Dimostrazioni formali di correttezza delle procedure ricorsive

Annotazioni relative alle impostazioni e ai passaggi logici
(vedi registrazioni delle lezioni per i commenti illustrativi)

```
;; (odd n)        →  ??

;; (unknown n)    →  ??


(define unknown   ; val: intero               x^2
  (lambda (x)     ; x:   intero non negativo
    (if (= x 0)
        0
        (+ (unknown (- x 1)) (odd x))
        )
    ))


(define odd     ; val: intero             2i-1
  (lambda (i)   ; i:   intero positivo
    (if (= i 1)
        1
        (+ (odd (- i 1)) 2)
        )
    ))
```

Proprietà generale da dimostrare:

$$\forall\, n \in \mathbf{N}^+ \,.\, (\texttt{odd}\ \underline{n}) \rightarrow \underline{2n-1}$$

Caso base:

$$(\texttt{odd}\ \underline{1}) \rightarrow \underline{2\cdot 1-1}$$

Ipotesi induttiva:

considero $k \in \mathbf{N}^+$ e assumo che

$$(\texttt{odd}\ \underline{k}) \rightarrow \underline{2k-1}$$

Passo induttivo:

per k considerato sopra, l'obiettivo è dimostrare che

$$(\texttt{odd}\ \underline{k+1}) \rightarrow \underline{2(k+1)-1}$$

$\forall\, n \in \mathbf{N}^+ \,.\; \texttt{(odd } \underline{n}\texttt{)} \xrightarrow{*} \underline{2n-1}$

dimostro il caso base:

$\texttt{(odd 1)} \xrightarrow{*} \underline{2-1} = 1$

$\texttt{(odd 1)} \xrightarrow{2} \texttt{(if (= 1 1) 1 ...)} \xrightarrow{2} 1$

ipotesi induttiva: considero $k \in \mathbf{N}^+$

assumo: $\texttt{(odd } \underline{k}\texttt{)} \xrightarrow{*} \underline{2k-1}$

dimostro il passo induttivo: per (quel) k

dimostro: $\texttt{(odd } \underline{k+1}\texttt{)} \xrightarrow{*} \underline{2(k+1)-1} = \underline{2k+1}$

[ho considerato $k \in \mathbf{N}^+$]

$\texttt{(odd } \underline{k+1}\texttt{)}$

$\xrightarrow{2} \texttt{(if (= } \underline{k+1} \texttt{ 1) 1}$
$\texttt{(+ (odd (- } \underline{k+1} \texttt{ 1)) 2))}$

$\xrightarrow{2} \texttt{(+ (odd (- } \underline{k+1} \texttt{ 1)) 2)}$

$\rightarrow \texttt{(+ (odd } \underline{k}\texttt{) 2)} \xrightarrow{*} \texttt{(+ } \underline{2k-1} \texttt{ 2)}$

$\rightarrow \underline{2k+1}$

proprietà generale da dimostrare:

$$\forall\, n \in \mathbf{N}\ .\ \texttt{(unknown}\ \underline{n}\texttt{)} \xrightarrow{*} \underline{n^2}$$

impostazione e dimostrazione del caso base:

$$\texttt{(unknown}\ 0\texttt{)} \xrightarrow{*} \underline{0^2} = 0$$

$$\begin{aligned}\texttt{(unknown}\ 0\texttt{)} &\xrightarrow{*} \texttt{(if (= } 0 \texttt{ 0) 0 ...)}\\ &\xrightarrow{*} \texttt{0}\end{aligned}$$

ipotesi induttiva: considero $n \in \mathbf{N}$

assumo: $\texttt{(unknown}\ \underline{n}\texttt{)} \xrightarrow{*} \underline{n^2}$

dimostrazione del passo induttivo: per n condiserato

dimostro: $\texttt{(unknown}\ \underline{n+1}\texttt{)} \xrightarrow{*} \underline{(n+1)^2}$

[focalizzandosi sul valore considerato $n \in \mathbf{N}$]

(unknown $\underline{n+1}$)

→ (if (= $\underline{n+1}$ 0) 0 (+ ...))

