Hybrid Automata and ε-Analysis on a Neural Oscillator

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In this paper we propose a hybrid model of a neural oscillator, obtained by partially discretizing a well-known continuous model. Our construction points out that in this case the standard techniques based on replacing sigmoids with step functions is not satisfactory. Then, we study the hybrid model through both symbolic methods and approximation techniques. This last analysis, in particular, allows us to show the differences between the considered approximation approaches. Finally, we focus on approximations via ε-semantics, proving how these can be computed in practice.

Introduction

Neural oscillations are rhythmic and repetitive electrical stimuli which play an important role in the activities of several brain regions. Some examples of brain locations in which it has been demonstrated the central role of neural oscillations are the hippocampus [23], the cortex [19], the thalamus [20], and the olfactory information processing [9]. With the aim of understanding neurophysiological activities, we propose the modeling of oscillatory phenomena exploiting hybrid automata.

A continuous model of a single oscillator based on an ordinary differential system has been proposed in [22]. Even if this is a simple model, its analysis and the analysis of its composition in multiple copies is limited due to the non-linearity of the ordinary differential system involved. For this reason, we are interested in the development of a piecewise affine hybrid automaton which correctly approximates the continuous model and on which automatic analysis and composition can be made.

Trying to linearize the non-linear components of the original continuous model, we first replace in the standard way the sigmoidal behaviours with sign functions [10, 14], approximating continuos signals with discrete off-on signals. Unfortunately, the behaviour of this model differs from the original one. For this reason, we propose a more sophisticated approximation of sigmoidals based on a piecewise linear function exploited in the development of a hybrid automaton which simulates a single oscillator.

It is well known that the reachability problem over hybrid automata is source of undecidability. Moreover, the exact computation of the reachable sets of hybrid automata, which represents the basis of the automatic analysis of the models, does not always reflect the behaviour of the real modeled systems. This is due to the fact that real systems are often subject to noise, thus their evolutions do not correspond to a single precise formalization. This has been noticed already in [22] in the specific case of the continuous model of the neural oscillator. For these reasons, we study our hybrid automaton exploiting different approximation techniques that introduce noise.

In the literature, several approximation techniques have been proposed (see, e.g., [8, 18, 11, 21, 4, 17]). Fränzle in [8] presents a model of noise over hybrid automata. The introduction of noise ensures in many cases the (semi-)decidability of the reachability problem. Another result of (semi-)decidability always based on the concept of perturbation and concerning the safety verification of hybrid systems is given by Ratschan in [18]. Furthermore, ε-(bi)simulation [11] relations, which are essentially relaxations on the infinite precision required by simulation and bisimulation, represent tools able to remove

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complexity and undecidability issues related to the analysis of the investigated model. Moreover, in [3] it is presented a different approach based on the reinterpretation of the standard semantics of the formulæ which compose hybrid automata. Exploiting this new class of semantics, called ε-semantics, the authors provide a result of decidability of the reachability problem over hybrid automata with bounded invariants.

In this paper, we focus precisely on the approximation approach based on the ε-semantics. In particular, we propose a translation that allows us to reduce the ε-semantics evaluation to the standard semantics evaluation, computable by exploiting tools for cylindrical algebraic decomposition. Then, we present some properties which have been automatically tested on the neural oscillator by applying such translation. Hence, in this work, we prove both that ε-semantics better represents the real behaviour of the neural oscillator, than the standard one, and that the approach is effective.

The paper is organized as follows: Section 1 gives some basic definitions concerning logics and hybrid automata; Section 2 is dedicated to the mathematical modeling of the neural oscillator; in Section 3 we present different approximation techniques based on noise, perturbation, approximate (bi)simulations, and ε-semantics. Section 4 exposes some considerations regarding the application of the previously presented approximation approaches to the investigated model. Finally, in Section 5, we first define a translation which make effectively computable the ε-semantics, then we experimentally exploit it in the analysis of the hybrid automaton which models the neural oscillator.

1 Hybrid Automata

1.1 Preliminaries

We formally define hybrid automata by using first-order languages and, because of that, we first need to introduce some basic notions and our notation.

We use X, X_i, Y, Y_i, W, and W_i to denote real variables and $\mathbf{X}$, $\mathbf{X_i}$, $\mathbf{Y}$, $\mathbf{Y_i}$, $\mathbf{W}$, and $\mathbf{W_i}$ to denote tuple of real variables. We always assume that all the variables that occur bound in a formula do not occur free and vice versa. This enables us to label variables, rather than occurrences, as free or bound. We write $\varphi[X_1,\dots,X_m]$ to stress the fact that X_1, ..., X_m are free in φ. By extension, $\varphi[\mathbf{X_1},\dots,\mathbf{X_n}]$ indicates that the components of vectors $\mathbf{X_1}$, ..., $\mathbf{X_n}$ are free in φ.

The formula obtained from $\varphi[X_1,\dots,X_m]$ by replacing X_i by s_0, where s_0 is either a constant or a variable, is denoted by $\varphi[\![X_i/s_0]\!]$. By extension, $\varphi[\![X_i \dots X_{i+n}/s_0 \dots s_n]\!]$ indicates the formula obtained from $\varphi[X_1,\dots,X_m]$ by simultaneously replacing all the variables $X_i \dots X_{i+n}$ by $s_0 \dots s_n$. If $\mathbf{X} = \langle X_i,\dots,X_{i+n}\rangle$, $\vec{s_0} = \langle s_0,\dots,s_n\rangle$, $\vec{s_1} = \langle s_{n+1},\dots,s_{2*n+1}\rangle$, and $\doteqdot$ is a relational symbol (e.g., $=$ or $\geq$), then we may write $\varphi[\![\mathbf{X}/\vec{s_0}]\!]$ in place of $\varphi[\![X_i \dots X_{i+n}/s_0 \dots s_n]\!]$ and $\vec{s_0} \doteqdot \vec{s_1}$ in place of $\bigwedge_{j\in[0,n]} \left(s_j \doteqdot s_{j+k*n}\right)$ (e.g., $\langle 7, X\rangle = \langle 2,3\rangle$ means $7 = 2 \wedge X = 3$). Finally, if $\varphi[\mathbf{X_1},\dots,\mathbf{X_i},\dots,\mathbf{X_n}]$ then we may denote the formula $\varphi[\![\mathbf{X_i}/\vec{s}]\!]$ by writing $\varphi[\![\mathbf{X_1},\dots,\vec{s},\dots,\mathbf{X_n}]\!]$.

The semantics of a formula is defined in the standard way (see [7, 16]). Given a set Γ of sentences and a sentence φ, we say that φ is a *logical consequence* of Γ (denoted, $\Gamma \models \varphi$) if φ is valid in any model $\mathscr{M}$ in which each formula of Γ is valid too ($\mathscr{M} \models \Gamma$). A *theory* $\mathscr{T}$ is a set of sentences such that if $\mathscr{T} \models \varphi$, then $\varphi \in \mathscr{T}$. A theory $\mathscr{T}$ admits the so-called *elimination of quantifiers*, if, for any formula φ, there exists in $\mathscr{T}$ a quantifier free formula ρ such that φ is equivalent to ρ with respect to $\mathscr{T}$. If there exists an algorithm for deciding whether a sentence φ belongs to $\mathscr{T}$ or not, we say that $\mathscr{T}$ is *decidable*.

Example 1. *Consider the formula* $\varphi \stackrel{def}{=} \exists X\,(a*X^2 + b*X + C = 0)$*. It is well known that* φ *is in the theory of reals with* $+$, $*$, *and* $\geq$ *if and only if the unquantified formula* $b^2 - 4ac \geq 0$ *holds.*

1.2 Syntax, Semantics, and Reachability

A hybrid automaton is an infinite state automaton that consists in a set of continuos variables and a finite directed graph. Each node of a graph is labelled by both an invariant condition and a dynamic law, while all the edges are tagged with an activation region and a reset map. The continuous variables evolve according to the dynamic law of the current node of the graph and the node's invariant condition must be satisfied along all the evolution. An edge is crossable if and only if the variable values are included the activation region and, when a hybrid automata jumps over it, the associated reset map is applied.

Definition 1 (Hybrid Automata - Syntax)**.** *A* hybrid automaton *H of dimension $d(H) \in \mathbb{N}$ is a tuple $\langle \mathbf{X}, \mathbf{X}', \mathcal{V}, \mathcal{E}, Inv, f., Act, Res\rangle$ where:*

- *$\mathbf{X} = \langle X_1, \ldots, X_n\rangle$ and $\mathbf{X}' = \langle X'_1, \ldots, X'_n\rangle$ are two vectors of variables ranging over the reals $\mathbb{R}$;*
- *$\langle \mathcal{V}, \mathcal{E}\rangle$ is a directed finite graph, i.e., $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Each element of $\mathcal{V}$ will be dubbed* location*;*
- *Each location $v \in \mathcal{V}$ is labelled by both a formula $Inv(v)[\mathbf{X}]$, called* invariant*, and a continuous function $f_v : \mathbb{R}^n \longrightarrow (\mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}^n)$, called* dynamics *or* flow function*. The dynamics may be specified either by differential equations, i.e., f_v is the solution of a given Cauchy problem, or by a logic formula. We use the formula $Dyn(v)[\mathbf{X}, \mathbf{X}', T]$, where T is a temporal variable ranging in $\mathbb{R}_{\geq 0}$, to denote the dynamics on v, i.e., $Dyn(v)[\mathbf{X}, \mathbf{X}', T] \stackrel{def}{=} \mathbf{X}' = f_v(\mathbf{X})(T)$;*
- *Each $e \in \mathcal{E}$ is labelled by the formulæ $Act(e)[\mathbf{X}]$ and $Res(e)[\mathbf{X}, \mathbf{X}']$ which are called* activation *and* reset*, respectively.*

If all the formulæ that define a hybrid automaton H belong to the same logical theory $\mathscr{T}$, then we say that H is *definable in* $\mathscr{T}$ or that H is a $\mathscr{T}$ *hybrid automaton*.

