Ehrenfeucht-Fraïssé Games: Applications and Complexity

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Outline

Introduction to EF-games

Inexpressivity results for first-order logic

Normal forms for first-order logic

Algorithms and complexity for specific classes of structures

General complexity bounds

Introduction to EF-games

Inexpressivity results for first-order logic

Normal forms for first-order logic

Algorithms and complexity for specific classes of structures

General complexity bounds

Background on finite model theory

Books

- H.-D. Ebbinghaus and J. Flum
  Finite Model Theory
  Springer, 2nd edition, 2005
- L. Libkin
  Elements of Finite Model Theory
  Springer, 2004

Why finite model theory?

Connections with computation - 1

- **Verification**
  finite structures can be coded as
  - words
  - trees
  - graphs

  and thus can be objects of computations

  finite structures can be used to describe finite runs of machines

Why finite model theory?

Connections with computation - 2

- **Database theory**
  the relational model identifies a database with a finite relational structure:

  formulas of a formal language can be viewed as programs in order to evaluate their meaning in a structure

  vice versa, one can express queries of a certain computational complexity in a given formal language

  Genuinely finite queries, e.g.,
  - Has the relation R even cardinality?

Why finite model theory?

Connections with computation - 3

- **Computational complexity**

  a logical account of complexity classes

  As an example, the problem $\mathrm{P} = \mathrm{NP}$ amounts to the question whether two fixed-point logics have the same expressive power in finite structures

Descriptive complexity is a branch of computational complexity theory and finite model theory that characterizes complexity classes by the type of logic needed to express the languages in them (decision problems as languages)

Most theorems fail, one method survives

We focus our attention on first-order (FO) logic

- Results of model theory often do not apply to the finite
    - Gödel's completeness theorem
    - Compactness theorem
    - Löwenheim-Skolem theorem
    - Definability and interpolation results
    - etc.
- Ehrenfeucht-Fraïssé games are an exception

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An application of the compactness theorem

Theorem (Compactness Theorem)

(i) *if ψ is a consequence of Φ, then ψ is a consequence of a finite subset of Φ*

(ii) *if every finite subset of Φ is satisfiable, then Φ is satisfiable*

- Connectivity is not FO-definable over the class of all graphs $\mathcal{G} = (G, E)$
  - The proof is via compactness
  - Assume ϕ defines connectivity
  - ψ_n: "there is no path of length $n + 1$ from c_1 to c_2"
  - Let $T = \{\psi_n \mid n > 0\} \cup \{c_1 \neq c_2, \neg E(c_1, c_2), \phi\}$
  - Every finite subset of T is satisfiable, but T is not

Compactness fails in the finite

- γ_n: "there are at least n distinct elements"
  - $\gamma_n \stackrel{\text{def}}{=} \exists x_1 \cdots \exists x_n \bigwedge_{1 \leqslant i < j \leqslant n} (x_i \neq x_j)$
- $\Gamma = \{ \gamma_n \mid n > 0 \}$
- **General case**: every finite subset of Γ is satisfiable and thus (compactness theorem) Γ is satisfiable, that is, it has an (infinite) model
- **Finite structures**: every finite subset of Γ is satisfiable (it has a finite model), but Γ has no finite model
- Is connectivity definable over all finite graphs? We cannot exploit the compactness theorem to answer the question

Isomorphic and elementarily equivalent structures

Definition (Isomorphic structures)

Two structures $\mathcal{A}$, $\mathcal{B}$, over the same finite vocabulary τ, are isomorphic ($\mathcal{A} \cong \mathcal{B}$) if there is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$, that is, a bijection $\pi : \mathcal{A} \mapsto \mathcal{B}$ preserving relations and constants

Theorem

Every finite structure can be characterized in FO logic up to isomorphism, that is, for every finite structure $\mathcal{A}$ there exists a FO sentence $\varphi_{\mathcal{A}}$ such that, for every $\mathcal{B}$, we have

$$\mathcal{B} \models \varphi_{\mathcal{A}} \text{ iff } \mathcal{A} \cong \mathcal{B}$$

Definition (Elementarily equivalent structures)

Two structures $\mathcal{A}$, $\mathcal{B}$ are elementarily equivalent ($\mathcal{A} \equiv \mathcal{B}$) if they satisfy the same FO sentences

Notation

- Vocabulary: finite set of relation symbols including $=$ (for the sake of simplicity, we restrict ourselves to a purely relational vocabulary; however, all results extend to vocabularies that have constant symbols)

- $\mathcal{A}$ and $\mathcal{B}$ structures on the same vocabulary

- $\vec{a} = a_1, \ldots, a_k \in \mathrm{dom}(\mathcal{A})$
- $\vec{b} = b_1, \ldots, b_k \in \mathrm{dom}(\mathcal{B})$

- $(\mathcal{A}, \vec{a})$: expansion of structure $\mathcal{A}$ by k elements from its universe
- $(\mathcal{B}, \vec{b})$: expansion of structure $\mathcal{B}$ by k elements from its universe

- Configuration: $(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$, with $|\vec{a}| = |\vec{b}|$
  - It represents the relation $\{ (a_i, b_i) \mid 1 \leqslant i \leqslant |\vec{a}| \}$

The notion of quantifier rank

Quantifier rank $\mathrm{qr}(\phi)$ of a FO-formula ϕ = maximum number of nested quantifiers in ϕ:

- if ϕ is atomic then $\mathrm{qr}(\phi) = 0$;
- $\mathrm{qr}(\neg\phi_1) = \mathrm{qr}(\phi_1)$; $\mathrm{qr}(\phi_1 \vee \phi_2) = \max(\mathrm{qr}(\phi_1), \mathrm{qr}(\phi_2))$;
- $\mathrm{qr}(\exists x\, \phi_1) = \mathrm{qr}(\phi_1) + 1$

Example

$\phi = \forall x\, (P(x) \rightarrow \exists y\, Q(x, y) \vee \exists y\, R(y))$ has $\mathrm{qr}(\phi) = 2$

A weakening of elementary equivalence: m-equivalent structures

Definition (m-equivalent structures)

Two structures $\mathcal{A}$ and $\mathcal{B}$ are *m-equivalent*, denoted $\mathcal{A} \equiv_m \mathcal{B}$, with $m \geqslant 0$, if they satisfy the same FO sentences of quantifier rank up to m

The notion of m-equivalence can be easily generalized to expanded structures: $(\mathcal{A}, \vec{a}) \equiv_m (\mathcal{B}, \vec{b})$ if they satisfy the same FO formulas of quantifier rank m with at most $|\vec{a}|$ free variables

A weakening of isomorphism: $\mathfrak{m}$-isomorphic structures - 1

Definition (partial isomorphisms)

$(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ is a partial isomorphism if it is an isomorphism of the substructures induced by $\vec{a}$ and $\vec{b}$, respectively

Let $I_1, \ldots, I_m$ be sets of partial isomorphisms such that, for every k, I_k contains partial isomorphisms which allow k-fold extensions

We can define a weakening of isomorphism: m-isomorphic structures

A weakening of isomorphism: $\mathfrak{m}$-isomorphic structures - 2

Definition (m-isomorphic structures)

Two pairs $(\mathcal{A}, \vec{a})$ and $(\mathcal{B}, \vec{b})$ are m-isomorphic, denoted $(\mathcal{A}, \vec{a}) \cong_m (\mathcal{B}, \vec{b})$, if there are nonempty sets $I_0, I_1, \ldots, I_m$ of partial isomorphisms, each of them extending the partial isomorphism $(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$, such that, for all $k = 1, \ldots, m$,

- (forth property)
  $\forall p \in I_k \forall a \in \mathrm{dom}(\mathcal{A}) \exists b \in \mathrm{dom}(\mathcal{B})(p \cup \{(a, b)\} \in I_{k-1})$
- (back property)
  $\forall p \in I_k \forall b \in \mathrm{dom}(\mathcal{B}) \exists a \in \mathrm{dom}(\mathcal{A})(p \cup \{(a, b)\} \in I_{k-1})$

Theorem (Fraïssé, 1954)

For $m \geqslant 0$, $(\mathcal{A}, \vec{a}) \equiv_m (\mathcal{B}, \vec{b})$ *iff* $(\mathcal{A}, \vec{a}) \cong_m (\mathcal{B}, \vec{b})$

Combinatorial Games

Ehrenfeucht-Fraïssé games are (logical) combinatorial games.

- Combinatorial games:
  - Two opponents
  - Alternate moves
  - No chance
  - No hidden information
  - No loops
  - The player who cannot move loses[1]

E. R. Berlekamp, J. H. Conway, and R. K. Guy
Winning Ways for your mathematical plays
A K Peters LTD, 2nd edition, 2001

[1] In Combinatorial Game Theory (CGT), this is called *normal play* (the opposite rule: "the player who cannot move wins" is called *misère play*, and it gives rise to quite a different theory)

Ehrenfeucht-Fraïssé games (EF-games)

- (Logical) combinatorial games
- The playground: two relational structures $\mathcal{A}$ and $\mathcal{B}$ (over the same finite vocabulary)
- Two players: **I** (Spoiler) and **II** (Duplicator)
- Perfect information
- Move by **I** : select a structure and pick an element in it
- Move by **II** : pick an element in the opposite structure
- Round: a move by **I** followed by a move by **II**
- Game: sequence of rounds
- **II** tries to imitate **I**
- A player who cannot move loses

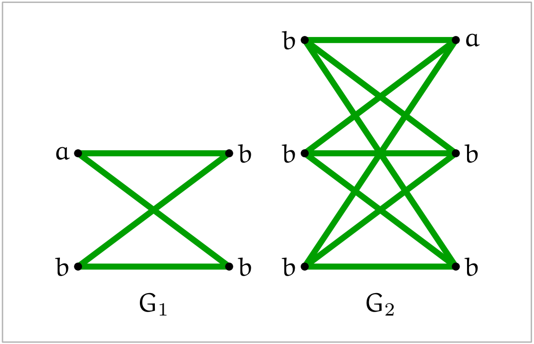
Winning strategies

- A play from $(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ proceeds by extending the initial configuration with the pair of elements chosen by the two players, e.g.,
  - if **I** picks c in $\mathcal{A}$
  - and **II** replies with d in $\mathcal{B}$
  - then the new configuration is $(\mathcal{A}, \vec{a}, c, \mathcal{B}, \vec{b}, d)$
- Ending condition: a player repeats a move or the configuration is not a partial isomorphism

Definition

II has a winning strategy from $(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ if every configuration of the game until an ending configuration is reached is a partial isomorphism, no matter how **I** plays.

An example on graphs



- **II** must respect the adjacency relation...
- ...and pick nodes with the same label as **I** does

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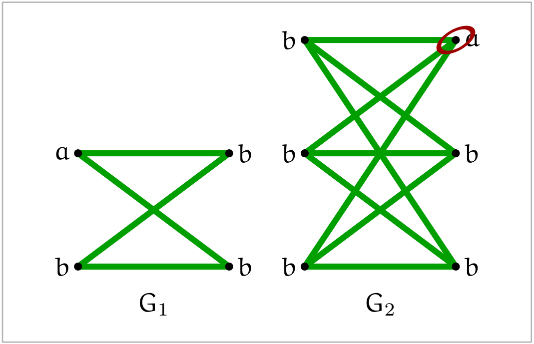

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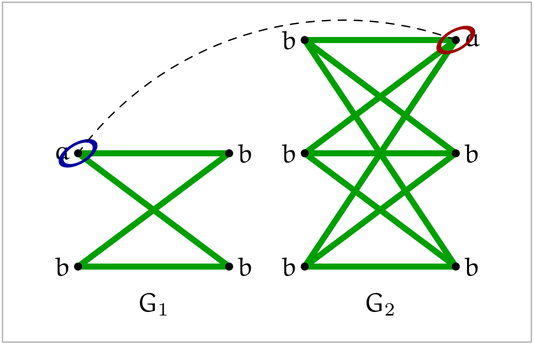

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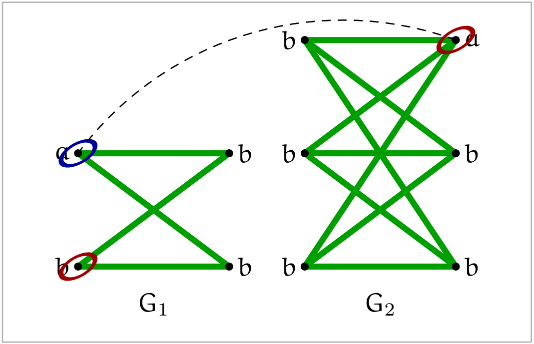

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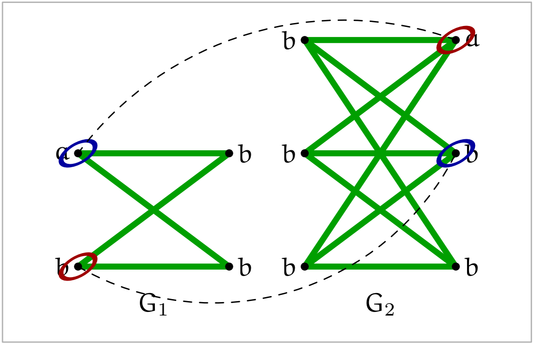

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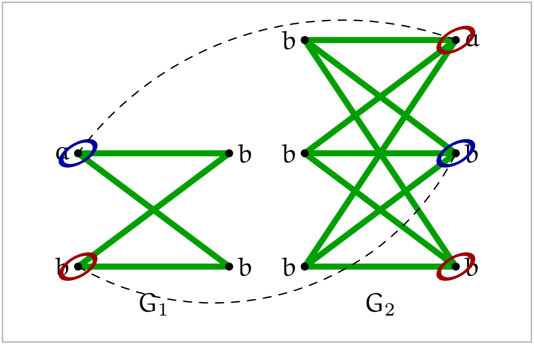


- **II** must respect the adjacency relation. . .
- . . . and pick nodes with the same label as **I** does

Bounded and unbounded games

How long does a game last?

