\Psi(x)\| \geq \|\Phi(x)\|$$

then γ is an appropriate value. However, the stabilizing control Φ could be not available. One possibility is that of achieving the so called λ–tracking by means of the following adaptive scheme [32] [33]

$$\begin{aligned} u(t) &= -\gamma(t)B^T\nabla\Psi(x) && (101)\\ \dot{\gamma}(t) &= \mu\sigma_\lambda(\Psi(x(t)) && (102)\\ \gamma(0) &= \gamma_0 \geq 0 && (103) \end{aligned}$$

where $\sigma_\lambda(\xi)$, with $\xi \geq 0$ is the following threshold function

$$\sigma_\lambda(\xi) = \begin{cases} 0 & \text{if} \quad 0 \leq \xi \leq \lambda \\ \xi - \lambda & \text{if} \quad \xi \geq \lambda \end{cases}$$

Basically this function is such that there is no adaptation as soon as

$$\Psi(x(t)) \leq \lambda$$

where λ represents a small tolerance. Now assume that there exist a *unknown* value $\bar{\gamma}$ such that (100) assures practical stability $\mathcal{N}[\Psi, \lambda]$ in the sense that

$$\dot{\Psi}(x,w) = \nabla\Psi(x)[f(x,w) + B(-\bar{\gamma}B^T\nabla\Psi(x))] \leq -\phi(\|x(t)\|), \quad \forall\ w \in \mathcal{W} \ \text{ and for } \ \Psi(x) \geq \lambda,$$

As we have seen the same inequality is verified for any $\gamma \geq \bar{\gamma}$. Then the adaptive scheme (101)–(102) converges in the sense that as $t \to \infty$

$$\gamma(t) \rightarrow \gamma_\infty < \infty, \quad (104)$$

$$\sigma_\lambda(\Psi(x(t))) \rightarrow 0. \quad (105)$$

then the gain is asymptotically finite and $\Psi(x(t))$ is asymptotically smaller or equal to λ.

The significance of this approach can be explained in the next points.

- It solves the problem due to the ignorance of $\bar{\gamma}$.
- It avoids too large values of γ since the adaptation stops as soon as the condition $\Psi(x) \leq \lambda$ is satisfied.

In particular the last condition is important because too large values of γ imply excessive control exploitation and, as it is know, can excite high–frequency neglected dynamics and produce instability.

This kind of scheme works under appropriate minimum–phase and relative degree assumptions in the case of output feedback.

Example 13.3 *Consider the following system*

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= w(t)sin(x_1(t)) + u(t)\end{aligned}$$

with

$$|w| \leq \bar{w}$$

and the control–Lyapunov function $\Psi(x_1, x_2) = \frac{1}{2}\, x^T P x$ *with*

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

the corresponding control is then

$$u(t) = -\gamma(t) B^T P x = -\gamma(t)(x_1 + 2x_2)$$

To show that Ψ *is a control Lyapunov function it is sufficient to consider its Lyapunov derivative. Simple computations yield*

$$\dot{\Psi}(x,\gamma) = -\gamma x_1^2 + (2-4\gamma)x_1x_2 - (4\gamma-1)x_2^2 + (x_1+2x_2)wsin(x_1) \quad (106)$$

$$\leq -\gamma x_1^2 + (2-4\gamma)|x_1||x_2| - (4\gamma-1)x_2^2 + \bar{w}x_1^2 + \bar{w}|x_1||x_2| \quad (107)$$

$$= -(\gamma-\bar{w})x_1^2 + 2(1+\bar{w}-2\gamma)|x_1||x_2| - (4\gamma-1)x_2^2 < 0 \quad (108)$$

for $\gamma >$ *large enough (the last inequality is simply derived by checking the nature of the quadratic form). Figure 8 shows the variables* x_1*,* x_2 *and the gain* $\gamma(t)$ *with* $\mu = 1$ *and* $\lambda = 0.01$*. In this case we obtained the limit value* $\kappa_\infty = 14.93$*.*

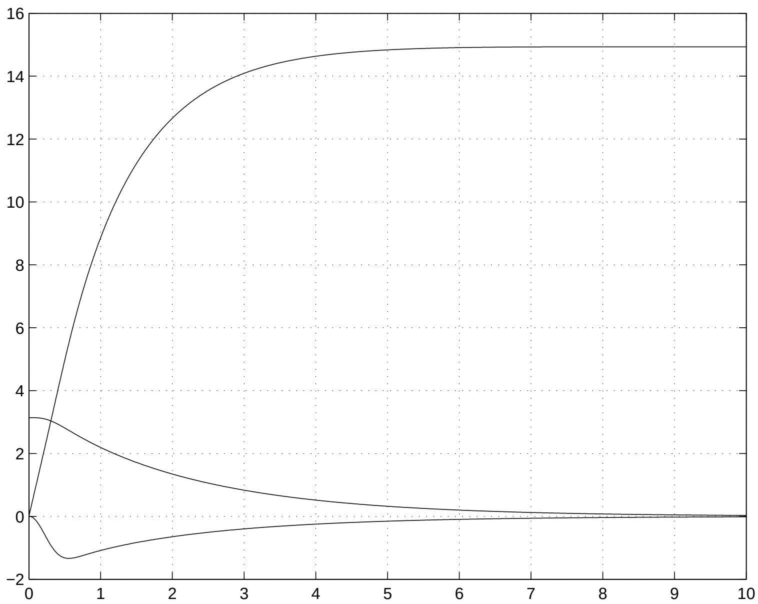


Figure 8: The gain evolution (upper curve) and the state evolution

13.5 Constrained control

It is well known that one of the most crucial problems in the applications is the presence of hard constraints on both the state and input variables. The Lyapunov theory offers an interesting point of view for the problem [37]. Assume that constraints of the form

$$x(t) \in \mathcal{X}, \qquad u(t) \in \mathcal{U},$$

with $\mathcal{X}$ and $\mathcal{U}$ including the origin as an interior point, are assigned. Given any stabilizing compensator, it is know that, in general, such constraints can be satisfied only if the state is close enough to the origin. Therefore the typical problem is that of evaluating the set of initial conditions from which the constraints are not violated. Assume that there exists a control Lyapunov function $\Psi(x)$ associated with a control $u = \Psi(x)$. Consider the control–admissible set

$$\mathcal{X}_u = \{x : \ \Phi(x) \in \mathcal{U}\}$$

and consider any value κ such that

$$\mathcal{N}[\Psi, \kappa] \subseteq \mathcal{X}_u \bigcap \mathcal{X}$$

then any state $x(0) \in \mathcal{N}[\Psi, \kappa]$ is such that

$$x(t) \in \mathcal{X}$$

$$u(t) \in \mathcal{U}$$

for all $t \geq 0$. In few words $\mathcal{N}[\Psi, \kappa]$ becomes a *safety set* in the sense that any trajectory starting inside it will not violate the constraints.

