Optimal Tableau Systems for Propositional Neighborhood Logic over All, Dense, and Discrete Linear Orders

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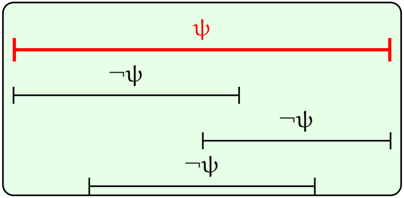
Outline

- Interval temporal logics
- The logic $A\overline{A}$ of temporal neighborhood
- A tableau-based decision procedure for $A\overline{A}$ over the class of all linear orders
- Conclusions and further work

Interval temporal logics

Truth of formulae is defined over intervals (not points).



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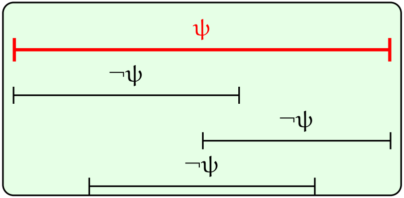


Interval temporal logics are very expressive (compared to point-based temporal logics).

In particular, formulas of interval logics express properties of pairs of time points rather than of single time points, and are evaluated as sets of such pairs, i.e., as binary relations.

Thus, in general there is no reduction of the satisfiability/validity in interval logics to monadic second-order logic, and therefore Rabin's theorem is not applicable here.

Binary Ordering Relations over intervals

The thirteen binary ordering relations between two intervals on a linear order (those below and their inverses) form the set of *Allen's interval relations*:

current interval:

equals:

ends :

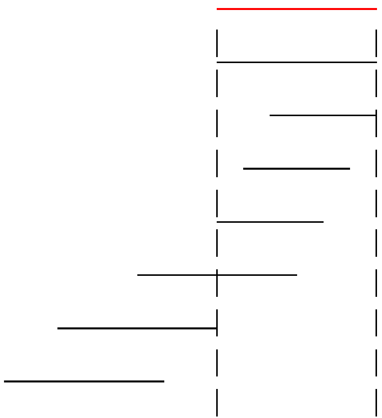
during:

begins:

overlaps:

meets:

before:



HS: the modal logic of Allen's interval relations

Allen's interval relations give rise to corresponding unary modalities over frames where intervals are primitive entities:
Halpern and Shoham's modal logic of time intervals HS (LICS 1986), interpreted over interval structures

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The satisfiability/validity problem for HS is highly undecidable over all standard classes of linear orders. What about its fragments?

More than four thousands fragments of HS (over the class of all linear orders) can be identified by choosing a different subset of the set of basic modal operators. However, 1347 genuinely different ones exist only.

D. Della Monica, V. Goranko, A. Montanari, and G. Sciavicco, *Expressiveness of the Interval Logics of Allen's Relations on the Class of all Linear Orders: Complete Classification*, IJCAI 2011

Decidability of HS fragments: main parameters

Decidability of HS fragments depends on two factors:

- the set of interval modalities;
- the linear order over which the logic is interpreted.

A real character: the logic D

The logic D of the subinterval relation (Allen's relation *during*) is quite interesting from the point of view of (un)decidability.

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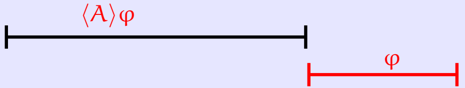
It is unknown, when D is interpreted over the class of all linear orders.

A well-behaved fragment: the logic $A\overline{A}$

Formulas of the logic are recursively defined by the following grammar:

$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle A\rangle\varphi \mid \langle\overline{A}\rangle\varphi$ ($[A] = \neg\langle A\rangle\neg$ as usual; same for $[\overline{A}]$)



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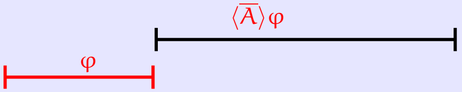

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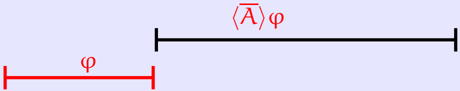
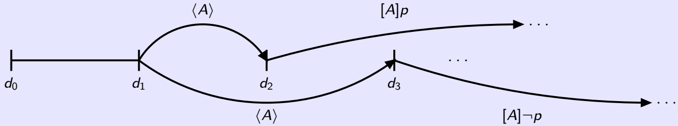


We cannot abstract way from any of the endpoints of intervals:

- Example: contradictory formulas can hold over intervals with the same right endpoint and a different left endpoint.

