

Chapter M

MRI

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Introduction to nuclear magnetic resonance imaging (NMR / MRI)

NMR principle discovered independently by Bloch and Purcell in 1946, led to Nobel Prize in physics [1–4].
First MR images in early 70's by Lauterbur [5].

Advantages of MRI

- Nonionizing radiation
- Good soft-tissue contrast
- High spatial resolution
- Flexible user control - many acquisition parameters allow optimization for specific imaging situations
- Negligible attenuation (although RF coil sensitivity is now a useful mechanism for fast scans)
- Image in arbitrary plane (or volume)
- Potential for chemically specified imaging
- Flow imaging

Disadvantages of MRI

- High cost
- Complicated “siting” due to large magnetic fields
- Low sensitivity
- Little signal from bone
- Fairly long scan duration

Naturally there is ongoing work to reduce or minimize the disadvantages of MRI (especially scan duration). For example, more recently “real-time” NMR images have been produced [6].

Physics overview

As usual, we begin by describing the general physical phenomena that underly the MR imaging modality [7].

Spins

Nuclei with an odd number of protons or neutrons possess an **angular momentum** vector \vec{J} , often simply called **spin**.
From a classical physics perspective, we can think of such nuclei as spinning charged spheres, called **spins**, that act as tiny dipoles.

In medical imaging, the most abundant spin is ^1H , a single proton, in water molecules.

There are over 100 stable nuclei that have an odd number of protons or neutrons.

Other particularly important nuclei in biology and medicine: ^2H , ^7Li , ^{13}C , ^{19}F , ^{23}Na , ^{31}P , ^{127}I

Associated with each spin is a **nuclear magnetic dipole moment** vector, or simply **magnetic moment** vector, $\vec{\mu} = (\mu_x, \mu_y, \mu_z) = \mu_x \vec{i} + \mu_y \vec{j} + \mu_z \vec{k}$ that is co-linear with direction of the spin's angular momentum vector.

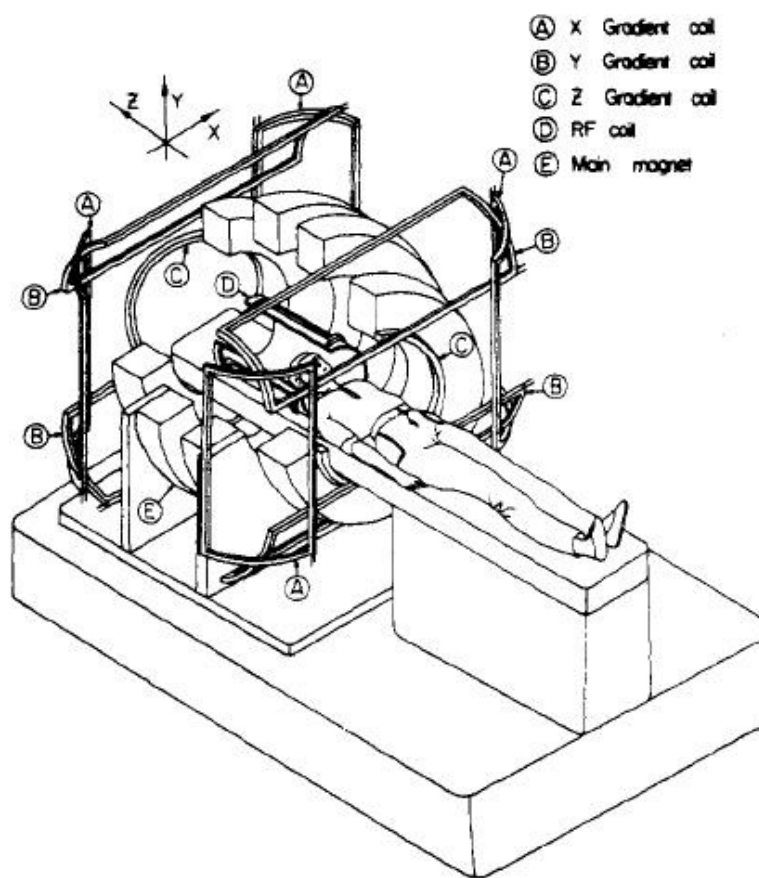
MRI involves the interaction of these spins with several types of magnetic fields. The following three of these fields are designed and controlled by us.

- Main field: \vec{B}_0 .
This field is (usually) aligned with the scanner axis (the “z” axis by convention) and is generated by superconducting coils or large permanent magnets. We select this (static) field strength when we design the system, e.g., “1.5T MR scanner”
- RF field: $\vec{B}_1(t)$, an amplitude-modulated pulse transmitted through a coil.
This field is usually (approximately) perpendicular to \vec{B}_0 .
There are many different ways to design this “excitation” field.
- Field gradients: $\vec{G}(t)$. These cause spatial variations in the relative strength of the z-oriented magnetic field.
Field gradients are also user-controlled, and are what separate MRI from NMR.
Paul Lauterbur and Peter Mansfield earned the 2003 Nobel Prize in Physiology or Medicine by developing, analyzing, and refining the use of gradient fields $\vec{G}(t)$.
- In addition to the above human-controlled fields, the presence of orbiting electrons near a nucleus also change the local apparent magnetic field. This effect is called **chemical shift** and will be described in more detail later. It can be both a source of image

artifacts and a source of information for distinguishing different types of tissue.

Fields

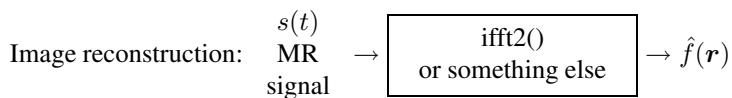
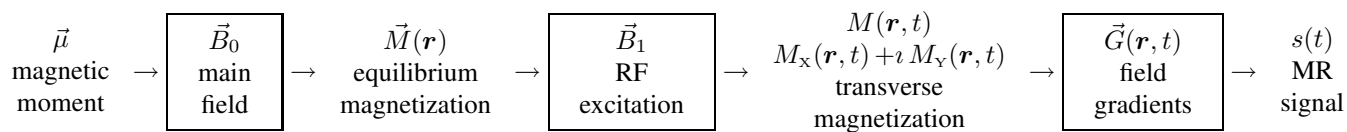
The three user-controlled fields described above each correspond to specific instrumentation components, as illustrated in the following diagram, taken from [8].



A typical superconducting magnet is based on niobium-titanium (NbTi) alloys, requiring a liquid helium cryogenic system that cools to about 4.2° K.

Block diagram

Image acquisition:



Spatial coordinates in 3D are denoted by $\mathbf{r} = (x, y, z)$.

Magnetic moments

The **magnetic moment** vector $\vec{\mu} = (\mu_x, \mu_y, \mu_z)$ is related to the **angular momentum** vector \vec{J} by

$$\vec{\mu} = \gamma \vec{J}.$$

The nuclei-dependent constant γ is called the **gyromagnetic ratio**.

Often we will work with the related quantity $\bar{\gamma} = \frac{\gamma}{2\pi}$ which has units Hz / Tesla.

Angular momentum \vec{J} has units J sec.

The *magnitude* of the magnetic moment is a fixed value determined by quantum physics:

$$|\vec{\mu}| = \sqrt{\mu_x^2 + \mu_y^2 + \mu_z^2} = h\bar{\gamma}\sqrt{I(I+1)}. \quad (\text{M.1})$$

- $\bar{\gamma} = \frac{\gamma}{2\pi}$ is called the **gyromagnetic ratio**. For ^1H , $\bar{\gamma} = 42.48$ MHz/Tesla.
- $h = 6.6 \cdot 10^{-34}$ J sec denotes **Planck's constant**.
- I denotes the **spin quantum number**

Hereafter we restrict attention to nuclei with two energy states, such as ^1H , for which $I = 1/2$. Thus $|\vec{\mu}| = h\bar{\gamma}\sqrt{3}/2$. Note that $\sqrt{3}/2$ is the distance from the center to the corner of the unit cube.

Although the $|\vec{\mu}|$ is fixed, normally the magnetic moment of any spin has a rapidly fluctuating random *orientation*.

Main field \vec{B}_0

Now suppose we apply a static and spatially uniform magnetic field $\vec{B}_0(\mathbf{r}) = \vec{B}_0$ along some direction.

By convention, the direction is taken to be z , *i.e.*, $\vec{B}_0 = \begin{bmatrix} 0 \\ 0 \\ B_{z0} \end{bmatrix} = B_{z0} \vec{k}$, where $\vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ denotes the unit vector along z .

The **main field** strength B_{z0} , technically the **magnetic flux density**, is about 1 Tesla = 10^4 Gauss = 1 Wb/m² = 1 kg / (s² A) typically. (The earth's magnetic field is about 0.5 Gauss).

Microscopic view

The static main field along \vec{k} causes **Zeeman splitting**: the **longitudinal component** μ_z of the magnetic moment is quantized:

$$\mu_z = \pm \frac{1}{2} h\bar{\gamma}, \quad (\text{M.2})$$

for ^1H , or more generally $\mu_z = m_z h\bar{\gamma}$ where $m_z \in \{-I, \dots, I-1, I\}$.

Define the **transverse component** of the magnetic moment as that lying in the **transverse plane**, *i.e.*, the x, y plane:

$$\vec{\mu}_{xy} \triangleq \mu_x \vec{i} + \mu_y \vec{j}.$$

By comparing (M.1) and (M.2), we see that (for ^1H):

$$|\vec{\mu}_{xy}| = \sqrt{|\vec{\mu}|^2 - |\mu_z|^2} = h\bar{\gamma}/\sqrt{2} = \sqrt{2} |\mu_z|.$$

More generally, $|\vec{\mu}_{xy}| = h\bar{\gamma}\sqrt{I(I+1) - m_z^2}$.

The angle θ between the magnetic moment $\vec{\mu}$ and the applied field direction \vec{k} is

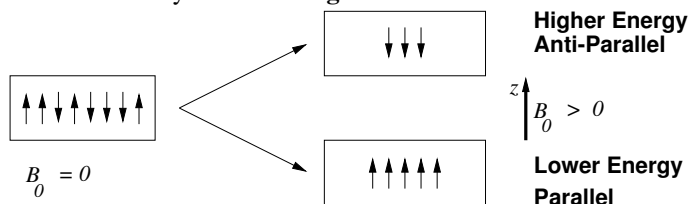
$$\arccos(\mu_z/|\vec{\mu}|) = \arccos(\pm 1/\sqrt{3}) \approx \{54.7^\circ, 180 - 54.7^\circ\}.$$

However, the *orientation* of the transverse component $\vec{\mu}_{xy}$ of the magnetic moment is random, *i.e.*, $\arctan(\mu_y/\mu_x)$ is uniformly distributed over $[0, 2\pi)$.

We can think of the two possibilities for the z -component μ_z as being **parallel** or **anti-parallel** to \vec{k} . The parallel state has slightly lower energy than the anti-parallel state. According to quantum theory, $E = -\vec{\mu} \cdot \vec{B}_0 = -\mu_z B_{z0} = -m_z h \gamma B_{z0}$. The energy difference between the two states is

$$\Delta\mathcal{E} = \mathcal{E}_- - \mathcal{E}_+ = h\gamma B_{z0}.$$

The energy difference is sometimes illustrated by **Zeeman diagram**:



At absolute zero, all nuclei would occupy the lower energy (parallel) state.

However, thermal agitation greatly exceeds $\Delta\mathcal{E}$.

The (equilibrium) ratio of anti-parallel (N_-) to parallel (N_+) nuclei is governed by **Boltzmann distribution**, for which:

$$\frac{E[N_+]}{N} = \frac{e^{-\mathcal{E}_+/(k_b T)}}{e^{-\mathcal{E}_+/(k_b T)} + e^{-\mathcal{E}_-/(k_b T)}}, \quad \frac{E[N_+]}{E[N_-]} = e^{\Delta\mathcal{E}/(k_b T)}.$$

- $N = N_+ + N_-$ is the total number of spins
- $k_b = 1.38 \cdot 10^{-23}$ J/°K (Boltzmann constant)
- T temperature in degrees Kelvin.

We will see that the signal recorded in MRI is proportional to the (expected) occupancy difference

$$E[N_+ - N_-] = N \frac{e^{-\mathcal{E}_+/(k_b T)}}{e^{-\mathcal{E}_+/(k_b T)} + e^{-\mathcal{E}_-/(k_b T)}} - N \frac{e^{-\mathcal{E}_-/(k_b T)}}{e^{-\mathcal{E}_+/(k_b T)} + e^{-\mathcal{E}_-/(k_b T)}} = N \frac{e^{\Delta\mathcal{E}/(k_b T)} - 1}{e^{\Delta\mathcal{E}/(k_b T)} + 1} \approx N \frac{\Delta\mathcal{E}}{2k_b T} = N \frac{h\gamma B_{z0}}{2k_b T}. \quad (\text{M.3})$$

Example. Consider a typical $B_{z0} = 1.5$ T scanner. At room temperature $T \approx 300^\circ$, so for ^1H :

$$\frac{h\gamma B_{z0}}{2k_b T} = \frac{(6.6 \cdot 10^{-34})(4.48 \cdot 10^6)(1.5)}{2(1.38 \cdot 10^{-23})300} \approx 0.5 \cdot 10^{-6}$$

Unfortunately, at room temperature, the difference in occupancy is *only about a part per million!*

Thus NMR is a fairly *insensitive* modality. (Fortunately there is a lot of ^1H , but imaging Na or P is much harder.)

How can one increase sensitivity (signal strength)?

- Decrease temperature (not clinically feasible, but fine for traditional spectroscopy).
- Increase field strength B_{z0} , hence today's trend towards 3T systems (and beyond).
However, there are imaging tradeoffs such as increased T_1 and increased field inhomogeneity, and practical tradeoffs such as more complicated siting.
- Do other things that affect the proportionality factor!
This is where much MRI research occurs, *e.g.*, using multiple receive coils.

Magnetization: Macroscopic view

The net effect of the magnetic moments of numerous spins is local **magnetization** of the sample. In MRI we are interested in manipulating and imaging this **magnetization**. We will denote the magnetization by $\vec{M}(\mathbf{r}, t)$ if it is varying with time, or $\vec{M}_0(\mathbf{r})$ if it is in an equilibrium,

The units of magnetization are Amperes per meter (A/m).