Let us consider now the case of a linear system

$$\dot{x}(t) = A(w(t))x(t) + B(w(t))u(t),$$

$$\begin{aligned} A(w) &= \sum_{i=1}^{s} w_i A_i, \qquad \sum_{i=1}^{s} w_i = 1, \quad w_i \geq 0, \\ B(w) &= \sum_{i=1}^{s} w_i B_i, \qquad \sum_{i=1}^{s} w_i = 1, \quad w_i \geq 0, \end{aligned}$$

And assume for instance that a linear quadratic stabilizing controller $u = Kx$ has been computed. Assume that $\mathcal{U}$ is the unit ball of the ∞–norm (namely that the constraints are $|u_i| \leq 1$. Then

$$\mathcal{X}_u = \{x : \quad \|K_i x\|_\infty \leq 1\}$$

It is easy to compute an ellipsoidal set of the form

$$\mathcal{E} = \{x : \ x^T P x \leq 1\}$$

which is invariant and included in $\mathcal{X}_u$. Indeed denoting by $Q = P^{-1}$ we have that $\mathcal{E} \subseteq \mathcal{X}_u$ if and only if

$$k_i Q k_i^T \leq 1 \tag{109}$$

Let $\hat{A}_i = A_i + B_i K$, then the invariance condition for $\mathcal{E}$ is

$$Q\hat{A}_i^T + \hat{A}_i Q < 0 \tag{110}$$

Then (109) and (110) form a convex set in the parameter Q. One therefore can optimize the size of this set for instance the volume (see [20]) for further details.

It is known that determining a domain of ellipsoidal domain of attractions is not efficient in terms of the accuracy of the representation since the actual stability domain can be quite larger than the derived one. Other type of domains of attraction can be determined by means of polyhedral Lyapunov functions using a procedure similar to that previously described. In this case, the advantage is that for linear (even uncertain) systems with linear constraint the largest admissible domain can be arbitrarily closely approximated by a polyhedron and therefore the polyhedral functions are non–conservative from this point of view. If we have uncertain nonlinear systems the domain of attraction is in general non–convex and in general very hard to determine. However, an approximated solution can be derived by considering absorbing equations as in the following example.

Example 13.4 *Let us consider the magnetic levitator of Figure 6 and the "absorbing" equation (94). The corresponding matrices are*

$$A = \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ -b \end{bmatrix}$$

with

$$1507 \leq a \leq 2177, \qquad 17.8 \leq b \leq 28$$

We have considered the control action

$$u = 120(0.025x_1 + 0.022x_2)$$

A feasible initial condition set corresponding to $|u| \leq 0.5$ *is the polygon whose vertices are the columns of the matrix* $[X, -X]$ *where* X *is reported next*

$$X = \begin{bmatrix} 0.005 & 0.005 \\ 0 & -0.398 \end{bmatrix}$$

In Figure (9) the estimated and the "true" safety set are reported. The "true" region is estimated for the nonlinear model (assuming known parameters) by integrating the full force trajectories.

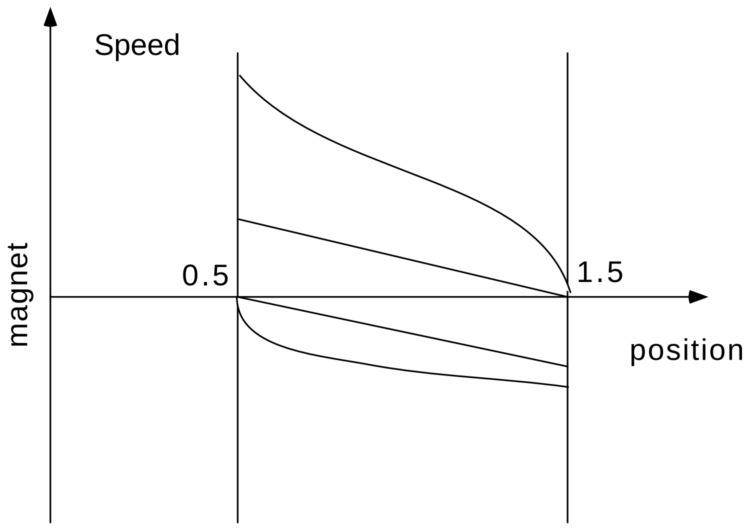


Figure 9: The estimated (polygonal) and the true and safety regions

13.6 Robot Manipulators

Robotics is one of the most popular application field of control theory. Before considering a specific application let us consider the following result concerning the presence of uncertainty in the input channel. Consider the system

$$\dot{x}(t) = f(x(t), w(t)) + B(I + \Delta_B)u \tag{111}$$

where $\Delta_B(x(t), w(t))$ represents a matched uncertainty. Assume that $\Psi(x)$ is a control Lyapunov function with the control

$$u = -\gamma B^T \nabla\Psi(x)^T$$

(possibly outside a certain neighborhood $\mathcal{S}$) for the system with $\Delta_B = 0$. Then it is also a Control Lyapunov function for the system (111) provided that

$$\Delta_B^T \Delta_B < \lambda^2 I, \quad \text{with} \quad \lambda < 1. \tag{112}$$

Indeed, consider the Lyapunov derivative

$$\begin{aligned} \dot{\Psi}(x) &= \nabla\Psi(x) f(x(t), w(t)) - \gamma\nabla\Psi(x) B B^T \nabla\Psi(x)^T - \gamma\nabla\Psi(x) B \Delta_B B^T \nabla\Psi(x)^T = \\ &= \nabla\Psi(x) f(x(t), w(t)) - (1-\lambda)\gamma\nabla\Psi(x) B B^T \nabla\Psi(x)^T \underbrace{-\gamma\nabla\Psi(x) B(\lambda I - \Delta_B) B^T \nabla\Psi(x)^T}_{\leq 0} \\ &\leq \nabla\Psi(x) f(x(t), w(t)) - \hat{\gamma}\nabla\Psi(x) B B^T \nabla\Psi(x)^T \end{aligned}$$

where $\hat{\gamma} = (1-\lambda)\gamma$. then, for large enough we have $\dot{\Psi}(x) \leq -\phi(\|x\|)$, for some κ–function ϕ by assumption. Therefore the prize to contrast Δ_B is to increase γ as $\gamma = \hat{\gamma}/(1-\lambda)$. It is not difficult to see that if $\gamma(x)$ (and then $\hat{\gamma}(x)$) are functions of x, the property still holds.

