Algebraic entropy for amenable semigroup actions

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Let A be an abelian group and $\phi : A \to A$ an endomorphism;
$\mathcal{P}_f(A) = \{F \subseteq A \mid F \neq \emptyset \text{ finite}\} \supseteq \mathcal{F}(A) = \{F \leq A \mid F \text{ finite}\}$.

For $F \in \mathcal{P}_f(A)$, $n > 0$, let $\underline{T_n(\phi, F) = F + \phi(F) + \ldots + \phi^{n-1}(F)}$.

The *algebraic entropy of ϕ with respect to F* is

$$\boxed{H_{alg}(\phi, F) = \lim_{n\to\infty} \frac{\log |T_n(\phi, F)|}{n}.}$$

[Adler–Konheim–McAndrew,M.Weiss] The *algebraic entropy of ϕ* is

$$\boxed{\mathrm{ent}(\phi) = \sup\{H_{alg}(\phi, F) \mid F \in \mathcal{F}(A)\}.}$$

[Peters, Dikranjan–GB] The *algebraic entropy of ϕ* is

$$\boxed{h_{alg}(\phi) = \sup\{H_{alg}(\phi, F) \mid F \in \mathcal{P}_f(A)\}.}$$

Clearly, $\mathrm{ent}(\phi) = \mathrm{ent}(\phi \restriction_{t(A)}) = h_{alg}(\phi \restriction_{t(A)}) \leq h_{alg}(\phi)$.

[Dikranjan–Goldsmith–Salce–Zanardo for ent, D–GB for h_{alg}]

Theorem (Addition Theorem)

If B is a ϕ-invariant subgroup of A, then

$$h_{alg}(\phi) = h_{alg}(\phi\restriction_B) + h_{alg}(\phi_{A/B}),$$

where $\phi_{A/B} : A/B \to A/B$ is induced by ϕ.

[Weiss for ent, Peters, Dikranjan–GB for h_{alg}]

Theorem (Bridge Theorem)

Denote $\widehat{A}$ the Pontryagin dual of A and $\widehat{\phi} : \widehat{A} \to \widehat{A}$ the dual of ϕ. Then

$$h_{alg}(\phi) = h_{top}(\widehat{\phi}).$$

Here h_{top} denotes the topological entropy for continuous selfmaps of compact spaces [Adler–Konheim–McAndrew].

Example

Let p a prime, $A = \bigoplus_{\mathbb{Z}} \mathbb{Z}(p)$ and
$\sigma : A \to A,\ (x_n)_{n\in\mathbb{Z}} \mapsto (x_{n-1})_{n\in\mathbb{Z}}$ the right Bernoulli shift.
Then $h_{alg}(\sigma) = \mathrm{ent}(\sigma) = \log p$.
(Here $\beta = \sigma^{-1}$ is the left Bernoulli shift and $h_{alg}(\beta) = h_{alg}(\sigma)$.)

Note that $\widehat{\mathbb{Z}(p)} = \mathbb{Z}(p)$, $\widehat{\bigoplus_{\mathbb{Z}} \mathbb{Z}(p)} = \prod_{\mathbb{Z}} \mathbb{Z}(p)$ and
$\widehat{\sigma} = \beta : \prod_{\mathbb{Z}} \mathbb{Z}(p) \to \prod_{\mathbb{Z}} \mathbb{Z}(p)$. Hence, $h_{alg}(\sigma) = h_{top}(\widehat{\sigma}) = \log p$.

Example

Let $k > 1$ be an integer and consider $\mu_k : \mathbb{Z} \to \mathbb{Z},\ x \mapsto kx$.
Then $h_{alg}(\mu_k) = \log k$.

Note that $\widehat{\mathbb{Z}} = \mathbb{T}$ and $\widehat{\mu_k} = \mu_k : \mathbb{T} \to \mathbb{T}$.