$\langle A \rangle [A] p \wedge \langle A \rangle [A] \neg p$ is satisfiable:



For any $d > d_3$, p holds over $[d_2, d]$ and $\neg p$ holds over $[d_3, d]$.

What do we already know about $A\overline{A}$?

Decidability (in fact, NEXPTIME-completeness) of the future fragment of $A\overline{A}$ (the future modality $\langle A \rangle$ only) over the natural numbers.

D. Bresolin and A. Montanari, A Tableau-based Decision Procedure for Right Propositional Neighborhood Logic, TABLEAUX 2005 (extended and revised version in *Journal of Automated Reasoning*, 2007)

Later extended to full $A\overline{A}$ over the integers (it can be tailored to natural numbers and finite linear orders).

D. Bresolin, A. Montanari, and P. Sala, An Optimal Tableau-based Decision Algorithm for Propositional Neighborhood Logic, STACS 2007

Expressive completeness of $A\overline{A}$ with respect to $\mathrm{FO}^2[<]$

Expressive completeness of $A\overline{A}$ (plus the modal constant π for point-intervals) with respect to the two-variable fragment of first-order logic for binary relational structures over various linearly-ordered domains $\mathrm{FO}^2[<]$ (M. Otto, Journal of Symbolic Logic, 2001).

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- It is far from being trivial to extract a decision procedure from Otto's proof.
- Some meaningful cases are missing (dense linear orders, weakly discrete linear orders).

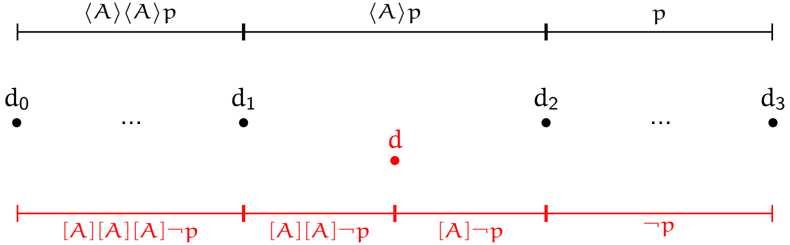
$A\overline{A}$ expressiveness - the formula $\mathrm{ImmediateSucc}$

It can be shown that $A\overline{A}$ is expressive enough to distinguish between satisfiability over the class of all linear orders and the class of dense (resp., discrete) ones.

Let $\mathrm{ImmediateSucc}$ be the $A\overline{A}$ formula

$$\langle A\rangle\langle A\rangle p \wedge [A][A][A]\neg p$$

$\mathrm{ImmediateSucc}$ is satisfiable over the class of all (resp., discrete) linear orders, but it is not satisfiable over dense linear orders.