Let us consider now the typical equation of a robotic manipulator.

$$M(q(t))\ddot{q}(t) + H(q(t),\dot{q}(t))\dot{q}(t) + G(q(t)) = \tau(t) \tag{113}$$

where $q(t)$ represents the free parameters m-vector, $\dot{q}(t)$ the corresponding velocity vector and τ is the torque vector. The mass matrix $M(q)$ is assumed invertible for all q A typical procedure to design a control is that of considering a pre–compensator of the form

$$\tau(t) = M(q)u(t) + H(q,\dot{q})\dot{q} + G(q) \tag{114}$$

so that the resulting system is

$$M(q(t))[\ddot{q}(t) - u(t)] = 0$$

In view of the invertibility of $M(q)$ this system is equivalent to the m decoupled equations

$$\ddot{q}_i(t) = u_i(t),$$

whose control is an elementary task.

The problem with this standard procedure is that the control (114) cannot be implemented because the model is uncertain and the true pre–compensator is based on a approximated model

$$\tau = \tilde{M}(q)u(t) + \tilde{H}(q,\dot{q})\dot{q} + \tilde{G}(q) \tag{115}$$

If we apply the control (115) the resulting equation becomes

$$\begin{aligned}\ddot{q}(t) &= u + \underbrace{M(q)^{-1}[\tilde{M}(q) - M(q)]}_{\Delta_B}u(t) + \\ &+ \underbrace{M(q)^{-1}[\tilde{H}(q,\dot{q}) - H(q,\dot{q})]\dot{q} + M(q)^{-1}[\tilde{G}(q) - G(q)]}_{\Delta_f}\end{aligned}$$

or, in a simpler form

$$\ddot{q}(t) = [I + \Delta_B]u(t) + \Delta_f \tag{116}$$

Now, under the assumption that $\tilde{M}(q)$ is a sufficiently accurate estimate of the mass matrix $M(q)$ we can assume that

$$\Delta_B = M^{-1}(\tilde{M} - M) \tag{117}$$

satisfies (112), along with the following boundedness condition for Δ_f

$$\|\Delta_f(q,\dot{q})\| \leq \alpha + \beta\|x\|. \tag{118}$$

Then the the next deign procedure can be used.

Procedure 13.1

1. *Assume $\Delta_B = 0$ and $\Delta_f = 0$ and consider nominal system whose state vector is $x = [q^T, \dot{q}^T]^T$ by setting*

$$A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

2. *Find a control Lyapunov function associated with the matrix $P > 0$, by considering the equation*
$$QA^T + AQ + R^T B^T + BR = -S < 0$$
where $Q = P^{-1}$ (see Subsection 8.2) and consider the state feedback controller
$$u(t) = -\gamma B^T P x(t)$$
which stabilizes the nominal system with $\gamma = 1$.

3. *Assure uniform ultimate boundedness for the actual plant by taking γ large enough.*

The fact that such a large γ, as required in the last step of the procedure, exist can be explained since in view of the bound (118) and (117). First assume $\Delta_B = 0$. By elementary analysis one can show that
$$\alpha \|B^T Sx\| - \gamma \|B^T Sx\|^2 \leq \frac{\alpha}{2\gamma}.$$
Fix any small ϵ and consider the ϵ-ellipsoid
$$\mathcal{E} = \{x : \quad x^T P x \leq \epsilon\}.$$
Let γ_ϵ such that, for all states in the complement $x \notin \bar{\mathcal{E}}$
$$\frac{x^T Sx}{2} \geq \frac{\alpha}{2\gamma_\epsilon}.$$
Denoting by $\sigma_{min}(S)$ the smallest eigenvalue of S, we get for $x \notin \mathcal{E}$
$$\begin{aligned}
\dot{\Psi}(x) &= -x^T Sx + x^T SB\Delta_f - \hat{\gamma} x^T SBB^T Sx \\
&\leq -x^T Sx + \alpha \|B^T Sx\| + \beta \|B^T Sx\| \|x\| - \hat{\gamma} \|B^T Sx\|^2 = \\
&= -x^T Sx/2 + \beta \|B^T Sx\| \|x\| - (\gamma - \gamma_\epsilon) \|B^T Sx\|^2 \\
&+ \underbrace{-x^T Sx/2 + \alpha \|B^T Sx\| - \hat{\gamma}_\epsilon \|B^T Sx\|^2}_{\leq 0} \\
&\leq -\sigma_{min}(S) \|x\|^2/2 + \beta \|B^T Sx\| \|x\| - (\hat{\gamma} - \gamma_\epsilon) \|B^T Sx\|^2.
\end{aligned}$$
The last expression is a quadratic form in the variables $\|x\|$ and $\|B^T Sx\|$ and by taking $\hat{\gamma}$ large enough, we can render it negative definite. Then if we have (117), we can take $\gamma = \hat{\gamma}/(1 - \lambda)$ as shown at the beginning of the subsection.

The practical problem is that of determining the bounds (118) (117). However it is fundamental to establish that, for γ large enough, the system is asymptotically driven in the ellipsoid, then adaptive scheme proposed in section 13.4 is very useful.

Let us finally consider an extended model of a manipulator with elastic joints. A typical approximation is to model the elasticity as linear springs at the joints. The resulting model is of the form

$$\begin{aligned}
D(q_2)\ddot{q}_2 + C(\dot{q}_1, q_1)\dot{q}_1 + g(q_1) + K(q_1 - q_2) &= 0 \\
J\ddot{q}_2 + K(q_2 - q_1) &= \tau
\end{aligned}$$

It is not difficult to see that this model, if written in terms of first order equations, is in a (multi–input) strict feedback form. For more details on this problem the reader is referred to the paper [53] and the reference inside.