The equilibrium spatial distribution of magnetization is essentially proportional to the local density $\rho(\mathbf{r})$ of ^1H , *i.e.*:

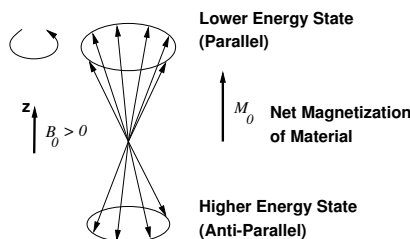
$$\vec{M}_0(\mathbf{r}) = \begin{bmatrix} 0 \\ 0 \\ M_{z0}(\mathbf{r}) \end{bmatrix} = M_{z0}(\mathbf{r}) \vec{k}, \quad M_{z0}(\mathbf{r}) = \rho(\mathbf{r}) (h\gamma)^2 \frac{1}{4k_b T} B_{z0},$$

where $\rho(\mathbf{r})$ denotes the **spin density**: the number of spins per unit volume.

Using (M.3), we can also write $M_{z0}(\mathbf{r}) = \rho(\mathbf{r}) h\gamma \frac{1}{2} E [N_+ - N_-] / N$.

Images proportional to $\rho(\mathbf{r})$ are of some interest, but barely scratch the surface of the potential of NMR imaging.

Why is the magnetization along \vec{k} ? Because the transverse component $\vec{\mu}_{xy}$ of each spin has a random orientation.



At $37^\circ \text{C} = 310^\circ \text{K}$, for a single proton in H_2O , $M_{z0} = 3.25 \cdot 10^{-3} B_{z0} \text{ A/m}$.

Microscopic view:

If $\vec{B} = \mathbf{0}$, the spins are equally likely to occupy either energy state, and the dipole orientations are random.

Macroscopic view:

If $\vec{B} = \mathbf{0}$ then the net **magnetization** is zero: $\vec{M}_0(\mathbf{r}) = \mathbf{0}$.

Precession

Microscopic view:

The transverse component of $\vec{\mu}$ experiences a torque due to the applied main field \vec{B}_0 , causing it **precess** about the direction of the applied field, *i.e.*, about \vec{k} or z , called **nuclear precession**. (Analogy: spinning top, gravitational pull.) The rate of this precession is given by the **Larmor equation**:

$$\boxed{\omega_0 = \gamma B_{z0}} \quad \text{or} \quad \boxed{f_0 = \gamma B_{z0} / 2\pi}$$

For a typical 1.5T main field, $f_0 \approx 63 \text{ MHz}$. This frequency is in the RF regime (lowest point on FM dial is about 88MHz).

A convenient way to describe this precession mathematically is to use complex notation:

$$\mu_{xy}(t) \triangleq \mu_x(t) + i\mu_y(t) = \mu_{xy}(0) e^{-i\omega_0 t},$$

where $\mu_{xy}(0)$ is a random initial phase. (In addition to precession there are also random fluctuations due to thermal agitation.)

The direction of precession is *clockwise* if viewed *against* the direction of \vec{B}_0 , *i.e.*, if we view the x, y plane from above. This corresponds to a **left hand rule**: with left thumb pointing along applied field \vec{B} , precession follows the direction of the fingers.

However, this precession in the transverse plane is not observable (without perturbing the system in some way to be described shortly) because neighboring spins have different random phases so the net magnetization in the transverse plane is zero.

Macroscopic view:

The phase of the precessing spins is random, so the net **transverse magnetization** is zero.

Because there are more spins in the parallel state, the net magnetization \vec{M}_0 is oriented along z (in equilibrium):

Units

Here is a quick check on the units of **magnetization**, using [http://en.wikipedia.org/wiki/Tesla_\(unit\)](http://en.wikipedia.org/wiki/Tesla_(unit)) in part.

Newton: force to give 1 kg mass a 1 m/s² acceleration. $N = \text{kg m} / \text{s}^2$

Joule: energy required for force of 1 Newton over 1 meter. $J = N \text{ m} = \text{kg m}^2 / \text{s}^2$

Ampere: current in two infinitesimal wires 1 m apart that produces a force of $2 \cdot 10^{-2}$ N per meters of length. $N \propto A \text{ m}$

Gauss: Maxwell / cm² (CGS unit)

Tesla = Wb/m² = kg / (s² A) (SI unit).

Thus $T \text{ A} / \text{m} \equiv N / \text{m}^2 \equiv J / \text{m}^3$ and $J / T = \text{m}^2 \text{ A}$.

Now, from Boltzmann distribution it is clear that $\Delta\mathcal{E}/(k_b T)$ is unitless, where $\Delta\mathcal{E} = h\bar{\gamma} B_{z0}$. Rewriting:

$$M_{z0}(\mathbf{r}) = \underbrace{\rho(\mathbf{r})}_{\text{spins} / \text{m}^3} \frac{1}{4} \underbrace{(\bar{\gamma} h)}_{J / T} \underbrace{\left(\frac{h\bar{\gamma} B_{z0}}{k_b T} \right)}_{\text{unitless}},$$

so M_{z0} has units: $J / T / \text{m}^3 = (\text{kg m}^2 / \text{s}^2) / (\text{kg} / (\text{s}^2 \text{ A})) / \text{m}^3 = \text{A} / \text{m}$, which is the usual SI units for magnetization.

RF field

The collection of precessing spins is **resonant** in the sense that one can induce oscillations by applying energy at the appropriate wavelength/frequency/energy. Note that $\Delta\mathcal{E} = \hbar\gamma B_{z0} = \hbar f_0$, which is exactly the formula for the energy in the quanta of an EM field with frequency f_0 . So a RF field tuned to f_0 will resonate with spins.

For **excitation** we apply an **RF field** *perpendicular* to the main field \vec{B}_0 , usually as an amplitude modulated sinusoid:

$$\vec{B}_1(t) = B_1 a_1(t) \begin{bmatrix} \cos(\omega_0 t + \phi_1) \\ -\sin(\omega_0 t + \phi_1) \\ 0 \end{bmatrix}, \quad (\text{M.4})$$

where $a_1(t)$ denotes a unitless pulse envelope that we can design under computer control. We normalize such that $\max_t |a(t)| = 1$. The duration of $a_1(t)$ is usually very short (≈ 1 ms), so this is called a **RF pulse**. The field $\vec{B}_1(t)$ is **circularly polarized**, generated using **quadrature** RF coils. It will also be convenient to represent this RF field in complex notation:

$$B_1(t) \triangleq B_{1,x}(t) + i B_{1,y}(t) = B_1 a_1(t) e^{-i(\omega_0 t + \phi_1)} = b_1(t) e^{-i\omega_0 t},$$

where $b_1(t) \triangleq B_1 a_1(t) e^{-i\phi_1}$. The RF field strength is $\|\vec{B}_1(t)\| = |B_1(t)| = |B_1|$, and typically is a fraction of a Gauss.

The above model is an idealization. In practice there are several departures that need not concern us now.

- Even if the RF coil were to produce a component in the z direction, this component would be negligible relative to B_{z0} .
- Ideally $|B_1(t)|$ would be spatially homogeneous, *i.e.*, independent of \mathbf{r} . In practice any coil design has a nonuniform field pattern, and this can be a complication in some types of MR scans, particularly at higher field strengths where the wavelengths are shorter or when multiple coils are used.
- Ideally the RF frequency ω_0 exactly equals the resonant frequency of the spins. In practice due to nonuniformity of the main field strength B_{z0} , the resonant frequency varies spatially so the RF frequency never matches the Larmor frequency everywhere. The effect of such **off resonance** is somewhat complicated [7, p. 87].

Microscopic view:

- The added RF energy can cause some of the spins to occupy the higher energy state.

What might happen to the longitudinal magnetization? ??

How much can it change? ??

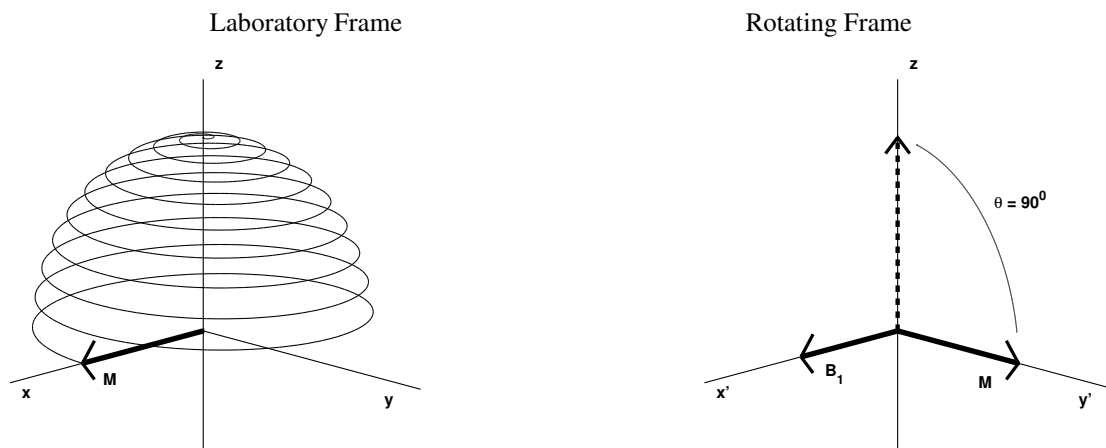
- The RF pulse causes the transverse components of the magnetic moments of some of the spins to become in phase (**coherence**). This creates a nonzero transverse magnetization component. The largest possible magnitude is $\|M_{xy}\| = M_{z0}$.
- The exact mechanisms are beyond the scope of this course. Fortunately, when $\Delta\mathcal{E} \ll k_b T$, as in NMR, classical and quantum mechanics agree, so from now on we take a classical viewpoint and only consider the net magnetization vector $\vec{M}(t)$.

Macroscopic view:

The magnetic moments and hence the bulk magnetization precess about the *overall* magnetic field $\vec{B}(t) = \vec{B}_0 + \vec{B}_1(t)$, which is time-varying.

Trying to imagine such precession directly is challenging. To simplify, we will later consider in detail a **rotating frame** that rotates at frequency ω_0 , having coordinates x', y', z . In this frame, the precessing cone due to \vec{B}_0 is “frozen.” If $\phi_1 = 0$ above, then the RF field is along the x' axis in the rotating frame, and the magnetization will precess around the x' axis (in this rotating frame) following the left hand rule. Due to Larmor equation, the frequency of this precession is $f_1 = \gamma B_1(t)$, which is much slower than f_0 because $|B_1| \ll B_{z0}$.

The following figures illustrate how the magnetization evolves during RF excitation. We will analyze this in detail later.



If the RF strength and duration is just right, then one can tip the magnetization into the transverse plane. (A 90° **flip angle**). Usually this type of excitation yields the largest signal (for a single excitation).

To describe the transverse component of the magnetization, we will sometimes use the vector notation

$$\vec{M}_{XY}(\mathbf{r}, t) \triangleq M_X(\mathbf{r}, t) \vec{i} + M_Y(\mathbf{r}, t) \vec{j}$$

but more often using the convenient complex mathematical notation:

$$M_{XY}(\mathbf{r}, t) \triangleq M_X(\mathbf{r}, t) + i M_Y(\mathbf{r}, t).$$

Often we will use $M_Z(\mathbf{r}, 0^-)$ and $M_Z(\mathbf{r}, 0^+)$ to denote the longitudinal magnetization immediately before and immediately after an RF excitation pulse at time $t = 0$ respectively. This notation reflects the fact that RF excitation pulses are very short.

We will similarly define $M_{XY}(\mathbf{r}, 0^-)$ and $M_{XY}(\mathbf{r}, 0^+)$.

For an on-resonance RF excitation pulse with flip angle α about the x' axis, the effect of the RF pulse when applied to an object at equilibrium is

$$\begin{aligned} M_Z(\mathbf{r}, 0^+) &= \cos(\alpha) M_Z(\mathbf{r}, 0^-) = \cos(\alpha) M_{Z0} \\ M_{XY}(\mathbf{r}, 0^+) &= i \sin(\alpha) M_Z(\mathbf{r}, 0^-) = i \sin(\alpha) M_{Z0}. \end{aligned}$$

In particular, if $\alpha = \pi/2$, then as illustrated in the figure above

$$\begin{aligned} M_Z(\mathbf{r}, 0^+) &= 0 \\ M_{XY}(\mathbf{r}, 0^+) &= i M_{Z0}. \end{aligned}$$

When we apply an RF pulse, it is called **forced precession** because \vec{M} is influenced by \vec{B}_1 .

After the RF pulse ends, the evolving magnetization is called **free precession**, because \vec{M}_{XY} precesses around \vec{B}_0 .

Relaxation

After we turn off the RF signal, the magnetization will continue to precess in the $x-y$ plane, but the system will begin to return to the equilibrium state.

After an RF pulse (at $t = 0$) with a 90° **tip angle**, the following occur.

The **longitudinal** component, along the z -axis, recovers to its equilibrium value as follows (for $t > 0$):

$$M_z(\mathbf{r}, t) = M_{z0}(\mathbf{r}) \left(1 - e^{-t/T_1(\mathbf{r})} \right) + M_z(\mathbf{r}, 0^+) e^{-t/T_1(\mathbf{r})}.$$

T_1 is the **spin-lattice** time constant; exchange of energy between spins and surrounding lattice of **electrons** etc.

T_1 typically from 100-1000 msec.

T_1 values lengthen with increasing B_{z0}

The magnitude of the **transverse** component decays as follows (for $t > 0$):

$$|M_{xy}(\mathbf{r}, t)| = |M_{xy}(\mathbf{r}, 0^+)| e^{-t/T_2(\mathbf{r})}.$$

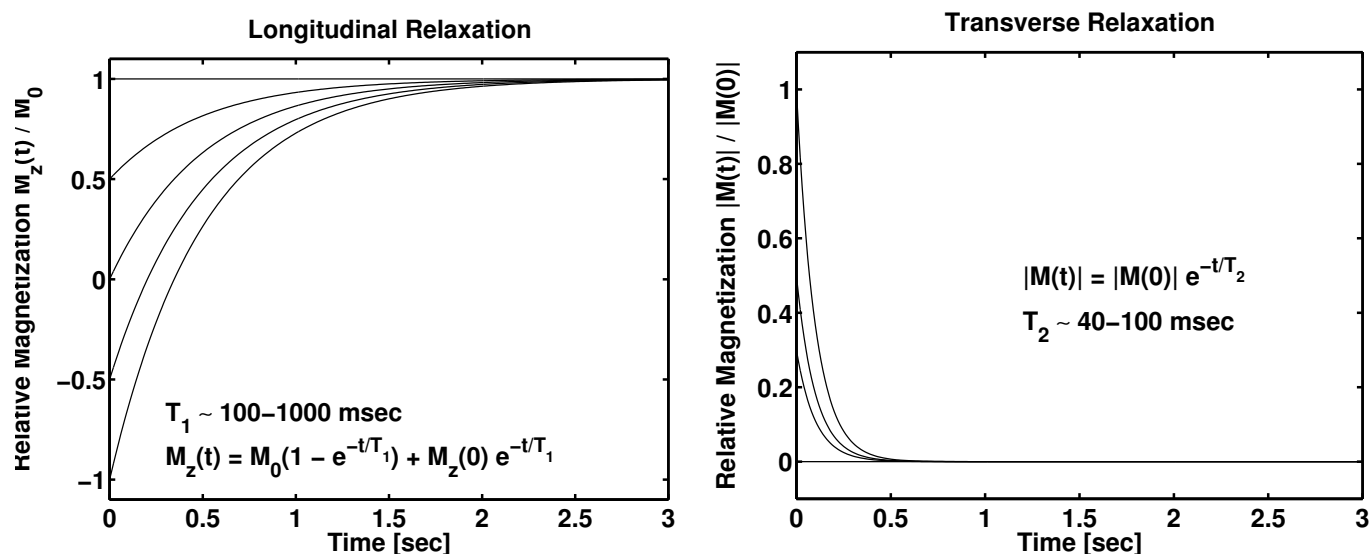
T_2 is the **spin-spin** time constant; loss of phase coherence due to interactions between **spins**.