The semantics of any hybrid automaton can be specified as a transition system that is composed by two different relations miming the double nature of the hybrid automaton itself: the *continuous reachability transition relation* and the *discrete reachability transition relation*.

Definition 2 (Hybrid Automaton - Semantics)**.** *A* state *a of H is a pair $\langle v, r\rangle$, where $v \in \mathcal{V}$ is a location and $r \in \mathbb{R}^{d(H)}$ is an assignment of values for the variables of $\mathbf{X}$. A state $\langle v, r\rangle$ is said to be* admissible *if $Inv(v)[\![r]\!]$ holds.*

The continuous transition relation *$\xrightarrow{t}_C$ between admissible states, where $t \geq 0$ denotes the transition elapsed time, is defined as follows:*

$$\langle v, r\rangle \xrightarrow{t}_C \langle v, s\rangle \iff r = f_v(r)(0),\ s = f_v(r)(t), \text{ and } Inv(v)[\![f_v(r)(t')]\!] \text{ hold for each } t' \in [0, t].$$

The discrete transition relation *$\xrightarrow{e}_D$ among admissible states is:*

$$\langle v, r\rangle \xrightarrow{e}_D \langle v, s\rangle \iff e = \langle v, v'\rangle \text{ and both } Act(e)[\![r]\!] \text{ and } Res(e)[\![r, s]\!] \text{ hold.}$$

We write $a \to_C a'$ and $a \to_D a'$ to mean that there exists a $t \in \mathbb{R}_{\geq 0}$ such that $a \xrightarrow{t}_C a'$ and that there exists an $e \in \mathcal{E}$ such that $a \xrightarrow{e}_D a'$, respectively.

Definition 3 (Hybrid Automata - Reachability)**.** *Let $\mathscr{I}$ be either $\mathbb{N}$ or an initial finite interval of $\mathbb{N}$. A* trace *of H is a sequence of admissible states $a_0, a_1, \ldots, a_j, \ldots$, with $j \in \mathscr{I}$, such that $a_{i-1} \to a_i$ holds for all $i \in [1, n]$ and either $a_{i-2} \to_C a_{i-1} \to_D a_i$, $a_{i-2} \to_D a_{i-1} \to_D a_i$, or $a_{i-2} \to_D a_{i-1} \to_C a_i$ for each $i \in I \setminus \{0\}$*[1].

The automaton H reaches *a state a_n from a state a_0 if there exists a trace $a_0, \ldots, a_n$. In such a case, we also say that a_n is* reachable *from a_0 in H.*

[1]This last condition supports not transitive dynamics. See [5] for a complete discussion.

The problem of deciding whether a hybrid automaton H reaches a set of states S from a set of states R is known as the *reachability problem* of S from R over H. A trace produced by an infinite sequence of discrete transitions during a bounded amount of time is called *Zeno trace* and every hybrid automaton allowing such kind of trace is said to have a *Zeno behaviour*.

Example 2. *Let us consider a hybrid automaton H_b modeling a* bouncing ball *whose collisions are inelastic. The automaton is equipped with two continuous variables X_1 and X_2 that represent ball's*

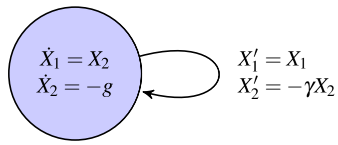


Figure 1: Bouncing ball hybrid automaton

elevation and velocity, respectively. The dynamics, resets, and discrete structure of H_b are presented in Fig. 1. The two coefficients g and γ are the standard gravity and the coefficient of restitution, respectively. The activation formula of the automaton edge is "$X_1 = 0$".

Imposing as starting height $h_0 = 10m$ and as coefficient of restitution $\gamma = 0.86$, the bounce peaks decrease at each iteration and the automaton H_b has a Zeno behaviour.

As the halting problem for the two counter machine can be reduced to the reachability problem of a particular class of hybrid automata, the reachability problem for hybrid automata itself is not always decidable [1]. However, if H is a $\mathscr{T}$-hybrid automaton and $\mathscr{T}$ is a first-order decidable theory, then the reachability through a bounded number of discrete transitions can be characterized with a first-order decidable formula (see e.g., [5]). In particular, in the case of automata defined through polynomials over the reals, we can use cylindrical algebraic decomposition tools can to decide bounded reachability.

2 Neural Oscillators: Continuous and Hybrid Models

Oscillatory electrical stimuli have been considered central for the activities of several brain regions since the begin of the '80s. It was shown that they play an important role in the olfactory information processing [9] and they were observed in the hippocampus [23], in the thalamus [20], and in the cortex [19]. Many studies suggested that, in the mammalian visual system, neurons signals may be group together through in-phase oscillations [12]. Because of this, the development and analysis of models representing oscillatory phenomena assume a great importance in understanding the neurophysiological activities.

A simple continuous model of a single oscillator has been proposed in [22]. The model describes the evolutions of one excitatory neuron (N_e) and one inhibitory neuron (N_i) by mean of the ordinary differential system.

$$f(\tau,\lambda): \begin{cases} \dot{X}_e = -\frac{X_e}{\tau} + \tanh(\lambda * X_e) - \tanh(\lambda * X_e) \\ \dot{X}_i = -\frac{X_i}{\tau} + \tanh(\lambda * X_i) + \tanh(\lambda * X_i) \end{cases}, \quad (1)$$

where X_e and X_i are the output of N_e and N_i, respectively, τ is a characteristic time constant, and $\lambda > 0$ is the amplification gain.

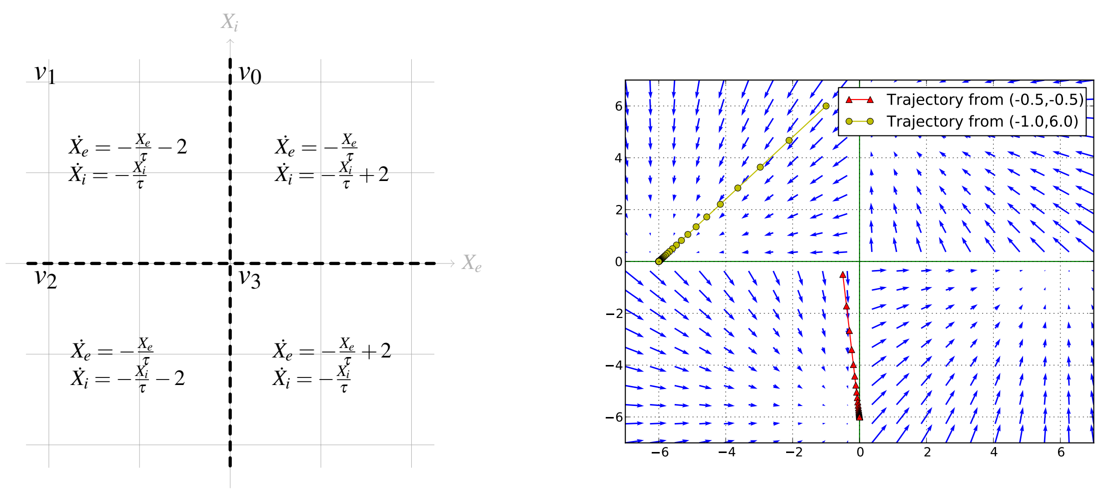
Hopf bifurcation characterizes a qualitative change in the evolution of $f(\tau,\lambda)$: if $\tau * \lambda \leq 1$, then the point $\langle 0, 0 \rangle$ is the unique global attractor of the system, if, otherwise, $\tau * \lambda > 1$, the origin is an unstable equilibrium and all the evolutions converge to a limit cycle attractor [2]. A simulation of $f(3,1)$ is represented in Fig. 5(a).

Even if $f(\tau,\lambda)$ is rather simple, the ability of analyzing a complex system obtained by composing multiple copies of this model is limited due to the non-linearity of $f(\tau,\lambda)$ itself. For this reason, we are interested in the development of a piecewise affine hybrid model whose behaviour fairly approximates System (1) and that can be automatically analyzed and composed.

Since the non-linear components in System (1) have the form $\tanh(\lambda * X)$, we try to linearize such function. In the case of genetic networks it is quite standard to approximate sigmoidal behaviours (e.g., tanh) through the sign function sgn [10, 14]. Such approximation replaces a continuous signal with a discrete off-on one (see Fig. 3(a)). In our case, by replacing $\tanh(\lambda * X)$ with $\mathrm{sgn}(X)$ in System (1), we obtain the following differential system:

$$\hat{f}(\tau): \begin{cases} \dot{X}_e = -\frac{X_e}{\tau} + \mathrm{sgn}(X_e) - \mathrm{sgn}(X_i) \\ \dot{X}_i = -\frac{X_i}{\tau} + \mathrm{sgn}(X_e) + \mathrm{sgn}(X_i) \end{cases}, \tag{2}$$

which corresponds to the piecewise hybrid model depicted in Fig. 2(a). Unfortunately, the behaviour of this model is quite different from that of System (1), as we can see comparing the simulation in Fig. 2(b) with that of Fig. 5(a). In particular, the model base on $\hat{f}(\tau)$ has four attractors with coordinates $\langle -2*\tau, 0\rangle$, $\langle 0, -2*\tau\rangle$, $\langle 2*\tau, 0\rangle$, and $\langle 0, 2*\tau\rangle$, it is not periodic, and its principal axes are stable.



(a) The automaton has 4 locations. The dashed line denote both the boundaries of the invariants and the activation regions. The resets are identify functions.

(b) Direction field and evolution of the automaton from the two points $\langle -\frac{1}{2}, -\frac{1}{2}\rangle$ and $\langle -1, 6\rangle$ when $\tau = 3$. The automaton has four attractors, is not periodic, and its principal axes are stable.

Figure 2: The piecewise hybrid automaton associated to the function $\hat{f}(\tau)$.