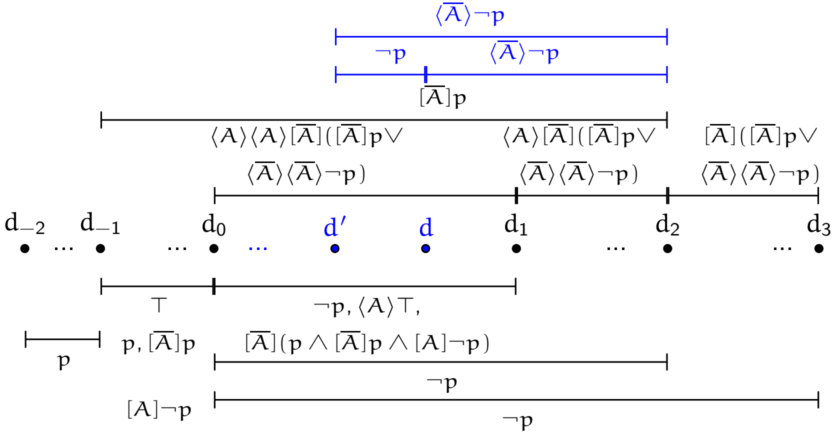


$A\overline{A}$ expressiveness - the formula NoImmediateSucc

Let $\mathrm{NoImmediateSucc}$ be the $A\overline{A}$ formula

$$\langle\overline{A}\rangle\top \wedge [\overline{A}](p \wedge [\overline{A}]p \wedge [A]\neg p) \wedge \langle A\rangle\langle A\rangle[\overline{A}]([\overline{A}]p \vee \langle\overline{A}\rangle\langle\overline{A}\rangle\neg p)$$

$\mathrm{NoImmediateSucc}$ is satisfiable over the class of all (resp., dense) linear orders, but it is not satisfiable over discrete linear orders.



Decidability of $A\overline{A}$ over the class of all linear orders

How to check an $A\overline{A}$ formula φ for satisfiability?

Outline of the proof:

- FROM existence of an interval model for φ
- TO existence of a (possibly infinite) φ-labeled interval structure (STANDARD)
- TO existence of a finite pseudo-model for φ (DIFFICULT)
- TO existence of a tableau for φ with a blocked branch (EASY)

Basic machinery

closure of φ: the set $\mathsf{CL}(\varphi)$ of all subformulae of φ and of their negations

temporal formulae of φ: the set $\mathsf{TF}(\varphi) \subseteq \mathsf{CL}(\varphi)$ of subformulae of the forms $\langle A\rangle\psi$, $[A]\psi$, $\langle\overline{A}\rangle\psi$, and $[\overline{A}]\psi$

maximal set of requests for φ: a subset of $\mathsf{TF}(\varphi)$ such that for every $\langle A\rangle\psi \in \mathsf{TF}(\varphi)$, $\langle A\rangle\psi \in S$ iff $\neg\langle A\rangle\psi \notin S$ (the same for $\langle\overline{A}\rangle\psi$)

φ-atom: a set $A \subseteq \mathsf{CL}(\varphi)$ such that (i) for every $\psi \in \mathsf{CL}(\varphi)$, $\psi \in A$ iff $\neg\psi \notin A$, and (ii) for every $\psi_1 \vee \psi_2 \in \mathsf{CL}(\varphi)$, $\psi_1 \vee \psi_2 \in A$ iff $\psi_1 \in A$ or $\psi_2 \in A$.

We denote by A_φ the set of all φ-atoms.

Interval models and φ-labeled interval structures

Let D be a set of points, $\mathbb{D} = \langle D, < \rangle$ be a linear order on it, and $\mathbb{I}(\mathbb{D})$ be the set of all intervals over $\mathbb{D}$

Interval model: a pair $\mathbf{M} = \langle \mathbb{D}, \mathcal{V} \rangle$, where $\mathbb{D} = \langle D, < \rangle$ and $\mathcal{V} : \mathbb{I}(\mathbb{D}) \mapsto 2^{AP}$

φ-labeled interval structure (φ-LIS): a pair $\mathbf{L} = \langle \mathbb{D}, \mathcal{L} \rangle$, where $\mathcal{L} : \mathbb{I}(\mathbb{D}) \mapsto \mathcal{A}_{\varphi}$ is such that, for every pair $[d_i, d_j], [d_j, d_k] \in \mathbb{I}(\mathbb{D})$ and every $[A]\psi \in CL(\varphi)$, if $[A]\psi \in \mathcal{L}([d_i, d_j])$, then $\psi \in \mathcal{L}([d_j, d_k])$ (the same for $[\overline{A}]\psi$)