- Bounded game: $\mathcal{G}_m(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ ($\mathcal{G}_m(\mathcal{A}, \mathcal{B})$ if $k = 0$)
  - the number of rounds is fixed: the game ends after m rounds have been played
- Unbounded game: $\mathcal{G}(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ ($\mathcal{G}(\mathcal{A}, \mathcal{B})$ if $k = 0$)
  - the game goes on as long as either a player repeats a move or the current configuration in not partial isomorphism
- **II** wins if and only if the ending configuration is a partial isomorphism

Unbounded games turn out to be useful to compare (finite) structures (comparison games): the remoteness (duration) of an unbounded game as a measure of structure similarity (the notion of remoteness will be formalized later)

Main result

First-order EF-games capture m-equivalence

Theorem (Ehrenfeucht, 1961)

***II** has a winning strategy in* $\mathcal{G}_m(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ *iff* $(\mathcal{A}, \vec{a}) \equiv_m (\mathcal{B}, \vec{b})$

Remarks.

- If two structures $\mathcal{A}$ and $\mathcal{B}$ are m-equivalent for every natural number m, then they are elementarily equivalent
- In finite structures, $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent if and only if they are isomorphic (in general, this is not the case: consider, for instance, $\mathbb{N}$ and the ordered sum $\mathbb{N} \triangleleft \mathbb{Z}$)

Definition (EF-problem)

The EF-problem is the problem of determining whether **II** has a winning strategy in $\mathcal{G}_m(\mathcal{A}, \mathcal{B})$, given $\mathcal{A}$, $\mathcal{B}$ and an integer m

Correspondence between games and formulas

EF-games have a natural logical counterpart which is based on the following simple properties of **II** winning strategies

Given two structures $\mathcal{A}$ and $\mathcal{B}$, a tuple $\vec{a}$ of elements of $\mathrm{dom}(\mathcal{A})$, and a tuple $\vec{b}$ of elements of $\mathrm{dom}(\mathcal{B})$, with $|\vec{a}| = |\vec{b}|$, and $m \geqslant 0$, we have that:

- **II** wins $\mathcal{G}_0(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ iff $(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ is a partial isomorphism
- for every $m > 0$, **II** wins $\mathcal{G}_m(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ iff
    - for all $a \in \mathrm{dom}(\mathcal{A})$, there exists $b \in \mathrm{dom}(\mathcal{B})$ such that **II** wins $\mathcal{G}_{m-1}(\mathcal{A}, \vec{a}, a, \mathcal{B}, \vec{b}, b)$
    - for all $b \in \mathrm{dom}(\mathcal{B})$, there exists $a \in \mathrm{dom}(\mathcal{A})$ such that **II** win $\mathcal{G}_{m-1}(\mathcal{A}, \vec{a}, a, \mathcal{B}, \vec{b}, b)$
- for every $m \geqslant 0$, if **II** wins the game $\mathcal{G}_m(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$, then **II** wins the game $\mathcal{G}'_m(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$, for all $m' < m$

From games to formulas: Hintikka formulas

Definition (Hintikka formulas)

Given a structure $\mathcal{A}$, a tuple $\vec{a}$ of elements of A, with $|\vec{a}| = k$, and a tuple $\vec{x}$ of variables $x_1, \ldots, x_k$, let

$$\varphi^0_{(\mathcal{A},\vec{a})}(\vec{x}) \stackrel{\text{def}}{=} \bigwedge_{\substack{\varphi(\vec{x}) \text{ atomic} \\ (\mathcal{A},\vec{a}) \models \varphi(\vec{x})}} \varphi(\vec{x}) \quad \wedge \quad \bigwedge_{\substack{\varphi(\vec{x}) \text{ atomic} \\ (\mathcal{A},\vec{a}) \models \neg\varphi(\vec{x})}} \neg\varphi(\vec{x})$$

and, for $m \geqslant 0$,

$$\varphi^{m+1}_{(\mathcal{A},\vec{a})}(\vec{x}) \stackrel{\text{def}}{=} \bigwedge_{a \in A} \exists x_{k+1}\, \varphi^m_{(\mathcal{A},\vec{a},a)}(\vec{x}, x_{k+1}) \wedge \forall x_{k+1} \bigvee_{a \in A} \varphi^m_{(\mathcal{A},\vec{a},a)}(\vec{x}, x_{k+1}).$$

For each m, $\varphi^m_{(\mathcal{A},\vec{a})}(\vec{x})$ is called the m-*Hintikka formula*.

From games to formulas: Hintikka formulas (cont.)

The Hintikka formula $\varphi^0_{(\mathcal{A},\vec{a})}(\vec{x})$ describes the isomorphism type of the substructure of $\mathcal{A}$ induced by $\vec{a}$.

In general, $\varphi^m_{(\mathcal{A},\vec{a})}(\vec{x})$ describes to which isomorphism types the tuple $\vec{a}$ can be extended in m steps by adding one element in each step. Since the vocabulary is finite, the above conjunctions and disjunctions are finite even if the structure is infinite.

Theorem (Ehrenfeucht, 1961 - cont.)

For any given $(\mathcal{A}, \vec{a})$, $(\mathcal{B}, \vec{b})$, *and* $m \geqslant 0$, *we have*

$$\textit{\textbf{II} has a winning strategy in } \mathcal{G}_m(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b}) \iff$$

$$(\mathcal{B}, \vec{b}) \models \varphi^m_{(\mathcal{A},\vec{a})}(\vec{x}) \iff (\mathcal{A}, \vec{a}) \equiv_m (\mathcal{B}, \vec{b})$$

Distributive normal form

Hintikka formulas are the basis of a normal form for FO formulas:

- the class of structures which satisfies a given FO formula $\varphi(\vec{x})$ of quantifier rank m must be a union of $\equiv_m$-classes
- each $\equiv_m$-class is defined by a Hintikka formula
- hence, $\varphi(\vec{x})$ is logically equivalent to the (finite) disjunction of those Hintikka formulas which define these $\equiv_m$-classes (distributive normal form for FO logic)

FO definability

A winning strategy for **I** in $\mathcal{G}_m(\mathcal{A}, \mathcal{B})$ can be converted into a FO sentence of quantifier rank at most m that is true in exactly one of $\mathcal{A}$ and $\mathcal{B}$ (the Hintikka formula $\varphi^m_{(\mathcal{A},\vec{a})}(\vec{x})$ or the Hintikka formula $\varphi^m_{(\mathcal{B},\vec{b})}(\vec{x})$).

A characterization of FO-definable (FO-axiomatizable) classes

- A class $\mathcal{K}$ of structures (on the same finite vocabulary) is FO-definable if and only if there is $m \in \mathbb{N}$ such that **I** has a winning strategy whenever $\mathcal{A} \in \mathcal{K}$ and $\mathcal{B} \notin \mathcal{K}$.

The same characterization holds in the finite case (classes of finite structures) – the same argument applies.

FO undefinability

FO-undefinable classes of structures

- A class $\mathcal{K}$ of structures is not FO-definable if and only if, for all $m \in \mathbb{N}$, there are $\mathcal{A} \in \mathcal{K}$ and $\mathcal{B} \notin \mathcal{K}$ such that **II** has a winning strategy in $\mathcal{G}_m(\mathcal{A}, \mathcal{B})$.

Example

Let $\mathcal{L}_k \stackrel{\text{def}}{=} (\{1, \ldots, k\}, <)$. It is possible to show that

$$n, p \geqslant 2^m - 1 \Rightarrow \textbf{II} \text{ wins } \mathcal{G}_m(\mathcal{L}_n, \mathcal{L}_p)$$

"The class of linear orderings of even cardinality is not FO-definable": given m, choose $\tilde{n} = 2^m$ and $\tilde{p} = 2^m + 1$; **II** wins $\mathcal{G}_m(\mathcal{L}_{\tilde{n}}, \mathcal{L}_{\tilde{p}})$ (i.e., $\mathcal{L}_{\tilde{n}} \equiv_m \mathcal{L}_{\tilde{p}}$).

Other applications will be given later (inexpressivity results for FO logic).

From differentiating formulas to games

- Let $\mathcal{A}$ and $\mathcal{B}$ be fixed
- Let ϕ be a formula with quantifier rank m
- Let $\mathcal{A} \models \phi$ but $\mathcal{B} \not\models \phi$
- Repeat m times:

❶ If $\phi = \forall x_1\, \psi$, let $\phi \leftarrow \neg\phi$ and swap $\mathcal{A}$ and $\mathcal{B}$

- So, ϕ holds in $\mathcal{A}$ but not in $\mathcal{B}$ and its first quantifier is $\exists$

❷ Let $\psi \leftarrow \psi\{x_1/\bar{c}_1\}$, with $\bar{c}_1$ a fresh constant symbol

❸ Let **I** pick a_1 in $\mathcal{A}$ such that $(\mathcal{A}, a_1) \models \psi[\bar{c}_1/a_1]$ (since $\mathcal{A} \models \phi$, such an a_1 must exist)

❹ Whatever b_1 **II** chooses in $\mathcal{B}$, $(\mathcal{B}, b_1) \not\models \psi[\bar{c}_1/b_1]$

❺ Let $\mathcal{A} \leftarrow (\mathcal{A}, a_1)$, $\mathcal{B} \leftarrow (\mathcal{B}, b_1)$ and $\phi \leftarrow \psi$

- Switching between models is encoded in ϕ as quantifier alternations (step 1)

Example

Consider the formula for density:
$\phi = \forall x_1 \forall x_2 \exists x_3\, (x_1 < x_2 \to x_1 < x_3 < x_2)$,
which holds in $(\mathbb{Q}, <)$ but not in $(\mathbb{Z}, <)$.

(step 1) $\phi \leftarrow \exists x_1 \exists x_2 \forall x_3\, (x_1 < x_2 \wedge \neg(x_1 < x_3 < x_2))$

(step 2) $\psi \leftarrow \exists x_2 \forall x_3\, (x_1 < x_2 \wedge \neg(x_1 < x_3 < x_2))\{x_1/\bar{c}_1\} =$
$\exists x_2 \forall x_3\, (\bar{c}_1 < x_2 \wedge \neg(\bar{c}_1 < x_3 < x_2))$

(step 3) **I** chooses z in $(\mathbb{Z}, <)$ such that
$(\mathbb{Z}, <, z) \models \psi\, [\bar{c}_1/z]$

(step 4) **II** replies q in $(\mathbb{Q}, <)$ such that
$(\mathbb{Q}, <, q) \not\models \psi\, [\bar{c}_1/q]$

(step 2) $\psi \leftarrow \forall x_3\, (\bar{c}_1 < x_2 \wedge \neg(\bar{c}_1 < x_3 < x_2))\{x_2/\bar{c}_2\} =$
$\forall x_3\, (\bar{c}_1 < \bar{c}_2 \wedge \neg(\bar{c}_1 < x_3 < \bar{c}_2))$

Example (cont.)

(step 3) **I** chooses $z+1$ in $(\mathbb{Z}, <, z)$ such that

$(\mathbb{Z}, <, z, z+1) \models \psi\,[\bar{c}_1/z, \bar{c}_2/z+1]$

(step 4) **II** replies with $q' > q$ in $(\mathbb{Q}, <, q)$ (otherwise it loses immediately) such that

$(\mathbb{Q}, <, q, q') \not\models \psi\,[\bar{c}_1/q, \bar{c}_2/q']$

(step 1) $\phi \leftarrow \exists x_3\,(\bar{c}_1 < \bar{c}_2 \rightarrow (\bar{c}_1 < x_3 < \bar{c}_2))$

(step 2) $\psi \leftarrow \bar{c}_1 < \bar{c}_2 \rightarrow (\bar{c}_1 < x_3 < \bar{c}_2)\{x_3/\bar{c}_3\} = \bar{c}_1 < \bar{c}_2 \rightarrow (\bar{c}_1 < \bar{c}_3 < \bar{c}_2)$

(step 3) **I** chooses $q + \frac{q'-q}{2}$ in $(\mathbb{Q}, <, q)$ such that

$(\mathbb{Q}, <, q, q', q + \frac{q'-q}{2}) \models \bar{c}_1 < \bar{c}_2 \rightarrow \bar{c}_1 < \bar{c}_3 < \bar{c}_2\,[\bar{c}_1/q, \bar{c}_2/q', \bar{c}_3/q+(\frac{q'-q}{2})]$

Example (cont.)

(step 4) Of course, whatever z' **II** chooses, we have $(\mathbb{Z}, <, z, z+1, z') \not\models \bar{c}_1 < \bar{c}_2 \to \bar{c}_1 < \bar{c}_3 < \bar{c}_2\, [\bar{c}_1/z, \bar{c}_2/z+1, \bar{c}_3/z']$

(game over) The resulting mapping from $\mathbb{Q}$ to $\mathbb{Z}$:

$$\begin{array}{rcl} q & \mapsto & z \\ q' & \mapsto & z+1 \\ q + \dfrac{q'-q}{2} & \mapsto & z' \end{array}$$

is not a partial isomorphism, so **I** wins

Labelled linear orders and congruence lemmas

A word w over an alphabet $\Sigma = \{a_1, \ldots, a_k\}$ can be represented by the structure $\underline{w} = (\{1, \ldots, |w|\}, <, P_1^w, \ldots, P_k^w)$, where P_i^w, for $i = 1, \ldots, k$, are unary relations such that $j \in P_i^w$ if and only if the j-th letter of w is a_i

A formal language $L \subseteq \Sigma^+$ is first-order definable if there exists a first-order sentence φ in the signature $\{<, P_1^w, \ldots, P_k^w\}$ such that $L = \{w \in \Sigma^+ \mid \underline{w} \models \varphi\}$

A formal language $L \subseteq \Sigma^+$ is star-free if it can be obtained from finite languages by applying Boolean operations and concatenation (no Kleene star $*$)

McNaughton-Papert theorem

Theorem (McNaughton-Papert)
A formal language is first-order definable iff it is star-free

The difficult direction of the proof of McNaughton-Papert theorem is from left to right

It can be proved by induction on the quantifier rank of the formula

The essential point in the induction step (dealing with the existential quantifier) is the proof of the congruence lemma

The congruence lemma

Lemma (Congruence lemma)

If $\underline{u} \equiv_m \underline{u'}$ and $\underline{v} \equiv_m \underline{v'}$, then $\underline{uv} \equiv_m \underline{u'v'}$

In game-theoretic terms, we need to prove that if Duplicator has winning strategies for games $\mathcal{G}_m(\underline{u}, \underline{u'})$ and $\mathcal{G}_m(\underline{v}, \underline{v'})$, then he/she has a winning strategy for the game $\mathcal{G}_m(\underline{uv}, \underline{u'v'})$