(In practice the signal decays with a faster time constant T_2^* because B_{z0} is nonuniform.)

T_2 typically from 10-100 msec

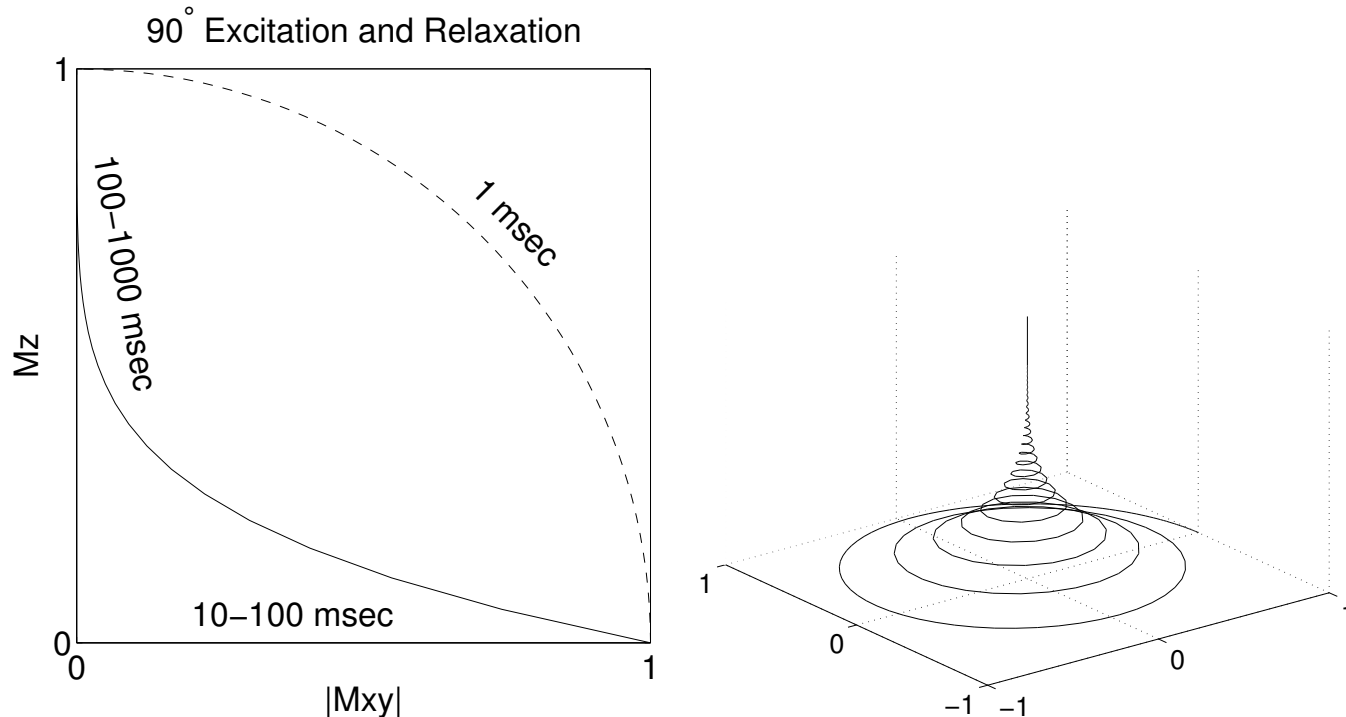
Transverse relaxation is caused by loss of phase coherence of the spins due to interactions between nearby microscopic dipoles causing a broadening of the spins' resonant frequencies. The details require quantum mechanics [9].

T_2 values are largely independent of B_{z0}



The values of T_1 and T_2 can vary considerably between different materials, providing very valuable **contrast** information.

Interestingly, because of the differences between T_1 and T_2 relaxation times, the length of the magnetization vector does not stay the same during relaxation.



Signal preview

As the system returns to equilibrium state, it emits energy that can be measured in the form of RF oscillations.

We can think of the body as a collection of tiny precessing magnetic dipoles. By Faraday's law, moving magnetic dipoles induce a voltage across an RF coil that is oriented to detect $x-y$ magnetization changes. The RF signal we measure is the superposition of the individual signals from all those oscillators.

The RF signal is a function of the key parameters: T_1 , T_2 , and ρ (the spin density), all of which are functions of spatial location \mathbf{r} . We would like to image the spatial distribution of one or more of those parameters from the RF signal.

The signal described thus far is called **free induction decay** (FID) ("free" because RF is off; we can also have steady-state systems).

Localization preview

The wavelength of RF is in meters, so we cannot easily selectively excite or view a desired slice or voxel using RF and B_{z0} alone.

As in ultrasound, MRI uses a sequence of excitation/reception steps. For each excitation/reception, we record a 1D RF signal. Because we want to form a 2D (or 3D) image, we need a sequence of excitation/reception steps.

Unlike ultrasound, in MRI one cannot form a useful image until after all data have been collected.

- In ultrasound, location (range) is related to time.
- In X-ray imaging, localization is defined via collimation and small source size (so detected photon position corresponds to a particular ray through the object).
- In MRI, location is related to temporal frequency.

Note that at typical RF wavelengths in MRI, there is relatively little attenuation in passing through the body (or through many organic materials, which is why radios work inside houses). See [10, Fig. 1.1].

Field gradients

If we applied only a perfectly uniform field \vec{B}_0 , then all spins would oscillate at the same frequency, so there would be no spatial localization ability. (The RF receiver responds to the whole volume (almost) equally.) The clever solution for localization is to use special coils to induce **field gradients**.

Example. By using suitable coils, we can induce an x gradient:

$$\vec{B}(\mathbf{r}) = \vec{B}(x, y, z) = \begin{bmatrix} 0 \\ 0 \\ B_{z0} + x G_x \end{bmatrix} = (B_{z0} + x G_x) \vec{k}.$$

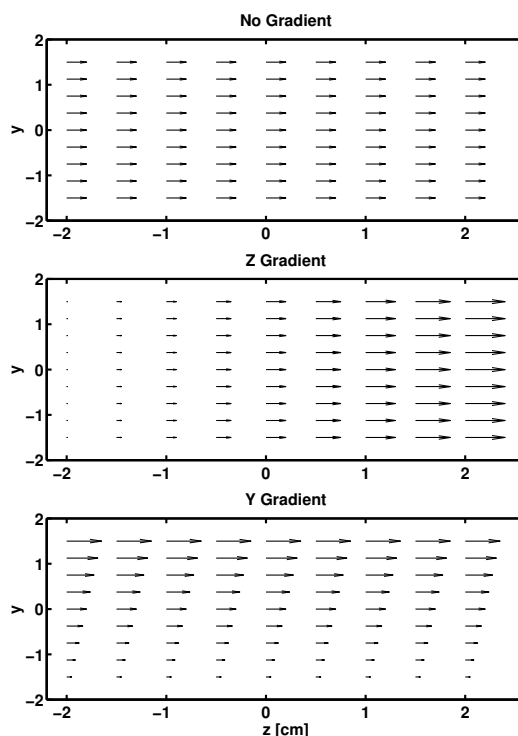
Note that the field is still oriented along the z direction! *But now the field strength is space-variant.*

G_x usually less than 1G/cm.

Because field-strength now depends on location, so does the resonant frequency, via the Larmor equation. So spatial location has been encoded in temporal frequency!

This was the key innovation needed to move from NMR to MRI.

Illustration of field gradients



Chemical shift

There is one more field that influences the spins. Orbital motion of electrons surrounding a nuclei perturbs the local magnetic field:

$$B_{\text{eff}} = B_{z0} (1 - \sigma) ,$$

where σ depends on local environment.

Because of the Larmor relationship, this field shift causes a shift in the resonant frequency:

$$\omega_{\text{eff}} = \omega_0 (1 - \sigma) .$$

All molecules may have this property, so to normalize things we define water to be a reference, and relate other chemical species to it in **parts per million (ppm)**:

$$\text{chemical shift} = \delta = \frac{\omega_S - \omega_R}{\omega_R} \cdot 10^6 = \frac{\omega_0(1 - \sigma_S) - \omega_0(1 - \sigma_R)}{\omega_0(1 - \sigma_R)} \cdot 10^6 = \frac{\sigma_R - \sigma_S}{1 - \sigma_R} \cdot 10^6 \approx (\sigma_R - \sigma_S) \cdot 10^6$$

Fat (lipids with many CH₂ groups) has resonant frequency about $\delta = 3.5$ ppm lower than water.

Example. Change in z position of fat due to chemical shift:

$$\Delta z = \frac{\Delta\omega}{\gamma G_z} = \frac{\gamma\omega_0}{\gamma G_z} = \frac{\delta\gamma B_{z0}}{\gamma G_z} = \delta \frac{B_{z0}}{G_z} = 3.5 \cdot 10^{-6} \frac{1.5 \cdot 10^4 \text{ G}}{1 \text{ G/cm}} = 0.525 \text{ mm}.$$

Misc

Other topics that could be mentioned: nonlinear field gradients, magnetization transfer.

Magnetic susceptibility

- diamagnetic materials decrease the field strength,
- paramagnetic materials slightly increase the field strength,
- ferromagnetic materials strongly increase the field strength.

Because carbon and molecular oxygen (O₂) are diamagnetic, the field in a body is slightly lower than in air. The transitions between air and soft tissue cause magnetic **field inhomogeneities** that can degrade image quality if left uncorrected [11].

Imaging overview

The object magnetization $\vec{M}(\mathbf{r}, t)$ is the quantity we wish to image; we do so by manipulating the applied magnetic field $\vec{B}(\mathbf{r}, t)$. An MR imaging **pulse sequence** typically alternates between two phases of manipulation.

- **Excitation:** apply RF to establish a pattern of transverse magnetization
- **Readout:** turn off RF transmitter and collect an RF signal that is related to that pattern, usually varying the field gradients \vec{G} .

Unfortunately, a single 1D RF signal in general will not contain enough information to describe a 2D or 3D magnetization pattern. Therefore, the sequence is repeated multiple times. In the simplest case to consider, the excitation part is identical for each repetition, and between excitations one allows the magnetization to return to steady state. Thus, following each excitation there is some pattern $\vec{M}(\mathbf{r}, 0)$, where we let $t = 0$ denote the time following excitation here. We would like to make an image of the transverse component of this pattern, *i.e.*, a picture of

$$m(x, y) \triangleq \int_{z_0 - \Delta_z/2}^{z_0 + \Delta_z/2} M_x(x, y, z, 0) + i M_y(x, y, z, 0) dz.$$

Typically we simply display the magnitude of this complex quantity, *i.e.*,

$$|m(x, y)| = \sqrt{\left(\int_{z_0 - \Delta_z/2}^{z_0 + \Delta_z/2} M_x(x, y, z, 0) dz \right)^2 + \left(\int_{z_0 - \Delta_z/2}^{z_0 + \Delta_z/2} M_y(x, y, z, 0) dz \right)^2}.$$

For each readout portion we vary $\vec{G}(t)$ so that we collect a signal related to different aspects of the magnetization so that eventually we collect enough information to make a picture of $|m(x, y)|$.

In some MR applications, important information is encoded in the phase of $m(x, y)$.

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Selective excitation preview

If only the main field B_{z_0} is present, then the whole volume is “resonant” at f_0 , so all spins are tipped by an applied RF field that oscillates at that frequency. This is called **non-selective or “hard” excitation**. Exciting the whole volume is useful for 3D imaging, but 3D imaging can be time consuming, as we will see later from our k -space analysis.

Therefore, usually we would like to excite just a single plane (or thin slab) to make a manageable 2D problem. This is called **selective or “soft” excitation** [12].

Solution: apply a linear field gradient, *e.g.*, along z direction (called **slice selection gradient**) to make traditional transaxial slices:

$$\vec{B}(\mathbf{r}) = \vec{B}(x, y, z) = \begin{bmatrix} 0 \\ 0 \\ B_{z_0} + z G_z \end{bmatrix} = (B_{z_0} + z G_z) \vec{k}.$$

Because MRI scanners have coils that produce field gradients in all three directions, the slice orientation is arbitrarily selectable.

With a slice selection gradient enabled, the Larmor frequency is space variant: $f_0(\mathbf{r}) = f_0(x, y, z) = \bar{\gamma}(B_{z_0} + z G_z)$ (*i.e.*, varies with slice position z). Thus to excite just plane z_1 , in principle we would apply an RF signal whose spectrum is concentrated at $f_1 = \bar{\gamma}(B_{z_0} + z_1 G_z)$. Such an RF signal would excite only those spins in slice z_1 .

What would be the corresponding RF signal? An infinite duration sinusoid, which is not practical.

Sinc pulse

Usually we content ourselves with exciting and imaging one or more thin slabs of thickness Δz . (A slab of finite thickness is necessary to have enough spins excited to get a measurable RF signal.) Then for a z -gradient G_z , the desired RF spectrum is $\text{rect}\left(\frac{f - f_1}{\Delta f}\right)$, where $\Delta f = \bar{\gamma} G_z \Delta z$. Thus the corresponding RF signal in the time domain would be $e^{i\omega_0 t} \text{sinc}(t \Delta f)$. In practice, time-limited approximations are used, so the “slice profile” is not perfectly rectangular. (Essentially a filter design problem.)

Note the relationship between spatial location (in this case z) and temporal frequency due to Larmor equation. This is key to localization in MRI.

Physics details

Now we delve deeper into the physics and the corresponding mathematics.

5.1

Bloch equation

In a heterogeneous object, all M 's and B 's depend on time and spatial location, *i.e.*, $\vec{M}(\mathbf{r}, t) = \vec{M}(x, y, z, t)$ where $\mathbf{r} = (x, y, z)$. We need a mathematical model how the input \vec{B} affects the object's magnetization \vec{M} .