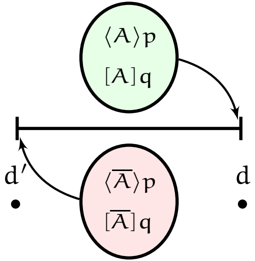
φ-LIS represent *candidate models* (they satisfy local conditions and universal temporal conditions). We must guarantee that existential temporal conditions are satisfied as well: fulfilling φ-LIS

Theorem. φ is satisfiable iff there exists a fulfilling φ-LIS $\mathbf{L} = \langle \mathbb{D}, \mathcal{L} \rangle$ with $\varphi \in \mathcal{L}([d_i, d_j])$ for some $[d_i, d_j] \in \mathbb{I}(\mathbb{D})$

How to make the notion of fulfilling φ-LIS effective?

Given a φ-LIS $\mathbf{L}$ and $d \in D$, we define the sets of future and past requests for d ($\mathrm{REQ}_f^{\mathbf{L}}(d)$ and $\mathrm{REQ}_p^{\mathbf{L}}(d)$, respectively)

Future requests for d



Past requests for d'

We say that a future request $\langle A\rangle\psi$ is fulfilled for d in $\mathbf{L}$ if there exists $\overline{d} \in D$ such that $\psi \in \mathcal{L}([d, \overline{d}])$ (the same for past requests).

We say that d is fulfilled in $\mathbf{L}$ if all its future and past requests ($\mathrm{REQ}^{\mathbf{L}}(d)$) are fulfilled.

The key notion of interval tuple

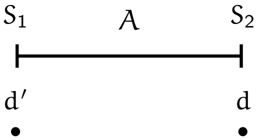
Let φ be an $A\overline{A}$ formula, $\mathcal{A}$ be a φ-atom, and $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathsf{TF}(\varphi)$ be two maximal sets of requests. We say that the triplet $\langle \mathcal{S}_1, \mathcal{A}, \mathcal{S}_2 \rangle$ is an interval-tuple if

(i) for every $[A]\psi \in \mathcal{S}_1$, $\psi \in \mathcal{A}$;

(ii) for every $\langle A \rangle \psi \in \mathsf{TF}(\varphi)$, $\langle A \rangle \psi \in \mathcal{A}$ iff $\langle A \rangle \psi \in \mathcal{S}_2$;

(iii) for every $\psi \in \mathcal{A}$ such that $\langle A \rangle \psi \in \mathsf{TF}(\varphi)$, $\langle A \rangle \psi \in \mathcal{S}_1$.

The same for past operators.



Let $\mathbf{L}$ be a φ-LIS for φ and $d, d' \in \mathbb{D}$. It can be easily shown that $\langle \mathsf{REQ}^{\mathbf{L}}(d), \mathcal{L}([d, d']), \mathsf{REQ}^{\mathbf{L}}(d') \rangle$ is an interval-tuple

From fulfilling φ-LISs to pseudo-models

Let $\mathbf{L}$ be a φ-LIS and $\langle R, A, R' \rangle$ be an interval-tuple. If there exists $[d, d']$ such that $\mathcal{L}([d, d']) = A$, $\mathsf{REQ}^{\mathbf{L}}(d) = R$, and $\mathsf{REQ}^{\mathbf{L}}(d') = R'$, we say that $\langle R, A, R' \rangle$ occurs in $\mathbf{L}$ (at $[d, d']$). Moreover, if $\langle R, A, R' \rangle$ occurs in $\mathbf{L}$ at $[d, d']$ and both d and d' are fulfilled in $\mathbf{L}$, we say that $\langle R, A, R' \rangle$ is fulfilled in $\mathbf{L}$ (via $[d, d']$).

Given a finite φ-LIS $\mathbf{L}$ for φ, we say that $\mathbf{L}$ is a pseudo-model for φ if every interval-tuple $\langle R, A, R' \rangle$ that occurs in $\mathbf{L}$ is fulfilled.

Being $\mathbf{L}$ is a pseudo-model for φ does not guarantee $\mathbf{L}$ to be fulfilling, since $\mathbf{L}$ can feature multiple occurrences of the same interval-tuple, associated with different intervals.