To win the game $\mathcal{G}_m(\underline{uv}, \underline{u'v'})$, it suffices to compose the two strategies for $\mathcal{G}_m(\underline{u}, \underline{u'})$ and $\mathcal{G}_m(\underline{v}, \underline{v'})$ in the obvious way: on $\underline{u}$ and $\underline{u'}$ use the strategy for $\mathcal{G}_m(\underline{u}, \underline{u'})$ and on $\underline{v}$ and $\underline{v'}$ use the strategy for $\mathcal{G}_m(\underline{v}, \underline{v'})$

Applications of EF-games

EF-games have been exploited to prove some basic results about (the expressive power of) FO logic:

- Hanf's theorem
- Sphere lemma
- Gaifman's theorem

EF-games have been extensively used to prove negative expressivity results (sufficient conditions that guarantee a winning strategy for **II** suffice)

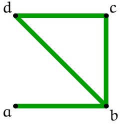
Gaifman's theorem and normal forms for FO logic

Gaifman graph

- Gaifman graph $G(\mathcal{A})$ of a structure $\mathcal{A}$: undirected graph $(\text{dom}(\mathcal{A}), E)$ where $(a, b) \in E$ iff a and b occur in the same tuple of some relation of $\mathcal{A}$
- If $\mathcal{A}$ itself is a (directed) graph, then $G(\mathcal{A})$ is (the undirected version of) $\mathcal{A}$ itself, plus all self-loops
- The *degree* of a node a is the number of nodes $b (\neq a)$ such that $(a, b) \in E$ (the degree of G is the maximum of the degrees of its nodes)
- $\delta(a, b)$: length of the shortest path between a and b in $G(\mathcal{A})$ (if there is not such a path, $\delta(a, b) = \infty$)

Example

$\mathcal{A} = (\{a, b, c, d\}, R, S)$, $R = \{(a, b)\}$, $S = \{(b, c, d)\}$
$\delta(a, c) = \delta(a, d) = 2$



r-sphere and r-neighbourhood

Definition (r-sphere)

Let $\mathcal{A}$ be a structure with domain A, $a \in A$, and $r \in \mathbb{N}$. The r-sphere of a (in $\mathcal{A}$), denoted $S_r^{\mathcal{A}}(a)$, is defined as follows:

$$S_r^{\mathcal{A}}(a) \stackrel{\text{def}}{=} \{\, b \in A \mid \delta(a, b) \leqslant r \,\}.$$

The notion of r-sphere can be extended to a vector $\vec{a} = a_1 \ldots a_s$ (r-sphere $S_r^{\mathcal{A}}(\vec{a})$):

$$S_r^{\mathcal{A}}(\vec{a}) \stackrel{\text{def}}{=} \{\, b \in A \mid \delta(\vec{a}, b) \leqslant r \,\} = S_r^{\mathcal{A}}(a_1) \cup \ldots \cup S_r^{\mathcal{A}}(a_s).$$

Definition (r-neighbourhood)

The r-neighbourhood $\mathcal{N}_r^{\mathcal{A}}(\vec{a})$ is the substructure of $\mathcal{A}$ induced by $S_r^{\mathcal{A}}(\vec{a})$.

If we restrict ourselves to graphs of degree $\leqslant d$ for some fixed d, there are, for any $r > 0$, only finitely many possible isomorphism types of r-spheres.

Hanf's theorem

- $\mathcal{A} \leftrightarrows_r \mathcal{B}$: there is a bijection $f\colon A \to B$ such that $\mathcal{N}_r^{\mathcal{A}}(a) \cong \mathcal{N}_r^{\mathcal{B}}(f(a))$ for every $a \in A$

The relation $\mathcal{A} \leftrightarrows_r \mathcal{B}$ states that *locally* $\mathcal{A}$ and $\mathcal{B}$ look the same.

Theorem (Hanf, 1965)

Let $\mathcal{A}$ and $\mathcal{B}$ be two structures such that, for any $r \in \mathbb{N}$, each r-sphere in $\mathcal{A}$ or $\mathcal{B}$ contains finitely many elements. Then, $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent if $\mathcal{A} \leftrightarrows_r \mathcal{B}$ for every $r \in \mathbb{N}$.

- Hanf's result does not hold if the Gaifman graph of (at least) one structure has infinite degree, e.g., the usual ordering relation on natural numbers

From the infinite case to the finite one

- Hanf's theorem is of interest only for infinite structures: as we already pointed out, two finite structures are elementarily equivalent if and only if they are isomorphic

- A weakened version of Hanf's theorem, called sphere theorem, provides a sufficient condition for $\mathfrak{m}$-equivalence (instead of a sufficient condition for elementary equivalence) and it turns out to be of interest for finite structures

- The proofs of both Hanf's theorem and sphere theorem use Fraïssé's theorem

Sphere theorem

- $\mathcal{A} \leftrightarrows_r^t \mathcal{B}$: isomorphic r-neighbourhoods occur the same number of times in both structures (that is, they have the same multiplicity) or they occur more than t times in both structures

Theorem (Sphere theorem)

Given $\mathcal{A}$ and $\mathcal{B}$ with degree at most d and $m \in \mathbb{N}$, if $\mathcal{A} \leftrightarrows_r^t \mathcal{B}$ for $r = 3^{m+1}$ and $t = m \cdot d^{3^{m+1}}$, then $\mathcal{A} \equiv_m \mathcal{B}$

- For all m there are r and t such that $\leftrightarrows_r^t$ is finer than $\equiv_m$ with respect to the class of structures with degree $\leqslant d$
- Strong hypotheses (it is a sufficient condition)
  - isomorphic neighbourhoods
  - uniform threshold for all neighbourhood sizes
  - scattering of neighbourhoods is not taken into account

Sphere theorem: a proof

Thanks to Fraïssé's theorem, it suffices to show that $(\mathcal{A}, \vec{a}) \cong_m (\mathcal{B}, \vec{b})$.

The required sequence of sets $I_0, \ldots, I_m$ of partial isomorphisms is defined as follows: $p = \{ (a_1, b_1), \ldots, (a_{m-k}, b_{m-k}) \} \in I_k$ iff

$$\mathcal{N}^{\mathcal{A}}_{3^k}(a_1, \ldots, a_{m-k}) \cong \mathcal{N}^{\mathcal{B}}_{3^k}(b_1, \ldots, b_{m-k})$$

To prove the forth property (a similar argument holds for the back property), we assume that such a condition holds for p and we show that, for every possible choice of $a (= a_{m-(k-1)}) \in A$, we can find $b (= b_{m-(k-1)}) \in B$ such that:

$$\mathcal{N}^{\mathcal{A}}_{3^{k-1}}(a_1, \ldots, a_{m-(k-1)}) \cong \mathcal{N}^{\mathcal{B}}_{3^{k-1}}(b_1, \ldots, b_{m-(k-1)})$$

Sphere theorem: a proof (cont.)

We must distinguish two cases:

- if $a \in S^{\mathcal{A}}_{2/3 \cdot 3^k}(a_i)$ for some a_i, then we may choose a corresponding b from $S^{\mathcal{B}}_{2/3 \cdot 3^k}(b_i)$ ($S^{\mathcal{A}}_{3^{k-1}}(a)$ is contained in $S^{\mathcal{A}}_{3^k}(a_i)$ and $S^{\mathcal{B}}_{3^{k-1}}(b)$ is contained in $S^{\mathcal{B}}_{3^k}(b_i)$, and thus $\mathcal{N}^{\mathcal{A}}_{3^{k-1}}(a) \cong \mathcal{N}^{\mathcal{B}}_{3^{k-1}}(b)$)
- otherwise, $S^{\mathcal{A}}_{3^{k-1}}(a)$ (of some isomorphism type σ) is disjoint from $S^{\mathcal{A}}_{3^{k-1}}(a_i)$, for $i = 1, \ldots, m-k$. From $\mathcal{A} \leftrightarrows^t_r \mathcal{B}$, with $r = 3^{m+1}$ and $t = m \cdot d^{3^{m+1}}$, it follows that the number of occurrences of spheres of type σ in $\mathcal{B}$ is large enough to guarantee that we may find one which is disjoint from $S^{\mathcal{B}}_{3^{k-1}}(b_i)$, for $i = 1, \ldots, m-k$

By sphere lemma and distributive normal form, any FO formula is equivalent (over graphs of degree $\leqslant d$) to a Boolean combination of statements of the form "there exist $\geqslant k$ occurrences of spheres of types σ": FO logic can only express local properties of graphs.

References for Hanf's and Sphere theorems

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  The Theory of Model, 1965
- W. Thomas
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  Proc. 18th ICALP, LNCS 510, 1991
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- R. Fagin, L. J. Stockmeyer, and M. Y. Vardi
  On monadic NP vs monadic co-NP
  Information and Computation, 1995

Introduction to EF-games

Inexpressivity results for first-order logic

Normal forms for first-order logic

Algorithms and complexity for specific classes of structures

General complexity bounds

Preliminaries

- From now on, we focus on (finite) structures over finite relational vocabularies
- σ-structure: (finite) structure over the vocabulary σ
- Every vocabulary implicitly contains $=$
- σ-formula: formula using extra-logical symbols from σ
- $\mathrm{STRUCT}[\sigma]$: set of all finite σ-structures
- All classes of structures we consider are assumed to be closed under isomorphism, that is, if $h\colon A \to B$ is an isomorphism between $\mathcal{A}$ and $\mathcal{B}$, then $\mathcal{A}$ belongs to a class of structures $\mathcal{K}$ if and only if $\mathcal{B} \in \mathcal{K}$

L. Libkin
Elements of Finite Model Theory, Springer, 2004

K. Doets
Introduction to Ehrenfeucht's Game, 2002

Theories

Definition

Let σ be a vocabulary. A (first-order) *(σ-)theory* T is a set of σ-sentences closed w.r.t. entailment (logical consequence), that is, T is a theory if and only if, for every σ-sentence ϕ,

$$T \models \phi \quad \Leftrightarrow \quad \phi \in T.$$

- The minimal σ-theory is the set of all valid sentences
- The maximal theory (in any language) is set of all sentences: it is the only inconsistent theory

Theories of classes of models

Definition

Let $\mathcal{K}$ be a class of σ-structures. The *theory of* $\mathcal{K}$ is the set

$$\mathrm{Th}\,\mathcal{K} \stackrel{\text{def}}{=} \{\, \phi \mid \forall \mathcal{A} \in \mathcal{K}.\, \mathcal{A} \models \phi \,\}.$$

We will write $\mathrm{Th}\,\mathcal{A}$ for $\mathrm{Th}\{\mathcal{A}\}$ when $\mathcal{A}$ is a single structure.

Lemma

$\mathrm{Th}\,\mathcal{K}$ *is a theory.*

- For a set of σ-sentences Γ, let $Mod(\Gamma)$ denote the set of all (finite and infinite) models of Γ
- Then, $\mathrm{Th}\, Mod(\Gamma)$ is the set of sentences that are true in every model of Γ
- $\mathrm{Cn}\,\Gamma \stackrel{\text{def}}{=} \{\, \phi \mid \Gamma \models \phi \,\} = \mathrm{Th}\, Mod(\Gamma)$
- A set of sentences Γ is a theory if and only if $\Gamma = \mathrm{Cn}\,\Gamma$

Complete theories

Definition

A theory T is *complete* if, for every sentence ϕ, either $\phi \in T$ or $\neg\phi \in T$.

- For every σ-structure $\mathcal{A}$, the theory $\mathrm{Th}\,\mathcal{A}$ is always complete
- In general, a theory $\mathrm{Th}\,\mathcal{K}$ of a class $\mathcal{K}$ of models is complete if and only if $\mathcal{A} \equiv \mathcal{B}$ for any $\mathcal{A}, \mathcal{B} \in \mathcal{K}$, that is, a theory T is complete if and only if any two of models of T are elementarily equivalent

Axiomatizable theories

Definition

A theory T is *(recursively) axiomatizable* if there is a decidable set of sentences Γ such that $\mathrm{T} = \mathrm{Cn}\,\Gamma$. A theory T is *finitely axiomatizable* if there is a finite set of sentences Γ_0 such that $\mathrm{T} = \mathrm{Cn}\,\Gamma_0$.

Lemma

A theory is axiomatizable if and only if it is r.e. If a theory is axiomatizable and complete, then it is decidable.

Definition

A theory T is *countably categorical* if any two countable models of T are isomorphic.

- If T is countably categorical, then, by the Löwenheim-Skolem theorem, any two models of T are elementarily equivalent, hence T is complete
- So, a countably categorical theory which is axiomatizable is decidable

Definability of relations

Definition

Let $\mathcal{A}$ be a σ-structure. A σ-formula ϕ with $n > 0$ free variables $x_1, \ldots, x_n$ *defines* the n-ary relation

$$R_\phi^\mathcal{A} = \{ (a_1, \ldots, a_n) \mid \mathcal{A} \models \phi\{x_1/a_1, \ldots, x_n/a_n\}\}.$$

- A relation S is *definable* in a σ-structure $\mathcal{A}$ if there is a σ-formula ϕ such that $R_\phi^\mathcal{A} = S$
- This form of definability is relative to a given structure

Example

The successor relation $S = \{ (n, n+1) \mid n \in \mathbb{N} \}$ is definable in $(\mathbb{N}, <)$

- Let $\phi(x, y) \stackrel{\text{def}}{=} (x < y) \wedge \neg \exists z \, (x < z \wedge z < y)$
- Then $S = R_{\phi(x,y)}^{(\mathbb{N},<)}$

Definability (of relations) in $(\mathbb{N}, <)$

Lemma

A set is definable in $(\mathbb{N}, <)$ iff it is finite or co-finite

$(\Leftarrow)$

- $\pi_n(x)$ ("x has exactly n predecessors") is definable
    - $\pi_0(x) \stackrel{\text{def}}{=} \neg\exists y\,(y < x)$, $\pi_1(x) \stackrel{\text{def}}{=} \exists z\,(\pi_0(z) \wedge s(z,x))$ (where s is a $<$-formula defining the successor), ...
- $A \subseteq_{\text{fin}} \mathbb{N}$ can be defined by $\phi_A(x) \stackrel{\text{def}}{=} \bigvee_{n \in A} \pi_n(x)$
- $\mathbb{N} \setminus A$ can be defined by $\neg\phi_A(x)$

Definability (of relations) in $(\mathbb{N}, <)$ (cont.)

