Entropy on totally disconnected locally compact groups

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Let G be a locally compact group, μ a right Haar measure on G and $\mathcal{C}(G)$ a local base of compact neighborhoods of 1.
Let $\phi : G \to G$ be a continuous endomorphism.
For every $U \in \mathcal{C}(G)$ and $n > 0$, the n-th ϕ-cotrajectory of U is

$$C_n(\phi, U) = U \cap \phi^{-1}(U) \cap \ldots \cap \phi^{-n+1}(U).$$

The topological entropy of ϕ with respect to U is

$$\boxed{H_{top}(\phi, U) = \limsup_{n\to\infty} -\frac{\log \mu(C_n(\phi, U))}{n}.}$$

(It does not depend on the choice of the Haar measure μ.)
The topological entropy of ϕ is

$$h_{top}(\phi) = \sup\{H_{top}(\phi, U) : U \in \mathcal{C}(G)\}.$$

Assume that G is also totally disconnected.

The family $\mathcal{B}(G) \subseteq \mathcal{C}(G)$ of all open compact subgroups of G is a local base of compact neighborhoods of 1 [van Dantzig].

For $U \in \mathcal{B}(G)$ and $n > 0$, $[U : C_n(\phi, U)]$ is finite,
and $\mu(U) = [U : C_n(\phi, U)] \cdot \mu(C_n(\phi, U))$.

Then $\log \mu(U) = \log[U : C_n(\phi, U)] + \log \mu(C_n(\phi, U))$,
so $-\log \mu(C_n(\phi, U)) = \log[U : C_n(\phi, U)] - \log \mu(U)$ and hence

$$\begin{aligned} H_{top}(\phi, U) &= \limsup_{n\to\infty} -\frac{\log \mu(C_n(\phi, U))}{n} \\ &= \limsup_{n\to\infty} \frac{\log[U : C_n(\phi, U)] - \log \mu(U)}{n} \\ &= \limsup_{n\to\infty} \frac{\log[U : C_n(\phi, U)]}{n} \end{aligned}$$

Let $\phi : G \to G$ be a topological automorphism. For $U \in \mathcal{B}(G)$,

$$\boxed{H_{top}(\phi, U) = \limsup_{n\to\infty} \frac{\log[U : C_n(\phi, U)]}{n}.}$$

For every $n > 0$ let $c_n := [U : C_n(\phi, U)]$. Then

- c_n divides c_{n+1} for every $n > 0$.

Let $\alpha_n := \frac{c_{n+1}}{c_n} = [C_n(\phi, U) : C_{n+1}(\phi, U)]$. Then

- $\alpha_{n+1} \leq \alpha_n$ for every $n > 0$;
- $\{\alpha_n\}_{n>0}$ stabilizes $(\exists n_0 > 0, \alpha > 0 \,:\, \alpha_n = \alpha \ \forall n \geq n_0)$;
- $H_{top}(\phi, U) = \log \alpha$.

Theorem (Limit-free formula)

For $U_+ = \bigcap_{n=0}^{\infty} \phi^n(U)$,

$$H_{top}(\phi, U) = \log[\phi(U_+) : U_+].$$

G totally disconnected locally compact group,
$\phi : G \to G$ topological automorphism.

Monotonicity: N closed normal subgroup of G, $\phi(N) = N$, $\overline{\phi} : G/N \to G/N$ induced by ϕ, then $h_{top}(\phi) \geq \max\{h_{top}(\phi \restriction_N), h_{top}(\overline{\phi})\}$.

Invariance under conjugation: $\xi : G \to H$ topological isomorphism, then $h_{top}(\xi\phi\xi^{-1}) = h_{top}(\phi)$.

Logarithmic law: $h_{top}(\phi^k) = k \cdot h_{top}(\phi)$ for every integer k.

Continuity: $G = \varprojlim G/G_i$ with G_i closed normal ϕ-invariant subgroup, then $h_{top}(\phi) = \sup_{i\in I} h_{top}(\phi \restriction_{G_i})$.

Additivity for direct products: $G = G_1 \times G_2$, $\phi_i : G_i \to G_i$ topological automorphism, $i = 1, 2$, then $h_{top}(\phi_1 \times \phi_2) = h_{top}(\phi_1) + h_{top}(\phi_2)$.