The **Bloch equation** provides a phenomenological description of time evolution of local magnetization:

$$\vec{M}(\mathbf{r}, t) = \begin{bmatrix} M_x(\mathbf{r}, t) \\ M_y(\mathbf{r}, t) \\ M_z(\mathbf{r}, t) \end{bmatrix}, \quad \frac{d}{dt} \vec{M} = \underbrace{\vec{M} \times \gamma \vec{B}}_{\text{precession}} - \underbrace{\frac{M_x \vec{i} + M_y \vec{j}}{T_2}}_{\text{relaxation}} - \underbrace{\frac{(M_z - M_{z0}) \vec{k}}{T_1}}_{\text{to equilibrium}},$$

where \vec{i} and \vec{j} are unit vectors in x and y directions respectively. Note that every quantity above varies with position \mathbf{r} .

This equation describes how the magnetization evolves over time in response to the (mostly) external input field \vec{B} as well as due to the internal relaxation processes.

The equation captures the three key phenomena:

- precession,
- transverse and longitudinal relaxation,
- equilibrium.

Ignoring chemical shift, the external input field \vec{B} is composed of

- main field $\vec{B}_0 = B_{z0} \vec{k}$,
- RF field $\vec{B}_1(t)$,
- longitudinal field strength gradients $(\mathbf{r} \cdot \vec{G}(t)) \vec{k} = (x G_x(t) + y G_y(t) + z G_z(t)) \vec{k}$.

The RF pulse $B_1(t)$ and gradient waveforms $\vec{G}(t)$ are user-controlled.

Recall that the cross product of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ is defined:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = \begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix} \vec{i} - \begin{vmatrix} u_x & u_z \\ v_x & v_z \end{vmatrix} \vec{j} + \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \vec{k} = -(\mathbf{v} \times \mathbf{u}) = \vec{e}_\perp \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta_{uv}$$

where \vec{e}_\perp is the unit vector perpendicular to \mathbf{u} and \mathbf{v} , and θ_{uv} is the smaller angle between \mathbf{u} and \mathbf{v} .

- Solutions to the Bloch equation are **time-shift invariant**, *i.e.*, $t = 0$ is arbitrary.
- Each point in space evolves independently. (This version ignores diffusion, flow, other motion [13].)

Why must we influence the magnetization on readout?

The RF receiver responds (essentially) to the entire volume due the long RF wavelengths, so there is very little spatial localization on transmit or receive. It would be nice conceptually if we could somehow excite each voxel sequentially and then “listen” for the RF signal returning from that voxel, the amplitude of which would be related to the spin density ρ for that voxel, and the exponential decay rate would be related to the T_2 for that voxel. We could thereby build a picture of $\rho(x, y, z_0)$ or of $T_2(x, y, z_0)$ for some slice z_0 one voxel at a time.

Because this is impractical, our principal goal now, as we focus on the readout part, is to understand how the user-controlled field gradients $\vec{G}(t)$ affect the time-evolution of the magnetization. This requires that we understand the relationship between spatial location and frequency components of the received signal. (The amplitude of each frequency component is proportional to the magnetization at some location(s).)

For now we focus on the *readout* phase of MR pulse sequence, and return to excitation later.

Solutions to the Bloch equation when applied RF=0

Unfortunately, there no *general* closed-form solution to the Bloch equation. However, when there is no applied RF, there are explicit solutions that are easily interpreted. When the applied magnetic field \vec{B} is oriented in the z direction, *i.e.*, $\vec{B}(\mathbf{r}, t) = B_z(\mathbf{r}, t) \vec{k}$, then by expanding the cross product in the Bloch equation we can write:

$$\frac{d}{dt} \vec{M}(\mathbf{r}, t) = \begin{bmatrix} -\frac{1}{T_2(\mathbf{r})} & \omega(\mathbf{r}, t) & 0 \\ -\omega(\mathbf{r}, t) & -\frac{1}{T_2(\mathbf{r})} & 0 \\ 0 & 0 & -\frac{1}{T_1(\mathbf{r})} \end{bmatrix} \vec{M}(\mathbf{r}, t) + \begin{bmatrix} 0 \\ 0 \\ \frac{M_{z0}(\mathbf{r})}{T_1(\mathbf{r})} \end{bmatrix},$$

where $\omega(\mathbf{r}, t) = \gamma B_z(\mathbf{r}, t)$ is the *local* Larmor frequency. Typically $B_z(\mathbf{r}, t) = B_{z0} + \mathbf{r} \cdot \vec{G}(t)$, so

$$\omega(\mathbf{r}, t) = \gamma B_z(\mathbf{r}, t) = \gamma (B_{z0} + \mathbf{r} \cdot \vec{G}(t)) = \omega_0 + \gamma \mathbf{r} \cdot \vec{G}(t).$$

Remarkably, the transverse and longitudinal components *separate* in this case, greatly simplifying the solution to the PDE.

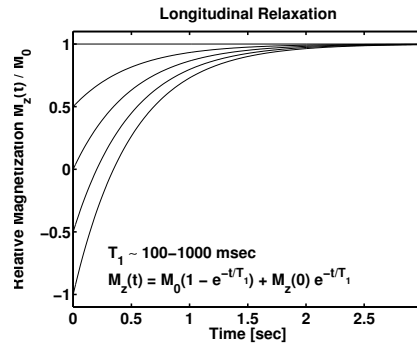
The differential equation for the **longitudinal component** of the magnetization is

$$\frac{d}{dt} M_z(\mathbf{r}, t) = -\frac{1}{T_1(\mathbf{r})} M_z(\mathbf{r}, t) + \frac{M_{z0}(\mathbf{r})}{T_1(\mathbf{r})} \quad \text{or:} \quad \frac{d}{dt} [M_z(\mathbf{r}, t) - M_{z0}(\mathbf{r})] = -\frac{1}{T_1(\mathbf{r})} [M_z(\mathbf{r}, t) - M_{z0}(\mathbf{r})].$$

A simple differential equation of the form $\frac{d}{dt} f(t) = a f(t)$ has solution $f(t) = f(0) e^{at}$ for $t \geq 0$. Thus, the solution to the differential equation for the longitudinal component is: $M_z(\mathbf{r}, t) - M_{z0}(\mathbf{r}) = (M_z(\mathbf{r}, 0) - M_{z0}(\mathbf{r})) e^{-t/T_1(\mathbf{r})}$, or equivalently:

$$M_z(\mathbf{r}, t) = M_{z0}(\mathbf{r}) (1 - e^{-t/T_1(\mathbf{r})}) + e^{-t/T_1(\mathbf{r})} M_z(\mathbf{r}, 0), \quad t \geq 0.$$

Note that (when RF=0), \vec{B} has no effect on the longitudinal component; that component simply relaxes back to its equilibrium value, as illustrated below.



We are particularly interested in the transverse components of the magnetization, because these induce signals in RF receive coils. The differential equation for the **transverse component** of the magnetization is

$$\frac{d}{dt} \begin{bmatrix} M_x(\mathbf{r}, t) \\ M_y(\mathbf{r}, t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{T_2(\mathbf{r})} & \omega(\mathbf{r}, t) \\ -\omega(\mathbf{r}, t) & -\frac{1}{T_2(\mathbf{r})} \end{bmatrix} \begin{bmatrix} M_x(\mathbf{r}, t) \\ M_y(\mathbf{r}, t) \end{bmatrix}.$$

This differential equation is easy to solve by combining the two transverse components using complex notation:

$$M(\mathbf{r}, t) \triangleq M_{xy}(\mathbf{r}, t) = M_x(\mathbf{r}, t) + \imath M_y(\mathbf{r}, t)$$

where $\imath = \sqrt{-1}$. (Note that this M is subscript free. It is the same as $M_{xy}(\mathbf{r}, t)$ defined earlier.) Note that the physical quantities involved, *i.e.*, M_x and M_y , are both real. We make the choice to define a complex quantity $M(\mathbf{r}, t)$ by combining these two real quantities because that choice simplifies the analysis. Using this representation, we can write the differential equation as simply:

$$\frac{d}{dt} M(\mathbf{r}, t) = \left(-\frac{1}{T_2(\mathbf{r})} - \imath \omega(\mathbf{r}, t) \right) M(\mathbf{r}, t).$$

This differential equation is solved very easily in the case where the applied field \vec{B} is **static**, because then $\omega(\mathbf{r})$ is independent of time t . In this case one can see directly that the solution is

$$M(\mathbf{r}, t) = M(\mathbf{r}, 0) e^{-i\omega(\mathbf{r})t} e^{-t/T_2(\mathbf{r})}, \quad t \geq 0.$$

More generally, note that a differential equation of the form $\frac{d}{dt}f(t) = a(t)f(t)$ has the solution for $t \geq 0$: $f(t) = f(0) e^{\int_0^t a(\tau) d\tau}$.

Here the “coefficient” of the differential equation is $a(t) = -\frac{1}{T_2(\mathbf{r})} - i\omega(\mathbf{r}, t)$.

Thus the general solution is

$$M(\mathbf{r}, t) = M(\mathbf{r}, 0) \exp\left(\int_0^t -\frac{1}{T_2(\mathbf{r})} - i\omega(\mathbf{r}, \tau) d\tau\right) = M(\mathbf{r}, 0) e^{-t/T_2(\mathbf{r})} e^{-i \int_0^t \gamma B_z(\mathbf{r}, \tau) d\tau}.$$

Case 1: Homogeneous object, static and uniform field

- $\vec{M}(\mathbf{r}, t) = \vec{M}(t)$ (independent of spatial position \mathbf{r})
- static: $\vec{B}(\mathbf{r}, t) = \vec{B}(\mathbf{r})$ independent of time
- uniform: $\vec{B}(\mathbf{r}) = \vec{B}_0$ has uniform field strength over volume, $\vec{B}(\mathbf{r}, t) = \vec{B}_0 = B_{z0} \vec{k}$

Solution:

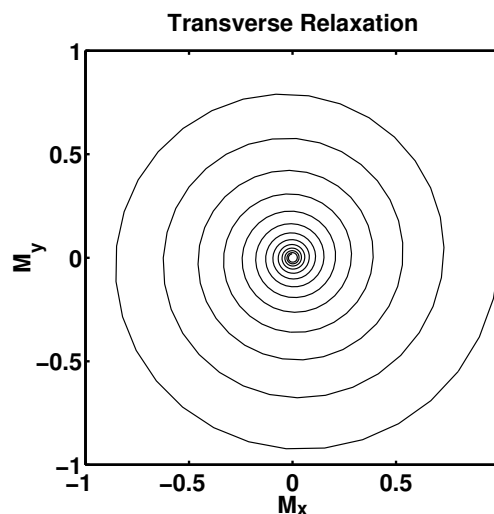
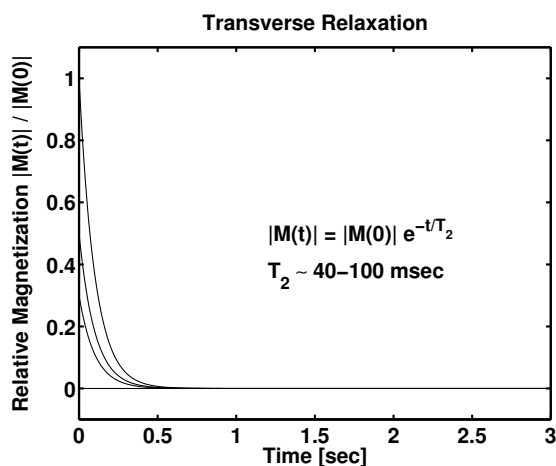
$$\vec{M}(t) = \begin{bmatrix} e^{-t/T_2} & & \\ & e^{-t/T_2} & \\ & & e^{-t/T_1} \end{bmatrix} \begin{bmatrix} \cos \omega_0 t & \sin \omega_0 t & 0 \\ -\sin \omega_0 t & \cos \omega_0 t & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{M}(0) + \begin{bmatrix} 0 \\ 0 \\ M_{z0}(1 - e^{-t/T_1}) \end{bmatrix},$$

where $\omega_0 = \gamma B_{z0}$ is the resonant frequency corresponding to the main field.

In the complex representation, for this case the magnetization is a decaying complex exponential:

$$M(t) = M(0) e^{-i\omega_0 t} e^{-t/T_2}, \quad t \geq 0.$$

- Initial condition (determined by RF excitation and local spin density)
- Precession at Larmor frequency (all spins at same rate because uniform field)
- Relaxation (decay of transverse component)



You might think of the relaxation functions described previously as “consequences” of this differential equation, but really it is the other way around: the differential equation was developed to match the required physical properties.

Case 2: Inhomogeneous object, static but nonuniform field

- Inhomogeneous object: $\vec{M}(\mathbf{r}, t)$
- Static field: $\vec{B}(\mathbf{r}, t) = \vec{B}(\mathbf{r})$
- Nonuniform field: $\vec{B}(\mathbf{r}) = \vec{B}_0 + (\mathbf{r} \cdot \vec{G})\vec{k}$.

Note that $\vec{B}(0) = \vec{B}_0$, i.e., a spin at the center of the scanner sees just the main field strength.

In complex notation, the transverse component of the solution to the Bloch equation is:

$$M(\mathbf{r}, t) = M(\mathbf{r}, 0) e^{-i\omega(\mathbf{r})t} e^{-t/T_2(\mathbf{r})}, \quad t \geq 0.$$

Because the field is nonuniform, each spin precesses at a rate corresponding to the Larmor frequency:

$$\omega(\mathbf{r}) = \gamma B_z(\mathbf{r}) = \gamma(B_{z0} + \mathbf{r} \cdot \vec{G}) = \omega_0 + \Delta\omega(\mathbf{r}), \quad \text{where } \Delta\omega(\mathbf{r}) \triangleq \gamma \mathbf{r} \cdot \vec{G}.$$

Thus it is also useful to write:

$$M(\mathbf{r}, t) = M(\mathbf{r}, 0) e^{-i\omega_0 t} e^{-i\Delta\omega(\mathbf{r})t} e^{-t/T_2(\mathbf{r})}, \quad t \geq 0,$$

where the term $e^{-i\Delta\omega(\mathbf{r})t}$ reflects the extra phase accrued by a spin at position \mathbf{r} relative to that of a spin at $(0, 0, 0)$.

Case 3: General solution for RF=0

- Inhomogeneous object: $\vec{M}(\mathbf{r}, t)$, $T_1(\mathbf{r})$, $T_2(\mathbf{r})$, ...
- Time-varying, space-variant (due to gradients) magnetic field.