13.7 Switching systems

A recent field in which the Lyapunov theory has been exploited is the analysis and control of switching systems [40]. A switching system is a system of the form

13.8 Switching systems

$$\dot{x}(t) = f(x(t), u(t), q(t))$$

with

$$q(t) \in \mathcal{Q},$$

where $\mathcal{Q}$ is a finite set. Typically the following three important cases have to be considered for the switching signal $q(t)$

- $q(t)$ is an uncontrolled (exogenously determined) unknown to the controller;
- $q(t)$ is an uncontrolled (exogenously determined) but known to the controller;
- $q(t)$ is a controlled switching signal;

Let us first consider the case in which the system is uncontrolled and there is no input signal $u(t)$. If we are interested in a Lyapunov function Ψ for the system

$$\dot{x}(t) = f(x(t), p(t))$$

the we have to cope with the condition

$$\dot{\Psi}(x, p) = \nabla\Psi(x) f(x, p) \leq -\phi(\|x\|). \quad \forall p \in \mathcal{P}$$

Without restriction assume that p is an index belonging to a finite set $\{1, 2, \ldots, N_P\}$. Then the function Ψ satisfies the previous condition if and only if it satisfies the condition

$$\dot{\Psi}(x, \alpha) = \sum_{p=1}^{N_p} \alpha_p \ \ \nabla\Psi(x) f(x, p) \leq -\phi(\|x\|)$$

with $\sum_{p=1}^{N_p} \alpha_p = 1$, and $\alpha_p \geq 0$. In particular, in the linear case, the stability of the switching system

$$\dot{x}(t) = A_p x(t)$$

is equivalent to the stability of the polytopic system $\dot{x}(t) = [\sum_{p=1}^{N_p} \alpha_p A_p]\, x(t)$, and then analyzed as previously described.

The situation is completely different when the switching is controlled. In this case the negativity of the Lyapunov derivative has not to be satisfied for all $p \in \mathcal{P}$ but for some p. A typical sufficient condition for this condition to be achieved is the following: assume that there exists a Lyapunov stable system in the convex hull of the points $f(x, p)$, precisely there exists a system

$$\dot{x}(t) = \bar{f}(x)$$

such that

$$\bar{f}(x) \in conv\{f(x, p),\ p = 1, 2, \ldots, N_p\}$$

and that admits a Lyapunov function

$$\nabla\Psi(x)\bar{f}(x) \leq -\phi(\|x\|)$$

then there exists a stabilizing switching strategy of the form

$$p = \Phi(x) = \arg\min_p \quad \nabla\Psi(x)f(x,p)$$

Indeed it is immediately seen that

$$\min_p \quad \nabla\Psi(x)f(x,p) \leq \nabla\Psi(x)\bar{f}(x) \leq -\phi(\|x\|)$$

In the case of a linear plant

$$\dot{x}(t) = A_i x(t), \quad 1 = 1, 2, \dots, r$$

the problem is easily solved if there exists

$$\tilde{A} \in conv\{A_i\}$$

which is stable. Consider any Lyapunov function for the system $\dot{x} = \tilde{A}x$, i.e. a symmetric positive definite matrix P such that

$$\tilde{A}^T P + P\tilde{A} \leq -x^T Q x, \qquad Q > 0$$

Then the switching law is

$$\Phi(x) = \arg\min_i \quad \dot{\Psi}(x) = \arg\min_i \quad x^T P A_i x,$$

which assures the condition

$$\dot{\Psi}(x) \leq -x^T Q x$$

Unfortunately the existence of such a stable element of the convex hull is not necessary. For instance the system given by the pair of matrices

$$A(w) = \begin{bmatrix} 0 & 1 \\ -1+w & -a \end{bmatrix}$$

where $a < 0$ is not stable for any value of a. $w = \pm\bar{w}$ is a switching parameter. However, if a is small enough then there exists a suitable stabilizing strategy.

Let us now consider the problem of stabilization of the continuous time linear system

$$\dot{x}(t) = A_i x(t) + B_i u(t)$$

Again, if the switching is uncontrolled (known or unknown to the controller), then the problem can be faced by finding a common control Lyapunov function. The case of *controlled switching* can be faced as follows. Assume that in the convex hull of the systems there is a stabilizable systems

$$[\tilde{A}, \tilde{B}] = [\sum_{i=1}^{s} w_i A_i, \sum_{i=1}^{s} w_i B_i], \quad \sum_{i=1}^{s} w_i = 1, \quad w_i \geq 0.$$

and consider a stabilizing feedback $u = \tilde{K}x$. Then

$$\tilde{A} + \tilde{B}\tilde{K} = \sum_{i=1}^{s} w_i [A_i + \tilde{K}B_i] = \sum_{i=1}^{s} w_i [\hat{A}_i]$$

is stable and therefore there exists a switching strategy which stabilizes the system. As we have observed before, this solution is conservative.

A typical example of application is a system with quantized control.

Example 13.5 *The following model represents the two–tank hydraulic system shown in Figure 10.*

$$\begin{aligned}\dot{h}_1(t) &= -\alpha\sqrt{x_1(t)+\bar{h}_1-x_2(t)-\bar{h}_2}+u(t)+\bar{q}\\ \dot{h}_2(t) &= \alpha\sqrt{x_1(t)+\bar{h}_1-x_2(t)-\bar{h}_2}-\beta\sqrt{x_2(t)+\bar{h}_2}\end{aligned}$$

where α and β are unknown positive parameters and where $\bar{h}_1$, $\bar{h}_2$ and $\bar{q}$ are the steady–state water levels and incoming flow satisfying the conditions

$$\bar{h}_1=\left(\frac{\bar{q}}{\alpha}\right)^2+\left(\frac{\bar{q}}{\beta}\right)^2, \qquad \bar{h}_2=\left(\frac{\bar{q}}{\beta}\right)^2.$$

Figure 10: The two–tanks hydraulic system

Let us now consider the candidate control Lyapunov function

$$\Psi(x)=\frac{1}{2}\left(x_1^2+x_2^2\right)$$

The corresponding Lyapunov derivative for $x\in\mathcal{N}[\Psi,\bar{h}_2^2/2]$ (this value is chosen in such a way that the ball is included in the positive region for the true levels $\bar{h}_i+x_i$)

$$\begin{aligned}&\dot{\Psi}(x,u)=\\ =\;&\underbrace{\alpha(x_1-x_2)\left(\sqrt{x_1(t)+\bar{h}_1-x_2(t)-\bar{h}_2}-\bar{q}\right)+x_2\left(\beta\sqrt{x_2(t)+\bar{h}_2}-\bar{q}\right)}_{\doteq\phi_N(\|x\|)}+x_1u\le\\ -\;&\phi_N(x_1,x_2)+x_1u\end{aligned}$$

Note that $-\phi_N(x_1,x_2)$, the natural derivative achieved for $u=0$, is negative definite, which confirms the system stability. Note that this is true for the nonlinear dynamics inside the considered region.

Now let us assume that the control input admits three admissible values

$$u(t) \in \{-\bar{q}; 0, \bar{q}\}$$

which correspond to the three cases in which none, just one or both the switching valves are open.