A more sophisticated approximation of $\tanh(\lambda * X)$ is the piecewise linear function:

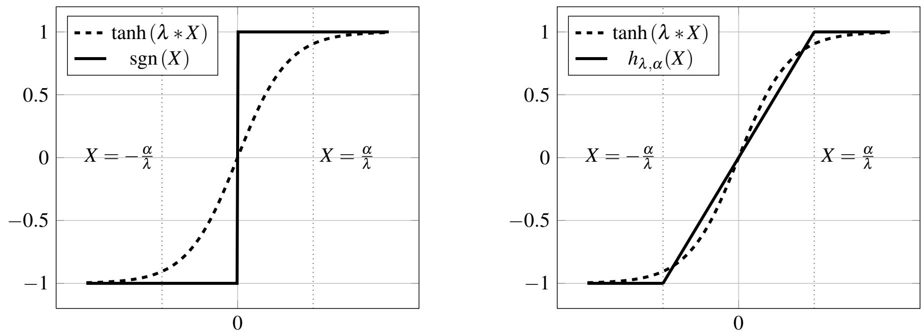
$$h_{\lambda,\alpha}(z) \stackrel{\text{def}}{=} \begin{cases} -1 & \text{if } z < -\frac{\alpha}{\lambda} \\ \frac{\lambda}{\alpha} * z & \text{if } -\frac{\alpha}{\lambda} \leq z < \frac{\alpha}{\lambda} \\ 1 & \text{if } z \geq \frac{\alpha}{\lambda} \end{cases}, \tag{3}$$

where α is the approximation coefficient which determines the slope of the central segment (see Fig. 3(b))

The substitution of $\tanh(\lambda * X)$ with $h_{\lambda,\alpha}(z)$ in System (1) leads to the system:

$$\tilde{f}_\alpha(\tau,\lambda): \begin{cases} \dot{X}_e = -\frac{X_e}{\tau} + h_{\lambda,\alpha}(X_e) - h_{\lambda,\alpha}(X_i) \\ \dot{X}_i = -\frac{X_i}{\tau} + h_{\lambda,\alpha}(X_e) + h_{\lambda,\alpha}(X_i) \end{cases} \tag{4}$$

whose corresponding hybrid automaton is depicted in Fig. 4.



(a) $\mathrm{sgn}(X)$ approximating $\tanh(\lambda * X)$.

(b) $h_{\lambda,\alpha}(X)$ approximating $\tanh(\lambda * X)$.

Figure 3: Both $h_{\lambda,\alpha}(X)$ and $\mathrm{sgn}(X)$ can be used as piecewise linear approximations of $\tanh(\lambda * X)$.

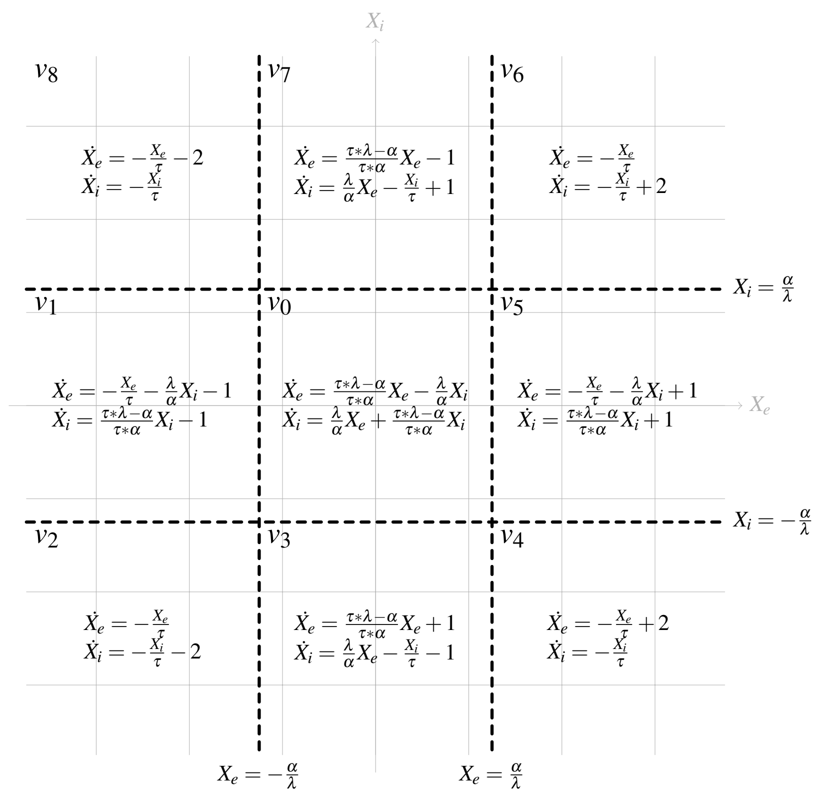
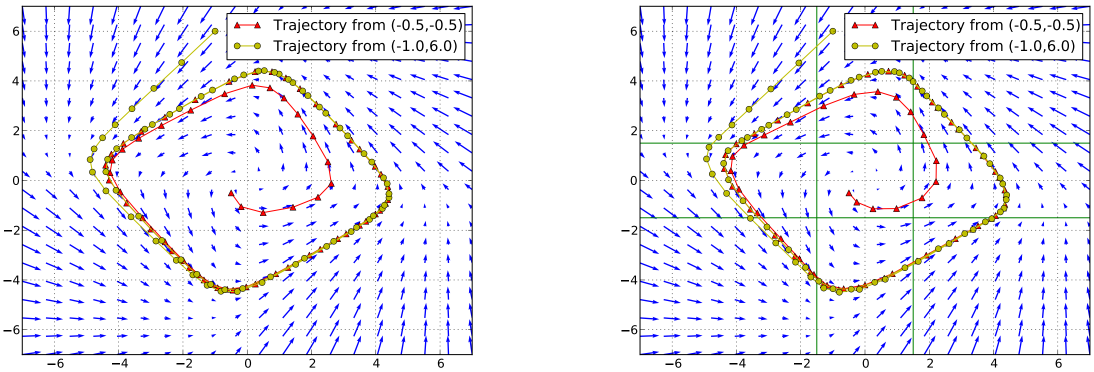


Figure 4: A graphical representation of the hybrid automaton $H_{\tilde{f}}$ associated to the function $\tilde{f}_\alpha(\tau,\lambda)$



(a) An evolution of $f(3,1)$

(b) An evolution of $\tilde{f}_2(3,1)$

Figure 5: Direction field and evolution of the models discussed in Section 2. The green lines in the figures depict the location invariant boundaries. The system $\tilde{f}_2(3,1)$ is a good approximation of $f(3,1)$.

In the rest of the paper we present some general techniques for studying hybrid automata and then we apply them to $H_{\hat{f}}$ to formally prove its properties.

3 Approximation Techniques

3.1 Noise and Disturbed Automata

The density of continuous variables provide an unbounded quantity of memory within a bounded region. As a matter of fact, the undecidability results proved in [13] are based on the possibility of embedding $\mathbb{N}$ in $(0,1] \subseteq \mathbb{R}$ through the function $f(n) = 2^{-n}$.

However, Fränzle in [8] observed that noise disturbes the trajectiories of real hybrid systems, augmenting the set of reachable points. Hence, in [8] a model of noise has been presented over hybrid automata. Remarkably, the introduction of noise ensures in many cases the (semi-)decidability of the reachability problem.

Our definitions of hybrid automata slightly differ from the ones in [8]. In particular, as far as the syntax is concerned, the formulæ $Act(e)[\mathbf{X}]$ and $Res(e)[\mathbf{X},\mathbf{X}']$ are glued together in a formula called $trans_e[\mathbf{X},\mathbf{X}']$. Moreover, our formulæ $Inv(v)[\mathbf{X}]$ and $Dyn(v)[\mathbf{X},\mathbf{X}',T]$ are replaced by a single formula $act_v[\mathbf{X},\mathbf{X}']$ whose meaning in our framework is:

$$\exists T(T \geq 0 \wedge \mathbf{X}' = f_v(\mathbf{X})(T) \wedge \forall T'(0 \leq T' \leq T \rightarrow Inv(v) f_v(\mathbf{X})(T')))$$

i.e., the formula $act_v[\mathbf{X},\mathbf{X}']$ syntactically ensures the existence of a continuous transition. Exploiting these relationships between our hybrid automata and the hybrid automata defined through the formulæ $act_v[\mathbf{X},\mathbf{X}']$ and $trans_e[\mathbf{X},\mathbf{X}']$, we can reformulate the results presented in [8] in our framework.

Definition 4. *Given a hybrid automaton* $H = \langle \mathbf{X}, \mathbf{X}', \mathcal{V}, \mathcal{E}, Inv, f_., Act, Res\rangle$ *we say that the hybrid automaton* $\widetilde{H} = \langle \mathbf{X}, \mathbf{X}', \mathcal{V}, \mathcal{E}, \widetilde{Inv}, \widetilde{f}_., Act, Res\rangle$ *is a* disturbed variant *of* H *if for each pair of states* a, a' *if* $a \rightarrow_C a'$ *in* H*, then* $a \rightarrow_C a'$ *in* $\widetilde{H}$*.*

Moreover, let δ *be a distance over* $\mathbb{R}^{d(H)}$ *and* $\varepsilon \in \mathbb{R}_{>0}$*.* $\widetilde{H}$ *is a* disturbance of noise level ε or more *if for each* s, s' *such that* $\delta(s,s') < \varepsilon$ *it holds that if* $\langle v,r\rangle \rightarrow_C \langle v,s\rangle$ *in* H*, then* $\langle v,r\rangle \rightarrow_C \langle v,s'\rangle$ *in* $\widetilde{H}$*.*

Intuitively, when there are no bifurcation behaviours, a small ε ensures that the dynamics of $\widetilde{H}$ are close to those of H.

In [8] it has been proved that in the case of bounded invariants there exists a finite computable index $i \in \mathbb{N}$ such that the reachability over H can be over-approximated with reachability within i discrete jumps over a disturbance of noise level ε or more of H.

