Algebraic entropy for non-torsion abelian groups

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Definition (Justin Peters)

G abelian group, $\phi : G \to G$ endomorphism, F finite subset of G, n positive integer.

- The *n-th ϕ-trajectory* of F is

$$T_n(\phi, F) = F + \phi(F) + \ldots + \phi^{n-1}(F).$$

- The *algebraic entropy of ϕ with respect to F* is

$$H(\phi, F) = \lim_{n \to \infty} \frac{\log |T_n(\phi, F)|}{n}.$$

- The algebraic entropy of $\phi : G \to G$ is

$$h(\phi) = \sup\{H(\phi, F) : F \text{ finite subset of } G\}.$$

Monotonicity: H ϕ-invariant subgroup of G, $\overline{\phi}: G/H \to G/H$ induced by ϕ; then $h(\phi) \geq \max\{h(\phi\restriction_H), h(\overline{\phi})\}$.

Invariance under conjugation: If $\phi = \xi^{-1}\psi\xi$, $\psi: H \to H$ endomorphism, $\xi: G \to H$ isomorphism
$$\begin{array}{ccc} G & \xrightarrow{\phi} & G \\ \xi\downarrow & & \downarrow\xi \\ H & \xrightarrow{\psi} & H \end{array},$$
then $h(\phi) = h(\psi)$.

Logarithmic law: $h(\phi^k) = k \cdot h(\phi)$ for every $k \geq 0$.

Continuity: If G is direct limit of ϕ-invariant subgroups $\{G_i : i \in I\}$, then $h(\phi) = \sup_{i \in I} h(\phi\restriction_{G_i})$.

Additivity for direct products: If $G = G_1 \times G_2$ and $\phi_i \in \mathrm{End}(G_i)$, $i = 1, 2$, then $h(\phi_1 \times \phi_2) = h(\phi_1) + h(\phi_2)$.

- $\mathrm{ent}(\phi) = \mathrm{ent}(\phi \restriction_{t(G)}) = h(\phi \restriction_{t(G)})$

Example

Let K be a non-trivial abelian group. The *right Bernoulli shift* $\beta_K : K^{(\mathbb{N})} \to K^{(\mathbb{N})}$ is defined by

$$\beta_K(x_0, x_1, x_2, \ldots) = (0, x_0, x_1, \ldots).$$

If K is finite, then:

- $h(\beta_K) = \mathrm{ent}(\beta_K) = \log |K|$.

On the other hand, if $K = \mathbb{Z}$, then:

- $\mathrm{ent}(\beta_{\mathbb{Z}}) = 0$;
- $h(\beta_{\mathbb{Z}}) = \infty$.

Example

$$h(id_{\mathbb{Z}}) = 0.$$

Indeed, let $F = \{f_1, \ldots, f_k\}$ be a finite subset of $\mathbb{Z}$.

For every natural number $n > 1$, we show that $|T_n(id_{\mathbb{Z}}, F)| = |\underbrace{F + \ldots + F}_{n}| \leq P_F(n)$, where $P_F(x) \in \mathbb{Z}[x]$:

if $x \in T_n(id_{\mathbb{Z}}, F)$, then $x = \sum_{i=1}^{k} m_i f_i$, where $\sum_{i=1}^{k} m_i = n$, $m_i \geq 0$.

Then $(m_1, \ldots, m_k) \in \{0, 1, \ldots, n\}^k$, and so

$$|T_n(id_{\mathbb{Z}}, F)| \leq (n+1)^k = (n+1)^{|F|} =: P_F(n).$$

Hence

$$H(id_{\mathbb{Z}}, F) = \lim_{n\to\infty} \frac{\log |T_n(id_{\mathbb{Z}}, F)|}{n} \leq \lim_{n\to\infty} \frac{k \log(n+1)}{n} = 0.$$

Theorem (Addition Theorem)

Let G be an abelian group, $\phi \in \mathrm{End}(G)$, H a ϕ-invariant subgroup of G and $\overline{\phi} : G/H \to G/H$ the endomorphism induced by ϕ.

$$\begin{array}{ccc} H & \xrightarrow{\ \phi\restriction_H\ } & H \\ \big\downarrow & & \big\downarrow \\ G & \xrightarrow{\ \phi\ } & G \\ \big\downarrow & & \big\downarrow \\ G/H & \xrightarrow{\ \overline{\phi}\ } & G/H \end{array}$$

Then

$$h(\phi) = h(\phi\restriction_H) + h(\overline{\phi}).$$

Reductions:

- G torsion-free abelian group;
- G countable;
- G divisible;
- G of finite rank;
- ϕ injective.

It remains the case of an automorphism

$$\phi : \mathbb{Q}^n \to \mathbb{Q}^n$$

for n a positive integer.

Theorem (Algebraic Yuzvinski Formula)

For $n \in \mathbb{N}$ an automorphism ϕ of $\mathbb{Q}^n$ is described by a matrix $A \in GL_n(\mathbb{Q})$. Then

$$h(\phi) = \log s + \sum_{|\lambda_i|>1} \log |\lambda_i|,$$

where λ_i are the eigenvalues of A and s is the least common multiple of the denominators of the coefficients of the (monic) characteristic polynomial of A.