However, to assure a certain degree of contractivity, one can use the following switching control law

$$u = -\bar{q}\ sgn(x_1)$$

which renders the convergence to the equilibrium much faster. Clearly if α *and* β *may vary, this produces an offset. It can be shown that, with the considered control, only the offset on the second tank* $x_2 \neq 0$ *is possible.*

As it is known, from a practical point of view, the discontinuous controller has to implemented with a threshold. We actually consider the function

$$u = \begin{cases} -\bar{q}\ sgn(x_1) & if \quad |x_1| > \epsilon \\ 0 & if \quad |x_1| \leq \epsilon \end{cases}$$

In Figures 11 and 12 show the experimental behavior with $\epsilon = 0.01$ *and* $\epsilon = 0.03$*, being the latter much subject to ripples as expected.*

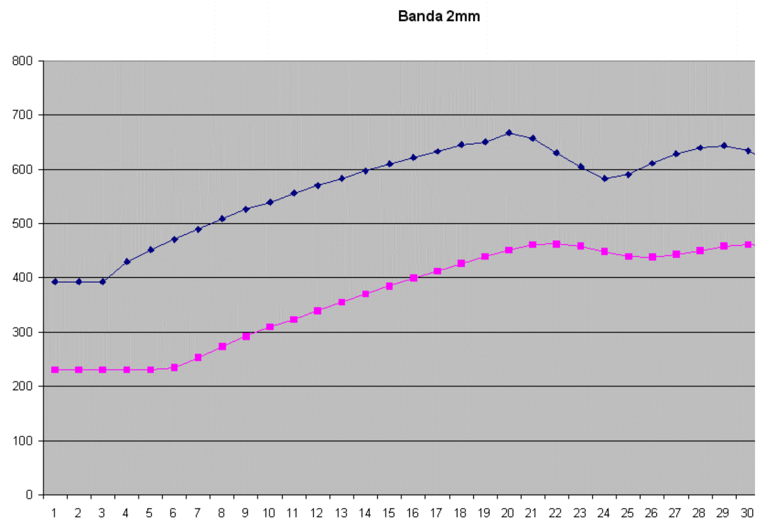


Figure 11: The experimental behavior of the plant with $\epsilon = 0.01$

As a final comment, the reader might be interested in understanding how the candidate Lyapunov functions has been chosen. The answer is that the linearized model in the equilibrium point admits a state matrix which is stable and symmetric. This implies that $\|x\|^2$ *is a Lyapunov function for the linearized natural dynamic.*

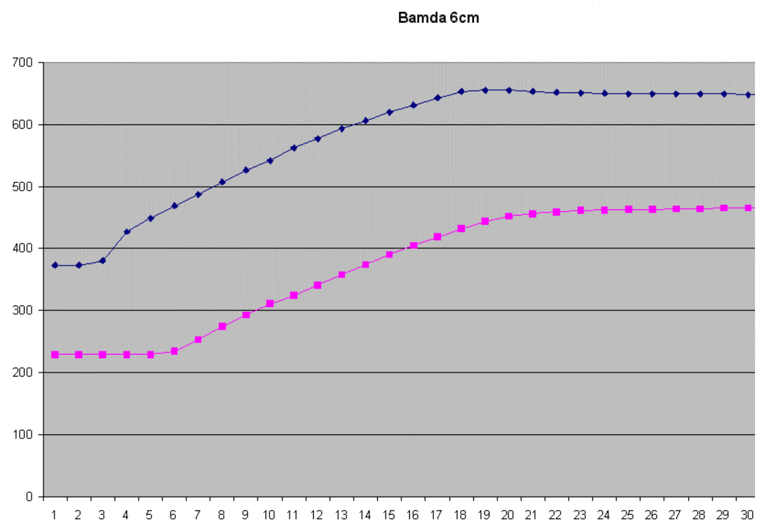


Figure 12: The experimental behavior of the plant with $\epsilon = 0.03$

13.9 Control of dynamical networks

Consider a class of dynamical networks modeled as follows

$$\dot{x}(t) = Sg(x(t)) + Rh(x(t)) + Bu(t) + d(t), \tag{119}$$

where $x(t) \in \mathbb{R}^n$ is the state, representing *buffer levels*, $u(t) \in \mathbb{R}^m$ is the control input, representing *controlled flows*, $d(t) \in \mathbb{R}^n$ is an exogenous signal, representing the *demand*. Vector functions g and h represent natural flows between two nodes within the system. Application examples include flowing of data, fluids and chemical reaction networks. A simple example is given by the two tank system previously considered.

Functions g and h have different physical meanings:

- g–type flows depend on the difference between the corresponding states
- h–type flows depend on the state of the starting node only.

For instance, in fluid systems, g–type flows between tanks depend on the fluid level in both tanks, while h–type flows depend on the fluid level in the upper tank only (the two cases are illustrated in Fig. 13). In the two–tank system, we have one g–flow between the two tanks, one h flow from the lower tank, one u–flow (controlled) from the external to the upper tank.

For each component of g and h, we have:

- $g_j = g_j(x_k - x_l)$, where $S_{kj} = -1$ and $S_{lj} = 1$;
- $h_j = h_j(x_k)$, where $R_{kj} = -1$.

A graph $\mathcal{N}$ with n nodes can be associated with the system; we assume that S, R and B are

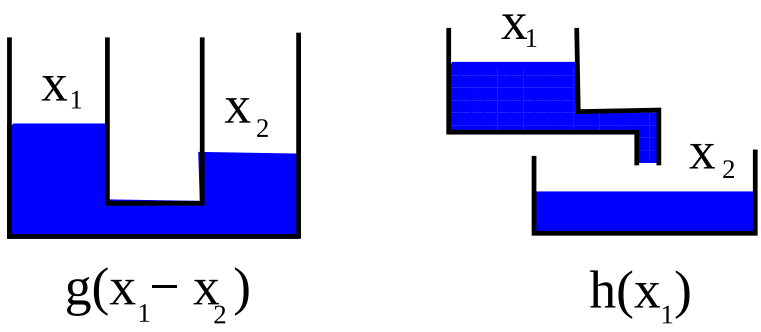


Figure 13: g–type (left) and h–type (right) flows in a fluid system.

incidence matrices: each of their columns has either two non–zero entries, equal to 1 and -1, or a single non–zero entry, equal either to 1 or to -1. These matrices correspond to the g flows, the h flows and the controlled u-flows.