Theorem 1. *[8] Let $\mathscr{T}$ be a decidable first-order theory. Let $H, \widetilde{H}$ be $\mathscr{T}$ hybrid automata, with $\widetilde{H}$ disturbance of noise level ε or more of H for some $\varepsilon \in \mathbb{R}_{>0}$. There exists i such that for all pairs of states a, a' if a reaches a' in H, then a reaches a' in $\widetilde{H}$ within i discrete transitions. Moreover, i can be effectively computed.*

Unfortunately, there are cases in which the over-approximation is always strict, no matter how small ε is. Such automata are called *fragile*, in contrast with *robust* automata, where if a does not reach a' in H, there exist $\varepsilon \in \mathbb{R}_{>0}$ and $\widetilde{H}$ disturbance of noise level ε such that a does not reach a' in $\widetilde{H}$. As a consequence, reachability is decidable over robust automata, while it is only semi-decidable over fragile automata. It is not possible to decide whether a hybrid automaton is robust or fragile. Intuitively, since real world systems are always subject to noise, a hybrid automaton is a reliable model of the real system only if it is robust. So, it is fundamental to develop and exploit design techniques which ensure robustness of the resulting hybrid automata.

3.2 Approximate Bisimulations and Simulations

Since the 90's, simulation and bisimulation have been successfully used to investigate hybrid automata However, due to the infinite precision required to relate different evolutions, there tools are able to remove neither the complexity nor the undecidability issues that may affect the analysis of the investigated model. The ε-(bi)simulation relations [11] relaxes these infinite precision requirements by relating system evolutions whose maximal distance is less than a given ε. This enables us to simplify both the dynamics and the resets of the investigated automaton. Moreover, provided an *observation map* $\langle\!\langle \cdot \rangle\!\rangle : \mathbb{R}^{d(H)} \longrightarrow \mathbb{R}^d$ that associates the internal status of an automaton H to the values measurable by an external observer, ε-(bi)simulations allow to relate the "visible" behaviours of H to the behaviours of an automaton whose dimensions is smaller that $d(H)$.

Any pair of hybrid automata H_1 to H_2 related by an ε-simulation must have the same discrete structure and share the same locations $\mathcal{V}$ and edges $\mathcal{E}$ by definition.

Definition 5. *Let $H_i = \langle \mathbf{X_i}, \mathbf{X_i}', \mathcal{V}, \mathcal{E}, Inv_i, f_{\cdot,i}, Act_i, Res_i \rangle$ be a hybrid automaton for each $i \in \{1,2\}$. Moreover, let ε be in $\mathbb{R}_{\geq 0}$. A relation $\mathscr{S}_\varepsilon \subseteq (\mathcal{V} \times \mathbb{R}^{d(H_1)}) \times (\mathcal{V} \times \mathbb{R}^{d(H_2)})$ is an* approximate simulation relation of H_1 by H_2 of precision ε *if, for all $\langle \langle v_1, r_1 \rangle, \langle v_2, r_2 \rangle \rangle \in \mathscr{S}_\varepsilon$:*

1. *$v_1 = v_2 = v$;*
2. *$\| \langle\!\langle r_1 \rangle\!\rangle_1 - \langle\!\langle r_2 \rangle\!\rangle_2 \| \leq \varepsilon$;*
3. *if $\langle v, r_1 \rangle \xrightarrow{t}_C \langle v, r_1' \rangle$ in H_1, there exists r_2' s.t. $\langle v, r_2 \rangle \xrightarrow{t}_C \langle v, r_2' \rangle$ in H_2 and $\langle \langle v, r_1' \rangle, \langle v, r_2' \rangle \rangle \in \mathscr{S}_\varepsilon$;*
4. *if $\langle v, r_1 \rangle \xrightarrow{e}_D \langle v', r_1' \rangle$ in H_1, there exists r_2' s.t. $\langle v, r_2 \rangle \xrightarrow{e}_D \langle v', r_2' \rangle$ in H_2 and $\langle \langle v', r_1' \rangle, \langle v', r_2' \rangle \rangle \in \mathscr{S}_\varepsilon$.*

The automaton H_2 approximately simulates H_1 with precision ε if there exists an approximate simulation relation of H_1 by H_2 of precision ε. An approximate simulation relation $\mathscr{S}_\varepsilon$ of H_1 by H_2 of precision ε is an *approximate bisimulation relation between H_1 and H_2 of precision ε* if the relation $\mathscr{S}_\varepsilon^{-1} = \{ \langle a_2, a_1 \rangle \mid \langle a_1, a_2 \rangle \in \mathscr{S}_\varepsilon \}$ is an approximate simulation relation of H_2 by H_1 of precision ε.

Many methods have been developed to automatically compute approximate simulation relations between systems such as constrained linear systems, autonomous nonlinear systems, and hybrid systems.

3.3 ε-Semantics

The undecidability of the reachability problem over hybrid automata having bounded invariants is a direct consequence of the ability of characterizing dense regions of arbitrarily small size. As noticed in [3], especially in the study of biological systems, such ability may result misleading. As a matter of the fact, the continuous quantities used in hybrid automata are very often abstractions of large, but discrete, quantities. In such cases, the ability of handling values with infinite precision is a model artifact rather than a real property of the original system.

In order to discretize the continuous space, we introduce the concept of ε-*sphere*. Given a set $\mathbb{S} \subseteq \mathbb{R}^n$, the ε-sphere $B(\mathbb{S}, \varepsilon)$ is the subset of $\mathbb{R}^n$ of points at distance less than ε from $\mathbb{S}$, i.e., $B(\mathbb{S}, \varepsilon) = \{q \in \mathbb{R}^n \mid \exists p \in \mathbb{S}(\delta(p,q) < \varepsilon)\}$, where δ is the standard euclidean distance over $\mathbb{R}^n$. Moreover, given a hybrid automaton H and an initial set of points $\mathbb{I} \subseteq \mathbb{R}^{d(H)}$, the set of points reachable from the set $\mathbb{I}$ by H, denoted by $RSet_H(\mathbb{I})$, is characterized by

$$RSet_H(\mathbb{I}) = \bigcup_{i \in \mathbb{N}} RSet^i_H(\mathbb{I}) = \lim_{i \to +\infty} RSet^i_H(\mathbb{I})$$

where $RSet^i_H(\mathbb{I})$ is the set of points reachable from $\mathbb{I}$ in at most i discrete transitions.

Theorem 2 ([3])**.** *Let $\mathscr{T}$ be a decidable first-order theory over reals and H be a $\mathscr{T}$ hybrid automaton with bounded invariants. If there exists $\varepsilon \in \mathbb{R}_{>0}$ such that, for each $\mathbb{I} \subseteq \mathbb{R}^{d(H)}$ and for each $i \in \mathbb{N}$, either $RSet^{i+1}_H(\mathbb{I}) = RSet^i_H(\mathbb{I})$ or there exists a $a_i \in \mathbb{R}^{d(H)}$ such that $B(\{a_i\}, \varepsilon) \subseteq RSet^{i+1}_H(\mathbb{I}) \setminus RSet^i_H(\mathbb{I})$, then there exists $j \in \mathbb{N}$ such that $RSet^i_H(\mathbb{I}) = RSet^j_H(\mathbb{I})$ and the reachability problem over H is decidable.*

This result finds applications when it makes no sense to distinguish measurements smaller than ε. Hence, since hybrid automata characterization is based on first-order fromulæ, it seems reasonable to reinterpret the semantics of semi-algebraic automata by giving to each formula a semantics of "dimension of at least ε". In [3] the authors introduce a new class of semantics for first-order formulæ, called ε-*semantics*, which guarantee the decidability of reachability in the case of hybrid automata with bounded invariants.

Definition 6. *Let $\mathscr{T}$ be a first-order theory and let $\varepsilon \in \mathbb{R}_{>0}$. For each formula ψ on $\mathscr{T}$ let $\{\!|\psi|\!\}_\varepsilon \subseteq \mathbb{R}^d$, where d is the number of free variables of ψ, be such that:*

- *(ε) either $\{\!|\psi|\!\}_\varepsilon = \emptyset$ or there exists $p \in \mathbb{R}^d$ such that $B(\{p\}, \varepsilon) \subseteq \{\!|\psi|\!\}_\varepsilon$;*
- *($\cap$) $\{\!|\phi \wedge \varphi|\!\}_\varepsilon \subseteq \{\!|\phi|\!\}_\varepsilon \cap \{\!|\varphi|\!\}_\varepsilon$;*
- *($\cup$) $\{\!|\phi \vee \varphi|\!\}_\varepsilon = \{\!|\phi|\!\}_\varepsilon \cup \{\!|\varphi|\!\}_\varepsilon$;*
- *($\forall$) $\{\!|\forall X \psi[X, \mathbf{X}]|\!\}_\varepsilon = \{\!|\bigwedge_{r \in \mathbb{R}} \psi[\![r, \mathbf{X}]\!]|\!\}_\varepsilon$;*
- *($\exists$) $\{\!|\exists X \psi[X, \mathbf{X}]|\!\}_\varepsilon = \{\!|\bigvee_{r \in \mathbb{R}} \psi[\![r, \mathbf{X}]\!]|\!\}_\varepsilon$;*
- *($\neg$) $\{\!|\psi|\!\}_\varepsilon \cap \{\!|\neg\psi|\!\}_\varepsilon = \emptyset$.*

*Any semantics satisfying the above conditions is said to be an ε-*semantics for $\mathscr{T}$.