$(\Rightarrow)$

- Let $\phi(x)$ define a set which is neither finite nor co-finite
- $(\mathbb{N}, <) \models \forall x \exists y\, (x < y \wedge \phi(y)) \wedge \forall x \exists y\, (x < y \wedge \neg\phi(y))$
- But $\mathbb{N} \equiv \mathbb{N} \triangleleft \mathbb{Z}$
- Hence, there is a in the $\mathbb{Z}$-part of $\mathbb{N} \triangleleft \mathbb{Z}$ that satisfies $\phi(x)$ and there is b in the $\mathbb{Z}$-part of $\mathbb{N} \triangleleft \mathbb{Z}$ that satisfies $\neg\phi(x)$
- But there is an automorphism of $\mathbb{N} \triangleleft \mathbb{Z}$ mapping a onto b
- A contradiction arises

Corollary

The set of even/odd natural numbers is not FO-definable in $(\mathbb{N}, <)$

Definability of σ-structures

Definition

A σ-sentence ψ *defines* the class $\mathcal{C}_\psi$ of models in which it is true, that is,

$$\mathcal{C}_\psi = \{ \mathcal{A} \mid \mathcal{A} \in \mathsf{STRUCT}[\sigma] \wedge \mathcal{A} \models \psi \}$$

- A given class of structures $\mathcal{D}$ is *definable* if there is a σ-sentence ψ such that $\mathcal{D} = \mathcal{C}_\psi$

Definition

A σ-sentence ψ *defines the class* $\mathcal{P}$ *relative to a class* $\mathcal{K}$ of σ-structures if (and only if)

$$\mathcal{K} \cap \mathcal{C}_\psi = \mathcal{P}$$

Queries

Definition

Let $m > 0$. An m-ary query on a class $\mathcal{K} \subseteq \mathrm{STRUCT}[\sigma]$ is a mapping Q that associates any σ-structure $\mathcal{A} \in \mathcal{K}$ with an m-ary relation over its universe A such that Q is closed under isomorphism, that is, if $h\colon A \to B$ is an isomorphism between $\mathcal{A}$ and $\mathcal{B}$ and Q is an m-ary query, then $(a_1, \ldots, a_m) \in Q(\mathcal{A})$ if and only if $(h(a_1), \ldots, h(a_m)) \in Q(\mathcal{B})$

Definition

A Boolean query on $\mathcal{K}$ is a mapping (closed under isomorphism) which assigns a value in $\{$*true*, *false*$\}$ to any given σ-structure $\mathcal{A} \in \mathcal{K}$

- Uniform definability over a class of structures
- Binary query $\neq$ Boolean query
- A Boolean query is a statement of a property of a class
    - E.g., connectivity of graphs

Queries: examples

Example

Let $\mathcal{G}$ be the class of finite graphs and let $G = (V, E) \in \mathcal{G}$ be a finite graph. The following are queries on $\mathcal{G}$:

1. "transitive closure of a graph" (binary query):

$$\mathrm{TC}(G) = \{\, (s, t) \in V \times V \mid \text{there is a path from } s \text{ to } t \,\};$$

2. "elements of degree m" (unary query):

$$D_m(G) = \{ v \in V \mid v \text{ has degree } m \};$$

3. "connectivity" (Boolean query):

$$\mathrm{CONN}(G) = \begin{cases} \textit{true} & G \text{ is connected;} \\ \textit{false} & \text{otherwise.} \end{cases}$$

Definability of queries

Definition

Let $m > 0$, $\mathfrak{L}$ be a logic, and $\mathcal{K}$ be a class of σ-structures. An m-ary query Q on $\mathcal{K}$ is $\mathfrak{L}$-*definable* if there is a σ-formula ϕ of $\mathfrak{L}$ with m free variables such that for every $\mathcal{A} \in \mathcal{K}$

$$Q(\mathcal{A}) = R_{\phi}^{\mathcal{A}}$$

Definition

Let $\mathfrak{L}$ be a logic and $\mathcal{K}$ be a class of σ-structures. A Boolean query Q on $\mathcal{K}$ is $\mathfrak{L}$-*definable* if there is a σ-sentence ψ of $\mathfrak{L}$ such that

$$\{\mathcal{A} \mid \mathcal{A} \in \mathcal{K} \land Q(\mathcal{A}) = \mathit{true}\} = \mathcal{C}_{\psi} \cap \mathcal{K}$$

First-order logic is too strong

- Any finite structure can be defined by a single sentence (up to isomorphism)

Example

Given a finite graph $G = (V, E)$, with $|V| = n$,

$$\exists x_1 \cdots \exists x_n \Big(\bigwedge_{i \neq j} \neg(x_i = x_j) \wedge (\forall y \bigvee_i (x_i = y))$$

$$\wedge \, (\bigwedge_{(v_i, v_j) \in E} E(x_i, x_j)) \wedge \bigwedge_{(v_i, v_j) \notin E} \neg E(x_i, x_j) \Big)$$

defines G.

- Every class of finite structures can be characterized by a set of sentences (up to isomorphism)
- Elementary equivalence is the same as isomorphism in the finite

First-logic is too strong (cont.)

Lemma

Let σ be a finite vocabulary. Every class $\mathcal{K}$ of finite σ-structures is definable by a set of σ-sentences.

- For every fixed $n > 0$, there is a only *finite* number of pairwise non-isomorphic σ-structures with n elements in $\mathcal{K}$ (because σ is finite)
- Let $\{\mathcal{A}_1, \ldots, \mathcal{A}_k\}$ be a maximal set of such structures
- Let $\phi_{\mathcal{A}_i}$ be the sentence that defines $\mathcal{A}_i$
- Let $\phi_{=n}$ be a σ-sentence that expresses the property "there are exactly n elements" in the domain
- Let $\psi_n \stackrel{\text{def}}{=} \phi_{=n} \rightarrow (\phi_{\mathcal{A}_1} \vee \cdots \vee \phi_{\mathcal{A}_k})$
- Then, $\mathcal{K}$ is precisely the class of models of $\{\, \psi_n \mid n > 0 \,\}$

First-order logic is too weak

- Natural properties cannot be expressed (such as, for instance, "the domain has even cardinality")
- "Weak" does not necessarily mean "bad"

"[...] weak expressive power can also be a good thing, as it implies transfer of properties across different situations. In non-standard arithmetic, one computes in the structure $\mathbb{N} \triangleleft \mathbb{Z}$ *using the infinite numbers to simplify calculations, and then transfers the outcome back to* $\mathbb{N}$*, provided it is a first-order statement about* $<$*."*

(van Benthem's course on logical games, Ch. 2, *Model Comparison Games*)

Example (Transfer of properties)

Assume that **II** has a winning strategy in $\mathcal{G}_3((A, R), (B, R'))$ and R is dense. Then, R' is also dense.

Definability and EF-games

Let $\mathcal{K}$ be a class of σ-structures and let Q be a Boolean query on $\mathcal{K}$

The following are equivalent (corollary of Ehrenfeucht theorem):

- Q is FO-definable on $\mathcal{K}$
- there is $m \in \mathbb{N}$ such that, for all $\mathcal{A}, \mathcal{B} \in \mathcal{K}$ such that $\mathcal{A}$ has property Q and $\mathcal{B}$ does not, **I** has a winning strategy in $\mathcal{G}_m(\mathcal{A}, \mathcal{B})$

How to prove an inexpressivity result?

- For every $m \in \mathbb{N}$, find $\mathcal{A}, \mathcal{B} \in \mathcal{K}$ such that
  1. $\mathcal{A}$ has property Q
  2. $\mathcal{B}$ has not property Q
  3. **II** has a winning strategy in $\mathcal{G}_m(\mathcal{A}, \mathcal{B})$

Soundness: the method above proves that Q is not definable
Completeness: if Q is not definable, the method above can (in principle) be used to prove it

Definability and EF-games (cont.)

Let $\mathcal{K}$ be a class of σ-structures and Q be an m-ary query on $\mathcal{K}$

The following are equivalent (corollary of Ehrenfeucht theorem):

- Q is FO-definable on $\mathcal{K}$
- there is $m \in \mathbb{N}$ such that, for every $\mathcal{A}, \mathcal{B} \in \mathcal{K}$ and m-tuples $\vec{a}, \vec{b}$ such that $\vec{a} \in Q(\mathcal{A})$ and $\vec{b} \notin Q(\mathcal{B})$, **I** has a winning strategy in $\mathcal{G}_m(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$

How to prove an inexpressivity result?

- For every $m \in \mathbb{N}$, find $\mathcal{A}, \mathcal{B} \in \mathcal{K}$ and m-tuples $\vec{a}, \vec{b}$ such that
  1. $\vec{a} \in Q(\mathcal{A})$
  2. $\vec{b} \notin Q(\mathcal{B})$
  3. **II** has a winning strategy in $\mathcal{G}_m(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$

Soundness: the method above proves that Q is not definable
Completeness: if Q is not definable, the method above can (in principle) be used to prove it

Games on sets

- Let $\mathcal{A}$ and $\mathcal{B}$ be two sets
- Empty vocabulary (only equality)

Lemma

If $|A|, |B| \geqslant m$, *then* $\mathcal{A} \equiv_m \mathcal{B}$

- Assume that, after i rounds ($i < m$), the mapping $\mathcal{A} \to \mathcal{B}$

$$(a_1, \ldots, a_i) \mapsto (b_1, \ldots, b_i)$$

  is a partial isomorphism
- At round $i + 1$, assume that **I** picks a_{i+1}
- W.l.o.g, $a_{i+1} \notin \{a_1, \ldots, a_i\}$
- **II** responds with $b_{i+1} \in B \setminus \{b_1, \ldots, b_i\}$
- $B \setminus \{b_1, \ldots, b_i\}$ is non-empty by hypothesis
- Dual reasoning for **I** playing in $\mathcal{B}$

Games on sets: applications

Even Cardinality:

$$\mathsf{EC}(\mathcal{A}) = \begin{cases} \textit{true} & A \text{ has even cardinality} \\ \textit{false} & \text{otherwise} \end{cases}$$

Lemma

EC is not FO-definable over sets

- $\{a_1, \ldots, a_m\} \equiv_m \{b_1, \ldots, b_{m+1}\}$ (see previous slide)

Games on sets: applications (cont.)

Finiteness:

$$\mathsf{FIN}(\mathcal{A}) = \begin{cases} \textit{true} & A \text{ has finite cardinality} \\ \textit{false} & \text{otherwise} \end{cases}$$

Lemma

FIN is not FO-definable over sets

- $\{a_1, \ldots, a_m\} \equiv_m \mathbb{N}$

Does $A \equiv_m B$ imply $|A|, |B| \geqslant m$?

Of course not! $A \equiv_m B \Leftrightarrow |A|, |B| \geqslant m \vee |A| = |B|$

Games on sets: applications (cont.)

Finiteness:

$$\mathsf{FIN}(\mathcal{A}) = \begin{cases} \textit{true} & A \text{ has finite cardinality} \\ \textit{false} & \text{otherwise} \end{cases}$$

Lemma

FIN is not FO-definable over sets

- $\{a_1, \ldots, a_m\} \equiv_m \mathbb{N}$

Does $A \equiv_m B$ imply $|A|, |B| \geqslant m$?

Of course not! $A \equiv_m B \Leftrightarrow |A|, |B| \geqslant m \vee |A| = |B|$

Games on sets: applications (cont.)

Finiteness:

$$\mathsf{FIN}(\mathcal{A}) = \begin{cases} \textit{true} & A \text{ has finite cardinality} \\ \textit{false} & \text{otherwise} \end{cases}$$

Lemma

FIN is not FO-definable over sets

- $\{a_1, \ldots, a_m\} \equiv_m \mathbb{N}$

Does $A \equiv_m B$ imply $|A|, |B| \geqslant m$?

Of course not! $A \equiv_m B \Leftrightarrow |A|, |B| \geqslant m \vee |A| = |B|$

Games on linear orderings

- $\mathcal{L}_n = (\{1, \ldots, n\}, <)$
- $\mathcal{L}_n^{<k} = (\{1, \ldots, k-1\}, <)$ and
  $\mathcal{L}_n^{>k} = (\{k+1, \ldots, n\}, <) \cong (\{1, \ldots, n-k\}, <)$
- For $m, n, t \in \mathbb{N}$, $m =_t n$ iff $m = n$ or $m, n \geqslant t$

Lemma

If $n, p \geqslant 2^m - 1$, *then* $\mathcal{L}_n \equiv_m \mathcal{L}_p$

- Direct proof by induction on m maintaining the following invariant: if $(a_1 \ldots, a_k) \mapsto (b_1, \ldots, b_k)$ is the mapping after k rounds, then, for every $1 \leqslant i, j \leqslant k$ and $t = 2^{m-k} - 1$,
  1. $a_i < a_j$ iff $b_i < b_j$
  2. $|a_i - a_j| =_t |b_i - b_j|$, $a_i =_t b_i$, and $n - a_i =_t p - b_i$
- Proof using the congruence of linear orderings:
  - $\mathcal{L}_n \equiv_{m+1} \mathcal{L}_p$ iff for every $i \in L_n$, there is $j \in L_p$ such that

$$\mathcal{L}_n^{<i} \equiv_m \mathcal{L}_p^{<j} \wedge \mathcal{L}_n^{>i} \equiv_m \mathcal{L}_p^{>j}$$

and for every $j \in L_p$, there is $i \in L_n$ such that

$$\mathcal{L}_n^{<i} \equiv_m \mathcal{L}_p^{<j} \wedge \mathcal{L}_n^{>i} \equiv_m \mathcal{L}_p^{>j}$$

Games on linear orderings: remarks

Lemma

EC is not FO-definable on the class of (finite) linear orderings.

- For every m, $\mathcal{L}_{2^m-1} \equiv_m \mathcal{L}_{2^m}$

Lemma

FIN is not FO-definable on the class of linear orderings.