However, it is possible to prove that any pseudo-model can be turned into a fulfilling LIS (for φ).

Decidability

Lemma 1. Given a pseudo-model $\mathbf{L}$ for φ, there exists a fulfilling LIS $\mathbf{L}'$ that satisfies φ.

Lemma 2. Given a formula φ and a fulfilling LIS $\mathbf{L}$ that satisfies it, there exists a pseudo-model $\mathbf{L}'$ for φ, with $|\mathrm{D}'| \leqslant 2 \cdot |\varphi| \cdot 2^{3 \cdot |\varphi| + 1}$.

Theorem. The satisfiability problem for $A\overline{A}$ over the class of all linear orders is decidable.

The decidability proof for $A\overline{A}$ over all linear orders can be tailored to the cases of dense linear orders and (weakly) discrete linear orders.

- dense: we force each point in a pseudo-model for φ to satisfy a *covering* condition which guarantees us that we can always insert a point in between any pair of consecutive points, thus producing a dense model for φ
- discrete: we force each point in a pseudo-model for φ to satisfy a *safety* condition which guarantees us that all points added during the construction of the fulfilling LIS get their (definitive) immediate successor and immediate predecessor in at most one step

How does the proof of Lemma 1 work?

Basic idea: we show how to obtain a fulfilling LIS $\mathbf{L}'$ starting from the pseudo-model $\mathbf{L}$ as the limit of a possibly infinite sequence of pseudo-models $\mathbf{L}_0(=\mathbf{L}), \mathbf{L}_1, \mathbf{L}_2, \ldots$ by fixing defects of points in the current pseudo-model (that is, existential temporal formulae whose requests are not fulfilled) in a principled way.

Points that must be checked for fulfillment are managed by a queue (this guarantees us that all defects are sooner or later fixed).

Initially, the queue consists of all and only the points $\mathrm{d} \in \mathrm{D}$ such that d is not fulfilled in the given pseudo-model $\mathbf{L}$.

A tableau system for $A\overline{A}$ over all linear orders

Basic notions.

A tableau for φ: a special decorated tree $\mathcal{T}$.

We associate a finite linear order $\mathbb{D}_B = \langle D_B, < \rangle$ and a request function $\mathcal{R}eq_B : D_B \mapsto \mathrm{REQ}_\varphi$ with every branch B of $\mathcal{T}$.

Every node n in B is labeled with a pair $\langle [d_i, d_j], A_n \rangle$ such that the triple $\langle \mathcal{R}eq_B(d_i), A_n, \mathcal{R}eq_B(d_j) \rangle$ is an interval-tuple.

The initial tableau for φ consists of a single node (a single branch B) labeled with a pair $\langle [d_0, d_1], A \rangle$, where $\mathbb{D}_B = \{d_0 < d_1\}$ and $\varphi \in A$.

Fulfilling conditions

Given a point $d \in D_B$ and a formula $\langle A \rangle \psi \in \text{REQ}_B(d)$, we say that $\langle A \rangle \psi$ is fulfilled in B for d if there exists a node $n' \in B$ such that n' is labeled with $\langle [d, d'], A_{n'} \rangle$ and $\psi \in A_{n'}$ (same for the past).

Given a point $d \in D_B$, we say that d is fulfilled in B if every $\langle A \rangle \psi$ (resp., $\langle \overline{A} \rangle \psi$) in $\mathcal{R}eq_B(d)$ is fulfilled in B for d.

Let $\mathcal{T}$ be a tableau and B be a branch of $\mathcal{T}$, with $\mathbb{D}_B = \{d_0 < \ldots < d_k\}$.

We denote by $B \cdot n$ the expansion of B with an immediate successor node n and by $B \cdot n_1 | \ldots | n_h$ the expansion of B with h immediate successor nodes $n_1, \ldots, n_h$.

Expansion rules

To expand B, we apply one of the following expansion rules:

$\langle A\rangle$-rule: there exist $d_j \in D_B$ and $\langle A\rangle\psi \in \mathrm{REQ}_B(d_j)$ such that $\langle A\rangle\psi$ is not fulfilled in B for d_j.