Let $f(x) = sx^n + a_{n-1}x^{n-1} + \ldots + a_0 \in \mathbb{Z}[x]$ be a primitive polynomial. The *Mahler measure* of f is

$$m(f) = \log s + \sum_{|\lambda_i|>1} \log |\lambda_i|,$$

where λ_i are the roots of f in $\mathbb{C}$.

Theorem (Algebraic Yuzvinski Formula)

Let $n > 0$, $\phi : \mathbb{Q}^n \to \mathbb{Q}^n$ an endomorphism and $f_\phi(x) = sx^n + a_{n-1}x^{n-1} + \ldots + a_0 \in \mathbb{Z}[x]$ the characteristic polynomial of ϕ. Then

$$h_{alg}(\phi) = m(f_\phi).$$

Let S be a cancellative semigroup.
S is *right-amenable* if and only if S admits a *right-Følner net*, i.e., a net $(F_i)_{i\in I}$ in $\mathcal{P}_f(S)$ such that, for every $s \in S$,

$$\lim_{i\in I} \frac{|F_i s \setminus F_i|}{|F_i|} = 0.$$

(analogously, left-amenable).

A map $f : \mathcal{P}_f(S) \to \mathbb{R}$ is:

1. *subadditive* if $f(F_1 \cup F_2) \leq f(F_1) + f(F_2)\ \forall F_1, F_2 \in \mathcal{P}_f(S)$;
2. *left-subinvariant* if $f(sF) \leq f(F)\ \forall s \in S\ \forall F \in \mathcal{P}_f(S)$;
3. *right-subinvariant* if $f(Fs) \leq f(F)\ \forall s \in S\ \forall F \in \mathcal{P}_f(S)$;
4. *unif. bounded on singletons* if $\exists M \geq 0,\ f(\{s\}) \leq M\ \forall s \in S$.

Let $\mathcal{L}(S) = \{f : \mathcal{P}_f(S) \to \mathbb{R} \mid (1), (2), (4) \text{ hold for } f\}$ and $\mathcal{R}(S) = \{f : \mathcal{P}_f(S) \to \mathbb{R} \mid (1), (3), (4) \text{ hold for } f\}$.

[Ceccherini-Silberstein–Coornaert–Krieger,
generalizing Ornstein–Weiss Lemma and Fekete Lemma]

Theorem

Let S be a cancellative semigroup which is right-amenable (respectively, left-amenable).

For every $f \in \mathcal{L}(S)$ (respectively, $f \in \mathcal{R}(S)$) there exists $\lambda \in \mathbb{R}_{\geq 0}$ such that

$$\boxed{\mathcal{H}_S(f) := \lim_{i \in I} \frac{f(F_i)}{|F_i|} = \lambda}$$

for every right-Følner (respectively, left-Følner) net $(F_i)_{i \in I}$ of S.

Let S be a cancellative left-amenable semigroup,
X a compact space and $\mathrm{cov}(X)$ the family of all open covers of X.
For $\mathcal{U} \in \mathrm{cov}(X)$, let $N(\mathcal{U}) = \min\{|\mathcal{V}| \mid \mathcal{V} \subseteq \mathcal{U}\}$.

Consider a left action $S \overset{\gamma}{\curvearrowright} X$ by continuous maps.
For $\mathcal{U} \in \mathrm{cov}(X)$ and $F \in \mathcal{P}_f(S)$, let

$$\boxed{\mathcal{U}_{\gamma,F} = \bigvee_{s \in F} \gamma(s)^{-1}(\mathcal{U}) \in \mathrm{cov}(X).}$$

$$f_{\mathcal{U}} : \mathcal{P}_{fin}(S) \to \mathbb{R}, \quad F \mapsto \log N(\mathcal{U}_{\gamma,F}).$$

Then $\underline{f_{\mathcal{U}} \in \mathcal{R}(S)}$.