Example 3 (The sphere semantics)**.** *Let $\mathscr{T}$ be a first-order theory over the reals and let $\varepsilon > 0$. The* sphere semantics *of ψ, $(\!|\psi|\!)_\varepsilon$, is defined by structural induction on ψ as follows:*

- $(\!|t_1 \circ t_2|\!)_\varepsilon \stackrel{def}{=} B(\{\!|t_1 \circ t_2|\!\}, \varepsilon)$, *for* $\circ \in \{=, <\}$;
- $(\!|\psi_1 \wedge \psi_2|\!)_\varepsilon \stackrel{def}{=} \bigcup_{B(\{p\},\varepsilon) \subseteq (\!|\psi_1|\!)_\varepsilon \cap (\!|\psi_2|\!)_\varepsilon} B(\{p\}, \varepsilon)$;
- $(\!|\psi_1 \vee \psi_2|\!)_\varepsilon \stackrel{def}{=} (\!|\psi_1|\!)_\varepsilon \cup (\!|\psi_2|\!)_\varepsilon$;

- $(\!|\forall X\, \psi\llbracket X, \mathbf{X}\rrbracket|\!)_\varepsilon \stackrel{def}{=} (\!|\bigwedge_{r\in\mathbb{R}} \psi\llbracket r, \mathbf{X}\rrbracket|\!)_\varepsilon$;
- $(\!|\exists X\, \psi\llbracket X, \mathbf{X}\rrbracket|\!)_\varepsilon \stackrel{def}{=} (\!|\bigvee_{r\in\mathbb{R}} \psi\llbracket r, \mathbf{X}\rrbracket|\!)_\varepsilon$;
- $(\!|\neg\psi|\!)_\varepsilon \stackrel{def}{=} \bigcup_{B(\{p\},\varepsilon)\cap(\!|\psi|\!)_\varepsilon=\emptyset} B(\{p\},\varepsilon)$.

ε-semantics are exploited in the reachability algorithm defined in [3]. In particular, describing sets of points through formulæ and, given a hybrid automaton H and an ε-semantics $\{\!|\cdot|\!\}_\varepsilon$, the computation of the algorithm proceeds as follows:

1. Given an initial set of points represented by a formula $I[\mathbf{X}]$, the formula $R[\mathbf{X}]$ is initialized putting $I[\mathbf{X}]$ in conjunction with the invariant conditions of H, while the formula $N[\mathbf{X}]$ is initialized as $\perp$;
2. $R[\mathbf{X}]$ is replaced with $R[\mathbf{X}] \vee N[\mathbf{X}]$ and $N[\mathbf{X}]$ becomes $\exists \mathbf{X}'(Reach_H\llbracket \mathbf{X}', \mathbf{X}\rrbracket \wedge R\llbracket \mathbf{X}'\rrbracket)$, where the formula $Reach_H[\mathbf{X}, \mathbf{X}']$ denotes automaton evolutions which perform at most one discrete transition, i.e., if $Reach_H\llbracket s, d\rrbracket$, then H reaches d from s with at most one discrete transition;
3. Step 2 is repeated until $\{\!|N[\mathbf{X}] \wedge \neg R[\mathbf{X}]|\!\}_\varepsilon$ is not empty.

Algorithm 1 Reachability($H, I[\mathbf{X}], \{\!|\cdot|\!\}_\varepsilon$)

$R[\mathbf{X}] \leftarrow I[\mathbf{X}]$
$N[\mathbf{X}] \leftarrow \perp$
repeat
  $R[\mathbf{X}] \leftarrow R[\mathbf{X}] \vee N[\mathbf{X}]$
  $N[\mathbf{X}] \leftarrow \exists \mathbf{X}'(Reach_H\llbracket \mathbf{X}', \mathbf{X}\rrbracket \wedge R\llbracket \mathbf{X}'\rrbracket)$
until $\{\!|N[\mathbf{X}] \wedge \neg R[\mathbf{X}]|\!\}_\varepsilon \neq \emptyset$
return $\{\!|R[\mathbf{X}]|\!\}_\varepsilon$

Intuitively, this means that reachability is computed incrementing at each iteration the number of allowed discrete transitions. New reachable sets of points are computed until they became too small to be captured by the ε-semantics. In the case of hybrid automata with bounded invariants, Algorithm 1 always terminates. Finally, notice that replacing the ε-semantics with the standard one, the above algorithm could not terminate even with bounded invariants, due to Zeno behaviours.

4 Approximated Analysis over Neural Oscillators

In this section we try to understand what happens when we apply the approximation techniques described in Section 3 to our neural oscillator hybrid model $H_{\tilde{f}}$ presented in Section 2.

4.1 ε-disturbance

The automaton $H_{\tilde{f}}$ presents two main behaviours: $(0,0)$ is an unstable equilibrium; each starting point different from $(0,0)$ reaches the limit cycle. For this reason we can prove that $H_{\tilde{f}}$ is fragile. As a matter of fact, if $\widetilde{H_{\tilde{f}}}$ is a disturbance of noise level ε of $H_{\tilde{f}}$, then $(0,0)$ in $\widetilde{H_{\tilde{f}}}$ reaches points different from $(0,0)$, while in $H_{\tilde{f}}$ it does not. In other words, if we consider backward reachability, $(0,0)$ is backward reachable in $\widetilde{H_{\tilde{f}}}$ from a region $R \neq \{(0,0)\}$, while in $H_{\tilde{f}}$ it is backward reachable from $\{(0,0)\}$. This is not due to the fact that $(0,0)$ is unstable, but to the presence of two limit behaviours over a connected region. We recall that in a piecewise hybrid automaton the invariants are connected disjoint regions whose union is connected and the resets are identities, i.e., the trajectories are continuos. We use the term *limit behaviour* of a hybrid automaton to denote both equilibria and limit cycles.

Theorem 3. *Let H be a piecewise hybrid automaton presenting at least two different limit behaviours. If from each point there is a unique possible evolution, then H is fragile.*

Proof. Let b_1 and b_2 be two different limit behaviours. Let I be the union of the invariants of H. Let $R_1 = \{r \mid r \text{ reaches } b_1 \text{ but not } b_2\}$ and $R_2 = \complement_I(R_1) = I \setminus R_1$. Moreover, let b_2 be the unsafe set and R_1 be the init set. H is safe since the points in R_1 do not reach b_2.

We have that $R_1 \neq \emptyset$, since the points belonging to b_1 reach b_1 and they cannot reach b_2, since the evolutions are uniquely defined from each starting point. Similarly $R_2 \neq \emptyset$. Hence, $\{R_1, R_2\}$ is a partition of I. Since I is connected, for each $\varepsilon \in \mathbb{R}_{>0}$ there are $p_1 \in R_1$ and $p_2 \in R_2$, such that $\delta(p_1, p_2) < \varepsilon$. So, p_1 reaches p_1 in H, implies that p_1 reaches p_2 in $\widetilde{H}$ disturbance of noise level ε or more of H. So, since $p_2 \in R_2$ reaches b_2 in H, we get that p_1 reaches b_2 in $\widetilde{H}$. Since, we made no assumption on ε, we get that all the disturbances of noise level ε or more of H are unsafe, this means that H is fragile. □

The above result points out that there are systems for which it is not possible to define robust models. In [18] a model is said to be safe only if it remains safe under small disturbances. In this terms our result show that there are systems which do not admit a safe model. This does not means that they are not interesting or that we need to remove some of their behaviours. This simply means that such systems have to be studied applying some form of disturbance or approximation. As the matter of facts, if we study them by applying standard semantics, we define a precise border between the points reaching different behaviours. Such precise border is not realistic.

4.2 ε-(bi)simulations

The automaton $H_{\tilde{f}}$ has both an unstable equilibrium in $\langle 0,0\rangle$ and a single limit cycle encompassing the origin of the axes. Because of that we are guaranteed that, during its evolutions, $H_{\tilde{f}}$ decreases the distance of its state from the limit cycle regardless of the starting state $s \neq \langle v_0, \langle 0,0\rangle\rangle$. Since all the differential equation defining the dynamics of $H_{\tilde{f}}$ are continuous, we can define an ε-simulation between states whose distance from the limit cycle is smaller than ε. This enable us to both approximate the non-linear differential System (1) with a linear differential system and reduce the complexity of the analysis.

However, if d is the maximum Euclidean distance between $\langle 0,0\rangle$ and the cycle limit, no ε-(bi)simulation, with $\varepsilon < d$, can relate $\langle v_0, \langle 0,0\rangle\rangle$ with any other state of $H_{\tilde{f}}$. As a matter of fact, the points belonging to any neighborhood of $\langle 0,0\rangle$ eventually converge to the limit cycle. It follows that $\langle v_0, \langle 0,0\rangle\rangle$ is a singularity of the model and, despite the original system always reaches a periodic evolution, any approximation of the proposed model by mean of ε-(bi)simulation does not manifest this property.

4.3 ε-semantics

In order to exploit ε-semantics for the study of $H_{\tilde{f}}$ the first step we have to perform is that of approximating through polynomials the solutions of the differential equations defining the semantics. This can be done, for instance, by using Taylor polynomials or more sophisticated numerical integration techniques. We do this in the next section, where we also apply cylindrical algebraic decomposition tools to automatically prove properties of our model. Here instead we try to infer some general results about the use of ε-semantics on $H_{\tilde{f}}$.

$H_{\tilde{f}}$ has an unstable equilibrium in $(0,0)$. This means that $(0,0)$ reaches $\{(0,0)\}$. However, when we compute the set of points reachable from $(0,0)$ through an ε-semantics we get either the empty set or a set having diameter at least ε. In particular, if our ε-semantics under-approximates the standard one, then we get the empty set. Otherwise, both cases are possible, depending on the ε-semantics. For instance, in the case of sphere semantics, no matter how we approximate the dynamics, we get that $(0,0)$ reaches a set having diameter at least ε.

Similarly, unless we use an under-approximation ε-semantics or some unusual metrics, the limit cycle is transformed into a limit flow tube. This means, that if we consider a point on the limit cycle and we compute the set of points reachable from such point, we do not only obtain the limit cycle, but at least a flow tube which includes the limit cycle. We will see some more details on this in the case of sphere semantics in Section 5.

All the other points, again, will reach either the empty set or a set having diameter at least ε. The result we would expect in this second case is that each point in the space reaches the flow tube including the limit cycle. We will see that this is true in the case of sphere semantics, even when we use the simplest Taylor polynomials of degree one.

These considerations already allow us to point out that sphere semantics better reflects the real system behaviour than the standard one.

5 Computing Sphere Semantics

In this section we show how sphere semantics can be computed exploiting tools for cylindrical algebraic decomposition. In particular, we introduce a translation from sphere semantics to standard semantics. Then, we apply the translation to study the "sphere" behaviour of our neural oscillator example.