- For all m, $\mathcal{L}_{2^m-1} \equiv_m \mathbb{N} \triangleleft \mathbb{N}^R$ ($\mathbb{N}^R$ is like a reversed copy of $\mathbb{N}$)
- Note that the class of finite linear orderings is axiomatizable, that is, a sentence is true on the class of linear orderings if and only if it is a logical consequence of the following axioms:
  - transitivity
  - trichotomy (exactly one among $a < b$, $b < a$, and $a = b$ holds)
  - existence of endpoints
  - discreteness (existence of successor/predecessor)

A game-theoretic proof of undefinability in $(\mathbb{N}, <)$

Lemma

The set of even natural numbers is not FO-definable in $(\mathbb{N}, <)$

- "The set of even natural numbers" is a unary query
- Let $\mathcal{A}$ be $(\mathbb{N}, <)$ and let $\mathcal{B}$ be $(\mathbb{N}, <)$
- Fix m
- Let a be any even number in $\mathcal{A} \geqslant 2^m$
- Let b be any odd number in $\mathcal{B} \geqslant 2^m - 1$
- Then, **II** has a winning strategy in $\mathcal{G}_m(\mathcal{A}, a, \mathcal{B}, b)$
  - $(\{0, \ldots, a-1\}, <) \equiv_m (\{0, \ldots, b-1\}, <)$
  - $(\{a+1, \ldots\}, <) \equiv (\{b+1, \ldots\}, <)$ (they are both $\equiv (\mathbb{N}, <)$)
  - Just compose the strategies

Undefinability on graphs

Lemma

The class of all (finite or infinite) connected graphs is not FO-definable.

- Compactness argument (first lesson)

Lemma

CONN is not FO-definable on the class of finite graphs.

- Given m, let $r = 3^{m+1}$
- Let $d > 2r + 1$
- Let $\mathcal{G}_1$ consist of a cycle of length $2d$
- Let $\mathcal{G}_2$ consist of two disjoint cycles of length d
- Every r-neighbourhood is a path of length $2r$
- By the Sphere theorem, $\mathcal{G}_1 \equiv_m \mathcal{G}_2$

Games on trees

- The class of finite trees is not FO-definable over the class of finite graphs
  - compare a path with a cycle
  - E.g., see Libkin 2004

K. Doets
On n-Equivalence of Binary Trees
Notre Dame Journal of Formal Logic, 1987

This note presents a simple characterization of the class of all trees which are n-elementary equivalent with B_m: the binary tree with one root all of whose branches have length m (for each pair of positive integers n and m). [...] Section 2 introduces the class $Q(n)$ of binary trees and proves that every tree in it is n-equivalent with B_m whenever $m \geqslant 2^n - 1$. Section 3 shows that, conversely, each n-equivalent of a B_m with $m > 2^n - 1$ belongs to $Q(n)$. Finally, all n-equivalents of B_m for $m < 2^n - 1$ are isomorphic to B_m.

Notions of locality

- Hanf's theorem (seen) and Gaifman's theorem (to be seen) are results about the *locality* of FO
- They have inspired a slightly different methodology to prove inexpressivity results

How to prove that query Q is not definable in logic $\mathfrak{L}$?

1. Provide a definition of locality for queries
2. Prove that every $\mathcal{L}$-definable query is local according to the given definition
3. Prove that Q is not local according to the same definition

Hanf-locality

- Recall that $\mathcal{A} \leftrightarrows_r \mathcal{B}$ means that there is a bijection $f \colon A \to B$ such that $\mathcal{N}_r^{\mathcal{A}}(a) \cong \mathcal{N}_r^{\mathcal{B}}(f(a))$ for every $a \in A$

Definition

A Boolean query Q on a class $\mathcal{K}$ of σ-structures is *Hanf-local* if, and only if, there is $r \in \mathbb{N}$ such that, for every $\mathcal{A}, \mathcal{B} \in \mathcal{K}$,

$$\text{if } \mathcal{A} \leftrightarrows_r \mathcal{B} \text{ then } (Q(\mathcal{A}) \Leftrightarrow Q(\mathcal{B}))$$

Example

CONN is not Hanf-local

- By contradiction, let CONN be Hanf-local for a given r
- Let $d > 2r + 1$
- Let $\mathcal{G}_1$ consist of a cycle of length $2d$
- Let $\mathcal{G}_2$ consist of two disjoint cycles of length d
- Let f an arbitrary bijection between $\mathcal{G}_1$ and $\mathcal{G}_2$
- Every r-neighbourhood is a path of length $2r$
- Then, $\mathcal{A} \leftrightarrows_r \mathcal{B}$; but, $\mathcal{G}_1$ is connected and $\mathcal{G}_2$ is not

Gaifman-locality

Definition

An m-ary query Q on a class $\mathcal{K}$ of σ-structures is *Gaifman-local* if, and only if, there is $r \in \mathbb{N}$ such that, for every $\mathcal{A} \in \mathcal{K}$ and m-tuples $\vec{a}_1, \vec{a}_2 \in A^m$,

$$\text{if } \mathcal{N}_r^{\mathcal{A}}(\vec{a}_1) \cong \mathcal{N}_r^{\mathcal{A}}(\vec{a}_2) \text{ then } (\vec{a}_1 \in Q(\mathcal{A}) \Leftrightarrow \vec{a}_2 \in Q(\mathcal{A})).$$

Example

Transitive Closure (TC) is not Gaifman-local

- E.g., consider $(\mathbb{Z}, \mathit{succ})$
- Given r, take a, b such that $b - a > 2r + 1$
- Then, $\mathcal{N}_r^{\mathcal{A}}(a) \cong \mathcal{N}_r^{\mathcal{A}}(b)$
- Since the neighbourhoods are not adjacent, then $\mathcal{N}_r^{\mathcal{A}}(a, b) \cong \mathcal{N}_r^{\mathcal{A}}(b, a)$
- However, $(a, b) \in \mathrm{TC}$ but $(b, a) \notin \mathrm{TC}$

Locality and first-order logic

- Hanf-locality can be applied only when $|A| = |B|$
- (The version of) Hanf-locality (generalized to m-queries) implies Gaifman-locality
- Every FO-definable query is Hanf-local
- Hence, every FO-definable query is Gaifman-local
- Exponential lower bounds on the locality ranks (the minimum integer such that the locality property holds) can be proved

Explicit definability

- Let σ be a purely relational vocabulary
- Let R be a relation symbol in σ
- Let σ_0 be $\sigma \setminus \{R\}$
- Let T be a set of σ-sentences (closed under entailment)

Definition

T *explicitly defines* R iff there is a σ_0-formula ϕ such that

$$T \models \forall \vec{x} \, (R(\vec{x}) \leftrightarrow \phi(\vec{x})).$$

Or, for a complete logic (as FO),

$$T \vdash \forall \vec{x} \, (R(\vec{x}) \leftrightarrow \phi(\vec{x})).$$

- Equivalently, ϕ *explicitly defines* R *relative to* T
- Syntactic notion of definability
- Obviously implies that any two models of T that agree on the interpretation of σ_0 must also agree on the interpretation of R

Explicit definability in FO: an example

- Let $\sigma = \{<, s\}$ and $\sigma_0 = \{<\}$
- Let T be the theory of linear orderings plus the following:
  1. $\forall x \forall y \forall y' \left((s(x, y) \wedge s(x, y')) \rightarrow y = y' \right)$
  2. $\forall x \forall y \left(s(x, y) \rightarrow x < y \right)$
  3. $\forall x \forall y \left(x < y \rightarrow \exists y' \left(y' \leqslant y \wedge s(x, y') \right) \right)$
- Then, s is explicitly definable relative to T:

$$\mathsf{T} \models \forall x \forall y \left(s(x, y) \leftrightarrow \phi(x, y) \right)$$

where

$$\phi(x, y) \equiv x < y \wedge \neg \exists w \left(x < w < y \right)$$

Explicit definability in FO: an example (cont.)

$(\rightarrow)$ $s(x, y)$ holds by hypothesis

- By (2), $s(x, y)$ implies $x < y$
- We need to prove that $s(x, y)$ implies $\neg \exists w \, (x < w < y)$
- For the sake of contradiction, assume that w exists such that $x < w < y$
- Then, by (3), $s(x, w')$ holds for some $w' \leqslant w$
- But then, by (1), $y = w' \leqslant w$, which contradicts $w < y$

$(\leftarrow)$ $x < y \wedge \neg \exists w \, (x < w < y)$ holds by hypothesis

- By (3), there is $y' \leqslant y$ such that $s(x, y')$ holds
- By (2), $x < y' \leqslant y$
- By hypothesis, no w exists such that $x < w < y$
- Hence, $y' = y$ and $s(x, y)$ holds

Implicit definability

- Let σ be a purely relational vocabulary
- Let R be a relation symbol in σ
- Let σ_0 be $\sigma \setminus \{R\}$
- Let T be a set of σ-sentences (closed under entailment)
- Let S be a fresh relation symbol with the same arity as R
- Let T' be like T with occurrences of R replaced by S

Definition

T *implicitly defines* R iff any σ_0-structure has *at most one expansion* to a model of T, i.e.,

$$T \cup T' \models \forall \vec{x}\,(R(\vec{x}) \leftrightarrow S(\vec{x})).$$

- I.e., every pair of models of T that agree on the interpretation of σ_0 also agree on the interpretation of R
- R can be characterized uniquely
- Semantic notion of definability

Beth theorem

- Explicit definability entails implicit definability
- What about the converse?

Definition (Beth Property)

A logic has the Beth property iff for every relation symbol $R \in \sigma$ and for every set of σ-sentences T, if T implicitly defines R then T explicitly defines R.

Theorem (Beth theorem)

First-order logic has the Beth property.

- A model-theoretic notion of definability coincides with a proof-theoretic notion of definability
- Good balance between syntax and semantics
- Unfortunately, FO interpreted over finite structures does not have the Beth property

Beth theorem fails in the finite

- Let $\sigma = \{<, P\}$ with P unary predicate
- Let T be the theory of linear orderings plus
  1. $\exists x\,(P(x) \wedge \forall y\,(x \leqslant y))$
  2. $\forall x \forall y\,(s(x,y) \to (P(x) \leftrightarrow \neg P(y))$, where $s(x,y)$ is a shorthand for "y is an immediate successor of x"

Lemma

T *implicitly defines* P *on finite models*

- Let $\mathcal{M}$ be a finite model of T
- $M = \{m_1, \ldots, m_k\}$ with $m_1 < m_2 < \cdots < m_k$
- According to T, $m_i \in P^{\mathcal{M}}$ iff i is odd
- Hence, the interpretation of P is uniquely determined
- Note that, on infinite models, this does not need to be the case
  - E.g., on $\mathbb{R}^+$ with $<$ interpreted as usual, any P containing 0 yields a model of T

Beth theorem fails in the finite (cont.)

Lemma

There is no explicit definition for P relative to T.

- For the sake of contradiction, suppose that a $\{<\}$-formula $\phi(x)$ that defines P exists ($\phi(x)$ means "x has odd index")
- Let k be the quantifier rank of ϕ
- Consider the formula

$$\psi \equiv \exists x \, (\phi(x) \wedge \forall y \, (y \leqslant x))$$

- ψ must be true in every finite model of T iff its cardinality is odd
- ψ has quantifier rank $k+1$
- We know that $\mathcal{L}_{2^{k+1}-1} \equiv_{k+1} \mathcal{L}_{2^{k+1}}$
- Hence, $\mathcal{L}_{2^{k+1}-1} \models \psi$ and $\mathcal{L}_{2^{k+1}} \not\models \psi$ is a contradiction

Explicit and implicit definability of queries

Definition

A m-ary query Q is *explicitly definable* iff there is a σ-formula $\phi(x_1, \ldots, x_m)$ such that, for every $\mathcal{A}$, $R_\phi^{\mathcal{A}} = Q(\mathcal{A})$

Definition

Let P be an m-ary relation symbol not occurring in σ. An m-ary query Q is *implicitly definable* iff there is a $(\sigma \cup \{P\})$-sentence ψ such that every σ-structure $\mathcal{A}$ has a unique expansion to a $(\sigma \cup \{P\})$-structure that satisfies ψ, namely $(\mathcal{A}, Q(\mathcal{A}))$

- Let Q be "the set of even elements of a finite linear ordering"
- Q is implicitly definable
  - See two slides before
  - $\psi \equiv \exists x\,(P(x) \wedge \forall y\,(x \leqslant y)) \wedge \forall x \forall y\,(s(x,y) \rightarrow (P(x) \leftrightarrow \neg P(y))$
- There is no explicit definition for Q
  - See the previous slide

Explicit and implicit definability: exercise

- Let $\sigma = \{s\}$ (interpreted as the successor relation)
- Let Q be binary query: "the transitive closure of s" ($<$ relation)

1. Using EF-games, prove that Q is not explicitly definable over the class of successor structures
2. Is Q implicitly definable?

A library of sufficient conditions

- Sufficient conditions allow us to prove negative expressivity results

R. Fagin and L. J. Stockmeyer and M. Y. Vardi
On monadic NP vs monadic co-NP
Information and Computation, 1995

T. Schwentick
On winning Ehrenfeucht games and monadic NP
Annals of Pure and Applied Logic, 1996

S. Arora and R. Fagin
On winning strategies in Ehrenfeucht-Fraïssé games
Theoretical Computer Science, 1997

Arora and Fagin's condition

- "Approximately" isomorphic neighbourhoods
- Still based on a multiplicity argument
- Neighborhoods must be tree-like structures

Definition (simplified for directed graphs)

- The $(m, 0)$-color of an element a is its label plus a description of whether it is a constant and whether it has a self-loop
- the $(m, r + 1)$-color of a is its (m, r)-color plus a list of triples, one for each possible (m, r)-color τ:
  1. the number of elements b with (m, r)-color τ such that $E(a, b)$ but not $E(b, a)$, counted up to m
  2. the number of elements b with (m, r)-color τ such that $E(b, a)$ but not $E(a, b)$, counted up to m
  3. the number of elements b with (m, r)-color τ such that $E(a, b)$ and $E(b, a)$, counted up to m

Arora and Fagin's condition (cont.)

Let the color of a directed edge be the ordered pair of colors of its nodes.

Theorem

Let $\mathcal{A} = (A, E)$ and $\mathcal{B} = (B, E)$ be two structures of degree at most d, and let $m \in \mathbb{N}$. If

- *there is a bijection $f : A \to B$ such that a and $f(a)$ have the same (m, r)-color, with $r = 3^{2m}$, for all $a \in A$,*
- *$\mathcal{A}$ and $\mathcal{B}$ do not have (undirected) cycles of length less than r,*
- *whenever $E^{\mathcal{A}}(a, b)$ holds but $E^{\mathcal{B}}(f(a), f(b))$ does not hold, or vice versa, then there are at least d^r edges in both structures having the same (m, r)-color as (a, b), (resp., $(f(a), f(b))$),*

*then **II** has a winning strategy in $\mathcal{G}_m(\mathcal{A}, \mathcal{B})$.*

Applications of Arora and Fagin's condition

- Directed reachability is not in monadic Σ_1^1 (a simpler proof of Ajtai and Fagin's result)
- Graph connectivity is not in monadic Σ_1^1
- Both results can be shown to hold even if the vocabulary is expanded with particular built-in relations of degree $n^{o(1)}$, where n is the size of the structure
- The requirement of the absence of small cycles can be relaxed at the expense of adding further hypotheses

Schwentick's extension theorem

Schwentick's work moves from the following question: Under which conditions can a "local" strategy be extended?