→ (+ (unknown (- $\underline{n+1}$ 1))

(odd $\underline{n+1}$))

→ (+ (unknown $\underline{n}$) (odd $\underline{n+1}$))

per l'ipotesi induttiva:

→ (+ $\underline{n^2}$ (odd $\underline{n+1}$))

per la proprietà di odd dimostrata precedentemente:

→ (+ $\underline{n^2}$ $\underline{2n+1}$)

→ $\underline{n^2+2n+1} = \underline{(n+1)^2}$

```
;; (power n k)  →  ??


(define power    ; val: intero                    x^y
  (lambda (x y)  ; x > 0, y: interi non negativi
    (if (= y 0)
        1
        (* x (power x (- y 1)))
        )
    ))
```

proprietà generale da dimostrare:

$$\forall\ m>0,\ n \in \mathbf{N}\ .\ (\texttt{power}\ \underline{m}\ \underline{n}) \xrightarrow{*} \underline{m^n}$$

dimostrazione per induzione sul valore di n:

dimostrazione dei casi base [infiniti!]:

$$\forall\ m \in \mathbf{N}^+\ .\ (\texttt{power}\ m\ 0) \xrightarrow{*} \underline{m^0}$$

ipotesi induttiva: considero $n \in \mathbf{N}$

assumo: $\forall m \in \mathbf{N}^+\ .\ (\texttt{power}\ \underline{m}\ \underline{n}) \xrightarrow{*} \underline{m^n}$

dimostrazione del passo induttivo: per n considerato mi propongo di dimostrare che

$$\forall m \in \mathbf{N}^+\ .\ (\texttt{power}\ \underline{m}\ \underline{n+1}) \xrightarrow{*} \underline{m^{n+1}}$$

[$\forall m \in \mathbf{N}^{+} \ldots$]

(power $\underline{m}$ $\underline{n+1}$)

$\rightarrow$ (if (= $\underline{n+1}$ 0) 1 (* $\underline{m}$ …))

$\rightarrow$ ... ...

Casi base

$m \in \{1, 2, 3, ... \}$ e $n = 0$:

$(\texttt{power}\ m\ n) \quad \rightarrow \quad m^n$ (*)

Dall'ipotesi induttiva al passo induttivo:

$m \in \{1, 2, 3, ... \}$ e $n = k$:

$(\texttt{power}\ m\ n) \quad \rightarrow \quad m^n$

$$\Downarrow \qquad (**)$$

$m \in \{1, 2, 3, ... \}$ e $n = k+1$

$(\texttt{power}\ m\ n) \quad \rightarrow \quad m^n$

Implicazioni:

$$m \in \{1, 2, 3, \ldots\} \text{ e } n = 0 \qquad (*)$$

$$(\texttt{power}\ m\ n) \quad \rightarrow \quad m^n$$

$$\Downarrow \qquad (**)$$

$$m \in \{1, 2, 3, \ldots\} \text{ e } n = 0+1 = 1$$

$$(\texttt{power}\ m\ n) \quad \rightarrow \quad m^n$$

$$\Downarrow \qquad (**)$$

$$m \in \{1, 2, 3, \ldots\} \text{ e } n = 1+1 = 2$$

$$(\texttt{power}\ m\ n) \quad \rightarrow \quad m^n$$

$$\Downarrow \qquad (**)$$

$$m \in \{1, 2, 3, \ldots\} \text{ e } n = 2+1 = 3$$

$$(\texttt{power}\ m\ n) \quad \rightarrow \quad m^n$$

$$\Downarrow \qquad (**)$$

$$\ldots \quad \ldots$$

Induttivamente si apprende che:

$m \in \{1, 2, 3, ...\}$ e $n \in \{0\}$

`(power` m n`)` $\rightarrow$ m^n

$$\Downarrow$$

$m \in \{1, 2, 3, ...\}$ e $n \in \{0, 1\}$

`(power` m n`)` $\rightarrow$ m^n

$$\Downarrow$$

$m \in \{1, 2, 3, ...\}$ e $n \in \{0, 1, 2\}$

`(power` m n`)` $\rightarrow$ m^n

$$\Downarrow$$

$m \in \{1, 2, 3, ...\}$ e $n \in \{0, 1, 2, 3\}$

`(power` m n`)` $\rightarrow$ m^n

$$\Downarrow$$

... ...

```
;; (mul-tr m n p)  →  ??

;; (mul m n)       →  ??


(define mul           ; val:  intero                 m*n
  (lambda (m n)       ; m, n: interi non negativi
    (mul-tr m n 0)
    ))


(define mul-tr        ; val:  intero                 m*n+p
  (lambda (m n p)     ; m, n, p: interi non negativi
    (cond ((= n 0)
           p)
          ((even? n)
           (mul-tr (* 2 m) (quotient n 2) p))
          (else
           (mul-tr (* 2 m) (quotient n 2) (+ m p)))
          )
    ))
```

