

THE MODULI SPACE OF GENUS 4 EVEN SPIN CURVES IS RATIONAL

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ABSTRACT. By the technique of 3-fold Mori theory, we prove that the moduli space whose general point parameterizes a couple (\mathcal{H}, θ) of a smooth curve \mathcal{H} of genus 4 and a halfcanonical divisor θ such that $h^0(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(\theta)) = 0$ is birational to \mathbb{P}^9 .

1. INTRODUCTION

Throughout the paper, we work over \mathbb{C} , the complex number field.

A *spin curve* is a couple (Γ, θ) , where Γ is a smooth projective curve of genus g and θ is a *theta characteristic*, that is, an element $\theta \in \text{Pic } \Gamma$ such that 2θ is the class of the canonical sheaf ω_{Γ} . There are 2^{2g} different kinds of spin curve structures for every smooth curve Γ and they are partitioned into two classes according to the parity of $h^0(\Gamma, \theta)$. A theta characteristic θ is said to be *even* or *odd* if $h^0(\Gamma, \theta)$ is even or odd respectively. Correspondingly we speak of *even* or *odd spin curves*.

There exists the moduli space \mathcal{S}_g which parameterizes smooth spin curves (Γ, θ) and by the forgetful map $\mathcal{S}_g \rightarrow \mathcal{M}_g$, where \mathcal{M}_g is the moduli space of curves of genus g , we see that \mathcal{S}_g is a disjoint union of two irreducible components \mathcal{S}_g^+ and \mathcal{S}_g^- of relative degrees $2^{g-1}(2^g + 1)$ and $2^{g-1}(2^g - 1)$ corresponding to even and odd spin curves respectively ([Mum1], [ACGH]). It was classically known that \mathcal{S}_2^+ is rational. The so called Scorza map gives a birational isomorphisms between \mathcal{S}_3^+ and \mathcal{M}_3 ([DK]), hence \mathcal{S}_3^+ is rational since so is \mathcal{M}_3 by [Ka] (see also [B]). Recently, Farkas [Fa] proved that a compactification $\overline{\mathcal{S}}_g^+$ of \mathcal{S}_g^+ is of general type for $g > 8$, and the Kodaira dimension of $\overline{\mathcal{S}}_g^+$ is negative for $g < 8$, and non-negative for $g = 8$.

In the previous papers [TZ1] and [TZ2], we discovered a method to study trigonal even spin curves of any genus by using biregular and birational geometries of the quintic del Pezzo 3-fold B . The 3-fold B is, by definition, a smooth projective threefold such that $-K_B = 2H$, where H is the ample generator of $\text{Pic } B$ and $H^3 = 5$. It is well known that the linear system $|H|$ embeds B into \mathbb{P}^6 .

We explain our method specializing to the genus 4 case, which is under consideration in this paper. In this case, our main ingredient is the Hilbert scheme \mathcal{H} of general sextic normal rational curves on B . We have shown that \mathcal{H} is irreducible (see [TZ1, Proposition 2.5.2]). For a general sextic normal rational curve C on B , we have constructed a smooth curve \mathcal{H}_1 of genus 4 and a theta characteristic θ on it. They come from the geometry of lines on B intersecting C . It is known that $\text{Aut } B$ is isomorphic to the automorphism group PGL_2 of the complex projective line (see [MU] and [PV]). From now on we set $G := \text{PGL}_2$. The G -action on B induces a G -action on \mathcal{H} . Thus we have a natural rational map $\pi_{\mathcal{S}_4^+} : \mathcal{H} \dashrightarrow \mathcal{S}_4^+$ which maps a

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general C to (\mathcal{H}_1, θ) and is constant on general G -orbits. By taking suitable compactifications of \mathcal{H} and of \mathcal{S}_4^+ , a resolution of indeterminacy of $\pi_{\mathcal{S}_4^+}$ and the Stein factorization, we have rational maps $p_{\mathcal{S}_4^+}: \mathcal{H} \dashrightarrow \tilde{\mathcal{S}}_4^+$ and $q_{\mathcal{S}_4^+}: \tilde{\mathcal{S}}_4^+ \dashrightarrow \mathcal{S}_4^+$ such that $\pi_{\mathcal{S}_4^+}$ is given by $q_{\mathcal{S}_4^+} \circ p_{\mathcal{S}_4^+}$, a general fiber of $p_{\mathcal{S}_4^+}$ is connected and $q_{\mathcal{S}_4^+}$ is generically finite. Then the G -orbit of a general point of \mathcal{H} is contained in a fiber of $p_{\mathcal{S}_4^+}$. In [TZ2, Theorem 4.0.2], we have proved that $\tilde{\mathcal{S}}_4^+$ is birational to \mathcal{S}_4^+ or to its double cover, and birationally parameterizes G -orbits in \mathcal{H} .

Farkas's result mentioned above and the rationality of \mathcal{M}_4 ([ShB1]) motivated us to deepen our understanding of \mathcal{S}_4^+ . Then we obtain the following result in this paper:

Theorem 1.1. \mathcal{S}_4^+ is rational.

Roughly speaking, the paper essentially consists of three parts; in the section 2, we review the results as for the biregular geometries of B . Especially, we review the biregular descriptions of B and the behaviour of lines on B and sextic rational curves on B . We also review the construction of the even spin curve (\mathcal{H}_1, θ) of genus 4 from a sextic rational curve on B . In the section 3, we study some special birational selfmap $B \dashrightarrow B$, which is one of our main ingredients to show the rationality of \mathcal{S}_4^+ . Finally, in the section 4, we prove the rationality of \mathcal{S}_4^+ as applications of the results in the section 2 and 3.

Now we explain an outline of the proof of the rationality of \mathcal{S}_4^+ . Among other things in the section 2, we remind the readers that a general sextic normal smooth rational curve on B has a natural marking, namely, its 6 distinct bi-secant lines on B (see Corollary 2.11). Therefore, noting the Hilbert scheme \mathcal{H}_1^B of lines on B is \mathbb{P}^2 (see the subsection 2.1), we can define the morphism $\Theta: \tilde{U}_0 \rightarrow (\mathbb{P}^2)^6/\mathfrak{S}_6$ mapping a sextic curve to the unordered set of its 6 bi-secant lines, where \tilde{U}_0 is the open subset of \mathcal{H} consisting of sextic curves with exactly six different bi-secant lines, and $(\mathbb{P}^2)^6$ is the Cartesian product of six copies of $\mathbb{P}^2 \simeq \mathcal{H}_1^B$ and \mathfrak{S}_6 is the permutation group on its factors. In Theorem 4.2, we show that Θ is birational.

Its proof requires the detailed study presented in the section 3 of the above mentioned birational selfmap $B \dashrightarrow B$ centered along a smooth sextic rational curve (Proposition 3.11). There we use techniques of the 3-fold explicit Mori theory, especially, properties of smooth flops and the classification of extremal contractions from smooth 3-folds. The selfmap $B \dashrightarrow B$ is decomposed as follows:

$$(1.1) \quad \begin{array}{ccc} & A \dashrightarrow A' & \\ f \swarrow & & \searrow f' \\ B & \dashrightarrow & B, \end{array}$$

where $A \dashrightarrow A'$ is one flop and both f and f' are the blow-ups along sextic normal rational curves C and C' on B , respectively. We remark that this diagram already appeared in [TZ2, the proof of Lemma 4.0.4] to show that the degree of $q_{\mathcal{S}_4^+}$ is at most two. Indeed, the rational deck transformation $J': \tilde{\mathcal{S}}_4^+ \dashrightarrow \tilde{\mathcal{S}}_4^+$ of the map $q_{\mathcal{S}_4^+}$ is induced from the correspondence between C and C' (if the pairs (B, C) and (B, C') were isomorphic up to the G -action, then $q_{\mathcal{S}_4^+}$ would be birational, hence J' would be the identity. We show, however, this is not the case).

It is easy to see that the morphism $\Theta: \tilde{U}_0 \rightarrow (\mathbb{P}^2)^6/\mathfrak{S}_6$ is G -equivariant. Thus we can translate the study of the rational map $p_{\mathcal{S}_4^+}: \mathcal{H} \dashrightarrow \tilde{\mathcal{S}}_4^+$ into the study of the quotient of $(\mathbb{P}^2)^6/\mathfrak{S}_6$ by

G . We carefully choose a G -invariant open subset of $(\mathbb{P}^2)^6/\mathfrak{S}_6$ such that its quotient by G exists and an involution J is induced on the quotient from J' through Θ (see the subsection 4.2 in detail). Only in this introduction, we denote by \mathcal{M} this quotient. The variety \mathcal{M}/J is birational to \mathcal{S}_4^+ . We can study \mathcal{M}/J relating it with the classically studied GIT quotient $\mathcal{Y} := (\mathbb{P}^2)^6//\mathrm{PGL}_3$, which is a compactification of the moduli space of ordered six distinct points on \mathbb{P}^2 . First, J has a nice interpretation. It is classically known that \mathcal{Y} has an involution called the association map. This involution descends to an involution j on $\mathcal{Y}/\mathfrak{S}_6$. In Lemma 4.5, we show that J is nothing but a lifting of j . Second, the G -action on $\mathcal{H}_1^B \simeq \mathbb{P}^2$ realizes G as a closed subgroup of the automorphism group PGL_3 of \mathbb{P}^2 . Indeed, G is the subgroup of PGL_3 consisting of elements which preserve one fixed conic on \mathbb{P}^2 , hence $\mathrm{PGL}_3/G \simeq \mathbb{P}^5$ (Proposition 2.2). This implies that \mathcal{M}/J is birationally a \mathbb{P}^5 -bundle over $(\mathcal{Y}/\mathfrak{S}_6)/j$ (Lemma 4.8 and the beginning of the proof of Theorem 4.15).

It is known that $(\mathcal{Y}/\mathfrak{S}_6)/j$ is rational. This is a classical result due to A. Coble, which easily follows from [DO, p.19 and 37]. Therefore, to obtain the rationality of \mathcal{M}/J , we have only to show that \mathcal{M}/J is birationally a *locally trivial* \mathbb{P}^5 -bundle over $(\mathcal{Y}/\mathfrak{S}_6)/j$. For this, we look for a sub \mathbb{P}^4 -bundle \mathcal{D} of $\mathcal{M} \dashrightarrow \mathcal{Y}/\mathfrak{S}_6$ which is invariant by J . Then \mathcal{D} descend to a sub \mathbb{P}^4 -bundle of $\mathcal{M}/J \dashrightarrow (\mathcal{Y}/\mathfrak{S}_6)/j$ and the local triviality of $\mathcal{M}/J \dashrightarrow (\mathcal{Y}/\mathfrak{S}_6)/j$ follows. To find the sub \mathbb{P}^4 -bundle \mathcal{D} , we go back from \mathcal{M} to $\tilde{\mathcal{S}}_4^+$, and we find the corresponding divisor on $\tilde{\mathcal{S}}_4^+$, which is defined by the class of sextic rational curves such that two of their 6 bi-secant lines intersect (see Lemmas 4.10–4.13). Now we have finished explanations of an outline of our proof of the rationality of \mathcal{S}_4^+ .

Finally, we would like to emphasize using geometries of B is natural and appropriate for the study of \mathcal{S}_4^+ . For, the birational \mathbb{P}^5 -bundle structure on \mathcal{S}_4^+ as above, which is indispensable to show its rationality, comes from the fact that the automorphism group of B is isomorphic to PGL_2 . Moreover, the Hilbert scheme \mathcal{H} of rational curves of degree 6 on B ties \mathcal{S}_4^+ and the moduli space of six points on \mathbb{P}^2 modulo the G -action, and the classically known association map has a good interpretation by the birational selfmap $B \dashrightarrow B$.

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2. RATIONAL CURVES ON THE QUINTIC DEL PEZZO 3-FOLD

2.1. Quintic del Pezzo 3-fold B .

Let $B \subset \mathbb{P}^6$ be the smooth quintic del Pezzo 3-fold. B is known to be unique and be realized as a linear section of $G(2, 5)$. There are several other characterizations of B . Here we give one of them, which is suitable for our purpose.

Let $\{F_2 = 0\} \subset \mathbb{P}^2$ be a smooth conic. Set

$$\mathrm{VSP}(F_2, 3)^o := \{(H_1, H_2, H_3) \mid H_1^2 + H_2^2 + H_3^2 = F_2\} \subset \mathrm{Hilb}^3 \check{\mathbb{P}}^2,$$

where $\check{\mathbb{P}}^2$ is the dual plane to \mathbb{P}^2 , thus linear forms H_i ($i = 1, 2, 3$) can be considered as points in $\check{\mathbb{P}}^2$. Mukai showed in [Muk] that B is isomorphic to the closed subset $\mathrm{VSP}(F_2, 3) := \overline{\mathrm{VSP}(F_2, 3)^o} \subset \mathrm{Hilb}^3 \check{\mathbb{P}}^2$. The variety $\mathrm{VSP}(F_2, 3)$ has the natural action of the subgroup G of

the automorphism group PGL_3 of \mathbb{P}^2 consisting of elements which preserve $\{F_2 = 0\}$. The group G is isomorphic to PGL_2 . By definition of $\mathrm{VSP}(F_2, 3)^\circ$, it is easy to see that G acts on $\mathrm{VSP}(F_2, 3)^\circ$ transitively. Thus B is a quasi-homogeneous G -variety.

2.2. Lines on B .

We summarize the known results about lines on B .

The dual plane $\check{\mathbb{P}}^2$ as above can be identified with the Hilbert scheme \mathcal{H}_1^B of lines on B , and, for a point $b := (H_1, H_2, H_3) \in \mathrm{VSP}(F_2, 3)^\circ \subset B$, the points $H_i \in \check{\mathbb{P}}^2$ ($i = 1, 2, 3$) represent three lines through b . By definition of $\mathrm{VSP}(F_2, 3)^\circ$ and transitivity of the action of G on $\mathrm{VSP}(F_2, 3)^\circ$, it is easy to show the following claim:

Claim 2.1. *G acts doubly transitively on the set of pairs of intersecting lines whose intersection points are contained in $\mathrm{VSP}(F_2, 3)^\circ$.*

Let \check{F}_2 be the dual quadratic form to F_2 and $\Omega := \{\check{F}_2 = 0\}$ is the associated conic in $\check{\mathbb{P}}^2$. Let l be a line on B . If $l \in \check{\mathbb{P}}^2 - \Omega$, then $\mathcal{N}_{l/B} = \mathcal{O}_l \oplus \mathcal{O}_l$. If $l \in \Omega$, then $\mathcal{N}_{l/B} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. Lines parameterized by Ω are called *special lines*. Let $\tilde{\Omega}$ be the symmetric bi-linear form associated to Ω . Then two lines l and m on B intersect if and only if $\tilde{\Omega}(l, m) = 0$, where $l, m \in \mathcal{H}_1^B$ are the points corresponding to l and m .