He develops a method that allows, under certain conditions, the extension of a winning strategy for **II** on some small parts of two finite structures to a global winning strategy.

- The structures must be isomorphic except for some small parts, for which local winning strategies exist by hypothesis
- The advantage is that there are no further constraints, either on the degree or on the internal characteristics of the substructures.

Schwentick's extension theorem (cont.)

- Let $\mathcal{C}$ and $\mathcal{D}$ be subtructures of $\mathcal{A}$ and $\mathcal{B}$, respectively
- Suppose that **II** has a winning strategy in $\mathcal{G}_m(\mathcal{C}, \mathcal{D})$ for some m
- **II** has a winning strategy in $\mathcal{G}_m(\mathcal{A}, \mathcal{B})$ if
  1. **II**'s strategy for $\mathcal{G}_m(\mathcal{C}, \mathcal{D})$ can be extended to a winning strategy in $\mathcal{G}_m(\mathcal{N}^{\mathcal{A}}_{2^m}(C), \mathcal{N}^{\mathcal{B}}_{2^m}(D))$, so that, at every round the two chosen elements have the same distance from C and D, respectively
  2. there is an isomorphism α: $(A \setminus C) \to (B \setminus D)$ such that $\delta(x, C) = \delta(\alpha(x), D)$ for all $x \in \mathcal{N}^{\mathcal{A}}_{2^m}(C) \setminus C$

Proof's idea

- Divide the domains of the structures into three regions:
- inner area: $I = C \cup D$
- outer area: $O = (A \setminus \mathcal{N}^{\mathcal{A}}_{2^m}(C)) \cup (B \setminus \mathcal{N}^{\mathcal{B}}_{2^m}(D))$
- the area in between
- At each round, the inner or outer areas may grow, according to the played moves
- Separation invariant: after round i the distance from every element in the inner area and every element in the outer area is greater than 2^{m-i}
- So, the winning strategy for **II** is guaranteed by the isomorphism α in the outer area, and by the extended winning strategy in the inner area and the area in between

Extensions

- different distance functions can be used
- winning strategies for several pairs of substructures can be combined
- The separation invariant may be required for some relations, but not for others (e.g., linear ordering), by adding a kind of homogeneity condition that guarantees that elements in the inner and outer areas behave in the same way with respect to the relations that do not satisfy the separation invariant

Applications of Schwentick's extension theorem

- Connectivity of finite graphs is not expressible in monadic Σ^1_1 in the presence of built-in relations of degree $n^{o(1)}$ (the same result as Arora and Fagin's) or even in the presence of a built-in linear ordering
- Monadic Σ^1_1 with a built-in linear ordering is more expressive than monadic Σ^1_1 with a built-in successor relation

Introduction to EF-games

Inexpressivity results for first-order logic

Normal forms for first-order logic

Algorithms and complexity for specific classes of structures

General complexity bounds

Gaifman's theorem (for sentences)

- r-local formula (around $\vec{x}$): has "bounded" quantifiers:

$$\exists y\,(\delta(\vec{x}, y) \leqslant r \wedge \phi)$$
$$\forall y\,(\delta(\vec{x}, y) \leqslant r \rightarrow \phi)$$

  where ϕ is either quantifier-free or r-local (around $\vec{x}$)

- $\delta(\vec{x}, y) \leqslant r$ is FO-definable
- existentially r-local sentence:

$$\exists x_1 \cdots \exists x_s \left(\bigwedge_{1 \leqslant i < j \leqslant s} \delta(x_i, x_j) > 2r \wedge \bigwedge_{1 \leqslant i \leqslant s} \phi_r^{(i)}(x_i) \right)$$

  where $\phi_r^{(i)}$ are r-local formulas around x_i

Theorem (Gaifman's theorem)

Every first-order sentence is logically equivalent to a boolean combination of existentially local sentences.

Gaifman's theorem: preliminary technicalities

To prove the theorem, we need the following results

Lemma

Let Φ be a set of first-order σ-sentences. If all the σ-structures that agree on Φ are elementarily equivalent, then any first-order σ-sentence is equivalent to a boolean combination of sentences of Φ.

Lemma (Relativization lemma)

Let σ be a purely relational vocabulary. For every $r \in \mathbb{N}$ and σ-formula $\phi(\vec{x}, \vec{y})$, with $|\vec{x}| = n$ and $|\vec{y}| = p$, there is an r-local formula $\phi^{S_r(\vec{x})}(\vec{x}, \vec{y})$ such that, for any σ-structure $\mathcal{A}$, $\vec{a} \in A^n$ and $\vec{b} \in \left(S_r^{\mathcal{A}}(\vec{a})\right)^p$,

$$\mathcal{A} \models \phi^{S_r(\vec{x})}\left\{\vec{x}/\vec{a}, \vec{y}/\vec{b}\right\} \qquad \Longleftrightarrow \qquad \mathcal{N}_r^{\mathcal{A}}(\vec{a}) \models \phi\left\{\vec{x}/\vec{a}, \vec{y}/\vec{b}\right\}.$$

For the proofs, see Ebbinghaus and Flum's *Finite Model Theory*

Gaifman's theorem: the idea of the proof

- We only prove the theorem for sentences, but it can be extended to arbitrary formulas
- We show that $\mathcal{A} \equiv \mathcal{B}$ iff they agree on all existentially local sentences
- The theorem then follows from the first lemma of the previous slide

- One direction is trivial
- We prove the non-trivial direction by showing that **II** has a winning strategy in an $\mathfrak{m}$-round EF-game for every $\mathfrak{m}$

Gaifman's theorem: invariant for the proof

- Let $f\colon \mathbb{N} \to \mathbb{N}$ be some function (to be defined)
  - constraints on f will emerge during the proof
  - in particular, it will turn out that f will be monotonically non-increasing
- Let $a_1, \ldots, a_i$ and $b_1, \ldots, b_i$ be the elements chosen in the first i rounds
- After i rounds, **II** will maintain

$$\mathcal{N}^{\mathcal{A}}_{7^{m-i}-1}(a_1 \cdots a_i) \equiv_{f(i)} \mathcal{N}^{\mathcal{B}}_{7^{m-i}-1}(b_1 \cdots b_i)$$

- This invariant ensures that $a_1 \cdots a_i \mapsto b_1 \cdots b_i$ is a partial isomorphism
- **Notation:** $r_i \stackrel{\text{def}}{=} 7^{m-i} - 1$
- So, the invariant after i rounds is

$$\mathcal{N}^{\mathcal{A}}_{r_i}(a_1 \cdots a_i) \equiv_{f(i)} \mathcal{N}^{\mathcal{B}}_{r_i}(b_1 \cdots b_i)$$

Relativized Hintikka formulas

- Let $\mathcal{H}_{\vec{a}}^{f(i)}(\vec{x})$ denote the $f(i)$-Hintikka formula for $\mathcal{N}_{r_i}^{\mathcal{A}}(\vec{a})$
- By the definition of Hintikka formulas,

$$\mathcal{N}_{r_i}^{\mathcal{A}}(\vec{a}) \models \mathcal{H}_{\vec{a}}^{f(i)}(x)\{\vec{x}/\vec{a}\}$$

- Let

$$\hat{\mathcal{H}}_{\vec{a}}^{i}(\vec{x}) \stackrel{\text{def}}{=} \left(\mathcal{H}_{\vec{a}}^{f(i)}(\vec{x})\right)^{S_{r_i}(\vec{a})}$$

  that is, $\hat{\mathcal{H}}_{\vec{a}}^{i}(\vec{x})$ is the relativized version (as in the relativization lemma) of $\mathcal{H}_{\vec{a}}^{f(i)}(x)$ with respect to $S_{r_i}(\vec{a})$
- By the relativization lemma,

$$\mathcal{A} \models \hat{\mathcal{H}}_{\vec{a}}^{i}(\vec{x})\{\vec{x}/\vec{a}\}$$

- By the relativization lemma and Ehrenfeucht theorem, for every structure $\mathcal{B}$ and $\vec{b} \in B^n$

$$\mathcal{B} \models \hat{\mathcal{H}}_{\vec{a}}^{i}(\vec{x})\{\vec{x}/\vec{b}\} \iff \mathcal{N}_{r_i}^{\mathcal{B}}(\vec{b}) \equiv_{f(i)} \mathcal{N}_{r_i}^{\mathcal{A}}(\vec{a})$$

The first round

- W.l.o.g, assume that **I** plays in $\mathcal{A}$
- Let **I** choose $a_1 \in A$
- $\mathcal{N}^{\mathcal{A}}_{r_1}(a_1)$ is characterized, up to $f(1)$-equivalence, by $\mathcal{H}^{f(1)}_{a_1}(x)$
- Since the relativized version $\hat{\mathcal{H}}^1_{a_1}(x)$ is r_1-local, then $\exists x\, \hat{\mathcal{H}}^1_{a_1}(x)$ is an existentially local sentence that holds in $\mathcal{A}$
- By the hypothesis, $\exists x\, \hat{\mathcal{H}}^1_{a_1}(x)$ must hold in $\mathcal{B}$ as well
- That is, there is $b_1 \in B$ such that $\mathcal{B} \models \hat{\mathcal{H}}^1_{a_1}(x)\{x/b_1\}$
- Hence, $\mathcal{N}^{\mathcal{B}}_{r_1}(b_1) \equiv_{f(1)} \mathcal{N}^{\mathcal{A}}_{r_1}(a_1)$
- But this is exactly the invariant after one round
- Therefore, b_1 is a suitable reply for **II**

The inductive step: first case

- Suppose that after i rounds, with $i < m$, the invariant holds
- Let **I** choose $a_{i+1} \in A$
- Two possibilities: either $\delta(a_{i+1}, \vec{a}) \leqslant 2r_{i+1} + 1$ or not
- Suppose that $\delta(a_{i+1}, \vec{a}) \leqslant 2r_{i+1} + 1$
- Recall that $r_{i+1} = 7^{m-(i+1)} - 1$
- Then, the $S^{\mathcal{A}}_{r_{i+1}}(a_{i+1}) \subseteq S^{\mathcal{A}}_{r_i}(\vec{a})$
- Therefore,

$$\mathcal{N}^{\mathcal{A}}_{r_i}(\vec{a}) \models \Theta\{\vec{x}/\vec{a}\}$$

where Θ is

$$\exists x_{i+1} \left(\delta(x_{i+1}, \vec{x}) \leqslant 2r_{i+1} + 1 \wedge \hat{\mathcal{H}}^{i+1}_{\vec{a}, a_{i+1}}(\vec{x}, x_{i+1})\right)$$

The inductive step: first case (cont.)

- First constraint on f: impose $f(i) \geqslant \mathsf{qr}(\Theta)$
- Then, by the invariant and the inductive hypothesis,

$$\mathcal{N}^{\mathcal{B}}_{r_i}(\overrightarrow{b}) \models \Theta\{\vec{x}/\vec{b}\}$$

- That is, there is $b_{i+1} \in B$ (with $b_{i+1} \leqslant 2r_{i+1} + 1$) such that (after applying the relativization lemma)

$$\mathcal{N}^{\mathcal{B}}_{r_{i+1}}(\overrightarrow{b}, b_{i+1}) \models \mathcal{H}^{f(i+1)}_{\vec{a}, a_{i+1}}(\overrightarrow{x}, x_{i+1})\{\vec{x}/\vec{b}, {x_{i+1}}/{b_{i+1}}\}$$

- Which implies

$$\mathcal{N}^{\mathcal{A}}_{r_{i+1}}(\overrightarrow{a}, a_{i+1}) \equiv_{f(i+1)} \mathcal{N}^{\mathcal{B}}_{r_{i+1}}(\overrightarrow{b}, b_{i+1})$$

- Therefore, the invariant is preserved after round $i + 1$

Inductive step: second case

- Suppose that $\delta(a_{i+1}, \vec{a}) > 2r_{i+1} + 1$
- Then, $\mathcal{N}^{\mathcal{A}}_{r_{i+1}}(\vec{a})$ and $\mathcal{N}^{\mathcal{A}}_{r_{i+1}}(a_{i+1})$ are not adjacent (i.e., there is no tuple in any relation of $\mathcal{A}$ which connects the two neighbourhoods)
- Hence, the disjoint union of $\mathcal{N}^{\mathcal{A}}_{r_{i+1}}(\vec{a})$ and $\mathcal{N}^{\mathcal{A}}_{r_{i+1}}(a_{i+1})$ is isomorphic to $\mathcal{N}^{\mathcal{A}}_{r_{i+1}}(\vec{a}, a_{i+1})$ (this will be used later)
- We will show that **II** is able to find an element b_{i+1} with the same $f(i+1)$-isomorphism type as a_{i+1} and such that $\delta(b_{i+1}, \vec{b}) > 2r_{i+1} + 1$
- Since $\mathcal{A}$ and $\mathcal{B}$ agree on all existentially local sentences, the existence of a suitable b_{i+1} will be guaranteed by the properties they express on the *scattering* of small neighbourhoods

Maximal sets of scattered neighbourhoods

- For $n \in \mathbb{N}$, let

$$\theta(x_1, \ldots, x_n) \stackrel{\text{def}}{=} \bigwedge_{1 \leqslant j < k \leqslant n} \delta(x_j, x_k) > 4r_{i+1} + 2 \wedge \bigwedge_{1 \leqslant j \leqslant n} \hat{\mathcal{H}}^{i+1}_{a_{i+1}}(x_j)$$

- "$\{x_1, \ldots, x_n\}$ is a $(4r_{i+1} + 2)$-scattered set of n elements whose r_{i+1}-neighbourhoods have the same $f(i+1)$-isomorphism type as a_{i+1}"
- Let t be the cardinality of a maximal subset of elements of $\mathcal{A}$ with the above property, that is, let t be such that

$$\mathcal{A} \models \exists x_1 \cdots \exists x_t \, \theta(x_1, \ldots, x_t)$$

but

$$\mathcal{A} \not\models \exists x_1 \cdots \exists x_{t+1} \, \theta(x_1, \ldots, x_{t+1})$$

- Note that $\exists \vec{x} \, \theta(\vec{x})$ is an existentially local sentence
- Hence, $\mathcal{B}$ agrees with $\mathcal{A}$ on the above sentences

Maximal sets of scattered neighbourhoods (cont.)