G totally disconnected locally compact group,
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The scale of ϕ is

$$s(\phi) = \min\{[\phi(U) : U \cap \phi(U)] : U \in \mathcal{B}(G)\}.$$

(Willis 2002, in 1994 only for inner automorphisms)

$U \in \mathcal{B}(G)$ is minimizing for ϕ if $s(\phi) = [\phi(U) : U \cap \phi(U)]$.

For $U \in \mathcal{B}(G)$ and $n > 0$, Willis considers

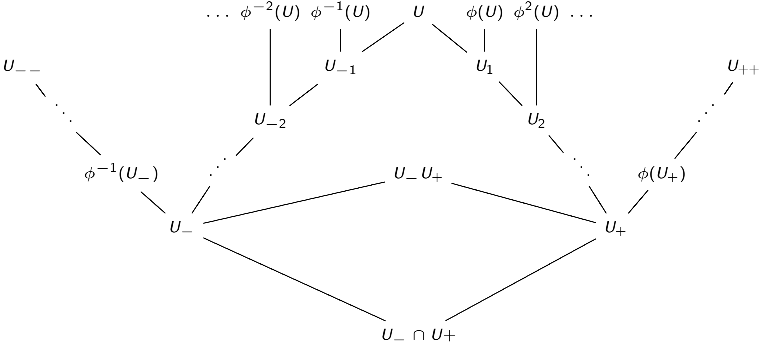
- $U_{-n} = U \cap \phi^{-1}(U) \cap \ldots \cap \phi^{-n}(U) = C_{n+1}(\phi, U)$
- $U_n = U \cap \phi(U) \cap \ldots \cap \phi^n(U) = C_{n+1}(\phi^{-1}, U)$
- $U_- = \bigcap_{n=0}^{\infty} \phi^{-n}(U)$ and $U_+ = \bigcap_{n=0}^{\infty} \phi^n(U)$
- $U_{--} = \bigcup_{n=0}^{\infty} \phi^{-n}(U_-)$ and $U_{++} = \bigcup_{n=0}^{\infty} \phi^n(U_+)$.

$$U_{-n} = U \cap \phi^{-1}(U) \ldots \cap \phi^{-n}(U) = C_{n+1}(\phi, U)$$

$$U_n = U \cap \phi(U) \ldots \cap \phi^{n}(U) = C_{n+1}(\phi^{-1}, U)$$

$$U_- = \bigcap_{n=0}^{\infty} \phi^{-n}(U) \text{ and } U_+ = \bigcap_{n=0}^{\infty} \phi^{n}(U)$$

$$U_{--} = \bigcup_{n=0}^{\infty} \phi^{-n}(U_-) \text{ and } U_{++} = \bigcup_{n=0}^{\infty} \phi^{n}(U_+).$$



G totally disconnected locally compact group,
$\phi : G \to G$ topological automorphism, $U \in \mathcal{B}(G)$.

- U is *tidy above* for ϕ if $U = U_- U_+$;
- U is *tidy below* for ϕ if U_{++} is closed;
- U is *tidy* for ϕ if it is tidy above and tidy below for ϕ.

Theorem (Willis)

$U \in \mathcal{B}(G)$ is minimizing for ϕ if and only if U is tidy for ϕ.
In this case

$$s(\phi) = [\phi(U_+) : (U_+)].$$

G totally disconnected locally compact group,
$\phi : G \to G$ topological automorphism.

By the limit-free formula

$$h_{top}(\phi) = \sup\{\log[\phi(U_+) : U_+] : U \in \mathcal{B}(G)\},$$

and by Willis' Theorem

$$\log s(\phi) = \min\{\log[\phi(U_+) : U_+] : U \in \mathcal{B}(G)\}.$$

This gives

Theorem

$$h_{top}(\phi) \geq \log s(\phi)$$

Equality holds when the tidy subgroups form a local base of neighborhoods of 1.

Problem (1)

Extend the limit-free formula to continuous endomorphisms.

Available in the compact case:

Theorem

Let K be a totally disconnected compact group, $\phi : K \to K$ a continuous endomorphism and $U \in \mathcal{B}(K)$ such that $[K : (\phi(K) \cdot U_-)] < \infty$. Then

$$H_{top}(\phi, U) = \log[\phi^{-1}(U_-) : U_-] - \log[K : \phi(K) \cdot U_-].$$

If K is abelian, then $[K : (\phi(K) \cdot U_-)] < \infty$ for every $U \in \mathcal{B}(K)$.

Problem (2)

Prove an analogous limit-free formula for the algebraic entropy.

Available for endomorphisms of discrete torsion groups.

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