# Topological entropy and algebraic entropy on locally compact abelian groups - The Bridge Theorem -

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**Topological entropy** [Adler-Konheim-McAndrew 1965]

$X$ compact topological space, $\psi : X \to X$ continuous selfmap.
$\mathcal{U}, \mathcal{V}$ open covers of $X$; $\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$.
$N(\mathcal{U})$ = the minimal cardinality of a subcover of $\mathcal{U}$.

- The topological entropy of $\psi$ with respect to $\mathcal{U}$ is
  $H_{top}(\psi, \mathcal{U}) = \lim_{n\to\infty} \frac{\log N(\mathcal{U} \vee \psi^{-1}(\mathcal{U}) \vee \ldots \vee \psi^{-n+1}(\mathcal{U}))}{n}$.
- The topological entropy of $\psi$ is
  $h_{top}(\psi) = \sup\{H_{top}(\psi, \mathcal{U}) : \mathcal{U} \text{ open cover of } X\}$.

$K$ totally disconnected compact abelian group,
$\psi\colon K \to K$ continuous endomorphism.

For $L \leq K$ open, $n > 0$, $C_n(\psi, L) = L \cap \psi^{-1}(L) \cap \ldots \cap \psi^{-n+1}(L)$.

Then $h_{top}(\psi) = \sup\{H^*_{top}(\psi, L)\colon L \leq K \text{ open}\}$,
where $H^*_{top}(\psi, L) = \lim_{n\to\infty} \frac{\log|L/C_n(\psi,L)|}{n} = \lim_{n\to\infty} \frac{\log|K/C_n(\psi,L)|}{n}$.

- $h_{top}(id_K) = 0$.
- The *left Bernoulli shift* ${}_K\beta\colon K^{\mathbb{N}} \to K^{\mathbb{N}}$ is defined by ${}_K\beta(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots)$. Then $h_{top}({}_K\beta) = \log|K|$.

Let $f(x)=sx^n+a_1x^{n-1}+\ldots+a_n \in \mathbb{Z}[x]$ be a primitive polynomial, and let $\{\lambda_i \colon i=1,\ldots,n\}$ be the roots of $f(x)$.

The *Mahler measure* of $f(x)$ is

$$m(f(x)) = \log|s| + \sum_{|\lambda_i|>1} \log|\lambda_i|.$$

**Yuzvinski Formula**: Let $n>0$ and $\psi \colon \widehat{\mathbb{Q}}^n \to \widehat{\mathbb{Q}}^n$ a topological automorphism. Then

$$h_{top}(\psi) = m(p_\psi(x)),$$

where $p_\psi(x)$ is the characteristic polynomial of $\psi$ over $\mathbb{Z}$.

$K$ compact abelian group, $\psi : K \to K$ continuous endomorphism.

**Invariance under conjugation**: $\psi\colon H \to H$ continuous endomorphism, $\xi\colon K \to H$ topological isomorphism and $\phi = \xi^{-1}\psi\xi$, then $h_{top}(\phi) = h_{top}(\psi)$.

**Logarithmic law**: $h_{top}(\psi^k) = k \cdot h_{top}(\psi)$ for every $k \geq 0$.

**Continuity**: $K = \varprojlim K/K_i$ with $K_i$ closed $\psi$-invariant subgroup, then $h(\psi) = \sup_{i\in I} h(\overline{\psi}_{K/K_i})$.

**Additivity for direct products**: $K = K_1 \times K_2$, $\psi_i\colon K_i \to K_i$ endomorphism, $i = 1, 2$, then $h(\psi_1 \times \psi_2) = h(\psi_1) + h(\psi_2)$.

**Addition Theorem**: $H$ closed $\psi$-invariant subgroup of $K$, $\overline{\psi}\colon K/H \to K/H$ induced by $\psi$. Then $h_{top}(\psi) = h_{top}(\psi \restriction_H) + h_{top}(\overline{\psi}_{K/H})$.

[Adler-Konheim-McAndrew 1965, Stojanov 1987, Yuzvinski 1968]

**Algebraic entropy** [Weiss 1974, Peters 1979, Dikranjan-GB 2009]

$G$ abelian group, $\phi\colon G \to G$ endomorphism.

$F \subseteq G$ non-empty, $n > 0$, $T_n(\phi, F) = F + \phi(F) + \ldots + \phi^{n-1}(F)$.

- The *algebraic entropy* of $\phi$ with respect to $F$ is
  $H_{alg}(\phi, F) = \lim_{n\to\infty} \frac{\log|T_n(\phi,F)|}{n}$.
- The algebraic entropy of $\phi$ is
  $h_{alg}(\phi) = \sup\{H_{alg}(\phi, F)\colon F \subseteq G \text{ non-empty finite}\}$.