5.1 A translation into standard semantics

If $\mathscr{T}$ is a first-order theory and δ is a distance definable in $\mathscr{T}$, then the sphere semantics of any formula in $\mathscr{T}$ is $\mathscr{T}$-definable in the standard semantics, i.e., for any formula $\varphi[\mathbf{X}] \in \mathscr{T}$ we can compute a formula $\widehat{(\varphi)_\varepsilon}[\mathbf{X}] \in \mathscr{T}$ such that $(\!|\varphi[\mathbf{X}]|\!)_\varepsilon = \left\{\!\left|\widehat{(\varphi)_\varepsilon}[\mathbf{X}]\right|\!\right\}$ for all $\varepsilon \in \mathbb{R}_{>0}$.

In order to achieve this goal, we need to distinguish two kind of variables: the variables of the original formula (named W, W_i, $\mathbf{W}$ and $\mathbf{W_i}$), whose evaluations follow the rules of the sphere semantics, and the auxiliary variables (named Y, Y_i, $\mathbf{Y}$ and $\mathbf{Y_i}$) that will be introduced to encode the sphere semantics into the standard one. From the point of view of the sphere semantics the later can seen as symbolic constants, even if they will be quantified in the formula $\widehat{(\varphi)_\varepsilon}$. In particular, we will use them to characterize sets of the form $(\!|\bigwedge_{r\in\mathbb{R}} \varphi[\![r,\mathbf{W}]\!]|\!)_\varepsilon$ and $(\!|\bigvee_{r\in\mathbb{R}} \varphi[\![r,\mathbf{W}]\!]|\!)_\varepsilon$ in the standard semantics.

Definition 7. *Let $\mathscr{T}$ be a first-order theory over the reals, $\varphi[\mathbf{Y}_,\mathbf{W}]$ be any first-order formula $\mathscr{T}$-definable, and $\varepsilon \in \mathbb{R}_{>0}$. We define $\widehat{(\varphi)_\varepsilon}[\mathbf{Y}_,\mathbf{W}]$ by structural induction on $\varphi[\mathbf{Y}_,\mathbf{W}]$ itself.*

1. $(\widehat{(t_1 \circ t_2)[\mathbf{Y},\mathbf{W}]})_\varepsilon \stackrel{def}{=} \exists \mathbf{W_0}((t_1 \circ t_2)[\![\mathbf{Y},\mathbf{W_0}]\!] \wedge \delta(\mathbf{W_0},\mathbf{W}) < \varepsilon)$, *for* $\circ \in \{=,<\}$;
2. $(\widehat{\phi \vee \psi})_\varepsilon \stackrel{def}{=} (\widehat{\phi})_\varepsilon \vee (\widehat{\psi})_\varepsilon$;
3. $(\widehat{\phi[\mathbf{Y},\mathbf{W}] \wedge \psi[\mathbf{Y},\mathbf{W}]})_\varepsilon \stackrel{def}{=} \exists \mathbf{W_0}(\forall \mathbf{W_1}(\delta(\mathbf{W_0},\mathbf{W_1}) < \varepsilon \rightarrow ((\widehat{\phi})_\varepsilon \wedge (\widehat{\psi})_\varepsilon)[\![\mathbf{Y},\mathbf{W_1}]\!]) \wedge \delta(\mathbf{W_0},\mathbf{W}) < \varepsilon)$;
4. $(\widehat{\forall W \phi[\mathbf{Y},W,\mathbf{W}]})_\varepsilon \stackrel{def}{=} \exists \mathbf{W_0}(\forall \mathbf{W_1}(\delta(\mathbf{W_0},\mathbf{W_1}) < \varepsilon \rightarrow \forall Y(\widehat{\phi[\![\mathbf{Y},Y,\mathbf{W_1}]\!]})_\varepsilon) \wedge \delta(\mathbf{W_0},\mathbf{W}) < \varepsilon)$;
5. $(\widehat{\exists W \phi[\mathbf{Y},W,\mathbf{W}]})_\varepsilon \stackrel{def}{=} \exists Y(\widehat{\phi[\![\mathbf{Y},Y,\mathbf{W_1}]\!]})_\varepsilon$;
6. $(\widehat{\neg\phi[\mathbf{Y},\mathbf{W}]})_\varepsilon \stackrel{def}{=} \exists \mathbf{W_0}(\forall \mathbf{W_1}(\delta(\mathbf{W_0},\mathbf{W_1}) < \varepsilon \rightarrow \neg(\widehat{\phi[\![\mathbf{Y},\mathbf{W_1}]\!]})_\varepsilon) \wedge \delta(\mathbf{W_0},\mathbf{W}) < \varepsilon)$.

Theorem 4 (Semantics Equivalence)**.** *Let $\mathscr{T}$ be any first-order theory and δ be a $\mathscr{T}$-definable distance. The sphere semantics $(\!|.|\!)_\varepsilon$ of $\mathscr{T}$ is $\mathscr{T}$-definable in the standard semantics and, in particular,* $(\!|\varphi[\mathbf{X}]|\!)_\varepsilon = \left\{\!\left|\widehat{(\varphi)}_\varepsilon[\mathbf{X}]\right|\!\right\}$ *for any formula $\varphi[\mathbf{X}] \in \mathscr{T}$ and all $\varepsilon \in \mathbb{R}_{>0}$.*

Proof. By structural induction on φ. We follow the indexes of the $\widehat{(\varphi)}_\varepsilon$'s definition.

$\varphi[\mathbf{Y},\mathbf{W}]$ is atomic.
By the definition of $(\!|\cdot|\!)_\varepsilon$, $(\!|\varphi[\mathbf{Y},\mathbf{W}]|\!)_\varepsilon = B(\{\!|\varphi[\mathbf{Y},\mathbf{W}]|\!\},\varepsilon)$. Since $\mathbf{Y}$ is a vector of auxiliary variables, we should threat them as symbolic constants and we must not consider them as free variables. Hence, $\langle\vec{y},\vec{w}\rangle \in (\!|\varphi[\mathbf{Y},\mathbf{W}]|\!)_\varepsilon$ if and only if there exists a $\vec{w_0}$ such that $\delta(\vec{w},\vec{w_0}) < \varepsilon$ and $\{\!|\varphi[\![\vec{y},\vec{w_0}]\!]|\!\}$. By the standard semantics, the later sentence holds if and only if $\exists \mathbf{W_0}(\varphi[\![\mathbf{Y},\mathbf{W_0}]\!] \wedge \delta(\mathbf{W_0},\mathbf{W}) < \varepsilon)$ does the same.

$\varphi[\mathbf{Y},\mathbf{W}]$ has the form $\phi[\mathbf{Y},\mathbf{W}] \vee \psi[\mathbf{Y},\mathbf{W}]$.
By the definition of sphere semantics, $(\!|\phi \vee \psi|\!)_\varepsilon = (\!|\phi|\!)_\varepsilon \cup (\!|\psi|\!)_\varepsilon$. By inductive hypothesis, both $(\!|\phi|\!)_\varepsilon = \left\{\!\left|\widehat{(\phi)}_\varepsilon\right|\!\right\}$ and $(\!|\psi|\!)_\varepsilon = \left\{\!\left|\widehat{(\psi)}_\varepsilon\right|\!\right\}$. From the standard semantics and the definition of $\widehat{(\cdot)}_\varepsilon$, we deduce the thesis.

$\varphi[\mathbf{Y},\mathbf{W}]$ has the form $\phi[\mathbf{Y},\mathbf{W}] \wedge \psi[\mathbf{Y},\mathbf{W}]$.
By the definition, $(\!|\phi[\mathbf{Y},\mathbf{W}] \wedge \psi[\mathbf{Y},\mathbf{X}]|\!)_\varepsilon \stackrel{\text{def}}{=} \bigcup_{B(\{p\},\varepsilon)\subseteq (\!|\phi|\!)_\varepsilon \cap (\!|\psi|\!)_\varepsilon} B(\{p\},\varepsilon)$, while, by inductive hypotheses, $(\!|\phi|\!)_\varepsilon = \left\{\!\left|\widehat{(\phi)}_\varepsilon\right|\!\right\}$ and $(\!|\psi|\!)_\varepsilon = \left\{\!\left|\widehat{(\psi)}_\varepsilon\right|\!\right\}$. From the standard semantics, we deduce that $(\!|\psi[\mathbf{Y},\mathbf{X}] \wedge \psi[\mathbf{Y},\mathbf{X}]|\!)_\varepsilon = \bigcup_{B(\{p\},\varepsilon)\subseteq \left\{\!\left|\widehat{(\phi)}_\varepsilon \wedge \widehat{(\psi)}_\varepsilon\right|\!\right\}} B(\{p\},\varepsilon)$. The righter term of the last equation is the union of all the ε-balls entirely included into the standard semantics of $\widehat{(\psi)}_\varepsilon \wedge \widehat{(\psi)}_\varepsilon$. Any w is included into such a union if and only if there exists a w_0 such that all the points included into the ε-ball centered in w_0 satisfy $\widehat{(\phi)}_\varepsilon[\mathbf{Y},\mathbf{W}] \wedge \widehat{(\psi)}_\varepsilon[\mathbf{Y},\mathbf{W}]$ and w is included into the ε-ball itself. By the standard semantics, the later sentence holds if and only if the formula

$$\exists \mathbf{W_0}(\forall \mathbf{W_1}(\delta(\mathbf{W_0},\mathbf{W_1}) < \varepsilon \rightarrow (\widehat{(\phi)}_\varepsilon \wedge \widehat{(\psi)}_\varepsilon)[\![\mathbf{Y},\mathbf{W_1}]\!]) \wedge \delta(\mathbf{W_0},\mathbf{W}) < \varepsilon)$$

does the same.