- There is not an interval-tuple $\langle \mathcal{R}eq_B(d_j), \mathcal{A}, \mathcal{S}\rangle$, with $\psi \in \mathcal{A}$. We *close* B.
- Let $\langle \mathcal{R}eq_B(d_j), \mathcal{A}, \mathcal{S}\rangle$ be such an interval-tuple. We take a new point d and we expand B with $h = k - j + 1$ immediate successor nodes $n_1, \ldots, n_h$ such that, for every $1 \leqslant l \leqslant h$, $\mathbb{D}_{B\cdot n_l} = \mathbb{D}_B \cup \{d_{j+l-1} < d < d_{j+l}\}$, $n_l = \langle [d_j, d], \mathcal{A}\rangle$, with $\mathrm{REQ}_{B\cdot n_l}(d) = \mathcal{S}$, and $\mathrm{REQ}_{B\cdot n_l}(d') = \mathrm{REQ}_B(d')$ for every $d' \in D_B$.

$\langle \overline{A}\rangle$-rule: symmetric to the $\langle A\rangle$-rule.

Fill-in rule:

- There exist d_i, d_j, with $d_i < d_j$, such that no node in B is decorated with $[d_i, d_j]$, but there exists an interval-tuple $\langle \mathrm{REQ}_B(d_i), \mathcal{A}, \mathrm{REQ}_B(d_j)\rangle$. We expand B with a node $n = \langle [d_i, d_j], \mathcal{A}\rangle$.
- Such an interval-tuple does not exist. We *close* B.

The notion of blocked branch

A node $n = \langle [d_i, d_j], A \rangle$ in a branch B is active if for every predecessor $n' = \langle [d, d'], A' \rangle$ of n in B, the interval-tuples $\langle \mathcal{R}eq_B(d_i), A, \mathcal{R}eq_B(d_j) \rangle$ and $\langle \mathcal{R}eq_B(d), A', \mathcal{R}eq_B(d') \rangle$ are different.
A point $d \in D_B$ is active if there exists an active node n in B such that $n = \langle [d, d'], A \rangle$ or $n = \langle [d', d], A \rangle$, for some $d' \in D_B$ and some atom A.

Let B be a non-closed branch. B is complete if for every $d_i, d_j \in D_B$, with $d_i < d_j$, there exists a node n in B labeled with $n = \langle [d_i, d_j], A \rangle$, for some atom A.
If B is complete, then the pair $\langle \mathbb{D}_B, \mathcal{L}_B \rangle$ such that, for every $[d_i, d_j] \in \mathbb{I}(\mathbb{D}_B)$, $\mathcal{L}_B([d_i, d_j]) = A$ if and only if there exists a node n in B labeled with $\langle [d_i, d_j], A \rangle$, is a LIS.

Let B be a non-closed branch. B is blocked if B is complete and, for every active point $d \in B$, d is fulfilled in B.

Expansion strategy

We start from an initial tableau for φ and we apply the expansion rules to all the non-blocked and non-closed branches B.

The expansion strategy is the following one:

1. Apply the *Fill-in rule* until it generates no new nodes in B.

2. If there exist an active point $\mathrm{d} \in \mathrm{D_B}$ and a formula $\langle A \rangle \psi \in \mathcal{R}eq_{\mathrm{B}}(\mathrm{d})$ such that $\langle A \rangle \psi$ is not fulfilled in B for d, then apply the $\langle A \rangle$*-rule* on d. Go back to step 1.

3. If there exist an active point $\mathrm{d} \in \mathrm{D_B}$ and a formula $\langle \overline{A} \rangle \psi \in \mathcal{R}eq_{\mathrm{B}}(\mathrm{d})$ such that $\langle \overline{A} \rangle \psi$ is not fulfilled in B for d, then apply the $\langle \overline{A} \rangle$*-rule* on d. Go back to step 1.

A tableau $\mathcal{T}$ for φ is final if and only if every branch B of $\mathcal{T}$ is closed or blocked.