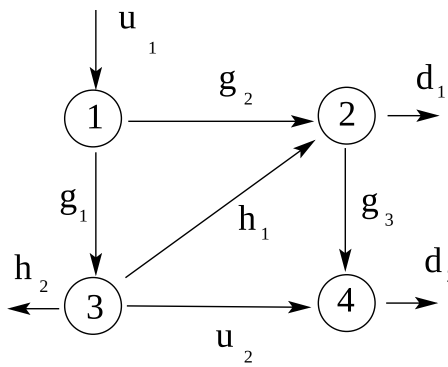


Figure 14: Graph of a network with natural dynamics (Example 13.6).

Matrix $[S\ R\ B]$ is the overall *incidence matrix* of the graph representing the network. In the network graph, we distinguish between *g–type*, *h–type*, *u–type* and *d–type* arcs, associated respectively with the components of vector g (*i.e.*, with the columns of S), of vector h (with the columns of R), of the control vector u (with the columns of B) and of the demand vector d. In general, any arc connected with a single node of the graph represents a connection with the external environment. This is the case, for instance, of arcs associated with vector d.

Denoting by M_j the jth column of a matrix M, we can write, according to the flow rules,

$$g_j = g_j(-S_j^\top x) \ \text{ and } \ h_j = h_j(-\tilde{R}_j^\top x), \tag{120}$$

where $[\tilde{R}]_{ij} = \min\{R_{ij}, 0\}$.

Example 13.6 *In the network graph in Fig. 14 there are g–type flows:*

$$g(x) = [g_1(x_1 - x_3) \quad g_2(x_1 - x_2) \quad g_3(x_2 - x_4)]^\top$$

h–type flows:

$$h(x) = [h_1(x_3) \quad h_2(x_3)]^\top$$

controlled flows:

$$u = [u_1 \quad u_2]^\top$$

and exogenous flows:

$$d = [d_1 \quad d_2]^\top$$

The system matrices are

$$S = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -1 & -1 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}, \; \tilde{R} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & -1 \\ 0 & 0 \end{bmatrix} \tag{121}$$

Remark 13.1 *All the results in this subsection are valid for any choice of matrices S, R, and B, which are not necessarily incidence matrices.*

We assume that functions $g_j(\cdot)$ and $h_j(\cdot)$ are smooth and have positive derivatives.

To analyze the stabilization problem, assume that the demand d is constant and that there exists an equilibrium $(\bar{x}, \bar{u})$ corresponding to such a demand

$$0 = Sg(\bar{x}) + Rh(\bar{x}) + B\bar{u} + d,$$

Without restriction we set $\bar{x} = 0$ $\bar{u} = 0$ and $d = 0$, a condition we can assure by a translation.

We assume that the control is component–wise bounded as

$$u^- \leq u \leq u^+. \tag{122}$$

with $u^- \leq 0 \leq u^+$.

As a first technical result, we need to observe the following. Each component of g can be regarded as a function of a single variable $g_j(\xi_j)$ where $\xi_j = x_k - x_l$ and $h_j(x_k)$. Then considering the derivative with respect to this variable [17]

$$g_j(\xi) = \left[\int_0^1 \frac{dg_j(\xi\sigma)}{d\xi} d\sigma\right] \xi = \delta_g(\xi)\ \xi$$

and

$$h_j(\xi) = \left[\int_0^1 \frac{dh_j(\xi\sigma)}{d\xi} d\sigma\right] \xi = \delta_h(\xi)\ \xi$$

where $\delta_g(\xi)$ and $\delta_h(\xi)$ are positive functions, we assume unknown, which are bounded above and below (by positive numbers) in any bounded ball centered in 0. With this in mind we can write

$$\dot{x}(t) = -S\Delta_g(x)S^\top x - R\Delta_h(x)\tilde{R}^\top x + Bu(t), \tag{123}$$

for some positive diagonal matrices $\Delta_g(x)$ and $\Delta_h(x)$.

We consider a saturated type of control. Define componentwise the *saturation function* $\text{sat}(v)$ as follows

$$[\text{sat}(v)]_j = \begin{cases} v_j^+ & \text{if} \quad v_j > v_j^+ \\ v_j & \text{if} \quad v_j^- \leq v_j \leq v_j^+ \\ v_j^- & \text{if} \quad v_j < v_j^- \end{cases} \tag{124}$$

We consider the following *saturated network–decentralized control* ([16, 17]):

$$u = \text{sat}(-\gamma B^\top x), \tag{125}$$

where γ is a positive gain (the higher γ, the stronger the control action). This control is suitable for applications in which the controlled flow in each link is decided based on the departure and arrival nodes.

A control similar to (125) is

$$v = \text{sat}(-\gamma \tilde{B}^\top z), \tag{126}$$

where $\tilde{B} = \min\{B, 0\}$, componentwise. In the strategy (126), the controlled flow in each link is decided based on the departure node only. This control is suitable for applications such as data transmission in which each node locally decides the outflows.

We have to notice that the saturation function, if we assume that the upper bound is positive and the lower one negative, can be written as follows

$$\text{sat}(\xi) = \delta_u(\xi)\xi$$

with positive $\delta_u(\xi)$. Hence

$$\text{sat}(-\gamma B^\top x) = -\Delta_u(x)\gamma B^\top x, \tag{127}$$

To recap, we have that in any ball centered in 0 all diagonal elements of the matrices $\Delta_g(x), \Delta_h(x), \Delta_u(x)$ are bounded by positive numbers above and below.

$$0 < \nu \leq [\Delta_g(x)]_k, [\Delta_h(x)]_j, [\Delta_u(x)]_h \leq \mu$$

We consider two cases. The first one in that in which all flows are of the g type

$$\dot{x}(t) = Sg(x(t)) + Bu(t),$$

for this case we can take as Lyapunov candidate $\Psi(x) = x^\top x/2$. Then we get the following

$$\dot{\Psi}(x) = -x^\top S\Delta_g(x)S^\top x + x^\top Bu$$

If we apply the saturated control

$$\begin{aligned} \dot{\Psi}(x) &= -x^\top S\Delta_g(x)S^\top x - x^\top B\Delta_u(x)\gamma B^\top x \\ &= -x^\top \begin{bmatrix} S & B \end{bmatrix} \begin{bmatrix} \Delta_g(x) & 0 \\ 0 & \Delta_u(x) \end{bmatrix} \begin{bmatrix} S^\top \\ B^\top \end{bmatrix} x \doteq -x^\top Q(x)x \end{aligned}$$

The matrix $Q(x)$ is symmetric positive semidefinite and it is positive definite if and only if

$$\text{rank} \begin{bmatrix} S & B \end{bmatrix} = n$$

Then we have asymptotic stability if and only if this condition holds. Sufficiency is obvious. Conversely, if the condition is not satisfied, take a vector $z \neq 0$ such that $z^\top \begin{bmatrix} S & B \end{bmatrix} = 0$. Hence considering the Lyapunov–like function $V(x) = z^\top x$ we get $\dot{V} = 0$ hence

$$V(x(t)) = z^\top x(t) = z^\top x(0)$$

so stability cannot hold. In the context of fluid networks, this basically means that, if there are no connections with the external environment, one can take $z^\top = \bar{1}^\top$ and then verify that the overall quantity of fluid remains constant.