proprietà generale da dimostrare:

$$\forall m, n, p \in \mathbf{N} \,.\, (\texttt{mul-tr}\ \underline{m}\ \ \underline{n}\ \ \underline{p}) \xrightarrow{*} \underline{mn+p}$$

proprietà da dimostrare per i casi base:

$$\forall\, m, p \in \mathbf{N} \,.\, (\texttt{mul-tr}\ \underline{m}\ \ 0\ \ \underline{p}) \xrightarrow{*} \underline{p}$$

ipotesi induttiva: considero $n \in \mathbf{N}$

assumo:

$$\forall m, p \in \mathbf{N} \,.\, (\texttt{mul-tr}\ \underline{m}\ \ \underline{n}\ \ \underline{p}) \to \underline{mn+p}$$

proprietà da dimostrare come passo induttivo:
(per n considerato)

$$\forall m, p \in \mathbf{N} \,.$$
$$(\texttt{mul-tr}\ \underline{m}\ \ \underline{n+1}\ \ \underline{p}) \to \underline{m(n+1)+p}$$

??

proprietà generale da dimostrare (senza dimenticare!):

$$\forall n \in \mathbf{N}\ .\ \forall m, p \in \mathbf{N}\ .\ (\texttt{mul-tr}\ \underline{m}\ \ \underline{n}\ \ \underline{p}) \xrightarrow{*} \underline{mn+p}$$

$$\forall n \in \mathbf{N}\ .\ \forall k \in [0,n] \subset \mathbf{N}\ .\ \forall m, p \in \mathbf{N}\ .\ (\texttt{mul-tr}\ \underline{m}\ \ \underline{k}\ \ \underline{p}) \xrightarrow{*} \underline{mk+p}$$

dimostrazione dei casi base:

$$\forall k \in [0,0] \subset \mathbf{N}\ .\ \forall m, p \in \mathbf{N}\ .\ (\texttt{mul-tr}\ \underline{m}\ \ \underline{k}\ \ \underline{p}) \xrightarrow{*} \underline{mk+p}$$

dimostro [dimostrazione immediata]:

$$\forall m,p \in \mathbf{N}\ .\ (\texttt{mul-tr}\ \underline{m}\ \ 0\ \ \underline{p}) \xrightarrow{*} \underline{m0+p} = p$$

ipotesi induttiva: considero $n \in \mathbf{N}$ e assumo che

$$\forall k \in [0,n] \subset \mathbf{N}\ .\ \forall m, p \in \mathbf{N}\ .\ (\texttt{mul-tr}\ \underline{m}\ \ \underline{k}\ \ \underline{p}) \xrightarrow{*} \underline{mk+p}$$

Induttivamente si apprende che:

$m, p \in \{0, 1, 2, 3, \ldots\}$ e $k \in \{0\}$

(`mul-tr` m k p) $\rightarrow$ $mk+p$

$$\Downarrow$$

$m, p \in \{0, 1, 2, 3, \ldots\}$ e $k \in \{0, 1\}$

(`mul-tr` m k p) $\rightarrow$ $mk+p$

$$\Downarrow$$

$m, p \in \{0, 1, 2, 3, \ldots\}$ e $k \in \{0, 1, 2\}$

(`mul-tr` m k p) $\rightarrow$ $mk+p$

$$\Downarrow$$

$m, p \in \{0, 1, 2, 3, \ldots\}$ e $k \in \{0, 1, 2, 3\}$

(`mul-tr` m k p) $\rightarrow$ $mk+p$

$$\Downarrow$$

$m, p \in \{0, 1, 2, 3, \ldots\}$ e $k \in \{0, 1, 2, \ldots, n\}$

(`mul-tr` m k p) $\rightarrow$ $mk+p$

$$\Downarrow$$

. . .   . . .