By the G -action on B , a line on B maps to a line on B , hence the G -action on B induces a G -action on \mathcal{H}_1^B .

Proposition 2.2. *The conic $\Omega \subset \check{\mathbb{P}}^2$ is invariant under the induced action of G on \mathcal{H}_1^B . Moreover, this G is exactly the closed subgroup of PGL_3 whose elements preserve Ω . In particular $\mathrm{PGL}_3/G \simeq \mathbb{P}_*H^0(\check{\mathbb{P}}^2, \mathcal{O}_{\check{\mathbb{P}}^2}(2)) \simeq \mathbb{P}^5$. If we take coordinates x, y, z of \mathcal{H}_1^B such that $\Omega = \{x^2 + y^2 + z^2 = 0\}$, then the map $\mathrm{PGL}_3 \rightarrow \mathbb{P}^5$ is induced by $g \in \mathrm{PGL}_3 \mapsto {}^tgg \in \mathbb{P}^5$, where we identify the vector space of symmetric matrices with the vector space of conics on $\check{\mathbb{P}}^2$.*

Proof. By the G -action on B , a special line is mapped to a special line, thus Ω is invariant by the induced G -action on \mathcal{H}_1^B .

By [FH, p.154], the closed subgroup of PGL_3 whose elements fix Ω is isomorphic to G .

Now we consider the induced action of PGL_3 on the space of conics $\mathbb{P}_*H^0(\check{\mathbb{P}}^2, \mathcal{O}_{\check{\mathbb{P}}^2}(2))$ on \mathcal{H}_1^B . Since PGL_3 acts on $\mathbb{P}_*H^0(\check{\mathbb{P}}^2, \mathcal{O}_{\check{\mathbb{P}}^2}(2))$ transitively and the kernel of the map

$$\begin{aligned} \mathrm{PGL}_3 &\rightarrow \mathbb{P}_*H^0(\check{\mathbb{P}}^2, \mathcal{O}_{\check{\mathbb{P}}^2}(2)) \\ g &\mapsto g \cdot \Omega \end{aligned}$$

is nothing but G , it holds that $\mathrm{PGL}_3/G \simeq \mathbb{P}^5$.

It is easy to see the last assertion. □

Now we collect the results obtained by Furushima and Nakayama [FN], which is based on another characterization of B by Mukai and Umemura [MU] as follows: let V be the vector space of binary sextic forms. The group PGL_2 acts on V by the law $g \cdot f_6(x, y) = f_6(ax + by, cx + dy)$, where f_6 is a binary sextic form with variable x and y , and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2$. Then B is isomorphic to the closure of the PGL_2 -orbit $\mathrm{PGL}_2[xy(x^4 - y^4)]$ in \mathbb{P}_*V .

Let $\pi: \mathbb{P} \rightarrow \mathcal{H}_1^B$ be the universal family of lines on B . Denote by $\varphi: \mathbb{P} \rightarrow B$ the natural projection. As we mentioned above, φ is a finite morphism of degree three (see also [FN, Lemma 2.3 (1)]).

Notation 2.3. For an irreducible curve C on B , denote by $M(C)$ the locus $\subset \mathcal{H}_1^B$ of lines intersecting C , namely, $M(C) := \pi(\varphi^{-1}(C))$ with reduced structure. Since φ is flat, $\varphi^{-1}(C)$ is purely one-dimensional. If $\deg C \geq 2$, then $\varphi^{-1}(C)$ does not contain a fiber of π , thus $M(C)$ is a curve. See Proposition 2.4 for the description of $M(C)$ in case C is a line.

Proposition 2.4. *It holds:*

- (1) *the union of special lines is the branched locus B_φ of $\varphi: \mathbb{P} \rightarrow B$. B_φ has the following properties:*
 - (1-1) $B_\varphi \in |-K_B|$,
 - (1-2) $\varphi^*B_\varphi = R_1 + 2R_2$, where $R_1 \simeq R_2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$, and $\varphi: R_1 \rightarrow B_\varphi$ and $\varphi: R_2 \rightarrow B_\varphi$ are injective, and
 - (1-3) *the pull-back of a hyperplane section of B to R_1 is a divisor of type $(1, 5)$,*
- (2) *the image of R_2 by $\pi: \mathbb{P} \rightarrow \mathcal{H}_1^B$ is the conic Ω ,*
- (3) *if l is a special line, then $M(l)$ is the tangent line to Ω at l . If l is not a special line, then $\varphi^{-1}(l)$ is the disjoint union of the fiber of π corresponding to l , and the smooth rational curve dominating a line on \mathcal{H}_1^B . In particular, $M(l)$ is the disjoint union of a line and the point l .*

By abuse of notation, we denote by $M(l)$ the one-dimensional part of $M(l)$ for any line l . Vice-versa, any line in \mathcal{H}_1^B is of the form $M(l)$ for some line l , and

- (4) *the locus swept by lines intersecting l is a hyperplane section T_l of B whose singular locus is l . For every point b of $T_l - l$, there exists exactly one line which belongs to $M(l)$ and passes through b .*

Proof. See [FN] and [II, §1]. □

By the proof of [FN] we see that B is decomposed into three G -orbits as follows:

$$B = (B - B_\varphi) \cup (B_\varphi - C_\varphi) \cup C_\varphi,$$

where C_φ is a smooth rational normal sextic and if $b \in B - B_\varphi$ exactly three distinct lines pass through it, if $b \in (B_\varphi - C_\varphi)$ exactly two distinct lines pass through it, one of them is special, and finally C_φ is the unique closed G -orbit and is the loci of $b \in B$ through which it passes only one line, which is special. Moreover, $B_\varphi - C_\varphi = \text{PGL}_2[xy^5]$ and $C_\varphi = \text{PGL}_2[y^6]$.

It also holds that $\text{VSP}^o(F_2, 3) = B - B_\varphi$.

2.3. Rational curves on B of degree $d \leq 6$.

In the rest of the section 2, We mainly review some of our results proved in [TZ1] or in [TZ2]. We point out the readers that for a general understanding of the content of this paper they only need to remind themselves only the statements and the definitions contained in this section. Moreover many of these preliminary results should be of easy geometrical intelligibility.

We denote by \mathcal{H}_d^B the union of irreducible components of the Hilbert scheme whose general points parameterize normal rational curves on B of degree $d \leq 6$.

Proposition 2.5. \mathcal{H}_d^B *is an irreducible variety of dimension $2d$. Moreover, a general normal rational curve $C_d \in \mathcal{H}_d^B$ is obtained as a smoothing of the union of a general normal rational curve $C_{d-1} \in \mathcal{H}_{d-1}^B$ and a general line l on B intersecting C_{d-1} .*

Proof. See [TZ1, Proposition 2.5.2]. To show this fact, we use the irreducibility of the Hilbert scheme of smooth rational curves on $G(2, 5)$ of degree d (see [P]). □

We investigate $\mathcal{H} := \mathcal{H}_6^B$ a bit more.

Proposition 2.6. *In the Hilbert scheme of curves of degree 6, the locus of C satisfying one of the following is a divisor of \mathcal{H} :*

- (1) C is the union of a general quintic normal rational curve C_5 and a general line l intersecting C_5 , or
- (2) C is a general sextic rational curve contained in a general hyperplane section of B .

Proof. If C satisfies (1), then C has one parameter smoothing to a sextic normal rational curve by [TZ1, the proof of Proposition 2.3.2], and, conversely, a general sextic normal rational curve is obtained in this way by Proposition 2.5. Thus $C \in \mathcal{H}$. By Proposition 2.5, \mathcal{H}_5^B is irreducible and is of dimension 10. Moreover, since $M(C_5)$ is a curve for a C_5 , such C 's form a divisor of \mathcal{H} .

Assume C satisfies (2). Let H be the hyperplane section of B containing C . Then, by $-K_H \cdot C = 6$, it holds that $(C^2)_H = 4$. By $H \cdot C = 6$, it holds that $\mathcal{N}_{C/B} \simeq \mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(6)$ or $\mathcal{O}_{\mathbb{P}^1}(5) \oplus \mathcal{O}_{\mathbb{P}^1}(5)$. In any case, the Hilbert scheme is smooth at C , and is 12-dimensional at C . On the other hand, sextic rational curves satisfying (2) form a 11-dimensional family. Indeed, once we fix a smooth hyperplane section, the family of smooth sextic rational curve on it is 5-dimensional, and hyperplane sections of B move in a 6-dimensional family. Thus $C \in \mathcal{H}$. \square

Now we recall some other results on intersections of lines with rational curves of degree $d \leq 6$.

Proposition 2.7. *A general element $C \in \mathcal{H}_d^B$ satisfies the following conditions:*

- (1) *there exist no k -secant lines of C on B with $k \geq 3$,*
- (2) *there exist at most finitely many bi-secant lines of C on B , and any of them intersects C simply, and*
- (3) *bi-secant lines of C are mutually disjoint.*

Proof. See [TZ1, Proposition 2.4.1]. \square

We describe some more relations of C with lines on B which can be translated into the geometry of \mathcal{H}_1^B . More explicitly, we prove that $M := M(C)$ is sufficiently general if C is general, where the readers have to remind the notations given in 2.3.

We denote by β_i ($1 \leq i \leq s$) the bi-secant lines of a general $C \in \mathcal{H}_d^B$.

Proposition 2.8. *A general element $C \in \mathcal{H}_d^B$ satisfies the following conditions:*

- (1) C intersects B_φ simply,
- (2) $M := M(C)$ is an irreducible curve of degree d with only simple nodes (if $d = 1$, then we remind the readers that, in Proposition 2.4 (3), we abuse the notation by denoting the one-dimensional part of $\pi(\varphi^{-1}(C))$ by $M(C)$),
- (3) for a general line l intersecting C , $M \cup M(l)$ has only simple nodes as its singularities,
- (4) $M \cup M(\beta_i)$ has only simple nodes as its singularities, and
- (5) for a general line α intersecting β_i , $M \cup M(\alpha)$ has only simple nodes as its singularities.

Proof. See [TZ1, Proposition 2.4.4]. \square

Corollary 2.9. *For a general $C \in \mathcal{H} := \mathcal{H}_6^B$, there are two lines α_{i1} and α_{i2} intersecting both C and β_i outside $C \cap \beta_i$ ($1 \leq i \leq s$). Moreover if $i \neq k$, then α_{ij} is disjoint from β_k , where $i, k = 1, \dots, s$ and $j = 1, 2$.*

Proof. This immediately follows from Proposition 2.8 (2) and (4). \square

2.4. Curve \mathcal{H}_1 parameterizing lines on the blow-up A of B along a sextic rational curve.

Though the argument in this subsection works also for other degrees d , we specialize to the degree 6 case. For readers' convenience, we repeat almost all the proofs.

We set $\mathcal{H} := \mathcal{H}_6^B$ for simplicity of notation as in the introduction.

2.4.1. Construction of \mathcal{H}_1 .

For a general $C \in \mathcal{H}$, we set

$$\mathcal{H}_1 := \varphi^{-1}C \subset \mathbb{P} \text{ and } M := M(C).$$

Proposition 2.10. *\mathcal{H}_1 is a smooth non-hyperelliptic trigonal curve of genus 4.*

Proof. By Propositions 2.4 (1) and 2.8 (1), it holds that \mathcal{H}_1 is smooth and the ramification for $\mathcal{H}_1 \rightarrow C$ is simple. Since $B_\varphi \in |-K_B|$, we can compute $g(\mathcal{H}_1)$ by the Hurwitz formula:

$$2g(\mathcal{H}_1) - 2 = 3 \times (-2) + 6 \times 2, \text{ equivalently, } g(\mathcal{H}_1) = 4.$$

□

Corollary 2.11. *The number of nodes of M is 6, whence C has 6 bi-secant lines on B .*

Proof. Note that a bi-secant line of C corresponds to a node of M . Thus, by Proposition 2.7 (2), the morphism $\pi|_{\mathcal{H}_1}: \mathcal{H}_1 \rightarrow M$ is birational. By Propositions 2.8 (2) and 2.10, it holds that $p_a(M) = \frac{(d-1)(d-2)}{2} = 10$ and the number of nodes of M is $10 - g(\mathcal{H}_1) = 6$. □

2.4.2. Lines on the blow-up A .

For a general $C \in \mathcal{H}$, we take the blow-up $f: A \rightarrow B$ along C . Let E be the f -exceptional divisor. We need to study the families of curves on A of degree one with respect to the anti-canonical sheaf of A to give another useful interpretation of the curve \mathcal{H}_1 .

Notation 2.12. For $i = 1, \dots, s$ and $j = 1, 2$, we set

- (1) $\{p_{i1}, p_{i2}\} = C \cap \beta_i \subset B$,
- (2) $\zeta_{ij} = f^{-1}(p_{ij}) \subset E \subset A$, and
- (3) $p'_{ij} = \beta'_i \cap \zeta_{ij}$.

Definition 2.13. We say that a connected curve $l \subset A$ is a *line* on A if $-K_A \cdot l = 1$ and $E \cdot l = 1$.

We point out that since $-K_A = f^*(-K_B) - E$ and $E \cdot l = 1$ then $f(l)$ is a line on B intersecting C . Based on this, we can classify lines on A as follows:

Proposition 2.14. *A line l on A is one of the following curves on A :*

- (i) *the strict transform of a uni-secant line of C on B , or*
- (ii) *the union $l_{ij} = \beta'_i \cup \zeta_{ij}$ ($i = 1, \dots, s, j = 1, 2$).*

In particular l is reduced and $p_a(l) = 0$.