$G$ torsion abelian group, $\phi\colon G \to G$ endomorphism.

Then

$$h_{alg}(\phi) = \sup\{H_{alg}(\phi, F)\colon F \leq G \text{ finite}\}$$

- $h_{alg}(id_G) = 0$.
- The *right Bernoulli shift* $\beta_G\colon G^{(\mathbb{N})} \to G^{(\mathbb{N})}$ is defined by $\beta_G(x_0, x_1, x_2, \ldots) = (0, x_0, x_1, \ldots)$. Then $h_{alg}(\beta_G) = \log |G|$.

**Algebraic Yuzvinski Formula**: Let $n > 0$ and $\phi : \mathbb{Q}^n \to \mathbb{Q}^n$ an endomorphism. Then

$$h_{alg}(\phi) = m(p_\phi(x)),$$

where $p_\phi(x)$ is the characteristic polynomial of $\phi$ over $\mathbb{Z}$.

[GB-Virili 2011]

$G$ abelian group, $\phi : G \to G$ endomorphism.

**Invariance under conjugation**: $\psi \colon H \to H$ endomorphism, $\xi \colon G \to H$ isomorphism and $\phi = \xi^{-1}\psi\xi$, then $h_{alg}(\phi) = h_{alg}(\psi)$.

**Logarithmic law**: $h_{alg}(\phi^k) = k \cdot h_{alg}(\phi)$ for every $k \geq 0$.

**Continuity**: $G = \varinjlim G_i$ with $G_i$ $\phi$-invariant subgroup, then $h_{alg}(\phi) = \sup_{i \in I} h_{alg}(\phi \restriction_{G_i})$.

**Additivity for direct products**: $G = G_1 \times G_2$, $\phi_i \colon G_i \to G_i$ endomorphism, $i = 1, 2$, then $h_{alg}(\phi_1 \times \phi_2) = h_{alg}(\phi_1) + h_{alg}(\phi_2)$.

**Addition Theorem**: $H$ $\phi$-invariant subgroup of $G$, $\overline{\phi} \colon G/H \to G/H$ induced by $\phi$. Then $h_{alg}(\phi) = h_{alg}(\phi \restriction_H) + h_{alg}(\overline{\phi}_{G/H})$.

[Weiss 1974, Dikranjan-Goldsmith-Salce-Zanardo 2009: torsion]
[Peters 1979, Dikranjan-GB 2009, 2011: general case]

## The connection of the algebraic and the topological entropy

**Theorem (Bridge Theorem [Dikranjan - GB 2012] )**

*$K$ compact abelian group, $\psi\colon K \to K$ continuous endomorphism.*
*Denote by $\widehat{K}$ the Pontryagin dual of $K$*
*and by $\widehat{\psi}\colon \widehat{K} \to \widehat{K}$ the dual endomorphism of $\psi$.*
*Then $h_{top}(\psi) = h_{alg}(\widehat{\psi})$.*

[Weiss 1974: torsion; Peters 1979: countable, automorphisms.]

- The torsion case was proved by Weiss.
- Reduction to the torsion-free abelian groups.
  [Addition Theorems]
- Reduction to finite-rank torsion-free abelian groups.
  [Bernoulli shifts, continuity for direct/inverse limits]
- Reduction to divisible finite-rank torsion-free abelian groups, that is, $\mathbb{Q}^n$.
  [Addition Theorems]
- Reduction to injective endomorphisms $\Rightarrow$ surjective.
- $\phi\colon \mathbb{Q}^n \to \mathbb{Q}^n$ automorphism, $\widehat{\phi}\colon \widehat{\mathbb{Q}}^n \to \widehat{\mathbb{Q}}^n$ topological automorphism.
  [Algebraic Yuzvinski Formula and Yuzvinski Formula]

## Topological and algebraic entropy for LCA groups

$G$ locally compact abelian group, $\mu$ Haar measure on $G$,
$\phi\colon G \to G$ continuous endomorphism;
$\mathcal{C}(G) =$ the family of compact neighborhoods of 0; $K \in \mathcal{C}(G)$.
For $n > 0$, $C_n(\phi, K) = K \cap \phi^{-1}(K) \ldots \cap \phi^{-n+1}(K)$.

- [Bowen 1971, Hood 1974]
  The topological entropy of $\phi$ is

$$h_{top}(\phi) = \sup\left\{\limsup_{n\to\infty} \frac{-\log \mu(C_n(\phi, K))}{n} : K \in \mathcal{C}(G)\right\}.$$

For $n > 0$, $T_n(\phi, K) = K + \phi(K) + \ldots + \phi^{n-1}(K)$.