dimostrazione del passo induttivo: per n considerato

$$\forall k \in [0,n+1] \subset \mathbf{N} \,.$$
$$\forall m, p \in \mathbf{N} \,.\ (\texttt{mul-tr}\ \underline{m}\ \ \underline{k}\ \ \underline{p}) \xrightarrow{*} \underline{mk+p}$$

$$\forall k \in [0,n] \subset \mathbf{N} \,.$$
$$\forall m, p \in \mathbf{N} \,.\ (\texttt{mul-tr}\ \underline{m}\ \ \underline{k}\ \ \underline{p}) \xrightarrow{*} \underline{mk+p}$$

e inoltre

$$\forall m, p \in \mathbf{N} \,.\ (\texttt{mul-tr}\ \underline{m}\ \ \underline{n+1}\ \ \underline{p})$$
$$\xrightarrow{*} \underline{m(n+1) + p}$$

$$(\texttt{mul-tr}\ \underline{m}\ \ \underline{n+1}\ \ \underline{p})$$
$$\xrightarrow{*} (\texttt{cond ((even?}\ \underline{n+1}\texttt{) ...) ...)}$$

a) suppongo $\underline{n+1}$ pari

$$\xrightarrow{*} (\texttt{cond ((even?}\ \underline{n+1}\texttt{) ...) ...)}$$
$$\xrightarrow{*} (\texttt{mul-tr (* 2}\ \underline{m}\texttt{)}$$
$$\texttt{(quotient}\ \underline{n+1}\ \texttt{2)}\ \underline{p}\texttt{)}$$
$$\xrightarrow{*} (\texttt{mul-tr}\ \underline{2m}\ \ \underline{(n+1)/2}\ \ \underline{p})$$

siccome $\underline{2m}$ e $\underline{p}$ sono naturali e $\underline{(n+1)/2} \in [0,n]$

[$(n+1)/2 > n$ solo se $n = 0$, ma allora $n+1$ non è pari]

posso applicare l'ipotesi induttiva:

$$\xrightarrow{*} \underline{2m\,(n+1)/2 + p} \;=\; \underline{m\,(n+1) + p}$$

b) suppongo $\underline{n+1}$ dispari

```
→* (cond (else ...))
→* (mul-tr (* 2 m)
           (quotient n+1 2)
           (+ m p))
→* (mul-tr 2m n/2 m+p)
```

[(quotient $\underline{n+1}$ 2) $\rightarrow$ $\underline{n/2}$ poiché $n+1$ dispari]

siccome $\underline{2m}$ e $\underline{m+p}$ sono naturali e $\underline{n/2} \in [0,n]$

posso applicare l'ipotesi induttiva:

$$\xrightarrow{*} \underline{2m\; n/2 + m+p} \;=\; \underline{m\,n + m + p}$$

$$= \underline{m\,(n+1) + p}$$

```
;; (paths i j)  →  ??


(define paths                      ; val:  intero
  (lambda (i j)                    ; i, j: interi non negativi
    (if (or (= i 0) (= j 0))
        1
        (+ (paths i (- j 1)) (paths (- i 1) j))
        )
    ))
```

proprietà generale da dimostrare:

$\forall k \in \mathbf{N} \,.\, \forall m,n \in \mathbf{N}$ t.c. m+n = k .

`(paths` $\underline{m}$ $\underline{n}$`)` $\rightarrow$ $\underline{(m+n)! \,/\, (\, m!\; n!\,)}$

dimostrazione del caso base:

$k = 0 \,.\, \forall m,n \in \mathbf{N}$ t.c. m+n = k .

`(paths` $\underline{m}$ $\underline{n}$`)` $\rightarrow$ $\underline{(m+n)! \,/\, (\, m!\; n!\,)}$

`(paths` 0 0`)` $\rightarrow$ $\underline{(0+0)! \,/\, (\, 0!\; 0!\,)} = 1$ [Ok]

ipotesi induttiva: considero $k \in \mathbf{N}$ e assumo che

$\forall m,n \in \mathbf{N}$ t.c. m+n = k .

`(paths` $\underline{m}$ $\underline{n}$`)` $\rightarrow$ $\underline{(m+n)! \,/\, (\, m!\; n!\,)}$

dimostrazione del passo induttivo: per *k* considerato

$\forall m,n \in \mathbf{N}$ t.c. m+n = k+1 .