Proof. This follows from easy computations on the Chow ring of A . □

Proposition 2.15. *The curve $\mathcal{H}_1 \subset \mathbb{P}$ is the Hilbert scheme of the lines of A .*

Proof. We only show that \mathcal{H}_1 parameterizes lines on A . See [TZ1, Corollary 4.1.8] for a rigorous proof.

By definition of \mathcal{H}_1 , we have $\mathcal{H}_1 = \{(l, t) \mid l \in M, t \in C \cap l\} \subset M \times C$, namely, \mathcal{H}_1 parameterizes the pairs of a line l and a point t in $C \cap l$. In [TZ1], these pairs are called marked lines. It is easy to see that there is one to one correspondence between marked lines and lines on A . Indeed, let m be a line on A . The line m satisfies (1) or (2) of Proposition 2.14. If m satisfies (1), then the image $f(m)$ of m on B is a uni-secant line, thus a marked line $(f(m), C \cap m)$ is uniquely determined from m . If $m = \beta'_i \cup \zeta_{ij}$, then we assign the marked line (β_i, p_{i3-j}) to m . Therefore \mathcal{H}_1 parameterizes lines on A . \square

By the proof of Proposition 2.15, we have the following:

Corollary 2.16. $\pi_{|\mathcal{H}_1}^{-1}(\beta_i) = \{l_{i1}, l_{i2}\}$, where $\beta_i \in M$.

2.4.3. *The theta characteristic on \mathcal{H}_1 .*

Via the interpretation of \mathcal{H}_1 recalled in subsection 2.4.2, we defined the following incidence correspondence in [TZ2, Section 3.1]:

$$(2.1) \quad I := \{(l_1, l_2) \in \mathcal{H}_1 \times \mathcal{H}_1 \mid l_1 \neq l_2 \text{ and } l_1 \cap l_2 \neq \emptyset\}.$$

We denote by δ the g_3^1 on \mathcal{H}_1 which defines $\varphi_{|\mathcal{H}_1}: \mathcal{H}_1 \rightarrow C$. Let l, l' and l'' be three lines on A such that $l + l' + l'' \sim \delta$. Then the images of l, l' and l'' are lines on B through one point of C . Set

$$(2.2) \quad \theta := (\pi_{|\mathcal{H}_1})^* \mathcal{O}_M(1) - \delta,$$

where $\pi: \mathbb{P} \rightarrow \mathcal{H}_1^B = \mathbb{P}^2$ is the natural projection of the universal family and $M = \pi(\mathcal{H}_1)$. Note that $\deg \theta = 3$.

Proposition 2.17. *The class of θ is an ineffective theta characteristic and $I = I_\theta$, where, by definition, $(x, y) \in I_\theta$ if and only if y belongs to the support of the unique effective divisor of $|\theta + x|$.*

Proof. See [TZ2, Proposition 3.1.2]. \square

3. BIRATIONAL SELFMAP OF B

We need to refine our understanding of the geometry of the blow-up $f: A \rightarrow B$ along a general $C \in \mathcal{H}$. The main result of this subsection is Proposition 3.11, in which we construct a birational selfmap $B \dashrightarrow B$ from f and describe it. This is a technical core of the proof of the main theorem. We recommend the readers to understand only the statement of this result first and go to the proof of the main theorem in the section 4.

For readers' convenience, we give the definition and basic properties of flops (the subsection 3.1), and descriptions of auxiliary birational maps which originate from B (the subsection 3.2).

3.1. Smooth 3-fold flops.

Definition 3.1. Let A be a smooth 3-fold. A projective morphism $g: A \rightarrow \overline{A}$ is called a *flopping contraction* if g is isomorphic outside the union γ of a finite number of curves (actually γ is a tree of smooth rational curves) and any irreducible component of γ is numerically trivial for K_A . An irreducible component of γ is called a *flopping curve*. If there exists a divisor D numerically negative for any irreducible component of γ , then g is called a *D -flopping contraction*. It is

well-known that, for a D -flopping contraction g , there exists a unique projective morphism $g': A' \rightarrow \overline{A}$ such that

- g' is isomorphic outside the union γ' of a finite number of curves and any irreducible component of γ' is numerically trivial for $K_{A'}$,
- the map $g'^{-1} \circ g: A \dashrightarrow A'$ gives an isomorphism between $A - \gamma$ and $A' - \gamma'$, and
- the strict transform D' on A' of D is numerically *positive* for any irreducible component of γ'

(see [Ko]). The map $g'^{-1} \circ g: A \dashrightarrow A'$ is called the D -flop for g and the morphism g' is called the D -flopped contraction. An irreducible component of γ' is called a *flopped curve*.

In case where $\rho(A/\overline{A}) = 1$ (for example, γ is irreducible), then the D -flop is independent of D and we say simply $A \dashrightarrow A'$ is the flop, g' is the flopped contraction, etc.

In Proposition 3.2, we summarize basic properties of flops, for which it is easy to find references in the literatures:

Proposition 3.2. *Let A be a smooth 3-fold and D a divisor on A . Let $g: A \rightarrow \overline{A}$ be a D -flopping contraction and γ the union of all the flopping curves. Let $A \dashrightarrow A'$ the D -flop and $g': A' \rightarrow \overline{A}$ the D -flopped contraction. Denote by γ' the union of all the D -flopped curves. Then*

- (1) A' is smooth,
- (2) g and g' is isomorphic analytically near γ and γ' . In particular, the numbers of irreducible components of γ and γ' are equal, and
- (3) if $\rho(A/\overline{A}) = 1$, then $G \cdot \gamma = -G' \cdot \gamma'$, where G is a divisor on A and G' is the strict transform on A' of G .

Proof. See [Ko]. □

Example 3.3 (Atiyah's flop). Here we describe the simplest flopping contraction. Actually, in the sequel, we mainly need only (composites of) flopping contractions of this type.

Let $g: A \rightarrow \overline{A}$ be a projective morphism whose exceptional curve γ is a smooth irreducible rational curve with $\mathcal{N}_{\gamma/A} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. It is easy to check that g is a flopping contraction. We can construct the flop $A \dashrightarrow A'$ as follows: let $p: \widehat{A} \rightarrow A$ be the blow-up of A along γ and E the p -exceptional divisor. Since $\mathcal{N}_{\gamma/A} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, it holds that $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$. There exists a morphism $q: \widehat{A} \rightarrow A'$ which is isomorphic outside E and $q|_E$ is the natural projection $E \simeq \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ different from $E \rightarrow \gamma$. It is easy to check that there exists a projective morphism $g': A' \rightarrow \overline{A}$ which is isomorphic outside $\gamma' := q(E)$ and $q \circ p^{-1}: A \dashrightarrow A'$ is the flop. The flop $A \dashrightarrow A'$ is called *Atiyah's flop*.

The following two results, Propositions 3.4 and 3.5, describe changes of intersection numbers by a flop. They are well-known for the experts but are not explicitly stated in the literatures. Therefore we decided to write their proofs in full details.

Proposition 3.4. *Let A be a smooth 3-fold and $g: A \rightarrow \overline{A}$ be a flopping contraction with $\rho(A/\overline{A}) = 1$. Denote by γ the union of all the g -exceptional curves. On A , take a divisor N and an irreducible projective curve $\delta \not\subset \gamma$. Let $A \dashrightarrow A'$ be the flop, and N' and δ' the strict transforms on A' of N and δ respectively. It holds:*

- (1) If $N \cdot \gamma = 0$, then $N^3 = N'^3$ and $N \cdot \delta = N' \cdot \delta'$.
- (2) If $N \cdot \gamma > 0$, then $N^3 > N'^3$ and $N \cdot \delta \leq N' \cdot \delta'$.

(3) If $N \cdot \gamma < 0$, then $N^3 < N'^3$ and $N \cdot \delta \geq N' \cdot \delta'$.

Proof. By Proposition 3.2, the inverse $A' \dashrightarrow A$ of $A \dashrightarrow A'$ is also the flop for the flopping contraction $g': A' \rightarrow \overline{A}$, thus we may assume that $N \cdot \gamma \geq 0$ by interchanging the roles of A and A' .

First we verify the inequality between N^3 and N'^3 . We learned the proof by [ShB2, Corollary 9.3], which originated from Mori. We write the proof for readers' convenience.

Since we assume N is g -gef, $\text{Bs } |m(N + H)| = \emptyset$ by Kawamata-Shokurov's base point free theorem ([KMM, Theorem 3-1-1]), where $m \gg 0$ and H is the pull-back of a sufficiently ample divisor on \overline{A} . Take $H_1, H_2, H_3 \in |H|$ and $N_1, N_2, N_3 \in |m(N + H)|$ such that H_i are disjoint from γ and N_i do not intersect each other on γ . For any divisor L on A , we denote by L' its strict transform on A' . It holds that $L_1 \cdot L_2 \cdot H_i = L'_1 \cdot L'_2 \cdot H'_i$ for any divisors L_1 and L_2 on A since $H_i \cap \gamma = \emptyset$ and $A \dashrightarrow A'$ is isomorphic outside γ . Then we have

$$\begin{aligned} m^3(N^3 - N'^3) &= m^3\{(N + H)^3 - (N' + H')^3\} = N_1 \cdot N_2 \cdot N_3 - N'_1 \cdot N'_2 \cdot N'_3 = \\ &= (N_1 \cdot N_2 \cdot N_3)_\gamma - (N'_1 \cdot N'_2 \cdot N'_3)_{\gamma'} = -(N'_1 \cdot N'_2 \cdot N'_3)_{\gamma'} \end{aligned}$$

If $N \cdot \gamma = 0$, then we may assume that $N_i \cap \gamma = \emptyset$, hence $N'_i \cap \gamma' = \emptyset$. This implies that $m^3(N^3 - N'^3) = -(N'_1 \cdot N'_2 \cdot N'_3)_{\gamma'} = 0$. If $N \cdot \gamma > 0$, then it holds that $N' \cdot \gamma' < 0$ by Proposition 3.2 (3). Thus $\text{Supp}(N'_2 \cap N'_3) = \gamma'$ and $m^3(N^3 - N'^3) = -(N'_1 \cdot N'_2 \cdot N'_3)_{\gamma'} = -N'_1 \cdot (N'_2 \cdot N'_3)_{\gamma'} > 0$.

Second we verify the inequality between $N \cdot \delta$ and $N' \cdot \delta'$. Take the following diagram:

$$(3.1) \quad \begin{array}{ccc} & \widehat{A} & \\ p \swarrow & & \searrow q \\ A & & A', \end{array}$$

where p and q are resolutions of A and A' respectively. By definition of the flop, $A \dashrightarrow A'$ is isomorphic outside γ and γ' . Therefore we may assume that p (resp. q) is isomorphic outside γ (resp. γ'). We can write $q^*N' = p^*N + R$, where R is a p -exceptional, hence is also a q -exceptional divisor. The divisor p^*N is q -nef since we assume that N is g -nef. By the negativity lemma (cf. [FA, Lemma 2.19]), it holds $R \geq 0$. The inequality $N \cdot \delta \leq N' \cdot \delta'$ follows from this fact.

Assume that $N \cdot \gamma = 0$. Then, by Proposition 3.2, it holds that $N' \cdot \gamma' = 0$. Therefore we can interchange the role of A and A' and we have also $R \leq 0$ by applying the negativity lemma to $p^*N = q^*N' - R$. Consequently, we have that $p^*N = q^*N'$ and then the equality $N \cdot \delta = N' \cdot \delta'$. \square

Now we specialize to Atiyah's flop and refine Proposition 3.4.

Proposition 3.5. *Let A be a smooth 3-fold and $g: A \rightarrow \overline{A}$ be a flopping contraction whose exceptional curve γ is irreducible. Assume that $\mathcal{N}_{\gamma/A} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Let N be a divisor on A and set $d := N \cdot \gamma$. Let δ be an smooth irreducible projective curve different from γ and set e to be the set-theoretic intersection number of δ and γ . Let $A \dashrightarrow A'$ be the flop, and N' and δ' the strict transforms on A' of N and δ respectively.*

It holds that $N^3 = (N')^3 + d^3$ and $N' \cdot \delta' \geq N \cdot \delta + de$. Moreover, if γ and δ intersect transversely at e points, then $N' \cdot \delta' = N \cdot \delta + de$.

Proof. Take the following diagram as in Example 3.3:

$$(3.2) \quad \begin{array}{ccc} & \widehat{A} & \\ p \swarrow & & \searrow q \\ A & & A', \end{array}$$

where p is the blow-up along γ and q is the blow-down of p -exceptional divisor $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$ in the other direction. We can write $q^*N' = p^*N + aE$ with some $a \in \mathbb{Z}$. We show that $a = d$. Indeed, for a fiber $\widehat{\gamma}$ of $E \rightarrow \gamma'$, which is mapped to γ by p , it holds

$$q^*N' \cdot \widehat{\gamma} = 0, \quad p^*N \cdot \widehat{\gamma} = N \cdot \gamma = d, \quad \text{and} \quad E \cdot \widehat{\gamma} = -1.$$

Therefore we have $a = d$.

Now we prove the inequality $N' \cdot \delta' \geq N \cdot \delta + de$. Let $\widehat{\delta}$ be the strict transform on \widehat{A} of δ . By definition of e , it holds that $E \cdot \widehat{\delta} \geq e$. By $q^*N' = p^*N + dE$, we have

$$N' \cdot \delta' = q^*N' \cdot \widehat{\delta} = (p^*N + dE) \cdot \widehat{\delta} \geq p^*N \cdot \widehat{\delta} + de = N \cdot \delta + de.$$

Moreover, if γ and δ intersect transversely at e points, then it holds that $E \cdot \widehat{\delta} = e$. Thus we have $N' \cdot \delta' = N \cdot \delta + de$.