The magnetic field still oriented in the z direction (by convention), but its strength can depend (linearly) on location:

$$\vec{B}(\mathbf{r}, t) = (B_{z0} + \mathbf{r} \cdot \vec{G}(t))\vec{k} = B_z(\mathbf{r}, t)\vec{k}.$$

Fortunately, the differential equation is solvable, because the transverse and longitudinal components separate.

The solution of the Bloch equation for the longitudinal component is identical to previous case.

The solution for the transverse component is called the **fundamental equation of NMR imaging**:

$$M(\mathbf{r}, t) = \underbrace{M(\mathbf{r}, 0)}_{\text{initial}} \underbrace{e^{-t/T_2(\mathbf{r})}}_{\text{relax}} \underbrace{e^{-i\omega_0 t}}_{\text{precess}} \underbrace{e^{-i\phi(\mathbf{r}, t)}}_{\text{encode}}$$

where the time-varying phase caused deliberately by field gradients is

$$\phi(\mathbf{r}, t) = \gamma \int_0^t \mathbf{r} \cdot \vec{G}(\tau) d\tau = \gamma \int_0^t (B_z(\mathbf{r}, \tau) - B_{z0}) d\tau.$$

Note that now the precession rate varies both with space and with time, and the phase term $\phi(\mathbf{r}, t)$ captures the local phase difference relative to an on-resonance spin at the scanner center.

To review again:

- $M(\mathbf{r}, t)$: local magnetization evolving over time (in complex notation to combine x and y components)
- $M(\mathbf{r}, 0)$: initial local magnetization (following excitation)
- Precession: $e^{-i\omega_0 t}$
- Transverse relaxation $M(\mathbf{r}, t) \rightarrow 0$ as $t \rightarrow \infty$
- Phase, due to the nonuniform field strength, means that spins precess at different rates (due to gradients).

We use the resonant frequency as a reference (due to baseband operation) and express other frequencies as a time-varying phase.

Summary

The above equation describes how the magnetization (ideally) evolves over time after RF turned off (which usually defines $t = 0$).

Key design parameters are $\vec{G}(t)$.

How can we manipulate $\vec{G}(t)$ to control signal to make image?

To answer that, we must first answer the question: What is the observed signal?

5.2

Signal equation

The time-varying magnetization $M(\mathbf{r}, t)$, that is evolving according to the fundamental equation of NMR, induces an **electromotive force (EMF)** in one or more neighboring receiver coils by Faraday's law of induction.

For simplicity, here we assume an ideal RF receiver coil, *i.e.*:

- uniformly sensitive over volume of interest, with sensitivity B_{1xy} ,
- detects flux changes in transverse direction.
- noiseless.

Then the received signal is

$$\begin{aligned} s_r(t) &\triangleq \iiint B_{1xy} \frac{\partial}{\partial t} M(\mathbf{r}, t) d\mathbf{r} = \iiint B_{1xy} \frac{\partial}{\partial t} M(x, y, z, t) dx dy dz \\ &= \iiint B_{1xy} \frac{\partial}{\partial t} \left[M(\mathbf{r}, 0) e^{-i\omega_0 t} e^{-i\phi(\mathbf{r}, t)} e^{-t/T_2(\mathbf{r})} \right] d\mathbf{r} \approx -i\omega_0 B_{1xy} e^{-i\omega_0 t} \iiint M(\mathbf{r}, 0) e^{-t/T_2(\mathbf{r})} e^{-i\phi(\mathbf{r}, t)} d\mathbf{r}, \end{aligned}$$

where the last approximation is essentially a **narrowband approximation**, which is reasonable because the time scales of relaxation and phase changes are *much* slower than the RF carrier frequency.

Next we **demodulate** the received signal to form a **baseband signal**. In practice this is done using I/Q channels just like in ultrasound. In the complex representation, demodulation is equivalent mathematically to multiplying by $e^{i\omega_0 t}$:

$$s(t) \triangleq \frac{e^{i\omega_0 t}}{-i\omega_0 B_{1xy}} s_r(t) = \iiint M(\mathbf{r}, 0) e^{-t/T_2(\mathbf{r})} e^{-i\phi(\mathbf{r}, t)} d\mathbf{r}.$$

The original received signal oscillates in the 10-100 MHz range whereas the baseband signal bandwidth is typically 1-10 kHz.

Now we make some further simplifications to this **signal equation** for analysis.

- Assume the **readout interval** where we record $s(t)$, say $[t_1, t_2]$, is small compared to T_2 . Typically somewhere in that interval will be a point in time where the signal is particularly large, called the **echo time**, denoted T_E .

Approximate the relaxation term by its value at the echo time: $e^{-t/T_2(\mathbf{r})} \approx e^{-T_E/T_2(\mathbf{r})}$ for $t_1 \leq t \leq t_2$. Thus

$$\begin{aligned} s(t) &\approx \iiint M(\mathbf{r}, 0) e^{-T_E/T_2(\mathbf{r})} e^{-i\phi(\mathbf{r}, t)} d\mathbf{r} \\ &= \iiint M(x, y, z, 0) e^{-t/T_2(x, y, z)} e^{-i2\pi[x k_X(t) + y k_Y(t) + z k_Z(t)]} dx dy dz, \end{aligned}$$

because we rewrite the gradient-induced phase as follows:

$$\begin{aligned} \phi(\mathbf{r}, t) &= \gamma \int_0^t \mathbf{r} \cdot \vec{G}(\tau) d\tau = \gamma \int_0^t x G_X(\tau) d\tau + \gamma \int_0^t y G_Y(\tau) d\tau + \gamma \int_0^t z G_Z(\tau) d\tau \\ &= 2\pi [x k_X(t) + y k_Y(t) + z k_Z(t)], \end{aligned}$$

where we define the **k-space trajectory** in terms of the gradient waveforms by:

$$\boxed{k_X(t) = \bar{\gamma} \int_0^t G_X(\tau) d\tau, \quad k_Y(t) = \bar{\gamma} \int_0^t G_Y(\tau) d\tau, \quad k_Z(t) = \bar{\gamma} \int_0^t G_Z(\tau) d\tau.}$$

- Now we focus on **2D imaging**, for which $G_Z = 0$ during the readout and hence $k_Z(t) = 0$. (See refocusing lobe later; that is part of excitation, not readout.) In other words, $\phi(\mathbf{r}, t)$ is independent of z for $t_1 \leq t \leq t_2$. This assumption is reasonable provided that we have used selective excitation to excite just a slice or then slab of the object. When $G_Z = 0$ we can rewrite the signal model above as follows:

$$s(t) = \iint \left[\int M(x, y, z, 0) e^{-t/T_2(x, y, z)} dz \right] e^{-i2\pi[x k_X(t) + y k_Y(t)]} dx dy.$$

To further simplify we define the 2D image $m(x, y)$ to be the bracketed term:

$$m(x, y) \triangleq \int M(x, y, z, 0) e^{-t/T_2(x, y, z)} dz \approx \int_{z_0 - \delta_z/2}^{z_0 + \delta_z/2} M(x, y, z, 0) e^{-T_E/T_2(x, y, z)} dz \approx \delta_z M(x, y, z_0, 0) e^{-T_E/T_2(x, y, z_0)}.$$

The exponential factor is called T_2 **weighting**.

Note that $m(x, y)$ is some function of $T_1(\mathbf{r})$, $T_2(\mathbf{r})$, and $\rho(\mathbf{r})$. It also depends on excitation parameters, especially timing T_R , T_E , and the slice selection process.

With this definition, the baseband signal model (for the 2D case where $G_z = 0$), and under all of the above approximations and simplifications, simplifies to the **signal equation**:

$$s(t) = \iint m(x, y) e^{-i2\pi[x k_x(t) + y k_y(t)]} dx dy = \mathcal{M}(k_x(t), k_y(t)),$$

where $\mathcal{M}(u, v)$ or equivalently $\mathcal{M}(k_x, k_y)$ denotes the 2D FT of $m(x, y)$ and we define the 2D **k-space trajectory** by:

$$k_x(t) = \bar{\gamma} \int_0^t G_x(\tau) d\tau, \quad k_y(t) = \bar{\gamma} \int_0^t G_y(\tau) d\tau.$$

In words: MRI directly measures information about the FT of the object's transverse magnetization!

MR signal at time t has amplitude (proportional to) the value of a spatial frequency component of the magnetization. Which spatial frequency component? $s(t) = \mathcal{M}(k_x(t), k_y(t))$, so which component is $(k_x(t), k_y(t))$, which depends on the user-selected gradient signals $G_x(t)$ and $G_y(t)$.

(A more general derivation including coil sensitivity effects and relaxation effects is given later in the notes.)

The following figure shows how the relative phases of the spins evolve as time progresses, when a y field gradient is used.

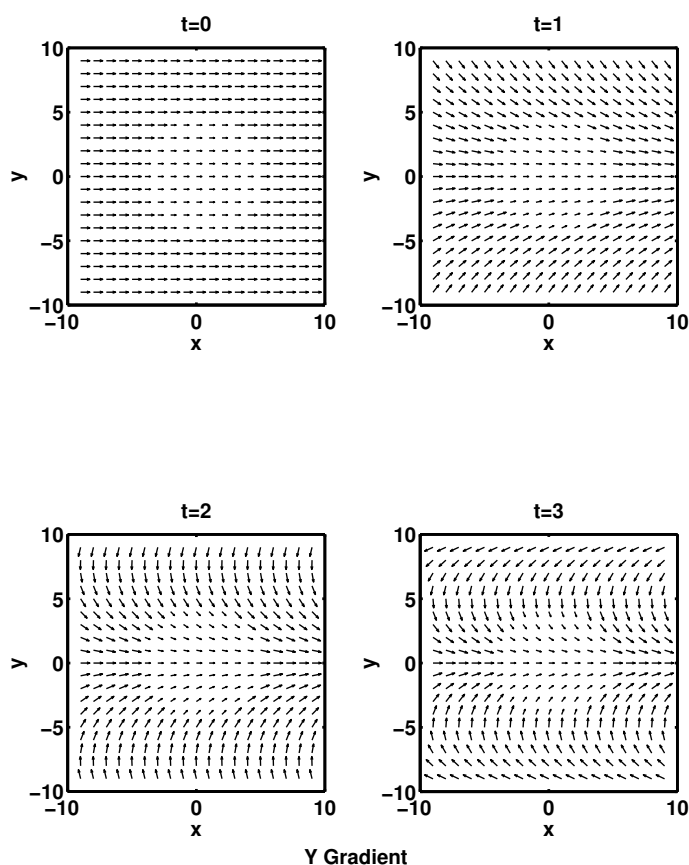


Image formation

In light of the above signal equation, we see that to form an image of $m(x, y)$ we must design the field gradient signals G_x and G_y so that we “scan through k -space.” Of course, in a finite amount of time we cannot cover *all* of k -space, but we must at least cover enough of it to collect sufficient information to make an image of the transverse magnetization $m(x, y)$ by some type of inverse 2D FT.

k -space trajectory

Note that $k_x(0) = k_y(0) = 0$. Thus the signal at time $t = 0$ is the “DC component” of the magnetization $m(x, y)$.

As time progresses, the signal value represents the values of the FT of the object along some trajectory in k space. Note that

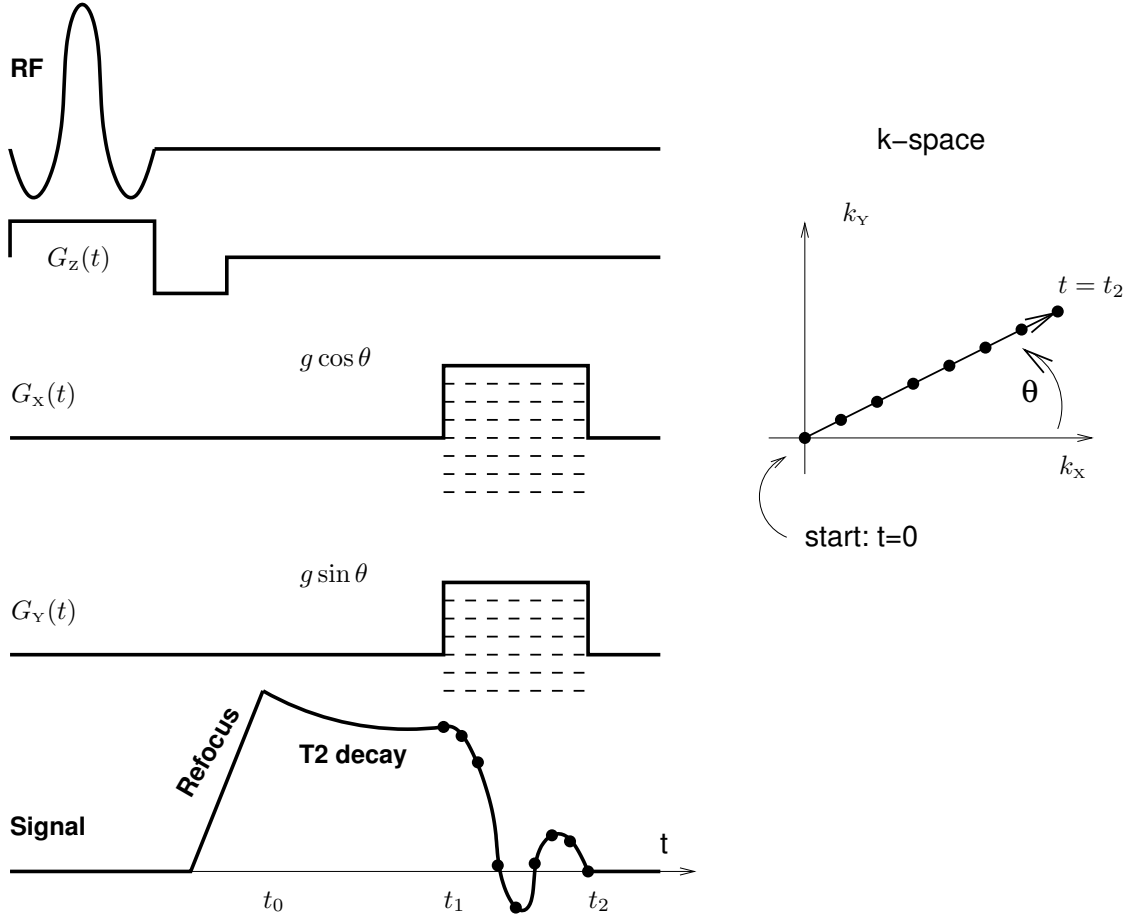
$$\frac{d}{dt} k_x(t) = \gamma G_x(t), \quad \frac{d}{dt} k_y(t) = \gamma G_y(t),$$

so the “velocity” at time t along the k -space trajectory is proportional to the gradient strength. So stronger gradients can mean faster scans. Modern high-end scanners have more powerful gradient amplifiers to help accelerate scanning.