Termination, soundness, and completeness

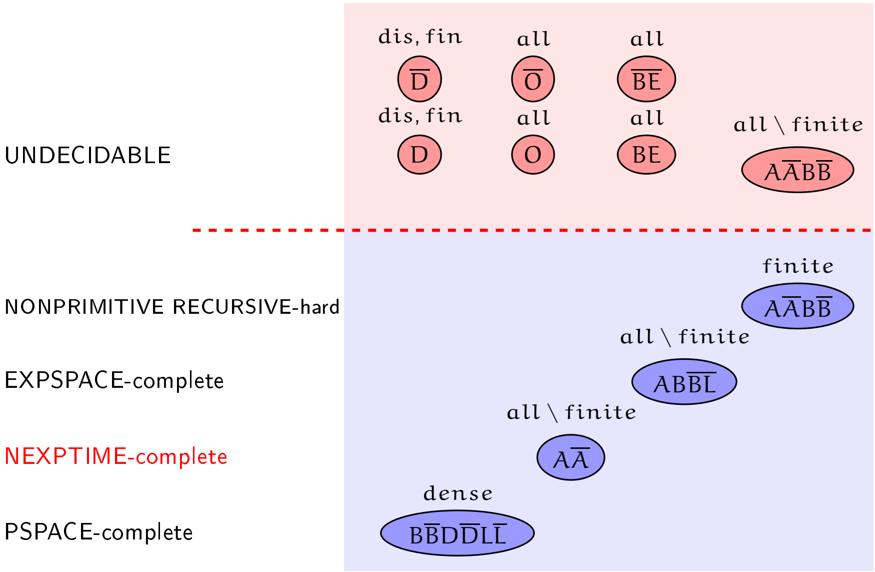
Termination.
Let $\mathcal{T}$ be a final tableau for φ and B be a branch of $\mathcal{T}$. We have that $|B| \leqslant (2 \cdot |\varphi| \cdot 2^{3 \cdot |\varphi| + 1}) \cdot (2 \cdot |\varphi| \cdot 2^{3 \cdot |\varphi| + 1} - 1)/2$.

Soundness.
Let $\mathcal{T}$ be a final tableau for φ. If $\mathcal{T}$ features one blocked branch, then φ is satisfiable over all linear orders.

Completeness.
Let φ be an $A\overline{A}$ formula which is satisfiable over the class of all linear orders. Then there exists a final tableau for it with at least one blocked branch.

The tableau system can be tailored to the dense and discrete cases.

Conclusions: $A\overline{A}$ is (nearly) maximal w.r.to decidability



Future work

- Implementation: a naïve implementation of the tableau system is easy, but computationally unfeasible

- The case of real numbers: there are $A\overline{A}$ formulae which are satisfiable over dense linear orders (in fact, rational numbers), but not over real numbers.

  Open problem: is the satisfiability problem for $A\overline{A}$ over the real numbers decidable?

The formula NoReal

Let NoReal be the $A\overline{A}$ formula

$$p \wedge \langle A\rangle\langle A\rangle q \wedge [G]((p \rightarrow \langle A\rangle p) \wedge (q \rightarrow \langle\overline{A}\rangle q) \wedge$$

$$(p \rightarrow [A]([\overline{A}]p \wedge [\overline{A}][\overline{A}]p)) \wedge (q \rightarrow [\overline{A}]([A]q \wedge [A][A]q)) \wedge$$

$$\neg(p \wedge q) \wedge (\neg p \wedge \neg q \rightarrow \langle\overline{A}\rangle p \wedge \langle A\rangle q)),$$

where $[G]$ is the *universal* operator defined as follows:

$$[G]\psi = \psi \wedge [\overline{A}][\overline{A}][A]\psi \wedge [\overline{A}][A][A]\psi \wedge [A][A][\overline{A}]\psi \wedge [A][\overline{A}][\overline{A}]\psi$$

NoReal is satisfiable over the class of dense linear orders, but it is not satisfiable over the real numbers