To prove the equality $N^3 = (N')^3 + d^3$, we compute $p^*N^2q^*N'$ in two ways. First, by applying the projection formula to p , we have $p^*N^2q^*N' = N^3$. Second, by the equality $p^*N = q^*N' - dE$, we have

$$p^*N^2q^*N' = (q^*N' - dE)^2q^*N' = (q^*N')^3 + d^2E^2q^*N' = (N')^3 + d^2N' \cdot q_*(E^2),$$

where it holds that $(q^*N')^2E = (q^*N'|_E)^2 = 0$ since E is a \mathbb{P}^1 -bundle over a curve and $q^*N'|_E$ is numerically a sum of its fibers. Thus we have $N^3 = (N')^3 + d^2N' \cdot q_*(E^2)$. It is easy to see that $-q_*(E^2) = \gamma'$ as a 1-cycle. Therefore $N'q_*(E^2) = -N' \cdot \gamma' = N \cdot \gamma = d$ by Proposition 3.2 (3). Consequently, we have the equality $N^3 = (N')^3 + d^3$. \square

3.2. Auxiliary birational maps originating from B .

Proposition 3.6. *Let l be a line on B . Then the projection of B from l is decomposed as follows:*

$$(3.3) \quad \begin{array}{ccc} & B_l & \\ \pi_{1l} \swarrow & & \searrow \pi_{2l} \\ B & & Q, \end{array}$$

where π_{1l} is the blow-up along l and $B \dashrightarrow Q$ is the projection from l and π_{2l} contracts onto a twisted cubic curve the strict transform of the locus T_l swept by the lines of B touching l . Moreover

$$(3.4) \quad -K_{B_l} = H_l + L_l,$$

where H_l and L_l are the pull backs of general hyperplane sections of B and Q respectively. We denote by E_l the π_{1l} -exceptional divisor.

Proof. This is well-known and explicitly stated in [Fu1] and [MM]. See also [TZ1, Proposition 3.1.1]. \square

As an application, we show the following, which we need in the section 4:

Corollary 3.7. *For a general $C \in \mathcal{H}$, the six points β_1, \dots, β_6 on \mathbb{P}^2 are in a general position.*

Proof. Assume by contradiction that there exists a line L through a set of 3 points $\beta_{i_j} \in \mathbb{P}^2$ ($1 \leq j \leq 3$). By Proposition 2.4 (3), there exists a line l on B such that $M(l) = L$. The above condition means that 3 bi-secant lines β_{i_j} ($1 \leq j \leq 3$) intersect l . Consider the successive linear projections $B \dashrightarrow Q \dashrightarrow \mathbb{P}^2$ first from β_{i_1} and then from the strict transform on Q of β_{i_2} . The image \tilde{C} of C on \mathbb{P}^2 is a line or a conic. If \tilde{C} is a line, then C is contained in a hyperplane section, a contradiction. Thus \tilde{C} is a conic and $C \dashrightarrow \tilde{C}$ is an isomorphism. However, the images of β_{i_2} and β_{i_3} on Q mutually intersect since l is contracted by $B \dashrightarrow Q$. Thus the image on Q of β_{i_3} is contracted by the projection $Q \dashrightarrow \mathbb{P}^2$. Moreover, the images of β_{i_2} and β_{i_3} on Q are bi-secant lines of the image of C , hence \tilde{C} must be singular at the image of β_{i_3} , a contradiction.

In the proof of [TZ2, Lemma 3.1.1], we have shown that there are no conic through the six points β_1, \dots, β_6 using the inductive construction of C . \square

Definition 3.8. Let b be a point of B . We call the rational map from B defined by the linear system of hyperplane sections singular at b the *double projection from b* .

Proposition 3.9. *For a point $b \in B - B_\varphi$, the double projection from b is described as follows:*

- (1) *the target of the double projection is \mathbb{P}^2 , and the double projection from b and the projection $B \dashrightarrow \overline{B}_b$ from b fit into the following diagram:*

$$(3.5) \quad \begin{array}{ccccc} & & B_b & \dashrightarrow & B'_b \\ & \swarrow \pi_{1b} & & & \searrow \pi_{2b} \\ B & & & & \mathbb{P}^2, \\ & \searrow & & \swarrow & \\ & & \overline{B}_b & & \end{array}$$

where π_{1b} is the blow-up of B at b , $B_b \dashrightarrow B'_b$ is the flop of the strict transforms of three lines through b , and $\pi_{2b}: B'_b \rightarrow \mathbb{P}^2$ is a (unique) \mathbb{P}^1 -bundle structure.

We denote by E_b the π_{1b} -exceptional divisor. For simplicity of notation, we denote the strict transforms on B'_b of divisors on B_b by the same notation.

- (2)

$$L_b = H_b - 2E_b \text{ and } -K_{B'_b} = H_b + L_b,$$

where H_b is the strict transform of a general hyperplane section of B , and L_b is the pull back of a line on \mathbb{P}^2 ,

- (3) *the strict transforms l'_i of three lines l_i through b on B_b have the normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. The flop $B_b \dashrightarrow B'_b$ is Atiyah's flop.*
- (4) *a fiber of π_{2b} not contained in E'_b is the strict transform of a conic through b , or the strict transform of a line $\not\ni b$ intersecting a line through b .*

Proof. This is well-known for the experts but is not explicitly stated in the literatures. See [TZ1, the proof of Proposition 3.2.2] for a sketch of its proof. \square

3.3. The birational selfmap of B .

The following is one of consequences of generality of a sextic normal rational curve.

Proposition 3.10. *Let C be a general sextic normal rational curve on B and β_i ($1 \leq i \leq 6$) its six bi-secant lines. Let $f: A \rightarrow B$ be the blow-up of B along C and β'_i the strict transforms on A of β_i . Then $\mathcal{N}_{\beta'_i/A} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.*

Proof. See [TZ1, Lemma 3.1.4]. □

Now we reach the main result of the section 3. The method of its proof we take was developed more or less by Takeuchi in the paper [T]. We write a proof in full details hoping that it becomes a good introduction to the readers of a method of the explicit 3-fold Mori theory.

Proposition 3.11. *Let C be a sextic normal rational curve on B and $f: A \rightarrow B$ the blow-up along C . There exists possibly a 5-dimensional locus \mathcal{S} in \mathcal{H} such that if $C \notin \mathcal{S}$, then we have the following diagram*

(3.6)

$$\begin{array}{ccc} & A \dashrightarrow A' & \\ f \swarrow & & \searrow f' \\ B & & B \end{array}$$

where $A \dashrightarrow A'$ is one flop and $f': A' \rightarrow B$ is also the blow-up along a sextic normal rational curve C' . Denote by E' the f' -exceptional divisor. For simplicity of notation, we denote the strict transforms on A' of curves and divisors on A by the same notation. It holds

(3.7)
$$L = 3H - 2E, \quad -2K_A = H + L \text{ and } E' = 4H - 3E,$$

where H (resp. L) is the strict transform of a general hyperplane section of the B on the left (resp. right) hand side.

Moreover, if C is general, then it holds:

- (1) all the flopping curves of $A \dashrightarrow A'$ are the six strict transforms $\beta'_1, \dots, \beta'_6$ of six bi-secant lines β_1, \dots, β_6 of C , and
- (2) a non-trivial fiber of f' is the strict transform of an irreducible tri-secant conic of C , or a line intersecting both C and a bi-secant line β of C outside $C \cap \beta$.

Proof. We divide the proof into several steps. For applying the Mori theory, the first step is to check A is a weak Fano 3-fold, namely, $-K_A$ is nef and big. Then we can carry on the so-called two-ray game (a special case of the minimal model program). In the present case, we have more; $|-K_A| = |2H - E|$ is base point free since C is the intersection of quadrics. The bigness of $-K_A$ follows by the calculation:

$$(-K_A)^3 = (f^*(-K_B) - E)^3 = (2H - E)^3 = 8H^3 + 6HE^2 - E^3 = 14 > 0,$$

where we use basic numerical equalities:

(3.8)
$$H^3 = 5, \quad H^2E = 0, \quad HE^2 = -6 \text{ and } E^3 = -10.$$

Let $g: A \rightarrow \bar{A}$ be the Stein factorization of the morphism defined by $|-K_A|$. We need to make a case division.

Case 1. g contracts a divisor F .

We show that such C 's satisfying this condition form at most a 5-dimensional family in \mathcal{H} .

We can write $F \sim aH - bE$, where $a, b \in \mathbb{Z}$. It holds that $(-K_A)^2 F = 0$. By $-K_A = 2H - E$ and (3.8), we have $(-K_A)^2 F = 14(a - b) = 0$. Thus $F = a(H - E)$. The image $g(F)$ of F is not a point since $-K_A F^2 = -4a^2 \neq 0$. For a fiber l of $F \rightarrow g(F)$, it holds that $F \cdot l = -1$ or -2 . If $F \cdot l = -1$, then $a = 1$ and $F \sim H - E$. This is impossible; $|H - E|$ is empty since C is not contained in a hyperplane section, Thus $F \cdot l = -2$ and $F \sim 2(H - E)$. Together with the equality $-K_A \cdot l = (2H - E) \cdot l = 0$, it holds that $H \cdot l = 1$ and $E \cdot l = 2$, namely, l is irreducible and is the strict transform of a bi-secant line of C . Now we consider the set-up as in the subsection 2.2. Let Λ be the curve in \mathcal{H}_1^B parametrizing lines which are the images on B of fibers of $F \rightarrow g(F)$. It holds that Λ is an irreducible conic since $f(F) \cdot m = 2$ for a general line m on B and there exists one image of a fiber of $F \rightarrow g(F)$ through one point of $f(F)$. We show that C is determined from Λ . Then we are done since conics in \mathcal{H}_B^1 form a 5-dimensional family. Let $F' \subset \mathbb{P}$ be the pull-back of Λ by $\pi: \mathbb{P} \rightarrow \mathcal{H}_B^1$. Then $\varphi(F') = f(F)$. Moreover, $\varphi|_{F'}: F' \rightarrow f(F)$ is birational since there exists one image of a fiber of $F \rightarrow g(F)$ through one point of $f(F)$. The natural morphism $F' \rightarrow f(F)$ is an isomorphism outside $F' \cap E$. For a fiber γ of $E \rightarrow C$, it holds that $F \cdot \gamma = 2(H - E) \cdot \gamma = 2$, thus $f(F)$ is singular along C . Therefore C is determined from Λ as the singular locus of the image of F' by φ .

From now on we assume that C does not belong to such a 5-dimensional family. Thus we fall into the following case:

Case. 2. g contracts only finite number of curves.

Then g is a flopping contraction. Moreover, it holds that $\rho(A/\bar{A}) = 1$ since $\rho(A) = 2$. Let $A \dashrightarrow A'$ be the flop. Since A' is rational, $K_{A'}$ is not nef. Therefore there exists an extremal contraction $f': A' \rightarrow B'$. The morphism f' is unique since $\rho(A') = 2$. For simplicity of notation, we denote the strict transforms on A' of curves and divisors on A by the same notation. We would like to determine the type of f' as in the statement of this theorem.

Step. 1. Let $L := 3H - 2E$. We show that L is nef on A' and f' is the Stein factorization of the morphism defined by some multiple of L .

We see that there exists no effective divisor $D \sim aH - bE$ on A such that $a > 0$ and $b \geq a$. Indeed, if such a D exists, then $(-K_A)^2 D \leq 0$ by (3.8), hence $(-K_A)^2 D = 0$ and D is the g -exceptional divisor since $-K_A$ is nef. This contradicts the assumption that g is a flopping contraction. Thus any nonzero effective divisor $D \sim aH - bE$ satisfies that $a = 0$ and $b < 0$, or $a > 0$ and $b < a$.

We show that $|L|$ has no fixed component. Assume by contradiction that $|L|$ has a fixed component. If E is a fix component, then $L - E \sim 3H - 3E$ is effective, a contradiction to the above consideration. If there exists a fixed component $D \sim aH - bH$ with $a > 0$ and $b < a$, then $L - (aH - bE) = (3 - a)H - (2 - b)E$ is effective, thus $3 - a > 0$ and $2 - b < 3 - a$. The inequality $b < a$ and $2 - b < 3 - a$ has no solution, a contradiction. Therefore $|L|$ has no fixed component.

We prove that $h^0(\mathcal{O}_A(L)) \geq 7$. Consider the exact sequence

$$(3.9) \quad 0 \rightarrow \mathcal{O}_A(L) \rightarrow \mathcal{O}_A(3H - E) \rightarrow \mathcal{O}_E(3H - E) \rightarrow 0.$$

$3H - E$ is nef since $3H - E = 2H - E + H = -K_A + H$, and $-K_A$ and H are nef. Thus, by the Kawamata-Viewheg vanishing theorem, $h^0(\mathcal{O}_A(3H - E)) = \chi(\mathcal{O}_A(3H - E)) = \frac{1}{12}(120H^3 + 49HE^2 - 6E^3) + \frac{1}{12}H \cdot c_2(A) + 3$. Let $H_0 \in |H|$ be a general member. By the exact sequence

$$0 \rightarrow T_{H_0} \rightarrow T_{A|H_0} \rightarrow \mathcal{O}_{H_0}(H) \rightarrow 0,$$

we can calculate $c_2(A) \cdot H = 18$. Thus, by (3.8), we have $h^0(\mathcal{O}_A(3H - E)) = 35$. Now we compute $h^0(\mathcal{O}_E(3H - E))$. Note that E is a \mathbb{P}^1 -bundle over $C \simeq \mathbb{P}^1$. Let l be a fiber of $E \rightarrow C$. Then $(3H - E) \cdot l = 1$. Thus $f_{|E^*} \mathcal{O}_E(3H - E) = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$, where $a + b = (3H - E)^2 E = 26$ and $a, b \geq 0$ since $3H - E$ is nef. Thus $h^0(\mathcal{O}_E(3H - E)) = 28$. Finally we have $h^0(\mathcal{O}_A(L)) \geq 7$ from (3.9).

Now we prove that L is nef on A' . Since $\rho(A') = 2$, it suffices to check that L is non-negative both for a flopped curve and a general curve in a general fiber of f' . First we check that L is positive for a flopped curve on A' . Indeed, for a flopping curve γ , it holds that $H \cdot \gamma > 0$ and $(2H - E) \cdot \gamma = -K_A \cdot \gamma = 0$. Thus $L \cdot \gamma = (3H - 2E) \cdot \gamma < 0$. Then, by Proposition 3.2, L is positive for a flopped curve on A' . Second we check L is non-negative for a general curve in a general fiber of f' . If f' is of fiber type, then curves in fibers cover A' whence L is non-negative for a general curve in a general fiber of f' since $|L| \neq \emptyset$. If f' is birational, then, again, L is non-negative for a general curve in a general fiber of f' since the f' -exceptional divisor is not a fixed component of $|L|$ on A' .