$\varphi[\mathbf{Y},\mathbf{W}]$ has the form $\forall W \phi[\mathbf{Y},W,\mathbf{W}]$.
By the definition, $(\!|\forall W \phi[\mathbf{Y},W,\mathbf{W}]|\!)_\varepsilon \stackrel{\text{def}}{=} (\!|\bigwedge_{r\in\mathbb{R}} \phi[\![\mathbf{Y},r,\mathbf{W}]\!]|\!)_\varepsilon$. Because of the $\wedge$-rule, this means that $(\!|\forall W \phi[\mathbf{Y},W,\mathbf{W}]|\!)_\varepsilon = \bigcup_{B(\{p\},\varepsilon)\subseteq \left\{\!\left|\bigwedge_{r\in\mathbb{R}} (\phi\widehat{[\![\mathbf{Y},r,\mathbf{W}]\!]})_\varepsilon\right|\!\right\}} B(\{p\},\varepsilon)$. The righter term of the last equation is the union of all the ε-balls entirely included into the standard semantics of the formula $\bigwedge_{r\in\mathbb{R}} (\phi\widehat{[\![\mathbf{Y},r,\mathbf{W}]\!]})_\varepsilon$. However, a point $\langle\vec{y},\vec{w}\rangle$ belongs to $\left\{\!\left|\bigwedge_{r\in\mathbb{R}} (\phi\widehat{[\![\mathbf{Y},r,\mathbf{W}]\!]})_\varepsilon\right|\!\right\}$ if and only if $\langle\vec{y},\vec{w}\rangle$ itself belongs to $\left\{\!\left|(\phi\widehat{[\![\mathbf{Y},y,\mathbf{W}]\!]})_\varepsilon\right|\!\right\}$ for all $y \in \mathbb{R}$. Hence, $\left\{\!\left|\bigwedge_{r\in\mathbb{R}} (\phi\widehat{[\![\mathbf{Y},r,\mathbf{W}]\!]})_\varepsilon\right|\!\right\}$ is equivalent to $\left\{\!\left|\forall Y(\phi\widehat{[\![\mathbf{Y},Y,\mathbf{W}]\!]})_\varepsilon\right|\!\right\}$. Let us notice that the new quantified variable Y should not be expanded since it was introduced exclusively to characterize the set $\left\{\!\left|\bigwedge_{r\in\mathbb{R}} (\phi\widehat{[\![\mathbf{Y},r,\mathbf{W}]\!]})_\varepsilon\right|\!\right\}$. By using the same argument used in the proof of the $\wedge$-case, we can conclude that $(\!|\forall W \phi[\mathbf{Y},W,\mathbf{W}]|\!)_\varepsilon$ is defined in the standard semantics by the formula

$$\exists \mathbf{W_0}(\forall \mathbf{W_1}(\delta(\mathbf{W_0},\mathbf{W_1}) < \varepsilon \rightarrow \forall Y(\phi\widehat{[\![\mathbf{Y},Y,\mathbf{W_1}]\!]})_\varepsilon) \wedge \delta(\mathbf{W_0},\mathbf{W}) < \varepsilon).$$

$\varphi[\mathbf{Y},\mathbf{W}]$ has the form $\exists W \phi[\mathbf{Y},W,\mathbf{W}]$.

By the definition, $(\!|\exists W \phi[\mathbf{Y},W,\mathbf{W}]|\!)_\varepsilon \stackrel{\text{def}}{=} (\!|\bigvee_{r\in\mathbb{R}} \phi[\![\mathbf{Y},r,\mathbf{W}]\!]|\!)_\varepsilon$. Because of the $\vee$-rule, this means that $(\!|\exists W \phi[\mathbf{Y},W,\mathbf{W}]|\!)_\varepsilon = \bigcup_{B(\{p\},\varepsilon)\subseteq\left\{\left|\bigvee_{r\in\mathbb{R}}(\phi\widehat{[\![\mathbf{Y},r,\mathbf{W}]\!]})_\varepsilon\right|\right\}} B(\{p\},\varepsilon)$. The righter term of the last equation is the union of all the ε-balls entirely included into the standard semantics of the formula $\bigvee_{r\in\mathbb{R}}(\phi\widehat{[\![\mathbf{Y},r,\mathbf{W}]\!]})_\varepsilon$. However, a point $\langle\vec{y},\vec{w}\rangle$ belongs to $\left\{\left|\bigvee_{r\in\mathbb{R}}(\phi\widehat{[\![\mathbf{Y},r,\mathbf{W}]\!]})_\varepsilon\right|\right\}$ if and only if there exists a y such that $\langle\vec{y},\vec{w}\rangle$ belongs also to $\left\{\left|(\phi\widehat{[\![\mathbf{Y},y,\mathbf{W}]\!]})_\varepsilon\right|\right\}$. Hence, $\left\{\left|\bigvee_{r\in\mathbb{R}}(\phi\widehat{[\![\mathbf{Y},r,\mathbf{W}]\!]})_\varepsilon\right|\right\}$ is equivalent to $\left\{\left|\exists Y(\phi\widehat{[\![\mathbf{Y},Y,\mathbf{W}]\!]})_\varepsilon\right|\right\}$. Hence, we can conclude that the two sets $(\!|\exists W \phi[\mathbf{Y},W,\mathbf{W}]|\!)_\varepsilon$ and $\left\{\left|\exists \mathbf{W_0}(\forall \mathbf{W_1}(\delta(\mathbf{W_0},\mathbf{W_1})<\varepsilon \rightarrow \exists Y(\phi\widehat{[\![\mathbf{Y},Y,\mathbf{W_1}]\!]})_\varepsilon)\wedge\delta(\mathbf{W_0},\mathbf{W})<\varepsilon)\right|\right\}$ are the same.

$\varphi[\mathbf{Y},\mathbf{W}]$ has the form $\neg\phi[\mathbf{Y},\mathbf{W}]$.

By the definition, $(\!|\neg\phi[\mathbf{Y},\mathbf{W}]|\!)_\varepsilon \stackrel{\text{def}}{=} \bigcup_{B(\{p\},\varepsilon)\cap(\!|\phi[\mathbf{Y},\mathbf{W}]|\!)_\varepsilon=\emptyset} B(\{p\},\varepsilon)$. By using the same argument used in the $\wedge$-case, the sets $\left\{\left|\exists \mathbf{W_0}(\forall \mathbf{W_1}(\delta(\mathbf{W_0},\mathbf{W_1})<\varepsilon \rightarrow \neg(\phi\widehat{[\![\mathbf{Y},\mathbf{W_1}]\!]})_\varepsilon)\wedge\delta(\mathbf{W_0},\mathbf{W})<\varepsilon)\right|\right\}$ and $(\!|\neg\phi[\mathbf{Y},\mathbf{W}]|\!)_\varepsilon$ are the same.

□

Corollary 1. *Let $\varphi[\mathbf{Y},\mathbf{W}]$ be any first order formula definable in Tarski algebra, and $\varepsilon\in\mathbb{R}_{>0}$. Satisfiability of $\widehat{(\varphi)}_\varepsilon[\mathbf{Y},\mathbf{W}]$ is decidable.*

Proof. Since $\widehat{(\varphi)}_\varepsilon[\mathbf{Y},\mathbf{W}]$ is a formula definable in Tarski algebra, and since it is well known that Tarski algebra is decidable, it immediately follows that satisfiability of $\widehat{(\varphi)}_\varepsilon[\mathbf{Y},\mathbf{W}]$ is decidable. □

Example 4. *Let us consider the formula $\varphi[X] \stackrel{def}{=} X>0\wedge X<2$. We have that $\widehat{(X>0)}_\varepsilon \equiv \exists X_0(X_0>0\wedge\delta(X_0,X)<\varepsilon)\equiv X_0+\varepsilon>0$. By applying the same rule, $\widehat{(X<2)}_\varepsilon\equiv\exists X_0(X_0<2\wedge\delta(X_0,X)<\varepsilon)\equiv X-2-\varepsilon<0$. Finally, since ε is a positive real, $\widehat{(X>0\wedge X<2)}_\varepsilon\equiv\exists X_0(\forall X_1(\delta(X_0,X_1)<\varepsilon\rightarrow X_1+\varepsilon>0\wedge X_1-2-\varepsilon<0)\wedge\delta(X_0,X)<\varepsilon)\equiv X>-\varepsilon\wedge X\leq 2+\varepsilon$.*

Let us notice that the application of the translation in Def. 7 to a formula, increases the evaluation complexity of such formula with respect to its untranslated version. This is mainly due to the possible introduction of quantifier operator alternations.

5.2 Experimental Results on the Neural Oscillator

Let us consider the hybrid automaton $H_{\tilde{f}}$ described in Section 2 for modeling a neural oscillator. We intend to study its behaviour through sphere semantics, exploiting cylindrical algebraic decomposition tools to automatically compute it.

First, we have to replace the differential equations representing the dynamics with semi-algebraic functions. We do this by exploiting their first-degree Taylor polynomials and obtaining the automaton $H'_{\tilde{f}}$ depicted in Figure 6. In order to keep the presentation simple, in this section we fix the parameters as follows $\tau=3$, $\lambda=1$, $\alpha=2$. Hence, the activations correspond to the axis $X_i=\pm 2$ and $X_e=\pm 2$.