Let us now consider the asymmetric case. In this case, the squared euclidean norm is not suitable. Indeed consider again the saturated control (125). We get

$$\dot{x}(t) = -S\Delta_g(x)S^\top x - R\Delta_h(x)\tilde{R}_j^\top x - B\Delta_u(x)\gamma B^\top x = -Q(x)x \tag{128}$$

The problem now is that the matrix

$$Q(x) = \begin{bmatrix} S & R & B \end{bmatrix} \begin{bmatrix} \Delta_g(x) & 0 & 0 \\ 0 & \Delta_h(x) & 0 \\ 0 & 0 & \Delta_u(x) \end{bmatrix} \begin{bmatrix} S^\top \\ \tilde{R}^\top \\ B^\top \end{bmatrix}$$

is not symmetric and $-x^\top Q(x)x$ can be positive. However it is not difficult to see that it is column-diagonally dominant. A good candidate is then the 1–norm [43, 17]. For a diagonally dominant matrix with negative diagonal elements, the 1–norm is a weak Lyapunov function.

Under proper connectivity conditions this function is a Lyapunov function whose derivative is negative definite [43, 17]. The reader is referred to [18] for results along this line for chemical networks. A switching control strategy, valid in the symmetric case, is proposed in [19].

13.10 Surge suppression in compression systems

Consider the system represented in Fig. 15 in which a compressor inflates a compressible fluid

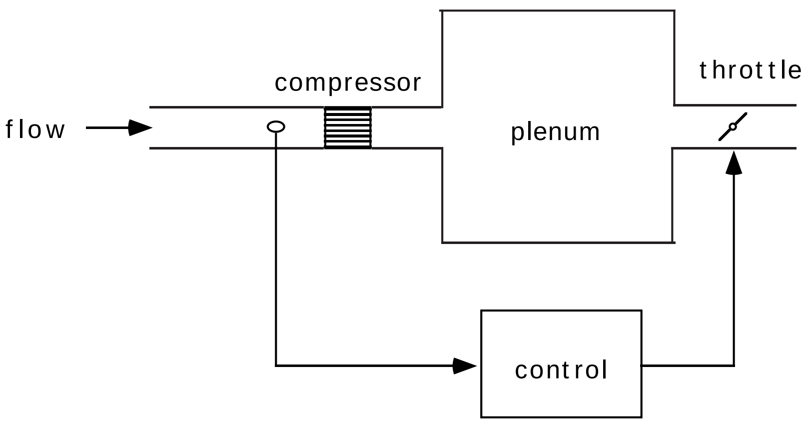


Figure 15: The compression system

in a plenum. We assume that the compressor works at constant angular speed and therefore impresses a pressure which is a function of the flow q in the pipe,

$$p_{comp} = \Psi(q)$$

The outgoing flow of fluid from the plenum is a function of the pressure in the plenum and of the throttle fraction opening a, having the form

$$q_{out} = \Gamma(p)a,$$

where $\Gamma(p)$ depends on the throttle characteristic. The overall dynamic model is

$$\begin{aligned} \dot{q} &= B[p_{comp} - p] = B[-p + \Psi(q)] \\ \dot{p} &= \frac{1}{B}[q - q_{out}] = \frac{1}{B}[q - \Gamma(p)a] \end{aligned}$$

where q and p represent the incoming flow in the plenum and the plenum pressure, respectively. B is a constant parameter, known as Helmholtz constant.

It turns out that for large values of a the system admits a stable equilibrium, but for small values, this point may become unstable and the system undergoes a limit cycle as shown in Fig. 16, left (blue curve), and in Fig. 16, right, where the time evolution is represented. In Fig. 16 left are also represented the typical compressor characteristics $p_{comp} = \Psi(q)$ (red curve) and the valve characteristic, $q_{out} = \Gamma(p)a$ (black curve), the intersection of which $(\bar{q}, \bar{p})$ is the equilibrium point.

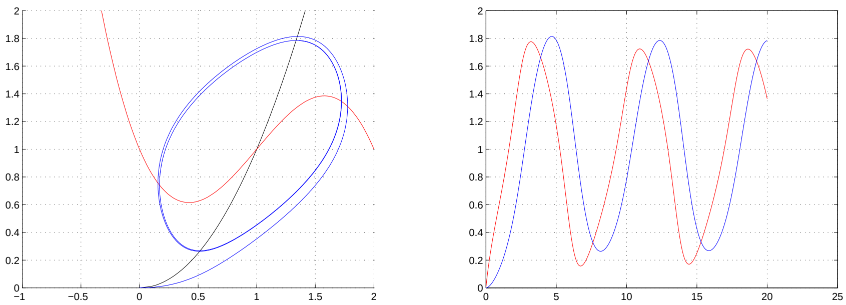


Figure 16: The compression system without control

Let us introduce the relative variables $x_1 = q - \bar{q}$ and $x_2 = p - \bar{p}$, and $u = a - \bar{a}$, where $\bar{q}$, $\bar{p}$ and $\bar{a}$ are the equilibrium values of the flow, the pressure and the valve fraction opening, respectively, which satisfy the equilibrium conditions

$$\begin{aligned} 0 &= B[-\bar{p} + \Psi(\bar{q})] \\ 0 &= \frac{1}{B}[\bar{q} - \Gamma(\bar{p})\bar{a}] \end{aligned}$$

If we subtract these expressions, the model becomes

$$\begin{aligned} \dot{x}_1 &= B[-x_2 + \Psi(x_1 + \bar{q}) - \Psi(\bar{q})] \\ \dot{x}_2 &= \frac{1}{B}[x_1 - \Gamma(x_2 + \bar{p})a + \Gamma(\bar{p})\bar{a}] \pm \Gamma(x_2 + \bar{p})\bar{a} \end{aligned}$$