Fourier transform interpretation: a FT of an object is just a “weighted” integral of that object where the weights are phase terms varying linearly with spatial position. This is exactly what happens physically to the magnetization in an MR scanner with linear gradients.

Next we give some concrete examples of MR imaging pulse sequences to illustrate k -space trajectories.

5.6.1

Example 1: 2D Projection-reconstruction MR sequence

The above **pulse sequence** illustrates the sequence of excitation/readout phases. The sequence is repeated for many θ values. For $t \in [t_0, t_1]$, there is (ideally) no change in the transverse magnetization (for $z = 0$) except for T_2 decay, which we have ignored in deriving the signal equation. So the baseband signal (which is not sampled during this time but could be) is just the DC component of $m(x, y)$, the transverse magnetization pattern established by the RF excitation.

At time $t = t_1$, we apply the field gradients, and for $t \in [t_1, t_2]$:

$$k_x(t) = \bar{\gamma} \int_0^t G_x(\tau) d\tau = (t - t_1) \bar{\gamma} g \cos \theta$$

$$k_y(t) = \bar{\gamma} \int_0^t G_y(\tau) d\tau = (t - t_1) \bar{\gamma} g \sin \theta$$

$$s(t) = \mathcal{M}(k_x(t), k_y(t)) = \mathcal{M}((t - t_1) \bar{\gamma} g \cos \theta, (t - t_1) \bar{\gamma} g \sin \theta).$$

In other words, samples of the signal $s(t)$ correspond to radial samples of the spectrum of $m(x, y)$.

Using the Fourier-slice theorem, which is discussed when we cover tomography, we can also write

$$s(t) = \mathcal{M}_{\text{polar}}((t - t_1) \bar{\gamma} g, \theta) = G_\theta((t - t_1) \bar{\gamma} g),$$

where $G_\theta = \mathcal{F}_1\{g_\theta\}$ and g_θ is the projection of $m(x, y)$ at angle θ .

If we make $|t_1|$ large, then there will be T_2 decay between the RF excitation and the readout, and we will get a T_2 **weighted** image, i.e., $m(x, y) e^{-t_1/T_2(x, y)}$. So to be more precise, we are ignoring T_2 decay *during the readout* (and, as we will see later, during the excitation too), but we can still account for T_2 decay between the RF excitation and the readout. In fact, good contrast in MR images often *requires* such decay.

Design issues

What are the parameters we can control?

- RF excitation (later)
- Delay between RF excitation and readout (T_2 contrast)
- Gradient strength g (limited to g_{\max} by hardware)
- Readout duration $t_{\text{daq}} = t_2 - t_1$
- Number of angles n_θ

Suppose the object magnetization pattern is (approximately) bandlimited to ρ_{\max} (cycles/cm).

What should the readout time t_{daq} and gradient amplitude g be? We want $\sqrt{k_x^2(t_2) + k_y^2(t_2)} = t_{\text{daq}} \bar{\gamma} g = \rho_{\max}$.

Two design parameters, one constraint; how do we choose? Minimize t_{daq} ! Let g go to g_{\max} .

What is a reasonable value for ρ_{\max} ? If we want $\Delta_x = 1.0$ mm spatial resolution, then by Nyquist sampling theory we need

$$\rho_{\max} = 1/(2\Delta_x) = 0.5 \text{ cycle/mm} = 5 \text{ cycles / cm.}$$

Thus the readout duration would be

$$t_{\text{daq}} = \frac{\rho_{\max}}{\bar{\gamma} g_{\max}} \approx \frac{5 \text{ cycles / cm}}{(42 \cdot 10^6 \text{ Hz} / 10^4 \text{ G}) 0.5 \text{ G/cm}} \approx 2.4 \text{ msec}$$

Was it reasonable to ignore T_2 relaxation during the readout when we derived the signal equation?

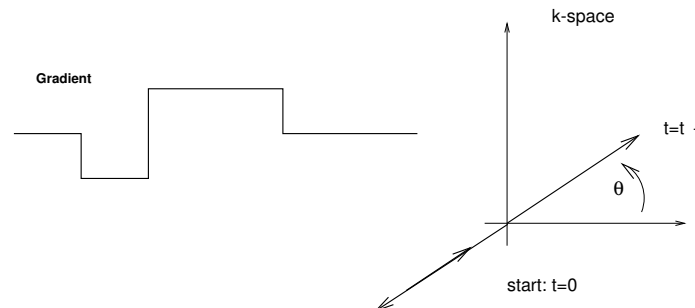
Yes, because T_2 is tens of msec, whereas the readout interval is only about 2 msec long. (But T_2^* ...)

The above pulse sequence must be repeated n_θ times to collect several radial lines through k -space.

What should n_θ be? Answer in tomography part.

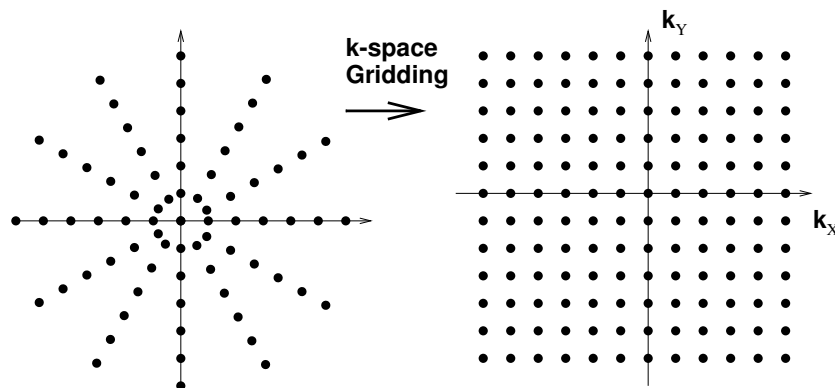
Can we complete a scan in just $n_\theta \cdot t_{\text{daq}}$ msec? Not quite, because we need time for T_1 recovery between excitations.

How can we reduce scan time? We can reduce scan time by about a factor of 2 by using a readout gradient that first goes negative and then positive, as illustrated below. (This is a nice example of systems engineering.)



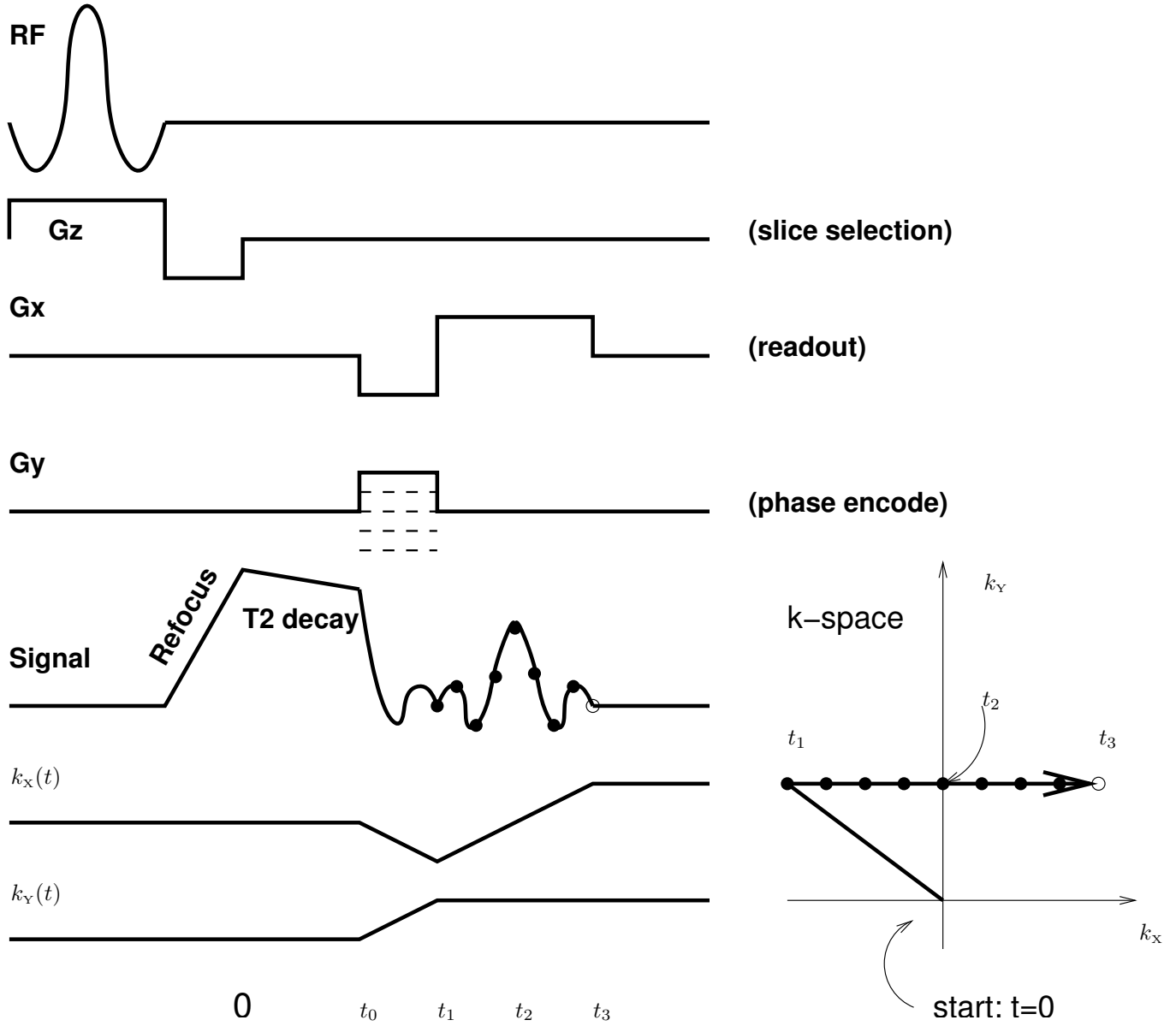
- This type of **radial sampling** is relatively robust to artifacts from angular under-sampling, so it is popular for dynamic studies.
- One can use a very short T_E with such a pulse sequence so it is useful for imaging materials with small T_2 values such as bone.

A practical problem with projection-reconstruction scanning is the need for **gridding**.



To avoid this complication, we can use the **spin-warp** or **2D FT** sequence described next.

5.6.2

Example 2: Spin-warp or 2DFT imaging (no analogy with CT)

For $t \in [t_1, t_3]$ (the readout interval) we have:

$$\begin{aligned}
 k_x(t) &= \bar{\gamma} \int_0^t G_x(\tau) d\tau = \bar{\gamma} G_x(t_2) (t - t_2) \\
 k_y(t) &= \bar{\gamma} \int_0^t G_y(\tau) d\tau = \bar{\gamma} G_y(t_0) (t_1 - t_0) \quad (\text{a constant for each readout}) \\
 s(t) &= \mathcal{M}(k_x(t), k_y(t)) = \mathcal{M}(\bar{\gamma} G_x(t_2)(t - t_2), \bar{\gamma} G_y(t_0)(t_1 - t_0)).
 \end{aligned}$$

Example. If $m(x, y) = \text{rect}(x/w_x) \text{rect}(y/w_y)$, then for $t \in [t_1, t_3]$:

$$s(t) = w_x w_y \text{sinc}(\bar{\gamma} G_x(t_2)(t - t_2)w_x) \text{sinc}(\bar{\gamma} G_y(t_0)(t_1 - t_0)w_y).$$

The actual raw data is samples of the signal, i.e., $s(t_1 + n\Delta_T)$, $n = 0, 1, \dots, N - 1$, where here $\Delta_T = (t_3 - t_1)/N$. Note that usually we use an even number of samples for an inverse FFT, so usually there is no sample at $t = t_3$.

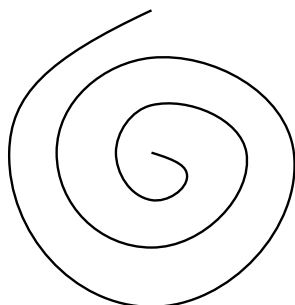
Gradient echoes

When the trajectory passes through the origin of k -space (DC), it is called a **echo**, because at that point ideally all of the spins are in phase, so their signal contributions add coherently and we get a very strong signal. (In fact, the dynamic range of the DC component compared to the high spatial frequency components is a challenge for A/D conversion.) Here it is called a **gradient-recalled** or **gradient-reversed** or **gradient-refocused** echo, or simply a **gradient echo**. Another name is a **field echo**.

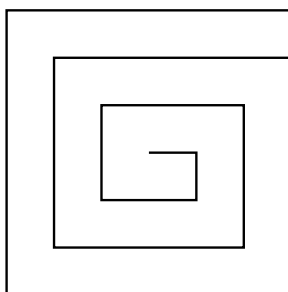
Although the “pulse / echo” terminology is similar to ultrasound, the physical mechanisms are quite different!

Fast imaging methods

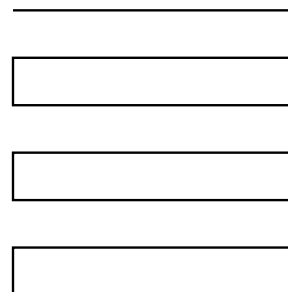
Spiral



Square Spiral



Echo Planar



For a spiral scan we want

$$k_x(t) \propto t \cos(\omega_s t)$$

$$k_y(t) \propto t \sin(\omega_s t)$$

so (working backwards now):

$$G_x(t) \propto \cos(\omega_s t) - t\omega_s \sin(\omega_s t).$$

Note the new perspective: we first determine the desired k -space trajectory and then compute the required gradient functions.

What are the practical problems with these approaches?