A simulation of $H'_{\tilde{f}}$ is presented in Figure 5, where we can notice that a limit cycle is still present, but it has a diamond-like shape. We are interested in studying this limit cycle. In particular, we are interested in proving, exploiting tools for symbolic computation, that if we apply sphere semantics, each point in the space reaches a bounded region which includes the limit cycle. Notice that in this example our

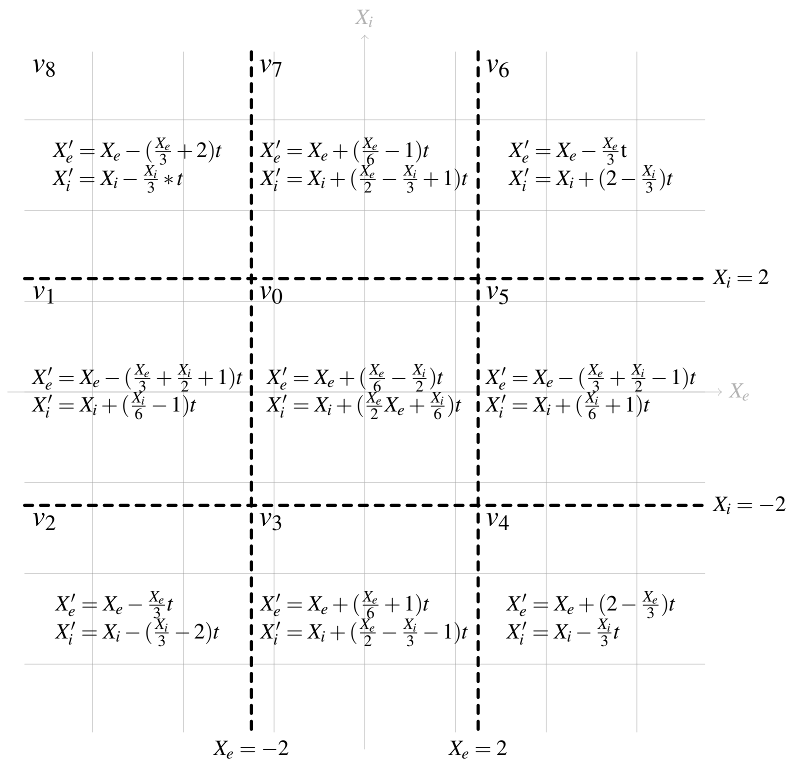


Figure 6: A graphical representation of the piecewise hybrid automaton associated to the function $f_2(3,1)$

automata have unbounded invariants, hence the termination of sphere semantics reachability algorithm is not guaranteed.

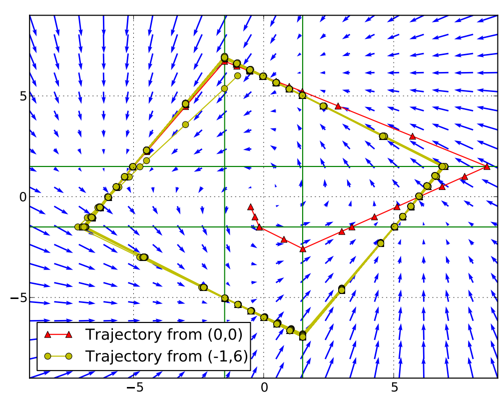


Figure 7: Two evolutions of the first degree approximation of the model proposed in Section 2.

We start computing the intersections of the limit cycle with the activation regions. Consider for instance the intersection $Q_0 = \langle x_{Q_0}, 2\rangle$ of the limit cycle with $X_i = 2$ and $X_e > 0$. We have that x_{Q_0} is the unique solution of the equation which describes the intersection of the diamond-like limit cycle with $X_i = 2$. Similarly, consider point $Q_1 = \langle 2, y_{Q_1} \rangle$ that in turn corresponds to the intersection of the limit cycle with $X_e = 2$ and $X_i > 0$. We effectively calculated all these intersections by using Maxima [15]. So, for instance, we get $x_{Q_0} = \frac{3526\sqrt{17}+14538}{495\sqrt{17}+2041}$ and $y_{Q_1} = \frac{190\sqrt{17}+786}{39\sqrt{17}+161}$. Notice that Q_0 and Q_1 are points which satisfy the activation formulæ wich regulate the discrete transitions between locations v_6 and v_5, and locations v_6 and v_7, respectively. Let us now consider a point P_0 located on $X_i = 2$, but which is such that its distance d_0 from Q_0 is at least 2ε, i.e., $P_0 = \langle x_{P_0}, 2 \rangle$ and $\delta(Q_0, P_0) = d_0 > 2\varepsilon$. Consider now any point P_1 on $X_e = 2$ resulting from the sphere semantics evaluation of the continuous evolution which starts in P_0 inside location v_6. Thus, let denote with d_1 the distance between such P_1 and Q_1, i.e., $\delta(Q_1, P_1) = d_1$.

If we could prove that d_1 is always smaller than d_0, then we would be able to conclude that all the points which start from a distance of at least 2ε from the limit cycle converge to a flow tube having diameter 2ε that includes the limit cycle. Of course, to obtain such conclusion, we need to prove this property on all locations.

We can formalize this concept through a first-order formula. We denote with r and s the straight lines $X_i = 2$ and $X_e = 2$, respectively, and with the notation $Q_0 \in r \cap C \cap X_e > 0$ the membership of Q_0 to the intersection of straight line r with limit cycle C and positive X_e semi-plane. Moreover, with the notation $(\!| P_0 \rightarrow_C P_1 |\!)_\varepsilon$ we denote the continuous transition from point P_0 to point P_1 performed exploiting sphere semantics. Thus, our desired property can be expressed as:

$$\begin{aligned}\forall Q_0 Q_1 \forall P_0 P_1 \big((Q_0 \in r \cap C \cap X_e > 0 \wedge Q_1 \in s \cap C \cap X_i > 0 \wedge P_0 \in r \cap X_e > 0 \wedge \\ P_1 \in s \cap X_e > 0 \wedge \delta(Q_0, P_0) > 2\varepsilon \wedge (\!| P_0 \rightarrow_C P_1 |\!)_\varepsilon) \twoheadrightarrow \delta(Q_1, P_1) < \delta(Q_0, P_0) \big)\end{aligned} \tag{5}$$

stating the convergence to the limit flow tube in location v_6. Such property can be easily rewritten for each location of the hybrid automaton, changing the roles of activation border lines r and s.

We automatically expanded such formula by using a Perl script that implements the Definition 7 to translate sphere semantics into the standard one. In particular, $(\!| P_0 \rightarrow_C P_1 |\!)_\varepsilon$, in the case of location v_6 becomes

$$\psi[\mathbf{X}] \stackrel{\text{def}}{=} \exists T (\exists \mathbf{X_0} (\forall \mathbf{X_1} (\delta(\mathbf{X_0}, \mathbf{X_1}) < \varepsilon \twoheadrightarrow (T > 0 \wedge \phi[\![\mathbf{X_1}, T]\!]))) \wedge \delta(\mathbf{X_0}, \mathbf{X}) < \varepsilon), \tag{6}$$

where

$$\begin{aligned}\phi[\mathbf{X}, T] \stackrel{\text{def}}{=} \exists \mathbf{X_0} (\forall \mathbf{X_1} (\delta(\mathbf{X_0}, \mathbf{X_1}) < \varepsilon \twoheadrightarrow \\ (\exists \mathbf{X_2} (\Pi_1 [\![\mathbf{X_2}, T]\!] \wedge \delta(\mathbf{X_2}, \mathbf{X_1}) < \varepsilon) \wedge \\ \exists \mathbf{X_2} (\Pi_2 [\![\mathbf{X_2}, T]\!] \wedge \delta(\mathbf{X_2}, \mathbf{X_1}) < \varepsilon))) \wedge \delta(\mathbf{X_0}, \mathbf{X}) < \varepsilon)\end{aligned} \tag{7}$$

and

$$\Pi_1[X_0, Y_0, X_1, Y_1, T] \stackrel{\text{def}}{=} 6 * X_1 = 6 * X_0 + (X_0 - 3 * Y_0) * T, \tag{8}$$

$$\Pi_2[X_0, Y_0, X_1, Y_1, T] \stackrel{\text{def}}{=} 6 * Y_1 = 6 * Y_0 + (Y_0 + 3 * X_0) * T. \tag{9}$$

However, we notice that, since Π_1 and Π_2 are closed and convex, ϕ can be simplified as:

$$\phi[\mathbf{X}, T] \equiv \exists \mathbf{X_0} (\Pi_1 [\![\mathbf{X_0}, T]\!] \wedge \Pi_2 [\![\mathbf{X_0}, T]\!] \wedge \delta(\mathbf{X_0}, \mathbf{X}) < \varepsilon). \tag{10}$$

Similarly, ψ becomes:

$$\psi[\mathbf{X}] \equiv \exists T (T > 0 \wedge \exists \mathbf{X_0} (\Pi_1 [\![\mathbf{X_0}, T]\!] \wedge \Pi_2 [\![\mathbf{X_0}, T]\!] \wedge \delta(\mathbf{X_0}, \mathbf{X}) < \varepsilon)). \tag{11}$$

So we plugged this last formula in Formula 5 and used REDLOG [6] to test it. The formula turns out to be true (the result is computed within few seconds), proving our conjectures.

Notice that we used Maxima to compute the exact coordinates of the points on the limit cycles since that computation does not require quantifier elimination. However, we could have used REDLOG.

As far as $\langle 0,0 \rangle$ is concerned it is immediate to prove through a first-order formula that it reaches points different from itself and, hence, it reaches the limit flow tube.

Other interesting properties that automatically verified express, for instance, the fact that applying the sphere semantics there are points that cross the limit cycle (in both directions). This is quite natural since points closer than ε to the limit cycle get expanded and cross it.

6 Conclusions

In this paper we have modeled a neural oscillator constructing a hybrid automaton whose components derive from the approximation of the continuous model presented in [22]. We have analyzed its behaviours considering the application of some approximation techniques for the introduction of noise, as already advocated in [22]. In particular, we focused on the approach based on the ε-semantics.

The simulation based on the application of the ε-semantics has revealed the any point which begins its evolution from a distance of at least 2ε from the limit cycle, converges to a flow tube which possesses a diameter equal to 2ε and that includes the limit cycle. Due to size of the formulæ which compose the hybrid automaton and the growth of such formalæ introduced by the translation of the ε-semantics evaluations, a direct computation of the reachable set would have high complexity and eventually returns results of difficult interpretation. For this reason, we have reformulated the problem in form of a closed property which guarantees the convergence of any point towards the limit cycle of the modeled system.

During the construction of the formula that describes the convergence to the limit cycle, some steps of simplification of the formulæ have been applied. In particular, we have reduced the complexities of translated formulæ, relying on the convexity of the sets characterized by some of their subformulæ. An interesting aspect to investigate is to determine whether these simplification steps can be automatically performed.

As future work, in order to analyze the behaviour of a group of neural oscillators, we plan to combine several hybrid automata and to study their evolutions always adopting the approximation approach based on the ε-semantics.

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