[Ceccherini-Silberstein–Coornaert–Krieger]
The *topological entropy of* γ *with respect to* $\mathcal{U}$ is

$$\boxed{H_{top}(\gamma, \mathcal{U}) = \mathcal{H}_S(f_{\mathcal{U}}).}$$

The *topological entropy of* γ is

$$\boxed{h_{top}(\gamma) = \sup\{H_{top}(\gamma, \mathcal{U}) \mid \mathcal{U} \in \mathrm{cov}(X)\}.}$$

Let S be a cancellative right-amenable semigroup.
Let A be an abelian group and
consider a left action $S \overset{\alpha}{\curvearrowright} A$ by endomorphisms.

For $X \in \mathcal{P}_f(A)$ and $F \in \mathcal{P}_f(S)$, let

$$\boxed{T_F(\alpha, X) = \sum_{s \in F} \alpha(s)(X) \in \mathcal{P}_f(A).}$$

$$f_X : \mathcal{P}_{fin}(S) \to \mathbb{R}, \quad F \mapsto \log |T_F(\alpha, X)|.$$

Then $\underline{f_X \in \mathcal{L}(S)}$.

The *algebraic entropy of* α *with respect to* X is

$$\boxed{H_{alg}(\alpha, X) = \mathcal{H}_S(f_X).}$$

[Fornasiero–GB–Dikranjan, Virili] The *algebraic entropy of* α is

$$\boxed{h_{alg}(\alpha) = \sup\{H_{alg}(\alpha, X) \mid X \in \mathcal{P}_f(A)\}.}$$

Moreover, $\boxed{\mathrm{ent}(\alpha) = \sup\{H_{alg}(\alpha, X) \mid X \in \mathcal{F}(A)\}.}$

Let S be a cancellative right-amenable semigroup.
Let A be an abelian group and
consider a left action $S \overset{\alpha}{\curvearrowright} A$ by endomorphisms.

Theorem (Addition Theorem)

If A is torsion and B is an α-invariant subgroup of A, then

$$h_{alg}(\alpha) = h_{alg}(\alpha_B) + h_{alg}(\alpha_{A/B}),$$

where $S \overset{\alpha_B}{\curvearrowright} B$ and $S \overset{\alpha_{B/A}}{\curvearrowright} B/A$ are induced by α.

Let S be a cancellative left-amenable semigroup.
Let K be a compact abelian group and
consider a left action $S \overset{\gamma}{\curvearrowright} K$ by continuous endomorphisms.

γ induces a right action $\widehat{K} \overset{\widehat{\gamma}}{\curvearrowleft} S$, defined by

$$\widehat{\gamma}(s) = \widehat{\gamma(s)} : \widehat{K} \to \widehat{K} \quad \text{for every } s \in S;$$

$\widehat{\gamma}$ is the *dual action of* γ.

Denote by $\widehat{\gamma}^{op}$ the left action $S^{op} \overset{\widehat{\gamma}^{op}}{\curvearrowright} \widehat{K}$ associated to $\widehat{\gamma}$ of the cancellative right-amenable semigroup S^{op}.

Theorem (Bridge Theorem)

If K is totally disconnected (i.e., A is torsion), then

$$h_{top}(\gamma) = h_{alg}(\widehat{\gamma}^{op}).$$

[Virili for group actions on locally compact abelian groups]

Let S be a cancellative left-amenable semigroup.
Let K be a compact abelian group and
consider a left action $S \overset{\gamma}{\curvearrowright} K$ by continuous endomorphisms.

Corollary (Addition Theorem)

If K is totally disconnected and L is a γ-invariant subgroup of K, then

$$h_{top}(\gamma) = h_{top}(\gamma_L) + h_{top}(\gamma_{K/L}),$$

where $S \overset{\gamma_L}{\curvearrowright} L$ and $S \overset{\gamma_{K/L}}{\curvearrowright} K/L$ are induced by γ.

Known in the case of compact groups for:

- $\mathbb{Z}^d$-actions on compact groups [Lind-Schmidt-Ward];
- actions of countable amenable groups on compact metrizable groups [Li].