- Let

$$\Lambda(n) \stackrel{\text{def}}{=} \exists x_1 \cdots \exists x_n \Big(\bigwedge_{1 \leqslant j \leqslant n} \delta(x_j, \vec{a}) \leqslant 2r_{i+1} + 1 \wedge \\ \wedge \theta(x_1, \ldots, x_n) \Big)$$

- "There is a $(4r_{i+1} + 2)$-scattered set of n elements of $S^{\mathcal{A}}_{2r_{i+1}+1}(\vec{a})$ whose r_{i+1}-neighbourhoods have the same $f(i+1)$-isomorphism type as a_{i+1}"
- Let s be the cardinality of a maximal subset of elements with the above property, that is, let s be such that

$$\mathcal{N}^{\mathcal{A}}_{r_i}(\vec{a}) \models \Lambda(s)$$

but

$$\mathcal{N}^{\mathcal{A}}_{r_i}(\vec{a}) \not\models \Lambda(s+1)$$

- Note that $\Lambda(n)$ is not an existentially local sentence

Inductive step: second case (cont.)

- There are no two $(4r_{i+1}+2)$-scattered points in a sphere of radius $2r_{i+1}+1$
- It is possible to choose at most one element for each $S^{\mathcal{A}}_{2r_{i+1}+1}(a_j)$, with $1 \leqslant j \leqslant i$
- Therefore, $s \leqslant i$
- Clearly, $s \leqslant t$, too (t may be ∞)
- By hypothesis,

$$\mathcal{B} \models \exists x_1 \cdots \exists x_t \, \theta(x_1, \ldots, x_t)$$

  and

$$\mathcal{B} \not\models \exists x_1 \cdots \exists x_{t+1} \, \theta(x_1, \ldots, x_{t+1})$$

- Second constraint on f: impose $f(i) \geqslant \mathrm{qr}(\Lambda(s))$
- Then, by the invariant,

$$\mathcal{N}^{\mathcal{B}}_{r_i}(\vec{b}) \models \Lambda(s)$$

  and

$$\mathcal{N}^{\mathcal{B}}_{r_i}(\vec{b}) \not\models \Lambda(s+1)$$

First sub-case: $s = t$

- Suppose that $s = t$
- Then, *any* element $e \in A$ with the same $f(i+1)$-Hintikka type of a_{i+1} is such that

$$\begin{aligned}\delta(e, \vec{a}) &\leqslant (2r_{i+1}+1)+(4r_{i+1}+2) = 6r_{i+1}+3 < \\ &< 7r_{i+1}+3 = 7\cdot 7^{m-(i+1)} - 4 < 7^{m-i} - 1 = r_i\end{aligned}$$

- In particular, the above implies that $\mathcal{S}^{\mathcal{A}}_{r_{i+1}}(e) \subseteq \mathcal{S}^{\mathcal{A}}_{r_i}(\vec{a})$
- This holds for a_{i+1}, too
- Therefore:

$$\mathcal{N}^{\mathcal{A}}_{r_i}(\vec{a}) \models \Pi\{\vec{x}/\vec{a}\}$$

  where Π is

$$\exists z\,(2r_{i+1}+1 < \delta(\vec{x}, z) \leqslant 6r_{i+1}+3 \wedge \hat{\mathcal{H}}^{i+1}_{a_{i+1}}(z) \wedge \hat{\mathcal{H}}^{i+1}_{\vec{a}}(\vec{x}))$$

- Third constraint on f: impose $f(i) \geqslant \mathrm{qr}(\Pi)$
- Then, by the invariant,

$$\mathcal{N}^{\mathcal{B}}_{r_i}(\vec{b}) \models \Pi\{\vec{x}/\vec{b}\}$$

First sub-case: $s = t$ (cont.)

- So, there is $b_{i+1} \in B$ such that

$$2r_{i+1} + 1 < \delta(\vec{x}, z) \leqslant 6r_{i+1} + 3$$

  and

$$\mathcal{N}^{\mathcal{A}}_{r_{i+1}}(a_{i+1}) \equiv_{f(i+1)} \mathcal{N}^{\mathcal{B}}_{r_{i+1}}(b_{i+1})$$

- Last constraint on f: impose $f(i) \geqslant f(i+1)$
- Then, the invariant implies

$$\mathcal{N}^{\mathcal{A}}_{r_{i+1}}(\vec{a}) \equiv_{f(i+1)} \mathcal{N}^{\mathcal{B}}_{r_{i+1}}(\vec{b})$$

- As the neighbourhood around a_{i+1} (resp., b_{i+1}) is not adjacent to the neighbourhood around $\vec{a}$ (resp., $\vec{b}$) we may take their disjoint union and conclude that

$$\mathcal{N}^{\mathcal{A}}_{r_{i+1}}(\vec{a}, a_{i+1}) \equiv_{f(i+1)} \mathcal{N}^{\mathcal{B}}_{r_{i+1}}(\vec{b}, b_{i+1})$$

Second sub-case: $s < t$

- Suppose that $s < t$
- Remember that $s < t$ holds in $\mathcal{B}$, too
- Then, there is an element $b_{i+1} \in B$ such that
  1. $\mathcal{N}^{\mathcal{B}}_{r_{i+1}}(b_{i+1})$ is not adjacent to $\mathcal{N}^{\mathcal{B}}_{r_{i+1}}(\overrightarrow{b})$, and
  2. $\mathcal{B}, \{x/b_{i+1}\} \models \hat{\mathcal{H}}^{i+1}_{a_{i+1}}(x)$
- Again, by applying the relativization lemma and the Ehrenfeucht theorem,

$$\mathcal{N}^{\mathcal{A}}_{r_{i+1}}(a_{i+1}) \equiv_{f(i+1)} \mathcal{N}^{\mathcal{B}}_{r_{i+1}}(b_{i+1})$$

- The thesis is then obtained as in the preceding case

Remarks on Gaifman's theorem

- First-order logic can only talk of *scattered small substructures*
- First-order logic can only express *local properties*
- Gaifman's normal form is effective
- Gaifman's proof uses EF-games to prove the invariant

$$\mathcal{N}^{\mathcal{A}}_{7^{m-i}-1}(a_1 \cdots a_i) \equiv_{f(i)} \mathcal{N}^{\mathcal{B}}_{7^{m-i}-1}(b_1 \cdots b_i)$$

- r-local formulas with $r \leqslant 7^{\mathsf{qr}(\phi)}$
- $f(i)$-equivalence instead of isomorphism as in the Sphere theorem
- Notion of *scattered* substructures
- No counting up to a threshold as in the Sphere theorem

Schwentick and Bartelmann's normal form

Theorem

Every first-order formula is logically equivalent to a formula of the form $\exists x_1 \cdots \exists x_n \forall y\, \phi$, where ϕ is r-local around y for some r.

- Consider a differentiating formula ψ
- By the theorem, $\psi \Leftrightarrow \exists x_1 \cdots \exists x_n \forall y\, \phi$, with $\mathrm{qr}(\phi) = k$
- In game-theoretic terms, there is winning strategy for **I** such that **I** plays n rounds by choosing elements in the same structure (the one that satifies ψ)
- Then, **I** plays a round in the opposite structure
- Finally, **I** plays k "local" rounds

T. Schwentick and K. Barthelmann
Local Normal Forms for First-Order Logic with Applications to Games and Automata
Discrete Mathematics and Theoretical Computer Science, 1999

Shrinking games

- Similar to Schwentick's extension theorem, but it works in the opposite direction, by shrinking the playground according to a sequence of "scattering parameters"
- The authors use Ehrenfeucht–Fraïssé type games with a shrinking horizon between structures to obtain a spectrum of normal form theorems of the Gaifman type
- They improve the bound in the proof of Gaifman's theorem from $7^{qr(\phi)}$ to $4^{qr(\phi)}$ and they provide bounds for other normal form theorems

H. J. Keisler and W. B. Lotfallah
Shrinking games and local formulas
Annals of Pure and Applied Logic, 2004

Shrinking games: preliminary definitions

- Let $\vec{s} = s_0, s_1, \ldots$ a possibly infinite sequence of natural numbers, called scattering parameters
- The sequence of local radii associated with $\vec{s}$ is defined as follows:

$$r_0 = 1$$
$$r_{n+1} = 2r_n + s_n$$

- A set C is s-scattered if $\delta(a, b) > s$ for all distinct $a, b \in C$
- A sequence $\vec{s}$ shrinks rapidly (towards s_0) if $s_i \geqslant 2r_i$ for all i
- Given $\vec{s} = s_0, s_1, \ldots$ that shrinks rapidly, if C is s_i-scattered then the r_i-neighbourhood around any $c \in C$ does not contain any other element of C

Shrinking games: local rounds

- Let $\vec{s} = s_0, s_1, \ldots$ be a sequence that shrinks rapidly
- For a given m, let $\mathcal{S}_m^{\vec{s}}(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ denote the current configuration of an $\vec{s}$-shrinking game

Definition ($\vec{s}$-shrinking game)

A round from $\mathcal{S}_m^{\vec{s}}(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ is played as follows: **I** chooses a structure *and* $1 \leqslant k < m$, and plays either a local or a scattered move

A local round is played as follows (assuming that **I** plays in $\mathcal{A}$):

1. **I** chooses $a \in \mathcal{N}_{r_k+s_k}^{\mathcal{A}}(\vec{a})$
2. **II** replies with $b \in \mathcal{N}_{r_k+s_k}^{\mathcal{B}}(\vec{b})$

- Note that $r_k + s_k < r_{k+1} \leqslant r_m$ (by the definition of the r_i's)
- Besides, the r_k-neighbourhood around **I**'s choice is inside $\mathcal{N}_{r_m}^{\mathcal{A}}(\vec{a})$ (by the definition of the r_i's)

Shrinking games: scattered rounds

A scattered round is played as follows:

1. **I** chooses a non-empty finite set of s_k-scattered elements $C \subseteq \mathcal{N}^{\mathcal{A}}_{r_k}(\vec{a})$ such that **II** has a winning strategy in $\mathcal{S}^{\vec{s}}_k(\mathcal{A}, c, \mathcal{A}, d)$ for all $c, d \in C$ (if $|\vec{a}| = 0$ then **I** chooses at most $m - k$ s_k-scattered elements in A)
2. **II** replies with a non empty set of s_k-scattered elements $D \subseteq \mathcal{N}^{\mathcal{B}}_{r_k}(\vec{b})$ such that $|C| = |D|$
3. **I** chooses $d \in D$
4. **II** chooses $c \in C$
5. The game proceeds from $\mathcal{S}^{\vec{s}}_k(\mathcal{A}, \vec{a}, c, \mathcal{B}, \vec{b}, d)$

The ending and winning conditions are as in standard EF-game

Properties of **II**'s winning strategies

- **I** has the freedom to shorten the game by choosing $k < i - 1$ at round i
- Hence, m is an upper-bound to the number of rounds
- This ensures that the set of scattered moves available to **I** increases as m increases

Lemma (Lemma 3.2 in the paper)

1. *If **II** has a winning strategy in* $\mathcal{S}_m^{\vec{s}}(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ *then **II** has a winning strategy in* $\mathcal{S}_k^{\vec{s}}(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ *for every* $k \leqslant m$
2. *"Having a winning strategy for **II**" is an equivalence relation*

The role of the local radii

Local radii allow us to establish the following congruence property:

Lemma (Lemma 3.3 in the paper)

If

- ***II*** *wins* $\mathcal{S}_m^{\vec{s}}(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$
- ***II*** *wins* $\mathcal{S}_m^{\vec{s}}(\mathcal{A}, \vec{c}, \mathcal{B}, \vec{d})$
- $\delta(\vec{a}, \vec{c}) > r_m$
- $\delta(\vec{b}, \vec{d}) > r_m$

then ***II*** *wins* $\mathcal{S}_m^{\vec{s}}(\mathcal{A}, \vec{a}, \vec{c}, \mathcal{B}, \vec{b}, \vec{d})$

Proof.
(Sketch) By induction on m, using the fact that $r_i + s_i \leqslant r_m$ for all $i < m$. $\square$

Note: the lemma holds even if $\vec{s}$ does not shrink rapidly

Shrinking games: main result

Shrinking games provide a sufficient condition for the existence of a winning strategy for **II** in a standard EF-game

Theorem (Corollary 4.3 in the paper)

Let $m \in \mathbb{N}$ *and let* $\vec{s} = s_0, s_1, \ldots$ *be a sequence that shrinks rapidly. For every* $(\mathcal{A}, \vec{a})$ *and* $(\mathcal{B}, \vec{b})$*, if*

- ***II*** *has a winning strategy in* $\mathcal{S}_m^{\vec{s}}(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$

then

- ***II*** *has a winning strategy in* $\mathcal{G}_m(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$*.*

Proof

- We write $\mathcal{S}_m(\cdots)$ instead of $\mathcal{S}_m^{\vec{s}}(\cdots)$, assuming that $\vec{s}$ has been fixed
- We will show that $\mathcal{S}_m(\mathcal{A}, \mathcal{B})$ implies $\mathcal{G}_m(\mathcal{A}, \mathcal{B})$
- Proof by induction on the number m of rounds of the EF-game
- Idea: maintain the following invariant after i rounds of the EF-game have been played and elements $a_1, \ldots, a_i$ and $b_1, \ldots, b_i$ have been chosen:

  ***II** has a winning strategy in*
  $\mathcal{S}_{m-i}(\mathcal{A}, a_1, \ldots, a_i, \mathcal{B}, b_1, \ldots, b_i)$

Induction base ($i = 1$):

- Suppose that **I** chooses a_1 in $\mathcal{A}$ in the first round of the EF-game
- This is always a local move in $\mathcal{S}_m(\mathcal{A}, \mathcal{B})$
- Hence, the winning strategy in $\mathcal{S}_m(\mathcal{A}, \mathcal{B})$, which exists by hypothesis permits to find b_1 in $\mathcal{B}$ such that the invariant holds for $i = 1$

Proof (cont.)