`(paths` $\underline{m}$ $\underline{n}$`)` $\rightarrow$ $\underline{(m+n)!\,/\,(\,m!\;n!\,)}$

[m+n = k+1]

`(paths` $\underline{m}$ $\underline{n}$`)`

$\rightarrow$ `(if (or (=` $\underline{n}$ `0) (=` $\underline{m}$ `0))...)`

a) suppongo $n = 0$

$\rightarrow$ `1 =` $\underline{(m+0)!\,/\,(\,m!\;0!\,)}$ [Ok]

b) suppongo $m = 0$

...

c) suppongo $m, n > 0$

$\rightarrow$ `(+ (paths` $\underline{m}$ `(-` $\underline{n}$ `1))`
`(paths (-` $\underline{m}$ `1)` $\underline{n}$`))`

$\rightarrow$ `(+ (paths` $\underline{m}$ $\underline{n\text{-}1}$`)`   [$\underline{m}+\underline{n\text{-}1} = k$]
`(paths (-` $\underline{m}$ `1)` $\underline{n}$`))`

$\rightarrow$ `(+` $\underline{(m + n\text{-}1)!\,/\,(\,m!\;(n\text{-}1)!\,)}$
`(paths (-` $\underline{m}$ `1)` $\underline{n}$`))` [$\underline{m\text{-}1+n} = k$]

$$\rightarrow \quad (+ \quad \underline{(m + n-1)! \,/\, (\, m!\, (n-1)!\,)}$$
$$\underline{(m-1 + n)! \,/\, (\, (m-1)!\, n!\,)}\ )$$

$$\rightarrow \quad \underline{(m + n-1)! \,/\, (\, m!\, (n-1)!\,) +}$$
$$\underline{(m-1 + n)! \,/\, (\, (m-1)!\, n!\,)}$$

$$= \quad \underline{(m + n-1)!\, (\, n + m\,) \,/\, (\, m!\, n!\,)}$$

$$= \quad \underline{(m + n)! \,/\, (\, m!\, n!\,)}$$

```scheme
;; UFO = Unidentified Flying prOcedure

;; (ufo n)  →  ??


(define ufo          ; val: intero
  (lambda (x)        ; x:   intero positivo
    (cond ((= x 1)
           1)
          ((even? x)
           (- (* 2 (ufo (quotient x 2))) 1))
          (else
           (+ (* 2 (ufo (quotient x 2))) 1))
          )
    ))
```

proprietà generale da dimostrare [$n = 2^k$] :

$$\forall k \in \mathbf{N}\,.\, (\texttt{ufo } \underline{2^k}) \xrightarrow{*} \underline{1}$$

dimostrazione del caso base [$k = 0$]:

$$(\texttt{ufo } \underline{2^0}) \rightarrow \underline{1} \qquad\qquad\qquad\qquad *$$

ipotesi induttiva: considero $k \in \mathbf{N}$

assumo:

$$(\texttt{ufo } \underline{2^k}) \xrightarrow{*} \underline{1}$$

dimostrazione del passo induttivo: per *k* considerato

dimostro:

$$\texttt{(ufo } \underline{2^{k+1}}\texttt{)} \xrightarrow{*} \underline{1}$$

dimostro il passo induttivo:

per *k* considerato

$$\texttt{(ufo } \underline{2^{k+1}}\texttt{)}$$

$$\xrightarrow{*} \texttt{(cond ((even? } \underline{2^{k+1}}\texttt{) ...) ...)}$$

$$\xrightarrow{*}$$

$$\texttt{(- (* 2 (ufo (quotient } \underline{2^{k+1}} \texttt{ 2))) 1)}$$

$$\rightarrow \texttt{(- (* 2 (ufo } \underline{2^{k}}\texttt{))) 1)}$$

$$\rightarrow \texttt{(- (* 2 } \underline{1}\texttt{) 1)} \qquad \text{[per l'ipotesi induttiva]}$$

$$\rightarrow \texttt{(- } \underline{2} \texttt{ 1)} \rightarrow \underline{1}$$

proprietà generale da dimostrare [$n = 2^k + j$] :

$$\forall k \in \mathbf{N}\ .$$
$$\quad \forall j \in [0, 2^k\text{-}1] \subset \mathbf{N}\ .\ \texttt{(ufo } \underline{2^k+j}\texttt{)} \xrightarrow{*} \underline{2j+1}$$

k	*j*	$n = 2^k + j$
0	[0]	[1]
1	[0, 1]	[2, 3]
2	[0, 1, 2, 3]	[4, 5, 6, 7]
3	[0, 1, 2, ..., 6, 7]	[8, 9, 10, ..., 14, 15]
4	[0, 1, 2, ..., 14, 15]	[16, 17, ..., 30, 31]
. . .	. . .	. . .