Finally we show that f' is defined by some multiple of L . For this we prove the existence of an irreducible k -secant conic of C with $k \geq 3$ by the double projection from a general point b of C . We may assume that C is not contained in B_φ . Indeed, if C is contained in B_φ , then the pull-back of C on R_1 is a divisor of type $(1, 1)$ by Proposition 2.4 (1-3) and $\deg C = 6$. Thus such C 's form 3-dimensional family (we do not prove the existence of such C 's). We may assume that C does not belong to this 3-dimensional family. Thus we may assume that $b \notin B_\varphi$ and then there are three lines l_1, l_2 and l_3 through b . We consider the double projection from b and we use the notation of Proposition 3.9. Since C has only finitely many bi-secant lines, we may assume that l_i are not bi-secant lines by generality of b . Thus the strict transforms C' and l'_i of C and l_i are disjoint on B_b . By $-K_{B_b} = \pi_{1b}^*(-K_B) - 2E_b$, it holds that $-K_{B_b} \cdot C' = 10$. Thus it holds that $H_b \cdot C' = 6$ on B'_b and $-K_{B'_b} \cdot C' = 10$, where we denote by C' the strict transform on B'_b of C abusing the notation. Hence $L_b \cdot C' = 4$ by Proposition 3.9 (2) and then the image of C' on \mathbb{P}^2 is a line, a conic or a quartic. This implies that π_b has a multi-secant fiber of C' . If it is the strict transform of a smooth conic q through b , then q is a k -secant conic of C with $k \geq 3$. Otherwise, the fiber is the strict transform of a bi-secant line of C intersecting one of l_i . We show this does not occur if b is general. If this occurs for general b 's, then C is contained in the locus of lines T_β intersecting one fixed bi-secant line β since there are a finite number of bi-secant lines of C . This is a contradiction since C is not contained in a hyperplane section. Therefore there exists an irreducible k -secant conic of C with $k \geq 3$.

Let q be a general irreducible k -secant conic of C with $k \geq 3$. Then $L \cdot q = 6 - 2k$ on A . Since a flopping curve of $A \dashrightarrow A'$ intersects L negatively, we have $L \cdot q \leq 6 - 2k$ on A' by Proposition 3.4 (3). Since L is nef on A' , we have $k = 3$ and $L \cdot q = 0$ on A' . Thus L is not ample. By Kawamata-Shokurov's base point free theorem ([KMM, Theorem 3-1-1]), some multiple of L defines a morphism, which is non-trivial since L is not ample. The extremal contraction f' is nothing but the Stein factorization of the morphism defined by some multiple of L .

To determine f' , we make a case division using the classification of extremal contractions from smooth 3-folds [Mo]. Note that L is the pull-back of a generator of $\text{Pic } B'$ since L is primitive.

Step 2. We exclude the case where f' is of fiber type.

Then $B' \simeq \mathbb{P}^1$ or \mathbb{P}^2 . We can derive this fact as follows: it is well-known that B' is smooth if f' is of fiber type [Mo]. Since A is rational, B' is covered by rational curves, thus is rational since $\dim B' \leq 2$. If $\dim B' = 1$, then $B' \simeq \mathbb{P}^1$. If $\dim B' = 2$, then $B' \simeq \mathbb{P}^2$ since the Picard number of B' is one. Thus L is the pull-back of a point or a line respectively. This is a contradiction since $h^0(L) \geq 7$ as in Step 1.

Step 3. Assuming that f' contracts a divisor E' to a curve C' , we show that f' is described as in the statement of the theorem.

B' is a smooth Fano 3-fold with $\rho(B') = 1$. By the classification of smooth Fano 3-folds with Picard number one, we may write $-K_{B'} = a\widehat{L}$ with $a = 1, 2, 3, 4$, where \widehat{L} is the image of L by f' . Equivalently, it holds that $f'^*(-K_{B'}) = aL$. If $a = 3$, then B' is the quadric 3-fold and if $a = 4$, then $B' \simeq \mathbb{P}^3$. These cases contradict $h^0(L) \geq 7$. If $a = 1$, then $-K_{A'} = f'^*(-K_{B'}) - E' = 2H - E$, thus $E' = H - E$, a contradiction since $h^0(H - E) = 0$. If $a = 2$, then, by the inequality $h^0(L) \geq 7$ and the classification of del Pezzo 3-folds (see [Fu2]), we have $h^0(\widehat{L}) = 7$ and $\widehat{L}^3 = 5$ on B' . Thus B' is also the quintic del Pezzo 3-fold. We can easily show that C' is a sextic normal rational curve. We check the equalities (3.7). By definition of L , we have the former two equalities. By $-K_{A'} = 2L - E'$, $-K_{A'} = 2H - E$ and $L = 3H - 2E$, we have the latter equality. Assuming C is general, we check the assertions (1) and (2). Actually, it suffices to assume that C has six mutually disjoint bi-secant lines β_i ($1 \leq i \leq 6$) (Proposition 2.7 (3) and Corollary 2.11), and $\mathcal{N}_{\beta'_i/A} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ for the strict transform β'_i of β_i (Proposition 3.10). Any β'_i is a g -exceptional curve. We show that β'_i ($1 \leq i \leq 6$) are the only g -exceptional curves. Passing to the analytic category and taking the algebraization, we can decompose the flop $A \dashrightarrow A'$ into a sequence of flops $A := A_1 \dashrightarrow A_2 \dashrightarrow \cdots \dashrightarrow A_n =: A'$ for some $n \in \mathbb{N}$, where $A_j \dashrightarrow A_{j+1}$ is the flop of the strict transform of β'_j if $1 \leq j \leq 6$, or the flop of the strict transform of an irreducible g -exceptional curve different from β'_i ($1 \leq i \leq 6$) if $6 < j \leq n - 1$. For simplicity of notation, we denote by the same notation the strict transforms of g -exceptional curves, L and H on each A_j . Noting $L = 3H - 2E$, we can easily compute that $L^3 = -1$ on A . Since L on A' is the pull-back of \widehat{L} , we have $L^3 = 5$ on A' . Note that the flop $A_j \dashrightarrow A_{j+1}$ ($1 \leq j \leq 6$) is Atiyah's flop. Thus by the equality $L \cdot \beta'_i = -1$ ($1 \leq i \leq 6$) on A , we see that $L^3 = -1 + 6 = 5$ on A_7 by Proposition 3.5. Assume by contradiction that there exists at least one g -exceptional curve different from β'_i 's, namely, $n > 7$. Since the strict transforms of all the other g -exceptional curves are still numerically negative for L on A_j ($j \geq 7$) by Proposition 3.4 (3), it holds that $L^3 > 5$ on $A' = A_n$ by Proposition 3.4 (3) again, a contradiction. Thus β'_i ($1 \leq i \leq 6$) are the only g -exceptional curves. Now we investigate non-trivial fibers of f' . Let γ be a non-trivial fiber of f' . Then it holds that

$$(3.10) \quad -K_{A'} \cdot \gamma = 1 \text{ and } L \cdot \gamma = 0.$$

If γ is disjoint from all the flopped curves on A' , then it holds also that $-K_A \cdot \gamma = 1$ and $L \cdot \gamma = 0$ on A since $A \dashrightarrow A'$ is isomorphic near γ . The equalities $-K_A = 2H - E$ and $L = 3H - 2E$ show that $H \cdot \gamma = 2$ and $E \cdot \gamma = 3$. This means that the image of γ on B' is an irreducible tri-secant conic. If γ intersect some flopped curve on A' , then γ intersect only one flopped curve β' at one point by Proposition 2.7 (3). Then, by applying Proposition 3.5 to the flop $A \dashrightarrow A'$, the equalities (3.10) and $L \cdot \beta' = -1$ imply that $-K_A \cdot \gamma = 1$ and $L \cdot \gamma = 1$ on A . Thus we have $H \cdot \gamma = 1$ and $E \cdot \gamma = 1$. Since γ intersect the flopping curve on A corresponding

to β' , the image of γ on B is a line as desired.

Step 4. We finish the proof by disproving the case where f' contracts a divisor E' to a point.

By [Mo], f' is the blow-up at a point b of B' and satisfies one of the following E_2 – E_5 :

E_2 : b is a smooth point of B' . $E' \simeq \mathbb{P}^2$ and $-K_{A'|E'} = \mathcal{O}_{\mathbb{P}^2}(2)$.

E_3 : B' is analytically isomorphic to $\{xy + zw = 0\} \subset \mathbb{C}^4$ near b . $E' \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and $-K_{A'} = f'^*(-K_{B'}) - E'$.

E_4 : B' is analytically isomorphic to $\{xy + z^2 + w^3 = 0\} \subset \mathbb{C}^4$ near b . E' is a singular quadric surface and $-K_{A'} = f'^*(-K_{B'}) - E'$.

E_5 : b is a $\frac{1}{2}(1, 1, 1)$ -singularity. $E' \simeq \mathbb{P}^2$ and $-K_{A'} = f'^*(-K_{B'}) - \frac{1}{2}E'$.

For the strict transform q of a general tri-secant conic of C , it holds that $-K_{A'} \cdot q = 1$. Therefore the case E_2 does not occur. If f' is of type E_3 or E_4 , then, by a similar consideration to Step 3, we see that B' is a (singular) quintic del Pezzo 3-fold. On the other hand, by $-K_{A'} = f'^*(-K_{B'}) - E'$ and $E'^3 = 2$, we have $(-K_{B'})^3 = (-K_{A'})^3 + 2 = (-K_A)^3 + 2 = 16$, a contradiction. If f' is of type E_5 , then, by $-K_{A'} = f'^*(-K_{B'}) - \frac{1}{2}E'$ and $E'^3 = 4$, we have $(-K_{B'})^3 = (-K_{A'})^3 + \frac{1}{2} = (-K_A)^3 + \frac{1}{2} = \frac{29}{2}$. By the classification of \mathbb{Q} -Fano 3-folds with only $\frac{1}{2}(1, 1, 1)$ -singularities (see [Sa1], [Sa2]) and $(-K_{B'})^3 = \frac{29}{2}$, the possible Fano index of B' is $\frac{1}{2}$, namely, $2(-K_{B'}) = \widehat{L}$. If the Fano index of B' is $\frac{1}{2}$, then it holds that $2(-K_{A'}) + E' = 2f'^*(-K_{B'}) = L = 3H - 2E$. By $-K_{A'} = 2H - E$, we have $E' \sim H$, a contradiction. \square

Remark. It is possible to prove the existence of \mathcal{S} as in the statement of Proposition 3.11 but we do not prove this since we do not need this in the sequel. We only mention that $C_\varphi \in \mathcal{S}$, where C_φ is the unique closed orbit of G -action on B .

3.4. The correspondence between lines on A and lines on A' .

The contents of this subsection is presented also in [TZ2, the proof of Lemma 4.0.5]; here we need a very detailed version of it for later usage. Let C be a general sextic normal rational curve on B and we consider the diagram (3.6) as in Proposition 3.11. Denote by β'_i the strict transform on A of β_i . Since $f' : A' \rightarrow B$ is also the blow-up of B along a general sextic normal rational curve, we can define the notion of lines on A' . For simplicity of notation, we denote by the same notation the strict transforms on A' of curves and divisors on A .

Proposition 3.12. *There exists a natural one to one correspondence between lines on A and on A' .*

Proof. Let l be a line on A . Since $E' = 4H - 3E$, we have $E' \cdot l = 1$ on A . If l is disjoint from any β'_i , then we have $-K_{A'} \cdot l = 1$ and $E' \cdot l = 1$ on A' , thus l is a line on A' . Assume that l intersects some flopping curve of $A \dashrightarrow A'$. By the classification of lines on A (Proposition 2.14), there are two cases:

(a) l is the strict transform of a line on B intersecting both C and one bi-secant line β_i outside $C \cap \beta_i$.

By Corollary 2.9, l is the strict transform of α_{i1} or α_{i2} . Since $E' \cdot \beta'_i = -2$ on A , it holds that $-K_{A'} \cdot l = 1$ and $E' \cdot l = -1$ on A' by Proposition 3.5. As in the proof of Proposition 3.11 (2), l is a fiber of $E' \rightarrow C'$. Hence the union of $l \cup \beta''_i$, where β''_i is the flopped curve corresponding to β'_i , is a line on A' of type (ii) as in Proposition 2.14.

In this case, l corresponds to the line $l \cup \beta''_i$ on A' .

(b) l is the union of the strict transform β'_i of one β_i and a fiber ζ_{ij} of E over one point p_{ij} of $C \cap \beta_i$.

Note that $-K_A \cdot \zeta_{ij} = 1$ and $E' \cdot \zeta_{ij} = 3$ on A . By $E' \cdot \beta'_i = -2$ on A and Proposition 3.5, it holds that $-K_{A'} \cdot \zeta_{ij} = 1$ and $E' \cdot \zeta_{ij} = 1$ on A' . Therefore ζ_{ij} is a line on A' . Moreover $f'(\zeta_{ij})$ is a line on B intersecting C' and the bi-secant line $\widehat{\beta}_i$ outside $C' \cap \widehat{\beta}_i$, where $\widehat{\beta}_i$ is the image of β'_i by f' .

In this case, l corresponds to the line ζ_{ij} on A' .

Thus, in any case, a line on A corresponds to the unique line on A' and vice versa by symmetry of the diagram (3.6). \square

We denote by \mathcal{H}'_1 the curve obtained from C' as a triple cover as in Proposition 2.10. The curve \mathcal{H}'_1 has an ineffective theta characteristic θ' as in Proposition 2.17.