Follows from Yuzvinski Formula for topological entropy of automorphisms of $\widehat{\mathbb{Q}}^n$ and from the "Bridge Theorem" by Peters:

Theorem

For an automorphism ϕ of a countable abelian group G, $h(\phi) = h_{top}(\widehat{\phi})$.

Uniqueness: The algebraic entropy of the endomorphisms of abelian groups is the unique collection

$$h = \{h_G : \mathrm{End}(G) \to \mathbb{R}_{\geq 0} \cup \{\infty\} : G \text{ abelian group}\}$$

satisfying:

- Invariance under conjugation;
- Algebraic Yuzvinski Formula;
- Addition Theorem;
- Continuity for direct limits;
- $h_{\mathbb{Z}(p)^{(\mathbb{N})}}(\beta_{\mathbb{Z}(p)}) = \log p$ for every prime p.

Let G be an abelian group and $\phi \in \mathrm{End}(G)$.

Definition

The Pinsker subgroup of G is the maximum ϕ-invariant subgroup $\mathbf{P}(G,\phi)$ of G such that $h(\phi \restriction_{\mathbf{P}(G,\phi)}) = 0$.

Addition Theorem $\Rightarrow$ existence of $\mathbf{P}(G,\phi)$.

Motivation:

- For a measure preserving transformation ϕ of a measure space $(X,\mathcal{B},\mu)$ the *Pinsker σ-algebra* $\mathfrak{P}(\phi)$ of ϕ is the maximum σ-subalgebra of $\mathcal{B}$ such that ϕ restricted to $(X,\mathfrak{P}(\phi),\mu\restriction_{\mathcal{B}})$ has entropy zero.
- If $\phi : K \to K$ is a homeomorphism of a compact Hausdorff space K, then ϕ admits a largest factor with zero topological entropy, called *topological Pinsker factor*.

Let G be an abelian group and $\phi \in \mathrm{End}(G)$.

Definition

The Pinsker subgroup of G is the maximum ϕ-invariant subgroup $\mathbf{P}(G,\phi)$ of G such that $h(\phi \restriction_{\mathbf{P}(G,\phi)}) = 0$.

- $\overline{\phi} : G/\mathbf{P}(G,\phi) \to G/\mathbf{P}(G,\phi)$ has strongly positive entropy (i.e., if $0 \neq H \leq G/\mathbf{P}(G,\phi)$ is $\overline{\phi}$-invariant, then $h(\overline{\phi} \restriction_H) > 0$).
- $\mathbf{P}(G/\mathbf{P}(G,\phi),\overline{\phi}) = 0$.

$x \in G$ is *quasi-periodic* if there exist $n > m$ in $\mathbb{N}$, $\phi^n(x) = \phi^m(x)$.

$$Q_1(G,\phi) = \{x \in G : (\exists n > m \text{ in } \mathbb{N})\ (\phi^n - \phi^m)(x) = 0\}.$$

Example

If G is a torsion abelian group, then $\mathbf{P}(G,\phi) = Q_1(G,\phi)$.

Let $Q_0(G,\phi)=0$, and for every $n\in\mathbb{N}$ let
$Q_{n+1}(G,\phi)=\{x\in G:(\exists n>m \text{ in } \mathbb{N})\ (\phi^n-\phi^m)(x)\in Q_n(G,\phi)\}$.

$$Q_0(G,\phi)\subseteq Q_1(G,\phi)\subseteq\ldots\subseteq Q_n(G,\phi)\subseteq\ldots.$$

Definition

The QP-subgroup of G is $\mathfrak{Q}(G,\phi)=\bigcup_{n\in\mathbb{N}}Q_n(G,\phi)$.

- Each $Q_n(G,\phi)$ and so $\mathfrak{Q}(G,\phi)$ is a ϕ-invariant subgroup of G.
- $\mathfrak{Q}(G/\mathfrak{Q}(G,\phi),\overline{\phi})=0$.

Theorem

$$\mathbf{P}(G,\phi)=\mathfrak{Q}(G,\phi).$$

To prove $h(id_{\mathbb{Z}}) = 0$, we used that $\forall\ n > 1$ and $F \subseteq \mathbb{Z}$ finite, $|T_n(id_{\mathbb{Z}}, F)| = |\underbrace{F + \ldots + F}_{n}| \leq P_F(n)$, where $P_F(x) = (x+1)^{|F|}$.

Let G be an abelian group and let $\phi \in \mathrm{End}(G)$. Then:

- $\phi \in \mathrm{Pol}_F$ for $F \subseteq G$ finite, if there exists $P_F(x) \in \mathbb{Z}[x]$, such that $|T_n(\phi, F)| \leq P_F(n)$ for every positive integer;
- $\phi \in \mathrm{Pol}$ if $\phi \in \mathrm{Pol}_F$ for every finite $F \subseteq G$.

Definition

Let $\mathrm{Pol}(G, \phi)$ be the maximum ϕ-invariant subgroup of G such that $\phi \restriction_{\mathrm{Pol}(G,\phi)} \in \mathrm{Pol}$.

Theorem

$$\mathbf{P}(G, \phi) = \mathrm{Pol}(G, \phi) = \mathfrak{Q}(G, \phi).$$