- irregular sampling in k -space so hard “gridding”
- T_2^* decay (ignored in derivation) means need short readout time so large gradients and slew rate.
- SNR decreases with decreasing scan time

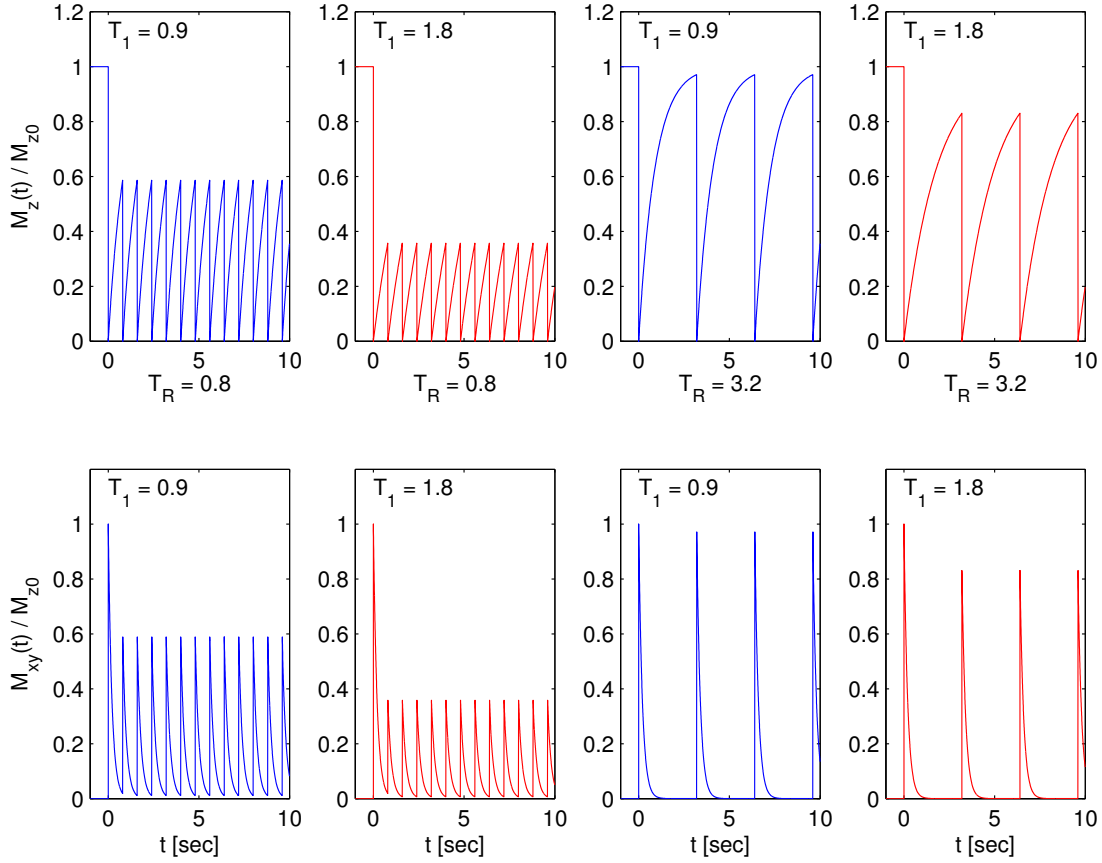
Compromise: interleaved spirals

T_1 contrast

In the PR and spin-warp pulse sequences described above, one must repeat the acquisition multiple times to acquire multiple lines in k-space.

The time interval between one excitation pulse and the next is called the **repetition time** T_R .

One way to induce T_1 contrast is to reduce T_R , as illustrated in the following figure.



The kind of sequence illustrated above is called **saturation-recovery** imaging; it assumes the transverse component has decayed essentially to zero before the next excitation, which requires that $T_R \gg T_2$.

In steady state (which happens after first 90):

$$m(x, y) \propto \rho(x, y) \left(1 - e^{-T_R/T_1(x, y)} \right).$$

If we center the readout interval at echo time T_E , then the magnetization at the echo time, in steady state, will be

$$m(x, y) \propto \underbrace{\rho(x, y)}_{\text{spin density}} \underbrace{(1 - e^{-T_R/T_1})}_{T_1 \text{ weighting}} \underbrace{e^{-T_E/T_2}}_{T_2 \text{ weighting}}$$

The spin density ρ has poor soft-tissue contrast. However, due to the exponentials, small differences in T_1, T_2 values can cause large signal changes, providing significant contrast.

We will see soon that in practice, proper T_2 contrast requires the use of a spin-echo.

Sometimes we want to avoid having much T_1 contrast, in which case we must use $T_R \gg T_1$.

5.7

Sampling and resolution issues for 2DFT sequence (Cartesian k-space sampling)

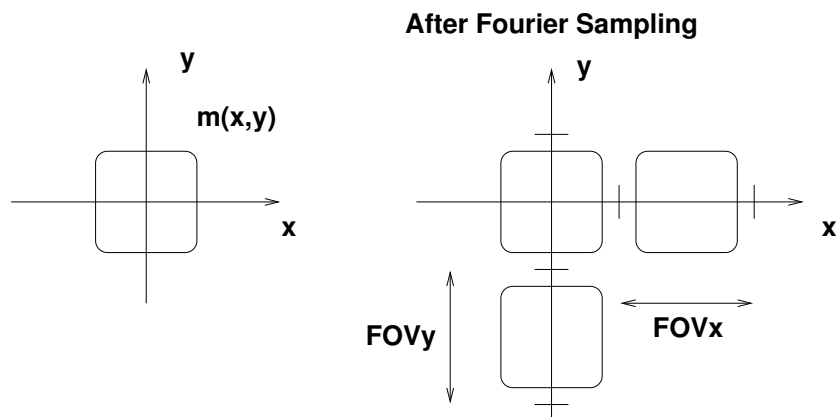
The received baseband signal $s(t)$ must be **sampled** (by A/D converter), hence k_x and k_y are sampled.

Sampling in *Fourier domain* causes replication in image domain.

Furthermore, the number of samples is **finite**.

FOV analysis

The effect of sampling in the Fourier domain follows from sampling theory and the duality property of the Fourier transform.



What is the spacing of the replications? $1/\Delta k_x$ and $1/\Delta k_y$

Thus $\text{FOV}_x = 1/\Delta k_x$, $\text{FOV}_y = 1/\Delta k_y$

What happens if k -space is undersampled?

Under sampling of k -space causes spatial aliasing of $m(x, y)$ or overlap of shifted replicates, *i.e.*, “wrap around” artifacts.

To prevent spatial aliasing, choose $\Delta k \leq 1/\text{FOV}$.

It is relatively easy to avoid spatial aliasing because objects are space limited!

Recall that for $t \in [t_1, t_3]$: $k_x(t) = \bar{\gamma} G_x(t_2)(t - t_2)$ so $\Delta k_x = k_x((n+1)\Delta_T) - k_x(n\Delta_T) = \bar{\gamma} G_x(t_2) \Delta_T$ or

$$\Delta_T = \frac{\Delta k_x}{\bar{\gamma} G_x(t_2)} = \frac{1}{\text{FOV}_x} \frac{1}{\bar{\gamma} G_x(t_2)}.$$

- For a larger FOV need finer time sampling (*cf.* higher bandlimited signal needs finer time sampling).
- Larger gradient requires finer time sampling (shorter readout).

Spatial resolution analysis

Because one can collect only a finite number of signal samples, we have **truncation** in Fourier domain.

Therefore the reconstructed image is a blurred version of the original object (*i.e.*, magnetization pattern).

The **PSF** of the blur is given by the **Dirichlet** response from HW1.

The Dirichlet response causes blur, ringing, and replication (spatial aliasing).

How can we reduce ringing? By apodizing.

What price do we pay? Decreased spatial resolution.

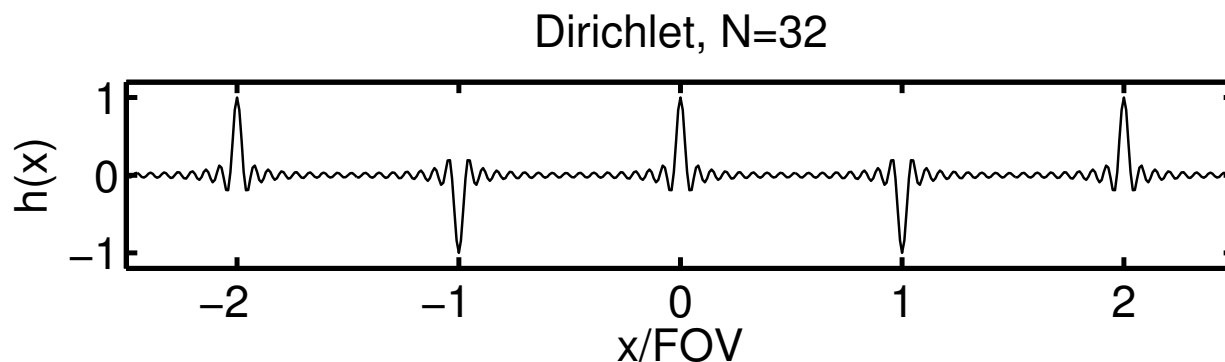
Tradeoff with scan time: collecting more of k -space decreases blur (improves spatial resolution), but may require more scan time (and more reconstruction time.)

One cannot achieve arbitrarily high resolution by increasing k -space sampling. Ultimately the spatial resolution will be limited by Lorentzian broadening due to T_2 decay and by diffusion of the molecules in the object during the data acquisition.

Example. If $\Delta k_x = 1/\text{FOV}$ then from HW1 the Dirichlet PSF is

$$h(x) = \frac{1}{\text{FOV}} \frac{\sin(N\pi x/\text{FOV})}{\sin(\pi x/\text{FOV})}.$$

The following figure shows the case $N = 32$.



How wide is the blur? The first zeros occur when $N\pi x/\text{FOV} = \pm\pi$, *i.e.*, when $x = \pm\text{FOV}/N$.

So the width of the PSF is approximately

$$\Delta_x = \frac{1}{2} 2 \frac{\text{FOV}}{N} = \frac{\text{FOV}}{N} = \frac{1}{N\Delta k_x}.$$

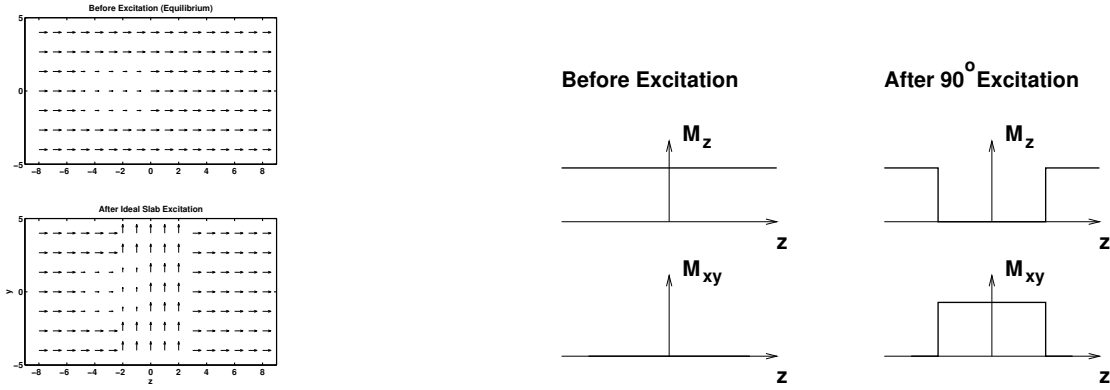
To improve resolution, *i.e.*, to decrease Δ_x , our only option is to increase N and hence increase scan time.

Could we improve resolution by increasing Δk_x ? Not really, because that would cause spatial aliasing.

Excitation

- **Nonselective:** RF only (no gradients). Excites entire volume if tuned to resonant frequency
- **Selective:** RF with gradient(s) (in z usually, but can be applied in any direction (*e.g.*, coronal and sagittal views). Excites spins whose (slice-dependent) resonant frequency lies within the nonzero part of RF spectrum.

Goal (ideal selective excitation, Fig. 6.7):



Design issues

- RF pulse amplitude function $b_1(t)$: shape, duration, strength
- $G_z(t)$: gradient signal shape / strength

We previously argued for a sinc pulse using qualitative arguments.

A “brute force” approach to pulse design is to perform simulations by discretizing the Bloch equation:

$$\frac{d}{dt} \vec{M} = \vec{M} \times \gamma \vec{B} - \frac{M_x \vec{i} + M_y \vec{j}}{T_2} - \frac{(M_z - M_{z0}) \vec{k}}{T_1},$$

and then trying various RF pulses to see which ones give good slice selection.

We would like to take an analytical approach, both to provide intuition as well as to give us an initial design that we could then refine. Therefore we make the following simplifications:

- Ignore relaxation. (RF pulse duration $\approx 1 \text{ msec} \ll T_1, T_2$.)
- Gradients constant during the RF pulse (for now):

$$\vec{B}(\mathbf{r}, t) = \underbrace{\vec{B}_1(\mathbf{r}, t)}_{\text{RF field}} + \left(\underbrace{B_{z0}}_{\text{static}} + \underbrace{\mathbf{r} \cdot \vec{G}}_{\text{gradients}} \right) \vec{k}.$$

- On-resonance RF excitation: oscillator tuned to ω_0 .
- Circularly polarized RF field (in x direction at $t = 0$) (note amplitude term and oscillation in xy plane term):

$$\vec{B}_1(\mathbf{r}, t) = \text{coil}(\mathbf{r}) b_1(t) \left(\vec{i} \cos \omega_0 t - \vec{j} \sin \omega_0 t \right),$$

where $\text{coil}(\mathbf{r})$ denotes the coil transmit pattern. (It will be larger close to the coil, and smaller further away. If the coil is large relative to the object, to first order we can consider it to be a constant.) The product $\text{coil}(\mathbf{r}) b_1(t)$ has units of field strength: Gauss or Tesla.

Transform magnetization into **rotating frame**:

$$\vec{M}_{\text{rot}}(\mathbf{r}, t) \triangleq R_z(-\omega_0 t) \vec{M}(\mathbf{r}, t), \quad R_z(\theta) = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2.27) \text{ rotation in } xy \text{ plane.}$$

(Note z component is unchanged.) From the appendix of Ch. 6, the Bloch equation in the rotating frame becomes:

$$\vec{M}_{\text{rot}}(\mathbf{r}, t) = \begin{bmatrix} M_{x'}(\mathbf{r}, t) \\ M_{y'}(\mathbf{r}, t) \\ M_z(\mathbf{r}, t) \end{bmatrix}, \quad \frac{d}{dt} \vec{M}_{\text{rot}} = \vec{M}_{\text{rot}} \times \gamma \vec{B}_{\text{eff}},$$

where now the “effective field” in the equation has only the RF and gradient terms:

$$\vec{B}_{\text{eff}}(\mathbf{r}, t) = (\text{coil}(\mathbf{r}) b_1(t)) \vec{i} + (\mathbf{r} \cdot \vec{G}) \vec{k}.$$

Note that the main-field component B_{z0} has been “eliminated” from the equations by the transformation! This form of the Bloch equation describes the time-evolution of the magnetization in rotating-frame coordinates.

Expanding the cross product in the rotating-frame Bloch equation yields:

$$\frac{d}{dt} \vec{M}_{\text{rot}}(\mathbf{r}, t) = \begin{bmatrix} 0 & \omega(\mathbf{r}) & 0 \\ -\omega(\mathbf{r}) & 0 & \omega_1(\mathbf{r}, t) \\ 0 & -\omega_1(\mathbf{r}, t) & 0 \end{bmatrix} \vec{M}_{\text{rot}}(\mathbf{r}, t),$$

where the above ω values are under the user’s control:

- $\omega(\mathbf{r}) = \gamma \mathbf{r} \cdot \vec{G}$ describes the *local* shift in resonant frequency caused by the gradients,
- $\omega_1(\mathbf{r}, t) = \gamma \text{coil}(\mathbf{r}) b_1(t)$ is Larmor frequency corresponding to RF field.