Proposition 3.13. *(\mathcal{H}_1, θ) and $(\mathcal{H}'_1, \theta')$ are isomorphic to each other as spin curves.*

Proof. Since \mathcal{H}_1 and \mathcal{H}'_1 are the Hilbert schemes of lines on A and A' respectively, we can naturally identify \mathcal{H}_1 and \mathcal{H}'_1 by Proposition 3.12. Moreover, we can identify also θ and θ' since the strict transforms of two general intersecting lines on A also intersect on A' and vice versa, and the theta characteristics are defined by the intersection of lines (see 2.4.3). \square

As we reviewed in the introduction, the natural rational map $\pi_{\mathcal{S}_4^+}: \mathcal{H} \dashrightarrow \mathcal{S}_4^+, C \mapsto (\mathcal{H}_1, \theta)$ is the composite of the rational maps $p_{\mathcal{S}_4^+}: \mathcal{H} \dashrightarrow \widetilde{\mathcal{S}}_4^+$ and $q_{\mathcal{S}_4^+}: \widetilde{\mathcal{S}}_4^+ \dashrightarrow \mathcal{S}_4^+$, where a general fiber of $p_{\mathcal{S}_4^+}$ is a PGL_2 -orbit in \mathcal{H} and $q_{\mathcal{S}_4^+}$ is birational or of degree two. From Proposition 3.13, we immediately obtain the following:

Corollary 3.14. *The rational map $q_{\mathcal{S}_4^+}$ exchanges the classes of C and C' on $\widetilde{\mathcal{S}}_4^+$.*

4. THE RATIONALITY PROOF

4.1. \mathcal{H} is birational to $(\mathbb{P}^2)^6/\mathfrak{S}_6$.

By Proposition 2.5, \mathcal{H} is an irreducible 12-dimensional variety. The G -action on B induces the G -action on \mathcal{H} . We construct a G -equivariant birational morphism $\Theta: \widetilde{U}_0 \rightarrow (\mathbb{P}^2)^6/\mathfrak{S}_6$, where \widetilde{U}_0 is the open subset of \mathcal{H} consisting of (possibly reducible) sextic curves with exactly six different bi-secant lines.

We remind the readers that the Hilbert scheme \mathcal{H}_1^B of lines on B is isomorphic to \mathbb{P}^2 and the Hilbert scheme \mathcal{H}_1 of lines on A is contained in the universal family \mathbb{P} of lines on B . We know that the restriction to $\mathcal{H}_1 \subset \mathbb{P}$ of the natural morphism $\pi: \mathbb{P} \rightarrow \mathcal{H}_1^B \simeq \mathbb{P}^2$ is the morphism $\pi|_{\mathcal{H}_1}: \mathcal{H}_1 \rightarrow M$, where, by Corollary 2.11, $M = \pi(\mathcal{H}_1)$ is a plane nodal sextic whose nodes are the points $\beta_1, \dots, \beta_6 \in \mathbb{P}^2$ corresponding to the six bi-secant lines $\beta_1, \dots, \beta_6 \subset B$ of C .

Then it remains defined a G -equivariant morphism

$$\Theta: \widetilde{U}_0 \rightarrow (\mathbb{P}^2)^6/\mathfrak{S}_6, C \mapsto (\beta_1, \dots, \beta_6).$$

Lemma 4.1. *The morphism Θ is dominant.*

Proof. We need to prove that, for six general lines on B , there is a sextic rational curve on B having them as its bi-secant lines. Let $\mathcal{G} \subset \widetilde{U}_0$ be the divisor whose general point parameterizes the union of a smooth quintic rational curve C_5 and a line l such that they intersect simply at only one point. First we prove that the restriction of Θ on \mathcal{G} is dominant over the divisor \mathcal{G}' consisting of 6-ples with three collinear points. To show this, let l, l_1, l_2, l_3 be four general lines

on B and m_1, m_2, m_3 three general lines intersecting l . We have only to prove there exists a C_5 such that $C_5 \cap l \neq \emptyset$, l_1, l_2, l_3 are three bi-secants of C_5 and m_1, m_2, m_3 intersect both C_5 and l . Consider the projection of B from l_1 . Recall that the divisor T_{l_1} swept by lines intersecting l_1 is mapped to a twisted cubic γ on Q by Proposition 3.6. The lines $l, l_2, l_3, m_1, m_2, m_3$ are mapped to lines $l', l'_2, l'_3, m'_1, m'_2, m'_3$ intersecting γ .

Let S be the smooth hyperplane section of Q spanned by l'_2 and l'_3 . Note that S is $\mathbb{P}^1 \times \mathbb{P}^1$, and l'_2 and l'_3 belong to the same ruling. Let n be a line in the other ruling. Then by a simple dimension count, there exists a twisted cubic $C' \sim 2n + l'_2$ passing through 5 points $l' \cap S, m'_1 \cap S, m'_2 \cap S, m'_3 \cap S$ and one point in $\gamma \cap S$. The strict transform on B of C' is a C_5 such that $C_5 \cup l$ is mapped by Θ to $(l_1, l_2, l_3, m_1, m_2, m_3)$.

We have proved that the divisor \mathcal{G}' in $(\mathbb{P}^2)^6/\mathfrak{S}_6$ given by six lines such that three of them intersect a line is dominated by \mathcal{G} . This is sufficient for the dominance of Θ . Indeed, for a general $C \in \tilde{U}_0$, the 6 points β_1, \dots, β_6 are in a general position by Corollary 3.7, hence $\text{Im } \Theta$ is not contained in \mathcal{G}' . Therefore, by the irreducibility of $(\mathbb{P}^2)^6/\mathfrak{S}_6$ the claim follows. \square

Theorem 4.2. *The morphism Θ is birational.*

Proof. Since $\dim \mathcal{H} = \dim(\mathbb{P}^2)^6/\mathfrak{S}_6 = 12$, it suffices to show that Θ is generically injective.

Let \mathcal{H}^o be the open set of \mathcal{H} consisting of sextic normal rational curves C which satisfy all the following conditions:

- (a) C has exactly six different bi-secant lines β_1, \dots, β_6 (Corollary 2.11).
- (b) $\beta_1, \dots, \beta_6 \in \mathbb{P}^2$ are in a general position (Corollary 3.7).
- (c) For the strict transform β'_i on A , it holds that $\mathcal{N}_{\beta'_i/A} = \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ (Proposition 3.10).
- (d) There are two lines α_{i1} and α_{i2} intersecting both C and β_i outside $C \cap \beta_i$ (Corollary 2.9).
- (e) For a general line α intersecting β_i , there exist four lines $\gamma_1, \dots, \gamma_4$ different from β_i and intersecting both C and α (Proposition 2.8 (5)).

To state the condition (f), we have the following remark: note that it is possible to define the diagram (3.6) as in Proposition 3.11 since C is normal and C has only a finite number of bi-secant lines by (a). Let C' be as in Proposition 3.11. If we consider the blow-up $f': A' \rightarrow B$ as the starting point of the diagram (3.6), then we obtain the diagram ending with $f: A \rightarrow B$. By this symmetry, if C is a general sextic normal rational curve, then so is C' .

- (f) C' is also contained in \mathcal{H}^o .

Let β_1, \dots, β_6 be general six lines on B . By Lemma 4.1, we may assume that there is a sextic normal rational curve C such that $C \in \mathcal{H}^o$ and β_1, \dots, β_6 are bi-secant lines of C . Let α_{ij} be as in the property (d).

Claim 4.3. *α_{ij} does not depend on C , namely, if Γ is another sextic normal rational curve such that $\Gamma \in \mathcal{H}^o$ and β_1, \dots, β_6 are also bi-secant lines of Γ , then α_{i1} and α_{i2} intersect Γ outside $\Gamma \cap \beta_i$.*

Proof of the claim. We take a general line α intersecting β_i . By the property (e), there exist four lines $\gamma_1, \dots, \gamma_4$ different from β_i and intersecting both C and α . Let α' and β'_i be the strict transforms of α and β_i on A . We consider the six lines $\alpha'_{i1}, \alpha'_{i2}$ and $\gamma'_1, \dots, \gamma'_4$ on A which are the strict transforms on A of the six lines α_{i1}, α_{i2} and $\gamma_1, \dots, \gamma_4$.

It holds that

$$(4.1) \quad \gamma'_1 + \cdots + \gamma'_4 = (\pi|_{\mathcal{H}_1})^*(M(\alpha)|_M) - l_{i1} - l_{i2}$$

and

$$(4.2) \quad \alpha'_{i1} + \alpha'_{i2} \sim (\pi|_{\mathcal{H}_1})^*(M(\beta_i)|_M) - (\delta - l_{i1}) - (\delta - l_{i2}),$$

where l_{ij} are lines on A as in Proposition 2.14 (ii). Summing up these two equalities, we obtain

$$(4.3) \quad \alpha'_{i1} + \alpha'_{i2} + \gamma'_1 + \cdots + \gamma'_4 \sim (\pi|_{\mathcal{H}_1})^*\mathcal{O}_M(2) - 2\delta = 2\theta \sim K_{\mathcal{H}_1},$$

namely, $\alpha'_{i1} + \alpha'_{i2} + \gamma'_1 + \cdots + \gamma'_4$ is a hyperplane section of $\mathcal{H}_1 \subset \mathbb{P}^4$.

Let $S \rightarrow \mathbb{P}^2$ be the blow-up at six points β_1, \dots, β_6 . Let $\lambda \subset S$ be the total transform of a line on \mathbb{P}^2 and ε_i the exceptional curve over the point β_i . Since $\varepsilon_i|_{\mathcal{H}_1} = l_{i1} + l_{i2}$, the equality (4.1) implies that

$$\gamma'_1 + \cdots + \gamma'_4 \sim (\lambda - \varepsilon_i)|_{\mathcal{H}_1}.$$

Thus, by the equality (4.3) and $K_{\mathcal{H}_1} \sim (3\lambda - \sum_{j=1}^6 \varepsilon_j)|_{\mathcal{H}_1}$, it holds that

$$\alpha'_{i1} + \alpha'_{i2} \sim \{(3\lambda - \sum_{j=1}^6 \varepsilon_j) - (\lambda - \varepsilon_i)\}|_{\mathcal{H}_1} = \{2\lambda - (\varepsilon_1 + \cdots + \varepsilon_i + \cdots + \varepsilon_6)\}|_{\mathcal{H}_1}.$$

Since \mathcal{H}_1 is not hyperelliptic, $\alpha'_{i1} + \alpha'_{i2}$ does not move, thus $\alpha'_{i1} + \alpha'_{i2}$ is cut out by the strict transform of the unique conic g_i on \mathbb{P}^2 passing through $\beta_1, \dots, \beta_i, \dots, \beta_6$ (note the property (b)). On the other hand, α_{i1}, α_{i2} belong to $M(\beta_i)$. Thus α_{i1} and α_{i2} are exactly two points of the intersection $g_i \cap M(\beta_i)$. In particular this does not depend on C . \square

Now we prove that $\Theta|_{\mathcal{H}^o}$ is of degree one. By contradiction assume that Γ is a sextic rational curve different from C such that $\Gamma \in \mathcal{H}^o$ and β_1, \dots, β_6 are bi-secant lines of Γ . By the remark just before the property (f), we can consider the diagram (3.6) as in Proposition 3.11 for C and we use the notation there freely. Let Γ' be the strict transform of Γ on A . For simplicity of notation, we denote by the same symbol the strict transforms on A and A' of curves on B . On B on the right hand side in the diagram (3.6), let $\widehat{\Gamma}$ be the strict transform of Γ and $\widehat{\beta}_i$ the image of the flopped curve corresponding to β_i .

Since $\deg \Gamma = 6$, we have $H \cdot \Gamma' = 6$ on A . By Proposition 3.4 (2), it holds that $H \cdot \Gamma' \geq 6$ on A' . Since L is nef on A' and Γ' is not a fiber of $A' \rightarrow B$ by Proposition 3.11 (2), it holds that $L \cdot \Gamma' \geq 1$. Thus it holds $-K_{A'} \cdot \Gamma' \geq 4$ by $-2K_{A'} = H + L$ (cf. (3.7)). By Proposition 3.4 (1), it holds $-K_A \cdot \Gamma' \geq 4$. On the other hand, $-K_B \cdot \Gamma = 12$ on B on the left hand side in the diagram (3.6). Therefore, since $-K_A = f^*(-K_B) - E$, Γ intersects C at less than or equal to 8 points. Thus, by the pigeon principle, for at least two bi-secant lines of C , say, β_1 and β_2 , Γ passes through at most one of $p_{11}, p_{12}, t_{11}, t_{12}$ and one of $p_{21}, p_{22}, t_{21}, t_{22}$, where $\{p_{i1}, p_{i2}\} := C \cap \beta_i$ and $t_{ij} := C \cap \alpha_{ij}$ ($i = 1, 2, j = 1, 2$).

This implies that $\widehat{\beta}_1$ and $\widehat{\beta}_2$ are at least 3-secant lines of $\widehat{\Gamma}$. Indeed, α'_{ij} on A' is a fiber of f' intersecting $\widehat{\beta}_i$ by Proposition 3.11 (2). In particular, this implies that $\widehat{\beta}_i$ ($i = 1, 2$) intersects $\widehat{\Gamma}$ at more than or equal to 3 points counted with multiplicities (if $\widehat{\beta}_i$ passes through a singular point of $\widehat{\Gamma}$, then we regard $\widehat{\beta}_i$ as a multi-secant line of $\widehat{\Gamma}$).

Now we show that $\deg \widehat{\Gamma} \leq 6$. Indeed, define the non-negative integer a by the equation $-K_A \cdot \Gamma' = 12 - a$, equivalently, C intersects Γ on B on the left hand side at a points counted with

multiplicities. Then Γ' intersects $\beta'_1, \dots, \beta'_6$ at more than or equal to $12 - a$ points depending on the common intersection points of C, Γ and β_i . This implies that $H \cdot \Gamma' \geq 6 + 12 - a$ on A' by Proposition 3.5. By (3.7) in Proposition 3.11, we have $L \cdot \Gamma' \leq 2(12 - a) - (18 - a) = 6 - a \leq 6$ on A' . Thus $\deg \widehat{\Gamma} \leq 6$.