The last added/subtracted term is a trick we need to arrive to a LDI. Let

$$\begin{aligned} \Psi(x_1 + \bar{q}) - \Psi(\bar{q}) &= \Delta_1(x_1)x_1 \\ \Gamma(x_2 + \bar{p})\bar{a} - \Gamma(\bar{p})\bar{a} &= \Delta_2(x_2)x_2 \\ \Gamma(x_2 + \bar{p})a - \Gamma(x_2 + \bar{p})\bar{a} &= \Delta_u(x_2)u \end{aligned}$$

Then we can write

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} B\Delta_1 & -B \\ \frac{1}{B} & -\frac{\Delta_2}{B} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{\Delta_u}{B} \end{bmatrix} u$$

We assume bounds

$$-a \leq \Delta_1 \leq b, \quad 0 < c \leq \Delta_2 \leq d, \quad 0 < e \leq \Delta_u \leq f$$

where a and b are positive. Note that Δ_1 can be positive, which explains why the system may be unstable. It is not difficult to see that if Δ_1 is negative ($\Delta_1 < 0$), then the equilibrium is stable, with Lyapunov function $\Psi(x_1, x_2) = (Bx_1^2 + x_2^2/B)2$. Unfortunately this is not a control Lyapunov function to adopt for stabilization if $\Delta_1 > 0$.

To stabilize this system we can adopt a backstepping approach. For brevity assume $B = 1$ (we can assure this condition by scaling the variables). As explained the first equation would be stabilized by the virtual control

$$\hat{x}_2 = \kappa_1 x_1 \tag{129}$$

because we would get

$$\dot{x}_1 = (-\kappa_1 + \Delta_1)x_1$$

stable as long as $\kappa_1 > b$. Since $\hat{x}_2$ is not a control, we use the actual control u to track $\hat{x}_2$, namely, to make $x_2 - \hat{x}_2$ small: $u = \kappa_2(x_2 - \hat{x}_2)$.

The proper change of variable is

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= x_2 - k_1 x_1 \end{aligned}$$

or

$$\begin{aligned} x_1 &= z_1 \\ x_2 &= z_2 + k_1 z_1 \end{aligned}$$

The control is selected in order to track the virtual control $\hat{x}_2 = \kappa_1 x_1$

$$u = \kappa_2(x_2 - \hat{x}_2) = \kappa_2(x_2 - \kappa_1 x_1) = \kappa_2 z_2 \tag{130}$$

We can compute the new equations of $\dot{z}_1$ and $\dot{z}_2$

$$\dot{z}_1 = \dot{x}_1 = (-\kappa_1 + \Delta_1)z_1 - z_2,$$

and $\dot{z}_2$

$$\begin{aligned} \dot{z}_2 &= \dot{x}_2 - k_1\dot{x}_1 = [x_1 - \Delta_2 x_2 - \Delta_u u] - \kappa_1[(-\kappa_1 + \Delta_1)z_1 - z_2] = \\ &= z_1 - \Delta_2(z_2 + k_1 z_1) + \kappa_1^2 z_1 - \kappa_1\Delta_1 z_1 + \kappa_1 z_2 - \Delta_u\kappa_2 z_2 \end{aligned}$$

Note that we used (130). Let $\Delta_3 \doteq 1 - \Delta_2 k_1 + \kappa_1^2 - \kappa_1\Delta_1$ and let $\Delta_4 \doteq -\Delta_2 + \kappa_1$. Both terms are bounded. Then we get

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -(\kappa_1 - \Delta_1) & -1 \\ \Delta_3 & -(\Delta_u\kappa_2 + \Delta_4) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

We assume that κ_1 has been fixed $\kappa_1 > b$ to virtually stabilize (129). This assures that $(\kappa_1 - \Delta_1) > 0$. Therefore we have only to decide κ_2. We take as candidate Lyapunov function

$$\Psi(z_1, z_2) = z_1^2 + z_2^2$$

then

$$\Psi(z_1, z_2) = \begin{bmatrix} z_1 & z_2 \end{bmatrix} \begin{bmatrix} -2(\kappa_1 - \Delta_1) & \Delta_3 - 1 \\ \Delta_3 - 1 & -2(\Delta_u \kappa_2 + \Delta_4) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

The $1, 1$ term of the symmetric matrix above is negative. This function is then negative if κ_2 is large enough to ensure

$$4(\kappa_1 - \Delta_1)(\Delta_u \kappa_2 + \Delta_4) - (\Delta_3 - 1)^2 > 2$$

The simulation are derived by assuming

$$\Psi(q) = -(q-1)^3 + (q-1) + 1$$

$$\Gamma(p) = \sqrt{(p)}$$

B=1, normalized equilibrium $\bar{q} = 1$, $\bar{p} = 1$ and $\bar{a} = 1$. Adopting the control gains $\kappa_1 = 1.2$ and $\kappa_2 = 4$ we achieve robust stability as shown in Fig. 17.

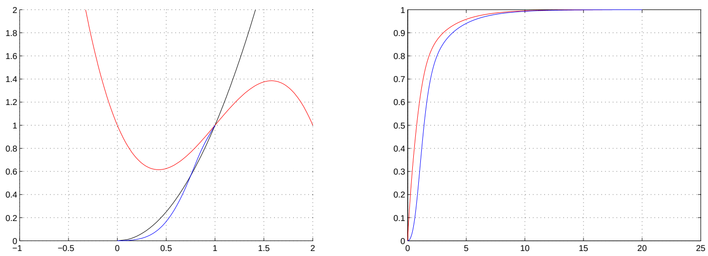


Figure 17: The compression system with control

13.11 Plant tuning

To be done.

14 Limits of the Lyapunov Theory

Among the several approaches adopted in the control of dynamical systems, that based on the Lyapunov theory has been the first one and undoubtedly one the most effective. This claim is evidenced by the fact that, although classical and well established in the literature, its tools are even today extensively exploited in many different practical and theoretical problems.

The Lyapunov theory presents several drawbacks that appear evident when it is applied to uncertain systems control. We can summarize the principal ones as follows.

- It is not always clear how to choose a candidate Lyapunov function.
- The theory basically works for state–feedback types of controls, the output feedback is still a very hard problem to be faced by means of these tools.

- The theory is conservative when we deal with constant uncertain parameter or slowly time–varying parameters.

However we still believe that Lyapunov approach is, after all, one of the most powerful to face the control problem of uncertain systems. The main advantages of the approach can be summarized as follows.

- Exploiting this theory is practically necessary when dealing with uncertain (especially nonlinear) systems with time–varying parameters.
- The theory proposes techniques that are effective and insightful.
- For important classes of problems and special classes of functions the theory is supported by efficient numerical tools such as those based on LMIs.

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