Let G be an amenable group, A an abelian group, $G \overset{\alpha}{\curvearrowright} A$.

For $H \leq G$ consider $H \overset{\alpha\restriction_H}{\curvearrowright} A$.

- If $[G:H] = k \in \mathbb{N}$, then $h_{alg}(\alpha \restriction_H) = k \cdot h_{alg}(\alpha)$.
  In particular, $h_{alg}(\alpha \restriction_H)$ and $h_{alg}(\alpha)$ are simultaneously 0.
- If H is normal, then $h_{alg}(\alpha) \leq h_{alg}(\alpha \restriction_H)$.

Conjecture

Let G be an amenable group, A an abelian group, $G \overset{\alpha}{\curvearrowright} A$. For every $H \leq G$,

$$h_{alg}(\alpha) \leq h_{alg}(\alpha \restriction_H).$$

Theorem

If H is normal and G/H is infinite,
$h_{alg}(\alpha \restriction_H) < \infty$ implies $h_{alg}(\alpha) = 0$.

Corollary

Let G and A be infinite abelian groups and $G \overset{\alpha}{\curvearrowright} A$.
If $g \in G \setminus \{0\}$ is such that $G/\langle g \rangle$ is infinite and $h_{alg}(\alpha(g)) < \infty$, then $h_{alg}(\alpha) = 0$.

Hence, for actions $\mathbb{Z}^d \overset{\alpha}{\curvearrowright} A$ with $\underline{d > 1}$,

- if $h_{alg}(\alpha(g)) < \infty$ for some $g \in \mathbb{Z}^d$, $g \neq 0$, then $h_{alg}(\alpha) = 0$; [Eberlein for h_{top}, Conze for h_μ]
- every action $\mathbb{Z}^d \overset{\alpha}{\curvearrowright} \mathbb{Q}^n$ has $h_{alg}(\alpha) = 0$. (Compare with the case $d = 1$, i.e., the Algebraic Yuzvinski Formula.)

Let G be an amenable group and A an abelian group.
Consider the action

$$\boxed{G \overset{\sigma_{G,A}}{\curvearrowright} A^G}$$

defined, for every $g \in G$, by

$$\sigma_{G,A}(g)(f)(x) = f(g^{-1}x)$$

for every $f \in A^G$ and $x \in G$.
In other words, for every $(a_x)_{x\in G} \in A^G$,

$$\boxed{\sigma_{G,A}((a_x)_{x\in G}) = (a_{g^{-1}x})_{x\in G}.}$$

If $G = \mathbb{Z}$, then $\sigma_{\mathbb{Z},A}(1) = \sigma$ is the right Bernoulli shift, that is,
$\sigma((a_n)_{n\in\mathbb{Z}}) = (a_{n-1})_{n\in\mathbb{Z}}$.

Let G be an amenable group and A an abelian group.
Consider the action

$$\boxed{G \overset{\beta_{G,A}}{\curvearrowright} A^G}$$

defined, for every $g \in G$, by

$$\beta_{G,A}(g)(f)(x) = f(xg)$$

for every $f \in A^G$ and $x \in G$.
In other words, for every $(a_x)_{x \in G} \in A^G$,

$$\boxed{\beta_{G,A}((a_x)_{x \in G}) = (a_{xg})_{x \in G}.}$$

If $G = \mathbb{Z}$, then $\beta_{\mathbb{Z},A}(1) = \beta$ is the left Bernoulli shift, that is, $\beta((a_n)_{n \in \mathbb{Z}}) = (a_{n+1})_{n \in \mathbb{Z}}$.

Consider the restrictions $G \overset{\bar{\beta}_{G,A}}{\curvearrowright} A^{(G)}$ and $G \overset{\bar{\sigma}_{G,A}}{\curvearrowright} A^{(G)}$.