Consider the projection $B \dashrightarrow Q$ from the line $\widehat{\beta}_1$ (see Proposition 3.6). Then the degree of the image $\widehat{\Gamma}'$ of $\widehat{\Gamma}$ is at most 3 since $\deg \widehat{\Gamma} \leq 6$ and $\widehat{\beta}_1$ is at least a 3-secant line of $\widehat{\Gamma}$. The lines $\widehat{\beta}_i$ ($i = 1, \dots, 6$) are the bi-secant lines of C' . It holds that $\widehat{\beta}_1 \cap \widehat{\beta}_2 = \emptyset$ since $C' \in \mathcal{H}^o$ by the property (f). Thus the image of $\widehat{\beta}_2$ on Q is at least 3-secant lines of $\widehat{\Gamma}'$. If $\deg \widehat{\Gamma}' = 1, 2$, then this is impossible. If $\deg \widehat{\Gamma}' = 3$, then $\widehat{\Gamma}'$ is a twisted cubic curve since a plane cubic curve does not exist on Q . Thus, again, $\widehat{\Gamma}'$ cannot have a 3-secant line. \square

4.2. Birational model of \mathcal{S}_4^+ .

Recall that \widetilde{U}_0 is the open subset of \mathcal{H} consisting of sextic curves with exactly six different bi-secant lines. Let $\widetilde{U}_1 \subset \widetilde{U}_0$ be the open subset such that Θ is an isomorphism on \widetilde{U}_1 . Clearly \widetilde{U}_1 is G -invariant. Let U_1 be the image of \widetilde{U}_1 on $(\mathbb{P}^2)^6/\mathfrak{S}_6$. Let $\widehat{U} \subset (\mathbb{P}^2)^6$ be the set of stable ordered six points with respect to the symmetric linearization of the action of PGL_3 , more explicitly, the set of ordered six points such that no two points coincide, or no four points are collinear (see [DO, p.23, Theorem 1]). By this explicit description, we see that \widehat{U} is \mathfrak{S}_6 -invariant. Note that the geometric quotient \widehat{U}/G exists. Indeed, let \mathcal{L} be the restriction of the PGL_3 -linearized line bundle to \widehat{U} . By restricting the PGL_3 -action to the G -action, \mathcal{L} is also G -linearized. We claim that \widehat{U} is the set of G -stable points. Indeed, let $x \in \widehat{U}$ be a point. The stabilizer group of x for the G -action is finite (actually trivial) since so is for the PGL_3 -action. There exists a PGL_3 -invariant section s of some multiple of \mathcal{L} such that $s(x) \neq 0$ and $\mathrm{PGL}_3 \cdot x$ is closed in $\widehat{U}_s := \{y \in \widehat{U} \mid s(y) \neq 0\}$. Since $G \subset \mathrm{PGL}_3$ is a closed subgroup, the same is true for G .

Set $U_2 = \widehat{U}/\mathfrak{S}_6 \subset (\mathbb{P}^2)^6/\mathfrak{S}_6$. Since the G -action and \mathfrak{S}_6 -action commutes, U_2/G also exists and $U_2/G \simeq (\widehat{U}/G)/\mathfrak{S}_6$. Let U'_3 be the open subset of $U_1 \cap U_2$ such that $C \in \Theta^{-1}(U'_3)$ is a sextic normal rational curve. Note that, if $C \in \Theta^{-1}(U_1)$ is a sextic normal rational curve, then we can define the diagram (3.6) as in Proposition 3.11 for C since $C \in \Theta^{-1}(U_1)$ has only a finite number of bi-secant lines. Let C' be as in Proposition 3.11. If C is a general sextic normal rational curve, then so is C' by the symmetry of the diagram (3.6). Thus $\Theta(C) \in U'_3$ with $C \in \Theta^{-1}(U'_3)$ such that $C' \notin \Theta^{-1}(U'_3)$ form a proper closed subset of U'_3 , which we denote by T . Set $U_3 := U'_3 - T$, namely, U_3 is the biggest open subset of $U_1 \cap U_2$ such that $C \in \Theta^{-1}(U_3)$ is a sextic normal rational curve, and the center C' of $f: A' \rightarrow B$ also belongs to $\Theta^{-1}(U_3)$. It is easy to see that U_3 is G -invariant since the diagram (3.6) is G -equivariant. Then by Corollary 3.14 and Theorem 4.2, the involution associated to the map $q_{\mathcal{S}_4^+}: \widetilde{\mathcal{S}}_4^+ \dashrightarrow \mathcal{S}_4^+$ is translated to an involution J on U_3/G satisfying $J: \Theta(C) \mapsto \Theta(C')$ since $\widetilde{\mathcal{S}}_4^+$ birationally parameterizes G -orbits in \mathcal{H} .

We can sum up the above discussion into the following:

Proposition 4.4. \mathcal{S}_4^+ is birational to $(U_3/G)/J$.

We investigate the variety $(U_3/G)/J$ relating it with the following classically well-studied variety:

$$\mathcal{Y} := (\mathbb{P}^2)^6 // \mathrm{PGL}_3,$$

where the GIT-quotient is taken with respect to the symmetric linearization of the action of PGL_3 ([DO, p.7, Proposition 1]). This is a compactification of the moduli space of ordered six points on \mathbb{P}^2 . Note that there exists a natural morphism $U_3/G \rightarrow \mathcal{Y}/\mathfrak{S}_6$ since G -action on $(\mathbb{P}^2)^6$ commutes with \mathfrak{S}_6 -action on $(\mathbb{P}^2)^6$.

4.3. A lifting of the association map on $(\mathbb{P}^2)^6/\mathfrak{S}_6$ modulo G .

We show that J is a lifting of the classical association map on $\mathcal{Y}/\mathfrak{S}_6$.

By [DO, p.37, Example 4], there exists an involution j' on \mathcal{Y} called the *(ordered) association map*. We do not give the definition of j' but only describe it on the open subset of \mathcal{Y} which parameterizes ordered six points in general positions (see [DO, p.118–120]).

Let $\Sigma \subset \mathbb{P}^3$ be a smooth cubic surface and $\sigma: \Sigma \rightarrow \mathbb{P}^2$ be the blow-up of \mathbb{P}^2 at six points p_1, \dots, p_6 . We consider ordered sets of six lines on Σ , equivalently, ordered sets of six points on \mathbb{P}^2 , while till now, we have considered only unordered sets of six points on \mathbb{P}^2 . The 27 lines on Σ can be grouped into three ordered subsets:

$$(l_1, \dots, l_6), (l'_1, \dots, l'_6), (m_{ij}) \quad (1 \leq i < j \leq 6),$$

where the lines l_i are the exceptional lines $\sigma^{-1}(p_i)$, the lines l'_i are the strict transforms of the conics $q_i \subset \mathbb{P}^2$ passing through $p_1, \dots, \check{p}_i, \dots, p_6$, and the lines m_{ij} are the strict transforms of the lines $\langle p_i, p_j \rangle$ joining the points p_i and p_j . The first two groups of lines (l_1, \dots, l_6) and (l'_1, \dots, l'_6) form a *double sixer*, which means that

$$l_j \cap l_j = \emptyset, \quad l'_i \cap l'_j = \emptyset, \quad l_i \cap l'_j \neq \emptyset \quad \text{if and only if} \quad i \neq j.$$

Every set of 6 disjoint lines on Σ can be included in a unique double sixer, from which Σ can be reconstructed uniquely. There are 36 double sixers of Σ . Every double sixer defines two regular birational maps $\sigma: \Sigma \rightarrow \mathbb{P}^2$, $\sigma': \Sigma \rightarrow \mathbb{P}^2$, each of which blows down one of the two sixes (sixtuples of disjoint lines) of the double sixers. The association map j' interchanges the two collections of ordered 6 points in \mathbb{P}^2 given by $(\sigma(l_1), \dots, \sigma(l_6))$ and $(\sigma'(l'_1), \dots, \sigma'(l'_6))$, namely, it holds that

$$(4.4) \quad j': (\sigma(l_1), \dots, \sigma(l_6)) \mapsto (\sigma'(l'_1), \dots, \sigma'(l'_6)).$$

We also remark that j' fixes ordered six points on a conic.

Since the symmetric group \mathfrak{S}_6 acts on the quotient \mathcal{Y} and its action commutes with j' , the map j' descends to an involution j on $\mathcal{Y}/\mathfrak{S}_6$. The map j is called the *(unordered) association map*.

Proposition 4.5. *The involution J is a lifting of j .*

Proof. It suffices to check the assertion at a general point of U_3/G . Let $C \in \mathcal{H}$ be a general point such that $\Theta(C) \in U_3$. By definition of Θ , it holds that $\Theta(C) = (\beta_1, \dots, \beta_6)$, where β_1, \dots, β_6 are six bi-secant lines of C . Now we compute $\Theta(C')$. By Corollary 2.9, there exist two lines α_{i1} and α_{i2} intersecting a bi-secant line β_i and C outside $C \cap \beta_i$ ($1 \leq i \leq 6$). Let α''_{i1} and α''_{i2} be the strict transforms of α_{i1} and α_{i2} on A' . Then α''_{i1} and α''_{i2} are the fibers of E' through $E' \cap \beta''_i$ by Proposition 3.11 (2), where E' is the f' -exceptional divisor and β''_i is the flopped curve corresponding to β_i . Thus, by Corollary 2.16, the two lines $\alpha''_{i1} \cup \beta''_i$ and $\alpha''_{i2} \cup \beta''_i$ on A' correspond to the node $\widehat{\beta}_i$ of $M(C')$, where $\widehat{\beta}_i$ is the image of β''_i by f' . Let β'_i , α'_{i1} and α'_{i2} be the strict transforms of β_i , α_{i1} and α_{i2} on A . By the proof of Proposition 3.12, the two lines $\alpha''_{i1} \cup \beta''_i$ and $\alpha''_{i2} \cup \beta''_i$ on A' correspond to the two lines α'_{i1} and α'_{i2} on A . By the proof of

Claim 4.3, $\alpha'_{i_1} + \alpha'_{i_2}$ on \mathcal{H}_1 is the divisor cut by the strict transform of the unique conic passing through all the nodes of $M(C)$ except β_i . Thus, by the above description of j' , J is a lifting of the association map j on $\mathcal{Y}/\mathfrak{S}_6$. \square

4.4. Rationality of the moduli space of double sixers on \mathbb{P}^2 .

By Proposition 4.4, we have only to show that $(U_3/G)/J$ is rational for the proof of the main theorem. Professor Igor Dolgachev kindly told us that a similar statement for PGL_3 is true, more precisely,

Proposition 4.6 (A. Coble). *The quotient variety $(\mathcal{Y}/\mathfrak{S}_6)/j$ is a rational variety.*

This is a classical result due to A. Coble, which easily follows from [DO, p.19 and 37].

This result is a bit subtle; it is not known if the moduli space $\mathcal{Y}/\mathfrak{S}_6$ of unordered six points on \mathbb{P}^2 is rational or not.

By the proof of Proposition 4.6, we see that the degree of the map $\mathcal{Y}/\mathfrak{S}_6 \rightarrow (\mathcal{Y}/\mathfrak{S}_6)/j$ is two, namely, we have

Corollary 4.7. *The map j is a non-trivial involution on $\mathcal{Y}/\mathfrak{S}_6$ whence so is J .*

4.5. Proof of the rationality of \mathcal{S}_4^+ .

The following diagram summerizes our construction above:

$$(4.5) \quad \begin{array}{ccccc} \tilde{U}_0 & \overset{p_{\mathcal{S}_4^+}}{\dashrightarrow} & \tilde{\mathcal{S}}_4^+ & \overset{q_{\mathcal{S}_4^+}}{\dashrightarrow} & \mathcal{S}_4^+ \\ \downarrow \Theta & & \downarrow \text{bir.} & & \downarrow \text{bir.} \\ U_3 & \longrightarrow & U_3/G & \longrightarrow & (U_3/G)/J \\ & & \downarrow e & & \downarrow \\ & & \mathcal{Y}/\mathfrak{S}_6 & \longrightarrow & (\mathcal{Y}/\mathfrak{S}_6)/j \end{array}$$

After proving some lemmas we show that $(U_3/G)/J$ is a rational variety.

We consider the following diagram:

$$(4.6) \quad \begin{array}{ccc} \hat{U} & \longrightarrow & \hat{U}/\mathfrak{S}_6 \\ \hat{\pi}_{\mathrm{PGL}_3} \downarrow & & \downarrow \pi_{\mathrm{PGL}_3} \\ \hat{U}/\mathrm{PGL}_3 & \xrightarrow{h} & (\hat{U}/\mathrm{PGL}_3)/\mathfrak{S}_6 \\ \downarrow & & \downarrow \\ \mathcal{Y} & \longrightarrow & \mathcal{Y}/\mathfrak{S}_6, \end{array}$$

where recall that $\hat{U} \subset (\mathbb{P}^2)^6$ is the set of stable ordered six points with respect to the symmetric linearization of the action of PGL_3 .

Lemma 4.8. *The natural projection π_{PGL_3} is a principal fiber bundle of PGL_3 over some non-empty open subset W_1 of $(\hat{U}/\mathrm{PGL}_3)/\mathfrak{S}_6$.*

Proof. By [DO, p.30, in the end of the proof of Theorem 2], $\hat{\pi}_{\mathrm{PGL}_3}$ is a principal fiber bundle of PGL_3 . We have seen that \mathcal{Y} is isomorphic to a quartic hypersurface in $\mathbb{P}(1^5, 2)$, hence its degree $\mathcal{O}_{\mathcal{Y}}(1)^4$ is equal to 2. By the equality (??) in proof of Proposition 4.6, $\mathcal{Y}' := \mathcal{Y}/\mathfrak{S}_6$ is a

hypersurface of degree 34 in $\mathbb{P}(2, 3, 4, 5, 6, 17)$. Then its degree $\mathcal{O}_{\mathcal{Y}'}(1)^4$ is equal to $\frac{2}{6!}$. Therefore the degree of the map h in the diagram (4.6) is $6!$, which is equal to the order of \mathfrak{S}_6 . Hence \mathfrak{S}_6 acts trivially on fibers of $\widehat{\pi}_{\mathrm{PGL}_3}$ over points in the open subset W'_1 of $\widehat{U}/\mathrm{PGL}_3$ where h is étale. By [Mum2, p.7, Proposition 0.2 and p.16, Proposition 0.9], π_{PGL_3} is a principal fiber bundle of PGL_3 over $W_1 := h(W'_1)$. \square

Now we consider the G -action. Let $\varrho: U_3/G \rightarrow \mathcal{Y}/\mathfrak{S}_6$ be the natural morphism. Set $V_1 := \varrho^{-1}(W_1) \cap J(\varrho^{-1}(W_1)) (\neq \emptyset)$. By definition, V_1 is invariant under J . Let $W_2 := \varrho(V_1)$ and $W'_2 := h^{-1}(W_2)$.

