Additivity of topological entropy for locally profinite groups

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Topological entropy

- Adler, Konheim, McAndrew 1965: for continuous selfmaps of compact spaces;
- Bowen 1971: for uniformly continuous selfmaps of metric spaces;
- Hood 1974: for uniformly continuous selfmaps of uniform spaces;
  in particular, for continuous endomorphisms of locally compact groups.

Let G be a locally compact group, μ a Haar measure on G, $\mathcal{C}(G)$ the family of all compact neighborhoods of 1 in G, $\phi : G \to G$ a continuous endomorphism.

- For $n > 0$, the *n-th ϕ-cotrajectory of $U \in \mathcal{C}(G)$* is

$$C_n(\phi, U) = U \cap \phi^{-1}(U) \cap \ldots \cap \phi^{-n+1}(U).$$

- The *topological entropy* of ϕ with respect to U is

$$H_{top}(\phi, U) = \limsup_{n\to\infty} \frac{-\log \mu(C_n(\phi, U))}{n}.$$

  (It does not depend on the choice of the Haar measure μ.)

- The *topological entropy* of ϕ is

$$h_{top}(\phi) = \sup\{H_{top}(\phi, U) : U \in \mathcal{C}(G)\}.$$

The main property of topological entropy is additivity.

Problem

Let G be a locally compact group, $\phi : G \to G$ a continuous endomorphism and N a ϕ-invariant closed normal subgroup of G. Is it true that

$$h_{top}(\phi) = h_{top}(\phi\restriction_N) + h_{top}(\bar{\phi}),$$

where $\bar{\phi} : G/N \to G/N$ is the endomorphism induced by ϕ?

$$\begin{array}{ccccc} N & \longrightarrow & G & \longrightarrow & G/N \\ \downarrow{\scriptstyle \phi\restriction_N} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \bar{\phi}} \\ N & \longrightarrow & G & \longrightarrow & G/N \end{array}$$

Yuzvinski 1965, Bowen 1971: Yes, for compact groups.

We consider the case when
G is a totally disconnected locally compact group
and $\phi : G \to G$ is a continuous endomorphism.

Let $\mathcal{B}(G) = \{U \leq G : U \text{ compact, open}\}$;
van Dantzig 1931: $\mathcal{B}(G)$ is a base of the neighborhoods of 1 in G.

- Then
$$h_{top}(\phi) = \sup\{H_{top}(\phi, U) : U \in \mathcal{B}(G)\},$$
- where
$$H_{top}(\phi, U) = \lim_{n\to\infty} \frac{\log[U : C_n(\phi, U)]}{n}.$$

Theorem

If N is a ϕ-stable closed normal subgroup of G with $\ker\phi \subseteq N$, then

$$h_{top}(\phi) = h_{top}(\phi\restriction_N) + h_{top}(\bar{\phi}),$$

where $\bar{\phi} : G/N \to G/N$ is the endomorphism induced by ϕ.

Main tool:

For $U \in \mathcal{B}(G)$,

$$H_{top}(\phi, U) = \log[\phi(U_+) : U_+],$$

where

$$U_0 = U,\ U_{n+1} = U \cap \phi(U_n)\ (n > 0),\ U_+ = \bigcap_{n=0}^{\infty} U_n.$$

If N is a ϕ-stable compact subgroup of G (not necessarily normal), then $G/N = \{xN : x \in G\}$ is a uniform space (and a locally compact 0-dimensional Hausdorff space)
and $\bar{\phi} : G/N \to G/N$ is a uniformly continuous map.

For $\pi : G \to G/N$ the canonical projection,

$$h_{top}(\bar{\phi}) = \sup\{H_{top}(\bar{\phi}, \pi U) : N \subseteq U \in \mathcal{B}(G)\}$$

and

$$H_{top}(\bar{\phi}, \pi U) = H_{top}(\phi, U).$$

Theorem

If N is a ϕ-stable compact subgroup of G such that $\ker\phi \subseteq N$, then

$$h_{top}(\phi) = h_{top}(\phi \restriction_N) + h_{top}(\bar{\phi}).$$

where $\bar{\phi} : G/N \to G/N$ is the map induced by ϕ.

Willis 1994, 2015:

- the scale of ϕ is

$$s(\phi) = \min\{[\phi(U) : \phi(U) \cap U] : U \in \mathcal{B}(G)\};$$

- $U \in \mathcal{B}(G)$ is *minimizing* if $s(\phi) = [\phi(U) : \phi(U) \cap U]$;
- $\mathrm{nub}\,\phi = \bigcap\{U \in \mathcal{B}(G) : U \text{ minimizing}\}$
  is a ϕ-stable compact subgroup of G.

Corollary

$$\log s(\phi) = h_{top}(\bar{\phi}),$$

where $\bar{\phi} : G/\mathrm{nub}\,\phi \to G/\mathrm{nub}\,\phi$ is the map induced by ϕ.

In particular, $h_{top}(\phi) = \log s(\phi)$ if and only if $\mathrm{nub}\,\phi = 1$

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