Theorem

If G is infinite, then

$$h_{alg}(\bar{\sigma}_{G,A}) = h_{alg}(\bar{\beta}_{G,A}) = \log |A|.$$

Consider

$$G \overset{\bar{\sigma}_{G,A}}{\curvearrowright} A^{(G)}.$$

Then the dual action is conjugated to

$$G \overset{\beta_{G,\widehat{A}}}{\curvearrowright} \widehat{A}^G,$$

and so

$$h_{alg}(\bar{\sigma}_{G,A}) = h_{top}(\beta_{G,\widehat{A}}).$$

Non-abelian case

Let G be a group and $\phi : G \to G$ an endomorphism.
Let $\mathcal{P}_f(G) = \{F \subseteq G \mid F \neq \emptyset \text{ finite}\}$.

For $F \in \mathcal{P}_f(G)$, $n > 0$, let $\underline{T_n(\phi, F) = F \cdot \phi(F) \cdot \ \ldots \ \cdot \phi^{n-1}(F).}$

The *algebraic entropy of* ϕ *with respect to* F is

$$\boxed{H_{alg}(\phi, F) = \lim_{n\to\infty} \frac{\log |T_n(\phi, F)|}{n}.}$$

[Dikranjan-GB] The *algebraic entropy of* ϕ is

$$\boxed{h_{alg}(\phi) = \sup\{H_{alg}(\phi, F) \mid F \in \mathcal{P}_f(G)\}.}$$

$G = \langle X \rangle$ finitely generated group ($X \in \mathcal{P}_f(G)$).

For $g \in G \setminus \{1\}$, $\ell_X(g)$ is the length of the shortest word representing g in $X \cup X^{-1}$, and $\ell_X(1) = 0$.

For $n \geq 0$, let $\boxed{B_X(n) = \{g \in G \mid \ell_X(g) \leq n\}.}$

The *growth function of G wrt X* is $\gamma_X : \mathbb{N} \to \mathbb{N},\ n \mapsto |B_X(n)|$.

The *growth rate of G wrt X* is $\lambda_X = \lim_{n\to\infty} \frac{\log \gamma_X(n)}{n}$.

For $\phi = id_G$ and $1 \in X$,

$\boxed{T_n(id_G, X) = B_X(n) \text{ and } H_{alg}(id_G, X) = \lambda_X.}$

[Milnor Problem, Grigorchuk group, Gromov Theorem]
There exists a group of intermediate growth.
G has polynomial growth if and only if G is virtually nilpotent.

Let G be a group, $\phi : G \to G$ an endomorphism and $X \in \mathcal{P}_f(G)$.
The *growth rate of ϕ wrt X* is $\gamma_{\phi,X} : \mathbb{N}_+ \to \mathbb{N}_+,\ n \mapsto |T_n(\phi, X)|$.

If $G = \langle X \rangle$ with $1 \in X \in \mathcal{P}_f(G)$, then $\boxed{\gamma_X = \gamma_{id_G,X}.}$

- ϕ has polynomial growth if $\gamma_{\phi,X}$ is polynomial $\forall X \in \mathcal{P}_f(G)$;
- ϕ has exponential growth if $\exists\ F \in \mathcal{P}_f(G)$, $\gamma_{\phi,X}$ is exp.;
- ϕ has intermediate growth otherwise.

$\boxed{\phi \text{ has exponential growth if and only if } h_{alg}(\phi) > 0.}$

The Addition Theorem does not hold for h_{alg}: let $G = \mathbb{Z}^{(\mathbb{Z})} \rtimes_\beta \mathbb{Z}$;

- G has exponential growth and so $h_{alg}(id_G) = \infty$;
- $\mathbb{Z}^{(\mathbb{Z})}$ and $\mathbb{Z}$ are abelian and hence $h_{alg}(id_{\mathbb{Z}^{(\mathbb{Z})}}) = 0 = h_{alg}(id_{\mathbb{Z}})$.

Theorem ([GB-Spiga, Dikranjan-GB for abelian groups, Milnor-Wolf in the classical setting])

No endomorphism of a locally virtually soluble group has intermediate growth.

Thank you for your attention!