k	$n = 2^k + j$	$2j + 1$ *(nell'ordine)*
0	[1]	[1]
1	[2, 3]	[1, 3]
2	[4, 5, 6, 7]	[1, 3, 5, 7]
3	[8, 9, 10, ..., 14, 15]	[1, 3, 5, ..., 13, 15]
4	[16, 17, ..., 30, 31]	[1, 3, 5, ..., 29, 31]
. . .	. . .	. . .

proprietà generale da dimostrare:

$$\forall k \in \mathbf{N} \,.$$
$$\forall j \in [0,2^k\text{-}1] \subset \mathbf{N} \,.\ (\texttt{ufo}\ \underline{2^k+j}) \xrightarrow{*} \underline{2j+1}$$

dimostrazione dei casi base [k = 0]:

$$\forall j \in [0,2^0\text{-}1] \subset \mathbf{N} \,.\ (\texttt{ufo}\ \underline{2^0+j}) \xrightarrow{*} \underline{2j+1}$$

ipotesi induttiva: considero $k \in \mathbf{N}$

assumo:

$$\forall j \in [0,2^k\text{-}1] \subset \mathbf{N} \,.\ (\texttt{ufo}\ \underline{2^k+j}) \xrightarrow{*} \underline{2j+1}$$

dimostrazione del passo induttivo: per k considerato

dimostro:

$$\forall j \in [0,2^{k+1}\text{-}1] \subset \mathbf{N} \,.\ (\texttt{ufo}\ \underline{2^{k+1}+j}) \xrightarrow{*} \underline{2j+1}$$

dimostro i casi base [k = 0]:

$\forall j \in [0,0] \subset \mathbf{N}$. (ufo $\underline{1+j}$) $\rightarrow$ $\underline{2j+1}$

(ufo $\underline{1}$) $\rightarrow$ $\underline{1}$

(ufo $\underline{1}$) $\rightarrow$ (cond ((= $\underline{1}$ 1) 1) ...)

$\rightarrow$ $\underline{1}$ [Ok]

dimostro il passo induttivo:

per k considerato e per $\forall j \in [0,2^{k+1}-1] \subset \mathbf{N}$

(ufo $\underline{2^{k+1}+j}$)

$\rightarrow$ (cond ((even? $\underline{2^{k+1}+j}$)
...) ...)

(a) *j* pari:

$\rightarrow$

```
(- (* 2 (ufo (quotient 2^(k+1)+j 2)))
   1)
```

$\rightarrow$ `(- (* 2 (ufo` $\underline{2^k+j/2}$`))) 1)`

$\rightarrow$ `(- (* 2` $\underline{j+1}$`) 1)` [per l'ipotesi induttiva]

$\rightarrow$ `(-` $\underline{2j+2}$ `1)` $\rightarrow$ $\underline{2j+1}$ [Ok (a)]

posso applicare l'ipotesi induttiva a $\underline{2^k+j/2}$

$\underline{2^k}$ [Ok]

$\underline{j/2} \in [0,2^k-1]$ $\Leftarrow$ so che $j \in [0,2^{k+1}-1]$

val. max. $j = 2^{k+1}-2$ perché *j* è pari

val. max. $j/2 = 2^k-1$

(b) *j* dispari:

$\longrightarrow$

`(+ (* 2 (ufo (quotient` $\underline{2^{k+1}+j}$ `2)))`
`1)`

$\longrightarrow$ `(+ (* 2 (ufo` $\underline{2^k+(j-1)/2}$`))) 1)`

$\longrightarrow$ `(+ (* 2` $\underline{j}$`) 1)` [per l'ip. ind.]

$\longrightarrow$ `(+` $\underline{2j}$ `1)` $\longrightarrow$ $\underline{2j+1}$ [Ok (b)]

posso applicare l'ipotesi indutt. a $\underline{2^k+(j-1)/2}$

$\underline{2^k}$ [Ok]

$\underline{(j-1)/2} \in [0,2^k-1]$ $\Leftarrow$ so che $j \in [1,2^{k+1}-1]$

val. max. $j = 2^{k+1}-1$ perché *j* è dispari

val. max. $(j-1)/2 = 2^k-1$