Given a specific $\vec{G}(t)$ and $\text{coil}(\mathbf{r}) b_1(t)$ one could plug into the rotating-frame Bloch equation, *e.g.*, discretized in Matlab, and observe the time-evolution of $\vec{M}_{\text{rot}}(\mathbf{r}, t)$ and try to adjust the pulses to make $\vec{M}_{\text{rot}}(\mathbf{r}, \tau)$ have the ideal slice profile pictured above. A more mathematical approach would be to solve the above differential equation for $\vec{M}_{\text{rot}}(\mathbf{r}, t)$. Hard in general!

Two easy cases:

- $\vec{G} = \mathbf{0}$ (no gradient) - non-selective
- **small tip angle**: “weak RF pulse” so $M_z(t) \approx M_{z0}$ and $\frac{d}{dt} M_z(t) \approx 0$

Non-selective excitation $\vec{G} = \mathbf{0}$

Because $\omega(\mathbf{r}) = 0$ in this case, the solution to rotating-frame simplified Bloch equation (6.23) becomes for $t \geq 0$:

$$\vec{M}_{\text{rot}}(\mathbf{r}, t) = R_x(\theta(\mathbf{r}, t)) \vec{M}_{\text{rot}}(\mathbf{r}, 0), \quad R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{bmatrix},$$

where the angle of rotation $\theta(\mathbf{r}, t)$ at time t is given by

$$\theta(\mathbf{r}, t) = \int_0^t \omega_1(\mathbf{r}, s) ds = \int_0^t \gamma \text{coil}(\mathbf{r}) b_1(s) ds.$$

Thus \vec{M}_{rot} precesses around x' axis according to a left-hand rule.



Example. Let $b_1(t)$ be a simple rectangular pulse of duration τ , *i.e.*, $b_1(t) = b_1 \text{rect}\left(\frac{t-\tau/2}{\tau}\right)$, and assume $\text{coil}(\mathbf{r}) = 1$.

Then the total rotation angle is $\theta(\tau) = \tau \gamma b_1$. The precession rate is $f_1 = \frac{d}{dt} \theta(t) / 2\pi = \frac{\omega_1}{2\pi} = \bar{\gamma} b_1$, where $\bar{\gamma} = 4260 \text{ Hz / G}$. A typical RF amplitude, $\text{coil}(\mathbf{r}) b_1(t)$, is 0.1 G, in which case $f_1 = \bar{\gamma} 0.1 \text{ G} = 426 \text{ Hz}$.

For a 90° pulse, need $\tau \gamma b_1 = \pi/2$ or $\tau \bar{\gamma} b_1 = \tau f_1 = 1/4$ so $\tau = 1/(4f_1) \approx 0.6 \text{ msec}$.

Was it reasonable to ignore relaxation? Yes, because $0.6 \text{ msec} \ll T_2$.

Selective excitation ($\vec{G} \neq 0$)

Unfortunately, there is no general analytical solution to the Bloch equation in this case.

Assume, in addition to previous assumptions, that

- **small tip angle:** ($\theta < 30^\circ$) “weak RF pulse” so $M_z \approx M_{z0}$ and $\frac{d}{dt} M_z \approx 0$,
- magnetization begins at equilibrium: $\vec{M}_{\text{rot}}(\mathbf{r}, 0) = (0, 0, M_{z0}(\mathbf{r}))$.

Then we can simplify the rotating-frame Bloch equation to the following:

$$\frac{d}{dt} \vec{M}_{\text{rot}}(\mathbf{r}, t) \approx \begin{bmatrix} 0 & \omega(\mathbf{r}) & 0 \\ -\omega(\mathbf{r}) & 0 & \omega_1(\mathbf{r}, t) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} M_{x'}(\mathbf{r}, t) \\ M_{y'}(\mathbf{r}, t) \\ M_{z0}(\mathbf{r}) \end{bmatrix},$$

where as before:

- $\omega(\mathbf{r}) = \gamma \mathbf{r} \cdot \vec{G}$ is the shift in resonant frequency caused by the gradients (still assumed to be constant),
- $\omega_1(\mathbf{r}, t) = \gamma \text{coil}(\mathbf{r}) b_1(t)$ is Larmor frequency corresponding to RF field.

Now the differential equation is solvable analytically. The solution separates into $x'y'$ and z components.

- The z component is constant: $M_z(\mathbf{r}, t) = M_z(\mathbf{r}, 0) = M_{z0}(\mathbf{r})$ by assumption.
- Express the transverse component (in this rotating frame) in complex representation:

$$M_R(\mathbf{r}, t) \triangleq M_{x'}(\mathbf{r}, t) + i M_{y'}(\mathbf{r}, t) = M(\mathbf{r}, t) e^{i\omega_0 t},$$

where from Ch. 5: $M(\mathbf{r}, t) = M_x(\mathbf{r}, t) + i M_y(\mathbf{r}, t)$.

The Bloch equation for the transverse component becomes

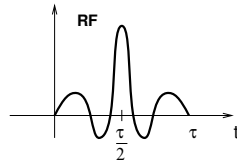
$$\frac{d}{dt} M_R(\mathbf{r}, t) = -i\omega(\mathbf{r}) M_R(\mathbf{r}, t) + i\omega_1(\mathbf{r}, t) M_{z0}(\mathbf{r}),$$

which has the following general solution:

$$M_R(\mathbf{r}, t) = i M_{z0}(\mathbf{r}) e^{-i\omega(\mathbf{r})t} \int_0^t e^{i\omega(\mathbf{r})s} \underbrace{\gamma \text{coil}(\mathbf{r}) b_1(s)}_{\omega_1(\mathbf{r}, s)} ds.$$

In a real system, $b_1(t)$ must have finite duration. Assume $[0, \tau]$.

We generally want symmetric slice profiles, so assume $b_1(t)$ is symmetric about $\tau/2$.



Also for simplicity assume only a z gradient is applied, *i.e.*, $\vec{G} = G_z \vec{k} = (0, 0, G_z)$. (This is traditional **slice selection**.) (One can also do spatially selective excitation using more complicated field gradients and RF [19].)

The transverse magnetization at the end of the RF excitation ($t = \tau$) is:

$$\begin{aligned} M_R(\mathbf{r}, \tau) &= i M_{z0}(\mathbf{r}) \text{coil}(\mathbf{r}) e^{-i\omega(\mathbf{r})\tau/2} \int_{-\tau/2}^{\tau/2} \gamma b_1(t + \tau/2) e^{i2\pi f(z)t} dt \\ &= i M_{z0}(\mathbf{r}) \text{coil}(\mathbf{r}) e^{-i\omega(\mathbf{r})\tau/2} \mathcal{F}\{\gamma b_1(t + \tau/2)\} \Big|_{f=-f(z)}, \end{aligned}$$

by change of variables $t' = s - \tau/2$, where the resonant frequency as a function of axial coordinate z is: $f(z) = \omega(z)/2\pi = \bar{\gamma}z G_z$.

We define the **slice profile** of the excitation pulse as:

$$\text{slice}(z) \triangleq \mathcal{F}\{\gamma b_1(t + \tau/2)\} \Big|_{f=-f(z)} = \int_{-\tau/2}^{\tau/2} \gamma b_1(t + \tau/2) e^{i2\pi f(z)t} dt.$$

Thus there is a Fourier transform relationship between recentered RF amplitude $b_1(t - \tau/2)$ and the **slice profile**!

What are the units of slice(\cdot)? ??

Therefore the magnetization at the end of the RF pulse is given by:

$$M_R(\mathbf{r}, \tau) = i M_{z0}(\mathbf{r}) \text{coil}(\mathbf{r}) e^{-i\omega(z)\tau/2} \text{slice}(z).$$

Why is there an i in the expression, *i.e.*, what does it mean physically? Precession about x' axis leaves us with y' component.

Can use this relationship to design $b_1(t)$ to achieve desired slice profile. Ideally both $b_1(t)$ and slice profile would have finite support, which is impossible. Thus choosing $b_1(t)$ is somewhat like the problem of designing a FIR filter (because $[0, \tau]$).

Sanity check: if $G_z = 0$, then $\omega(z) = 0$, so

$$M_R(\mathbf{r}, \tau) = i M_{z0}(\mathbf{r}) \theta(\mathbf{r}) \approx i M_{z0}(\mathbf{r}) \sin(\theta(\mathbf{r})),$$

using $\sin \theta \approx \theta$, where the tip angle is:

$$\theta(\mathbf{r}) \triangleq \text{coil}(\mathbf{r}) \int_{-\tau/2}^{\tau/2} \gamma b_1(t + \tau/2) dt = \text{coil}(\mathbf{r}) \int_0^{\tau} \gamma b_1(t) dt.$$

6.2.3

Refocusing

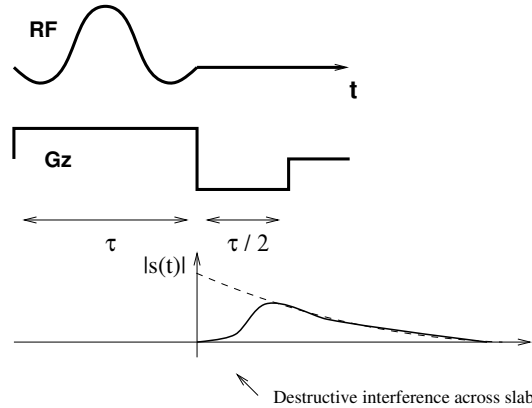
Note $e^{-i\omega(z)\tau/2}$ term in magnetization at τ (after RF excitation), where $\omega(z) = \gamma z G_z$.

Thus there is a z -dependent phase across the slab.

Reason: spins at “top” of slab precessed faster than those at “bottom” during RF excitation.

This would cause undesirable destructive interference on readout.

Solution: apply a z gradient of opposite polarity for half the time of the RF pulse.



How do we know it works? Could look at Bloch equation. Easier way: look at Fundamental Equation of MR (no RF):

$$M_R(\mathbf{r}, 3\tau/2) = M_R(\mathbf{r}, \tau) e^{-(\tau/2)T_2(\mathbf{r})} e^{-i\phi(\mathbf{r})} \quad \text{where} \quad \phi(\mathbf{r}) = \int_{\tau}^{3\tau/2} \gamma \mathbf{r} \cdot \vec{G} dt = \frac{\tau}{2} \gamma z (-G_z) = -\frac{\tau}{2} \omega(z),$$

so the extra phase accrued by the spins during the time interval $[\tau, 3\tau/2]$ will “refocus” the spins to be back in phase across the slab.

The resulting pattern of transverse magnetization is:

$$M_R\left(\mathbf{r}, \frac{3}{2}\tau\right) = i M_{z0}(\mathbf{r}) \text{coil}(\mathbf{r}) \text{slice}(z),$$

which has an amplitude that is the equilibrium longitudinal magnetization pattern modulated by the coil transmit pattern $\text{coil}(\mathbf{r})$ and by the slice profile (the FT of the RF pulse amplitude function).

Example: truncated sinc pulse

$$b_1(t) = b_1 \operatorname{sinc}\left(\frac{t - \tau/2}{\tau/4}\right) \operatorname{rect}\left(\frac{t - \tau/2}{\tau}\right).$$

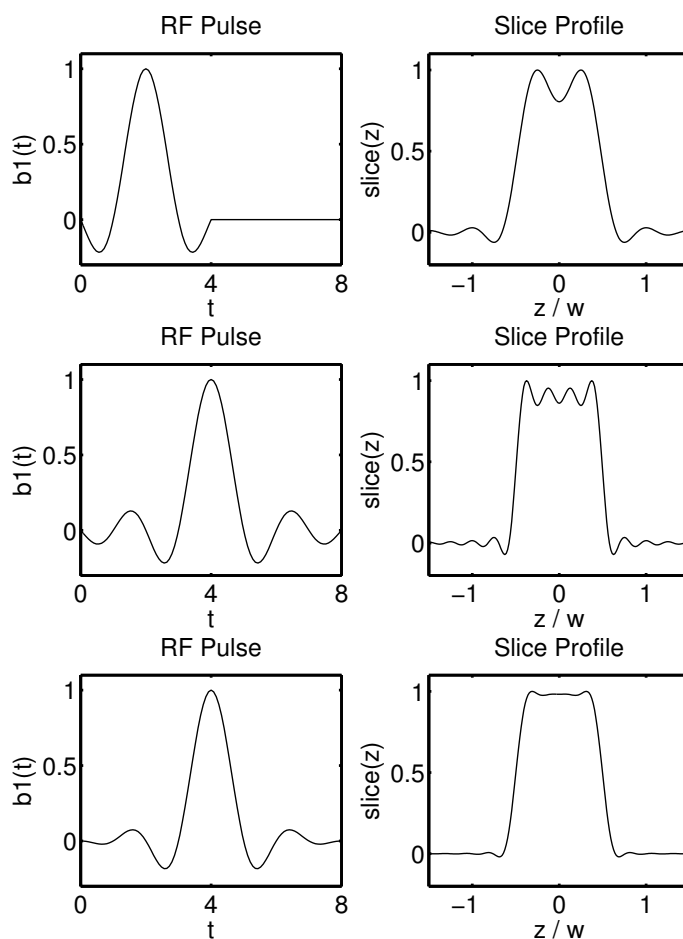
Then

$$\operatorname{slice}(z) \propto \operatorname{rect}\left(\frac{\tau}{4} \gamma z G_z\right) * \operatorname{sinc}(\tau \gamma z G_z),$$

which is the ideal rectangular slice profile smeared out by a sinc function caused by truncation in time of the RF pulse.

Define $w = \frac{4}{\tau \gamma G_z}$ to be the nominal slice width (for the ideal untruncated sinc pulse). Then we have

$$\operatorname{slice}(z) \propto \operatorname{rect}\left(\frac{z}{w}\right) * \operatorname{sinc}\left(\frac{z}{w/4}\right).$$



How applicable is the **small tip angle** assumption? Actually FT approach works well even up to 90° tips!

So we can use Fourier methods (filter design) to choose $b_1(t)$.

For 180° tips, we need more sophisticated approaches, including control theory [20], filter design [21, 22], or iterative methods [18, 19].

Summary

By applying a gradient along z direction, we map spins in different slices to different frequencies, can apply RF pulse with appropriate spectrum to excite (primarily) those spins in slices of interest. Using Bloch equation and simplifying assumptions, we found a FT relationship between the RF amplitude and the slice profile for selective excitation.