From the diagram (4.6) and the proof of Lemma 4.8, we obtain the following diagram:

$$(4.7) \quad \begin{array}{ccc} \widehat{\pi}_{\mathrm{PGL}_3}^{-1}(W'_2)/G & \longrightarrow & \pi_{\mathrm{PGL}_3}^{-1}(W_2)/G \supset V_1 \\ \downarrow & & \downarrow \varrho' \\ W'_2 & \longrightarrow & W_2 \\ \downarrow & & \downarrow \\ \mathcal{Y} & \longrightarrow & \mathcal{Y}/\mathfrak{S}_6 \end{array}$$

Lemma 4.9. *The natural projection ϱ' is a \mathbb{P}^5 -bundle.*

Proof. A fiber of ϱ' is isomorphic to PGL_3/G , which is isomorphic to \mathbb{P}^5 by Proposition 2.2. \square

Set $\mathcal{V} := \pi_{\mathrm{PGL}_3}^{-1}(W_2)/G$. We are going to find a sub \mathbb{P}^4 -bundle of $\rho': \mathcal{V} \rightarrow W_2$.

Recall that $\Omega \subset \mathcal{H}_B^1 = \mathbb{P}^2$ is the G -invariant conic (Proposition 2.2), and, for the symmetric bi-linear form $\widetilde{\Omega}$ associated to Ω , it holds that two lines l and m on B intersect if and only if $\widetilde{\Omega}(l, m) = 0$, where $l, m \in \mathcal{H}_1^B$ are the points corresponding to l and m . Let $\mathcal{D}' \subset (\mathbb{P}^2)^6/\mathfrak{S}_6$ be the closure of the set of unordered six points two of which are polar with respect to $\widetilde{\Omega}$. Clearly \mathcal{D}' is G -invariant.

Lemma 4.10. *The locus \mathcal{D}' is an irreducible divisor of $(\mathbb{P}^2)^6/\mathfrak{S}_6$.*

For a general point $(l_1, \dots, l_6) \in \mathcal{D}'$, it holds that

- (1) *only two of six lines l_1, \dots, l_6 intersect on B , and*
- (2) *six points $l_1, \dots, l_6 \in \mathbb{P}^2$ are in a general position.*

Proof. \mathcal{D}' is the image of the locus \mathcal{D}'' defined by ordered six points $(l_1, \dots, l_6) \in (\mathbb{P}^2)^6$ such that $\widetilde{\Omega}(l_5, l_6) = 0$. Since (l_1, \dots, l_5) moves freely, the 5-plets (l_1, \dots, l_5) are parameterized by $(\mathbb{P}^2)^5$. Once we fix l_5 , the points l_6 are parameterized by the line $\widetilde{\Omega}(l_5, *) = 0$. Then \mathcal{D}'' is birational to a \mathbb{P}^1 -bundle over $(\mathbb{P}^2)^5$. In particular \mathcal{D}'' is an irreducible divisor and so is \mathcal{D}' .

Similarly, we can show that the sublocus in \mathcal{D}'' consisting of 6-plets (l_1, \dots, l_6) not satisfying (1) nor (2) is 4-dimensional. Thus the latter assertion follows. \square

Lemma 4.11. *A general point of \mathcal{D}' is the image by Θ of a sextic normal rational curve C such that it is possible to define the diagram (3.6) as in Proposition 3.11 for C .*

Proof. Once we show that a general point of \mathcal{D}' is the image by Θ of a sextic normal rational curve C , then it is possible to define the diagram (3.6) as in Proposition 3.11 for C since such a C has only a finite number of bi-secant lines.

First we show that a general point of \mathcal{D}' is in the image of Θ . Let H be a smooth hyperplane section of B . Then H is the blow-up of \mathbb{P}^2 at four points. It is easy to show that, if C_o is the strict transform of a general smooth conic on \mathbb{P}^2 , then $C_o \in \mathcal{H}$ by Proposition 2.6 (2), C_o has six b-secant lines, and there exist three pairs of intersecting lines among the six bi-secant lines of C_o . In particular $C_o \in \tilde{U}_0$ and $\Theta(C_o) \in \mathcal{D}'$. Since Θ is dominant by Lemma 4.1, a general point of \mathcal{D}' is also in the image of Θ .

Take $C \in \tilde{U}_0$ such that $\Theta(C)$ is a general point of \mathcal{D}' . By contradiction assume that C is contained in a hyperplane section H of B . We may assume that H is smooth since even for the special sextic rational curve C_o as above, the hyperplane section containing it is smooth. Let β_1, \dots, β_6 be the six bi-secant lines of C . By generality of C and Lemma 4.10 (1), we may assume that only β_5 and β_6 intersect. Note that β_i ($i = 1, \dots, 6$) are contained in H . We have a contraction $H \rightarrow \mathbb{P}^2$ of β_1, \dots, β_4 since they are disjoint, a contradiction since the image of β_5 and β_6 on \mathbb{P}^2 are still (-1) -curves. \square

Denote by $\mathcal{D} \subset \mathcal{V} = \pi_{\text{PGL}_3}^{-1}(W_2)/G$ the image of $\mathcal{D}' \cap \pi_{\text{PGL}_3}^{-1}(W_2)$ by the quotient map.

Lemma 4.12. $\mathcal{D} \cap V_1 \neq \emptyset$ and $\mathcal{D} \cap V_1$ is invariant under J .

Proof. Recall the notation as in the subsection 4.2. By Lemma 4.11, \mathcal{D}' intersects the image of Θ . Moreover, since $\mathcal{D}' \subset (\mathbb{P}^2)^6/\mathfrak{S}_6$ is a divisor, Θ is isomorphic over a general point of \mathcal{D}' , namely, $U_1 \cap \mathcal{D}' \neq \emptyset$. By Lemmas 4.10 (2) and 4.11, $U_3' \cap \mathcal{D}' \neq \emptyset$. By Lemma 4.11, we can define the diagram (3.6) as in Proposition 3.11 for $C \in \Theta^{-1}(U_3' \cap \mathcal{D}')$. Let β_1 and β_2 be two intersecting bi-secant lines of C . Let $\beta_i' \subset A$ and $\beta_i'' \subset A'$ ($i = 1, 2$) be the flopping and flopped curve corresponding to β_i respectively. Since $\beta_1' \cap \beta_2' \neq \emptyset$, it holds that $\beta_1'' \cap \beta_2'' \neq \emptyset$ by Proposition 3.2 (2). Thus two of bi-secant lines of C' which is the images of β_1'' and β_2'' intersect. This implies that $\Theta(C') \in \mathcal{D}'$. By generality of C , we may assume that C' is also general. In particular, we may assume that $\Theta(C) \in \mathcal{D}' \cap U_3$. Then it holds that $J(\Theta(C)) = \Theta(C')$ on U_3/G . This implies that \mathcal{D} is invariant by the action of J . \square

Lemma 4.13. The restriction to \mathcal{D} of $\varrho': \mathcal{V} \rightarrow W_2$ gives a \mathbb{P}^4 -bundle structure on \mathcal{D} .

Proof. Let m_1, \dots, m_6 be six lines on B such that $(m_1, \dots, m_6) \in (\mathbb{P}^2)^6/\mathfrak{S}_6$ is mapped to a point w of W_2 . We show that the restriction of \mathcal{D} to the fiber F of ϱ' over the point w is isomorphic to \mathbb{P}^4 . By Claim 2.1, G acts doubly transitively on the set of general unordered pairs of intersecting lines. Thus, for any $1 \leq i < j \leq 6$ and $1 \leq k < l \leq 6$, it holds that

$$\{(g_1(m_1), \dots, g_1(m_6)) \in (\mathbb{P}^2)^6/\mathfrak{S}_6 \mid g_1 \in \text{PGL}_3, g_1(m_i) \cap g_1(m_j) \neq \emptyset\} = \\ \{(g_2(m_1), \dots, g_2(m_6)) \in (\mathbb{P}^2)^6/\mathfrak{S}_6 \mid g_2 \in \text{PGL}_3, g_2(m_k) \cap g_2(m_l) \neq \emptyset\} \text{ modulo } G$$

since there exists $h \in G$ such that $h\{g_1(m_i), g_1(m_j)\} = \{g_2(m_k), g_2(m_l)\}$ by the double transitivity. Therefore, a point of $F \cap \mathcal{D}$ is the image of a point $(g(m_1), \dots, g(m_6)) \in (\mathbb{P}^2)^6/\mathfrak{S}_6$, where $g \in \text{PGL}_3$ and $\tilde{\Omega}(g(m_5), g(m_6)) = 0$. Now we choose a coordinate of \mathbb{P}^2 such that $\Omega = \{x^2 + y^2 + z^2 = 0\}$. Set $m_5 = (a_1 : a_2 : a_3)$ and $m_6 = (b_1 : b_2 : b_3)$. Then $\tilde{\Omega}(g(m_5), g(m_6)) = 0$ if and only if

$$(4.8) \quad (a_1 \ a_2 \ a_3) {}^t g g \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = 0.$$

Recall that by Proposition 2.2 the map $\mathrm{PGL}_3 \rightarrow \mathbb{P}_*H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)) \simeq \mathbb{P}^5$ is defined by $g \mapsto {}^tgg$, where a conic on \mathbb{P}^2 is identified with a 3×3 symmetric matrix. Since the condition (4.8) is linear, $F \cap \mathcal{D}$ is a hyperplane in $F \simeq \mathbb{P}^5$. \square

Lemma 4.14. *The involution J on V_1 extends on \mathcal{V} .*

Proof. By [H, III Corollary 12.9], $\mathcal{F} = \varrho'_* \mathcal{O}_{\mathcal{V}}(\mathcal{D})$ is a locally free sheaf of rank 6 on W_2 and $\varrho'_* \mathcal{O}_{\mathcal{V}}(\mathcal{D}) \otimes k(w) \simeq H^0(\varrho'^{-1}(w), \mathcal{O}_{\mathcal{V}}(\mathcal{D})|_{\varrho'^{-1}(w)})$ for $w \in W_2$. Consider the following diagram:

$$(4.9) \quad \begin{array}{ccc} \varrho'^* \mathcal{F} & \longrightarrow & \mathcal{O}_{\mathcal{V}}(\mathcal{D}) \\ \downarrow & & \downarrow \\ \varrho'^* \mathcal{F}|_{\varrho'^{-1}(w)} & \longrightarrow & \mathcal{O}_{\mathcal{V}}(\mathcal{D})|_{\varrho'^{-1}(w)} \\ \downarrow & & \downarrow \text{id} \\ H^0(\varrho'^{-1}(w), \mathcal{O}_{\mathcal{V}}(\mathcal{D})|_{\varrho'^{-1}(w)}) \otimes \mathcal{O}_{\varrho'^{-1}(w)} & \longrightarrow & \mathcal{O}_{\mathcal{V}}(\mathcal{D})|_{\varrho'^{-1}(w)}. \end{array}$$

The map $H^0(\varrho'^{-1}(w), \mathcal{O}_{\mathcal{V}}(\mathcal{D})|_{\varrho'^{-1}(w)}) \otimes \mathcal{O}_{\varrho'^{-1}(w)} \rightarrow \mathcal{O}_{\mathcal{V}}(\mathcal{D})|_{\varrho'^{-1}(w)}$ is surjective since $\varrho'^{-1}(w) \simeq \mathbb{P}^5$ and $\mathcal{O}_{\mathcal{V}}(\mathcal{D})|_{\varrho'^{-1}(w)} \simeq \mathcal{O}_{\mathbb{P}^5}(1)$. Thus, by the Nakayama lemma, $\varrho'^* \mathcal{F} \rightarrow \mathcal{O}_{\mathcal{V}}(\mathcal{D})$ is surjective. Then by [H, II Proposition 7.12] it remains defined a morphism $\gamma: \mathcal{V} \rightarrow \mathbb{P}(\mathcal{F})$ over W_2 . Since γ is fiberwise an isomorphism then it is an isomorphism by the Zariski main theorem.

Let W^o be any open subset of W_2 . Since \mathcal{D} is invariant under the rational involution J , it holds that $\Gamma(W^o, \mathcal{F}) \simeq \Gamma(j(W^o), \mathcal{F})$, which induces an isomorphism $\mathcal{F} \simeq j^* \mathcal{F}$. Thus J extends to the involution $\mathcal{V} \simeq \mathbb{P}(j^* \mathcal{F}) \rightarrow \mathbb{P}(\mathcal{F}) = \mathcal{V}$. \square

We still denote by J the extension of J to \mathcal{V} . Set $\mathcal{R} := \mathcal{V}/J$ and $\mathcal{W} := W_2/j$. Now we can prove the main result:

Theorem 4.15. *\mathcal{R} is a rational variety.*

Proof. The action of J is trivial on the fiber of ϱ' since j acts non-trivially on W_2 by Corollary 4.7. Thus ϱ' descends to a \mathbb{P}^5 -bundle $p: \mathcal{R} \rightarrow \mathcal{W}$. Moreover, the sub- \mathbb{P}^4 -bundle \mathcal{D} of \mathcal{V} descends to a sub- \mathbb{P}^4 -bundle \mathcal{T} of \mathcal{R} since it is invariant under J by Lemma 4.12. Set $\mathcal{E} := p_* \mathcal{O}_{\mathcal{R}}(\mathcal{T})$. As in the proof of Lemma 4.14, we can show that $\mathcal{R} \simeq \mathbb{P}(\mathcal{E})$. In particular, \mathcal{R} is a locally trivial \mathbb{P}^5 -bundle over \mathcal{W} . Consequently \mathcal{R} is rational since so is \mathcal{W} by Proposition 4.6. \square

Corollary 4.16. *S_4^+ is a rational variety.*

Proof. It follows by Proposition 4.4 and Theorem 4.15 since \mathcal{R} is birationally equivalent to $(U_3/G)/J$. \square

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