- [Peters 1981, Virili 2010]
  The algebraic entropy of $\phi$ is

$$h_{alg}(\phi) = \sup\left\{\limsup_{n\to\infty} \frac{\log \mu(T_n(\phi, K))}{n} : K \in \mathcal{C}(G)\right\}.$$

Does the Bridge Theorem extend to all LCA groups?

**Theorem ([Peters 1981; Virili 20??])**

*Let $G$ be a locally compact abelian group and $\psi\colon G \to G$ a topological automorphism. Then $h_{top}(\psi) = h_{alg}(\widehat{\psi})$.*

**Theorem (Bridge Theorem [Dikranjan - GB 2014])**

*Let $G$ be a totally disconnected locally compact abelian group and $\psi\colon G \to G$ a continuous endomorphism. Then $h_{top}(\psi) = h_{alg}(\widehat{\psi})$.*

$G$ totally disconnected locally compact abelian group,
$\psi\colon G \to G$ continuous endomorphism,
$\mathcal{B}(G) = \{U \leq G\colon U \text{ compact open}\} \subseteq \mathcal{C}(G)$.

Then $\mathcal{B}(G)$ is a base of the neighborhoods of 0 in $G$ and
$h_{top}(\psi) = \sup\{H^*_{top}(\psi, U)\colon U \in \mathcal{B}(G)\}$, where
$H^*_{top}(\psi, U) = \lim_{n\to\infty} \frac{\log |U/C_n(\psi,U)|}{n}$.

Moreover, $\mathcal{B}(\widehat{G})$ is cofinal in $\mathcal{C}(\widehat{G})$ and
$h_{alg}(\widehat{\psi}) = \sup\{H^*_{alg}(\widehat{\psi}, V)\colon V \in \mathcal{B}(\widehat{G})\}$,
where $H^*_{alg}(\widehat{\psi}, V) = \lim_{n\to\infty} \frac{\log |T_n(\widehat{\psi},V)/V|}{n}$.

$K$ be a compact Hausdorff space, $\psi\colon K \to K$ homeomorphism.

- The topological Pinsker factor of $(K, \psi)$ is the largest factor $\overline{\psi}$ of $\psi$ with $h_{top}(\overline{\psi}) = 0$.

[Blanchard-Lacroix 1993]

$G$ abelian group, $\phi : G \to G$ endomorphism.

- The Pinsker subgroup of $G$ is the largest $\phi$-invariant subgroup $\mathbf{P}(G, \phi)$ of $G$ such that $h_{alg}(\phi \restriction_{\mathbf{P}(G,\phi)}) = 0$.

[Dikranjan-GB 2010]

Let $G$ be an abelian group, $\phi : G \to G$ an endomorphism, $K = \widehat{G}$ and $\psi = \widehat{\phi}$; $\mathbf{P} = \mathbf{P}(G, \phi)$, $\mathbf{E} = \mathbf{E}(K, \psi) := \mathbf{P}^{\perp}$.

$$\begin{tikzcd}
\mathbf{P} \arrow[r, hook] \arrow[d, "\phi\restriction_{\mathbf{P}}"'] & G \arrow[r, two heads] \arrow[d, "\phi"'] & G/\mathbf{P} \arrow[d, "\overline{\phi}"] \\
\mathbf{P} \arrow[r, hook] & G \arrow[r, two heads] & G/\mathbf{P}
\end{tikzcd}$$

$$\begin{tikzcd}
\widehat{\mathbf{P}} \arrow[r, equal] & K/\mathbf{E} & K \arrow[l, two heads] & \mathbf{E} \arrow[l, hook'] \arrow[r, equal] & \widehat{G/\mathbf{P}} \\
\widehat{\mathbf{P}} \arrow[r, equal] \arrow[u, "\widehat{\phi\restriction_{\mathbf{P}}}"'] \arrow[ur, phantom, "\cong"] & K/\mathbf{E} \arrow[u, "\overline{\psi}"] & K \arrow[l, two heads] \arrow[u, "\psi"] & \mathbf{E} \arrow[l, hook'] \arrow[r, equal] \arrow[u, "\psi\restriction_{\mathbf{E}}"'] \arrow[ur, phantom, "\cong"] & \widehat{G/\mathbf{P}} \arrow[u, "\widehat{\overline{\phi}}"]
\end{tikzcd}$$

$\overline{\psi}\colon K/\mathbf{E} \to K/\mathbf{E}$ is the topological Pinsker factor of $(K, \psi)$.

- THE END -

Thank you for the attention