Induction step ($i > 1$):

- Suppose that **I** chooses a_{i+1} in $\mathcal{A}$ at round $i+1$ of the EF-game
- Let $\vec{a} = a_1, \ldots, a_i$ and $\vec{b} = b_1, \ldots, b_i$ be the elements chosen so far
- By the inductive hypothesis, **II** wins $\mathcal{S}_p(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$, where $p = m - i$
- We distinguish two cases

Case 1: $a_{i+1} \in \mathcal{N}^{\mathcal{A}}_{r_{p-1}+s_{p-1}}(\vec{a})$

- **I**'s move is a local move in $\mathcal{S}_p(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$
- By the inductive hypothesis, **II** can find $b_{i+1} \in \mathcal{N}^{\mathcal{B}}_{r_{p-1}+s_{p-1}}(\vec{b})$ such that the invariant still holds for $i+1$

Proof (cont.)

Case 2: $a_{i+1} \notin \mathcal{N}^{\mathcal{A}}_{r_{p-1}+s_{p-1}}(\vec{a})$

- We will show that **II** is able to find $d \notin \mathcal{N}^{\mathcal{B}}_{r_{p-1}}(\vec{b})$ such that

  $$\textbf{II} \text{ has a winning strategy in } \mathcal{S}_{p-1}(\mathcal{A}, a_{i+1}, \mathcal{B}, d) \qquad (1)$$

- Let **II** choose $b_{i+1} = d$
- Then, the invariant after round $i + 1$ is obtained by Lemma 3.2 and Lemma 3.3 (congruence)
- For the sake of contradiction, assume that such d does not exist
- Then, any d satisfying (1) must be inside $\mathcal{N}^{\mathcal{B}}_{r_{p-1}}(\vec{b})$
- At least one d satisfying (1) exists (otherwise, **I** would win by choosing $k = p - 1$ and a_{i+1} in the first round)

Proof (cont.)

- Let D be a maximal s_{p-1}-scattered set of elements in $\mathcal{N}^{\mathcal{B}}_{r_{p-1}}(\vec{b})$ such that (1) holds for every $d \in D$
- Note that D cannot be empty (see previous slide)
- Since $\vec{s}$ shrinks rapidly, that is, $s_{p-1} \geqslant 2r_{p-1}$, the set D contains at most one element for each $\mathcal{N}^{\mathcal{B}}_{r_{p-1}}(b_j)$, hence $|D| \leqslant |\vec{b}| = i$
- Let **I** play a scattered round in the shrinking game as follows: **I** chooses $k = p - 1$ and the set D above
- Let **II** reply with a set C according to her winning strategy
- C consists of $|D|$ elements with disjoint r_{p-1}-neighbourhoods all "equivalent" to a_{i+1}, chosen from $\mathcal{N}^{\mathcal{B}}_{r_{p-1}}(\vec{b})$
- Note that C does not contain a_{i+1}

Proof (cont.)

- Now, consider another play of the shrinking game starting from $\mathcal{S}_m(\mathcal{A}, \mathcal{B})$ (which **II** wins by hypothesis)
- In the first round, **I** sets $k = p - 1$ and plays a scattered move by choosing the set $C \cup \{a_{i+1}\}$
- $C \cup \{a_{i+1}\}$ is s_{p-1}-scattered (because C is s_{p-1}-scattered by construction, and a_{i+1} is "far" from C by hypothesis)
- This is a legal move because $|C \cup \{a_{i+1}\}| \leqslant i + 1 = m - k$
- By hypothesis, **II** can find an s_{p-1}-scattered set in $\mathcal{B}$ with the same cardinality
- All such elements have disjoint r_{p-1}-neighbourhoods
- They are all "equivalent" to a_{i+1}
- They are all inside $\mathcal{N}^{\mathcal{B}}_{r_{p-1}}(\overrightarrow{b})$
- But this contradicts the maximality of D

Shrinking formulas

- Hierarchy of FO formulas corresponding to shrinking games
- Hierarchy that depends on a given sequence $\vec{s}$

Definition

The set $SF_m(\vec{x})$ of *shrinking formulas with free variables in* $\vec{x}$ *of rank at most* m is defined inductively as follows:

- $SF_0(\vec{x})$ is the set of all quantifier-free formulas in $\vec{x}$
- for each k<m, $SF_{k+1}(\vec{x})$ is the set of all finite Boolean combinations of formulas in $SF_k(\vec{x})$ and formulas of the form:

$$\exists y\,(\delta(y,\vec{x}) \leqslant r_k + s_k \wedge \psi(\vec{x},y))$$

and

$$\exists y_1 \cdots \exists y_n \Big(\bigwedge_{1 \leqslant i < j \leqslant n} \delta(y_i, y_j) > s_k \wedge \bigwedge_{1 \leqslant i \leqslant n} (\delta(y_i,\vec{x}) \leqslant r_k \wedge \theta(y_i)) \Big)$$

where $\psi(\vec{x},y) \in SF_k(\vec{x},y)$, $\theta(y) \in SF_k(y)$.

Shrinking sentences

Definition

The set $\mathcal{SF}_m$ of *shrinking sentences of rank at most* m is defined inductively as follows:

- for each $k < m$, $\mathcal{SF}_{k+1}$ is the set of all finite Boolean combinations of sentences in $\mathcal{SF}_k$ and sentences of the form:

$$\exists y_1 \cdots \exists y_n \Big(\bigwedge_{1 \leqslant i < j \leqslant n} \delta(y_i, y_j) > s_k \wedge \bigwedge_{1 \leqslant i \leqslant n} \theta(y_i) \Big)$$

where $\theta(y) \in \mathcal{SF}_k(y)$ and $n \leqslant m - k$.

Theorem (Lemma 5.3 in the paper)

If $(\mathcal{A}, \vec{a})$ *and* $(\mathcal{B}, \vec{b})$ *agree on all shrinking formulas of rank at most* m*, then* $(\mathcal{A}, \vec{a}) \equiv_m (\mathcal{B}, \vec{b})$*.*

Theorem (Theorem 5.4 in the paper)

If $\mathcal{A}$ *and* $\mathcal{B}$ *agree on all shrinking sentences of rank at most* m*, then* $\mathcal{A} \equiv_m \mathcal{B}$*.*

Normal forms of the Gaifman type

Definition

An *s-scattered r-local sentence of width k* is a sentence of the form

$$\exists x_1 \cdots x_k \Big(\bigwedge_{1 \leqslant i < j \leqslant k} \delta(x_i, x_j) > s \wedge \bigwedge_{1 \leqslant i \leqslant k} \phi(x_i) \Big)$$

where $\phi(x)$ is r-local.

Theorem

Fix a scattering sequence $\vec{s}$ shrinking rapidly. Then each FO-sentence ψ with qr$(\psi) \leqslant m$ is logically equivalent to a finite Boolean combination of sentences each of which is s_k-scattered and $(r_k - 1)$-local of width at most $m - k$, for some $k < m$.

Corollary

ψ is logically equivalent to a finite Boolean combination of sentences each of which is $2 \cdot 4^m$-scattered and $(4^m - 1)$-local of width at most $m - k$, for some $k < m$.

Introduction to EF-games

Inexpressivity results for first-order logic

Normal forms for first-order logic

Algorithms and complexity for specific classes of structures

General complexity bounds

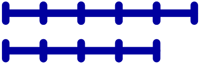
Sufficient vs. "iff" conditions

$$\mathcal{L}_k \stackrel{\text{def}}{=} (\{1, \ldots, k\}, <)$$

We know that

$$n, p \geqslant 2^m - 1 \Rightarrow \textbf{II} \text{ wins } \mathcal{G}_m(\mathcal{L}_n, \mathcal{L}_p).$$

- Given $\mathcal{L}_5$ and $\mathcal{L}_6$, does **II** win $\mathcal{G}_3(\mathcal{L}_5, \mathcal{L}_6)$?



No!

In fact,

$$n = p \text{ or } n, p \geqslant 2^m - 1 \Leftrightarrow \textbf{II} \text{ wins } \mathcal{G}_m(\mathcal{L}_n, \mathcal{L}_p)$$

- Complete characterizations are needed to exploit games algorithmically

Sufficient vs. "iff" conditions

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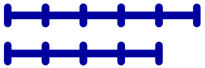

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In fact,

$$n = p \text{ or } n, p \geqslant 2^m - 1 \Leftrightarrow \textbf{II} \text{ wins } \mathcal{G}_m(\mathcal{L}_n, \mathcal{L}_p)$$

- Complete characterizations are needed to exploit games algorithmically

Winning vs. optimal strategies

Winning strategy $\neq$ Optimal strategy

The distinction between winning and optimal strategies is essential in unbounded games:

- In unbounded EF-games on finite structures, **I** wins unless $\mathcal{A} \cong \mathcal{B}$
- "Play randomly" is a winning strategy for **I**
- But, how far actually is the end of a game?
- What are the *best* moves for **I** (and **II**)?

Remoteness

Optimal strategies (in combinatorial games $\mathcal{G}$) can be characterized in terms of remoteness (*rem*$(\mathcal{G})$):

- Current player has no legal moves from (the current configuration of) $\mathcal{G} \Rightarrow rem(\mathcal{G}) = 0$
- Current player can move to a configuration with even remoteness $\Rightarrow rem(\mathcal{G}) = 1 +$ least even remoteness

  Win Quickly!

- Current player can only move to configurations with odd remoteness $\Rightarrow rem(\mathcal{G}) = 1 +$ greatest odd remoteness

  Lose Slowly!

- The parity of the remoteness tells the winner

Win quickly, lose slowly!

Remoteness in EF-games:

- For EF-games, remoteness in terms of rounds, not moves
- Remoteness of $\mathcal{G}$: the minimum m such that **I** wins $\mathcal{G}_m$ (simplified definition under the hypothesis $\mathcal{A} \not\equiv \mathcal{B}$)
- Optimal **I** 's move: given a configuration $\mathcal{G}$, a move by **I** is *optimal* if and only if, whatever **II** replies, the remoteness of the resulting configuration is less than or equal to $\mathit{rem}(\mathcal{G}) - 1$.
- Optimal **II** 's move: given a configuration $\mathcal{G}$ and a move by **I** , a reply by **II** is *optimal* if and only if the remoteness of the resulting position is
  - $\mathit{rem}(\mathcal{G}) - 1$, if **I** 's move is optimal
  - $\mathit{rem}(\mathcal{G})$, otherwise

Solving Games

Example

$$n = p \text{ or } n, p \geqslant 2^m - 1 \Leftrightarrow \textbf{II} \text{ wins } \mathcal{G}_m(\mathcal{L}_n, \mathcal{L}_p)$$

How many rounds are needed to **I** to win?
Assume $n < p$. Then:

1. The remoteness can be computed as:

$$\mathit{rem}(\mathcal{G}(\mathcal{L}_n, \mathcal{L}_p)) = \lfloor \log_2(n+1) \rfloor + 1$$

2. A move of **I** from $\mathcal{G}(\mathcal{L}_n, \mathcal{L}_p)$ is optimal if and only if **I** chooses
   - an element in $[n - 2^{\lfloor \log_2(n+1) \rfloor} + 2, 2^{\lfloor \log_2(n+1) \rfloor} - 1]$ in L_n, or
   - an element in $[n - 2^{\lfloor \log_2(n+1) \rfloor} + 2, p - n + 2^{\lfloor \log_2(n+1) \rfloor} - 1]$ in L_p
3. Similarly, the set of **II**'s optimal replies can be computed

Algorithmic and complexity results

B. Khoussainov and J. Liu,
On Complexity of Ehrenfeucht-Fraïssé Games
Annals of Pure and Applied Logic, 2009

A. Montanari and A. Policriti and N. Vitacolonna,
An Algorithmic Account of Winning Strategies in Ehrenfeucht Games on Labeled Successor Structures (extended version: Ehrenfeucht Games on Labelled Successor Structures: Remoteness and Optimal Strategies, RR 02/09, Dept. of Math. and CS, University of Udine, Feb. 2009)
Proc. 12th LPAR, LNCS 3835, 2005

E. De Maria, A. Montanari, N. Vitacolonna,
Games on Strings with a Limited Order Relation
Proc. LFCS 2009, LNCS 5407

E. Pezzoli,
Computational Complexity of Ehrenfeucht-Fraïssé Games on Finite Structures
Proc. CSL 1998, LNCS 1584

EF-games on specific classes

- Equivalence relations (with/without colors)
- Embedded equivalence relations
- Trees (with level predicates)
- Labelled successor structures
- Labelled linear structures with a bounded ordering

Remark: equivalence relation(s) and the Gaifman graph

Equivalence relations: local strategy

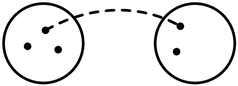
Definition

Structures $\mathcal{A} = (A, E)$, where E is an equivalence relation on A.

Definition

- For $m, n, t \in \mathbb{N}$, $m =_t n$ iff $m = n$ or both $m, n \geqslant t$
- $(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ is t-locally safe iff
  1. $\vec{a} \to \vec{b}$ is a partial isomorphism, and
  2. if $t > 0$, then $|[a_i] \setminus \{a_1, \ldots, a_k\}| =_t |[b_i] \setminus \{b_1, \ldots, b_k\}|$ for $i = 1, \ldots, k$.

When a position is t-locally safe, there is not incentive for **I** to play in a class that has already been chosen, in a game with at most t rounds.



1-locally safe, but not 2-locally safe

Equivalence relations: "small disparity"

- $q_t^{(\mathcal{A},\vec{a})}$: number of classes of size t in $\mathcal{A}$ not containing any a_i (free classes)
- Let $\Delta^{(\mathcal{A},\vec{a})}_{(\mathcal{B},\vec{b})} = \{\, t \mid q_t^{(\mathcal{A},\vec{a})} \neq q_t^{(\mathcal{B},\vec{b})} \,\}$
- Let $q_t = \min\{q_t^{(\mathcal{A},\vec{a})}, q_t^{(\mathcal{B},\vec{b})}\}$

Lemma

Given $(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ *and* $t \in \Delta^{(\mathcal{A},\vec{a})}_{(\mathcal{B},\vec{b})}$, ***I*** *can reach a position that is not* t*-locally safe after* $q_t + 1$ *rounds.*

Corollary

I *has a winning strategy in* $\leqslant q_t + 1 + t$ *rounds, with* $t \in \Delta^{(\mathcal{A},\vec{a})}_{(\mathcal{B},\vec{b})}$.

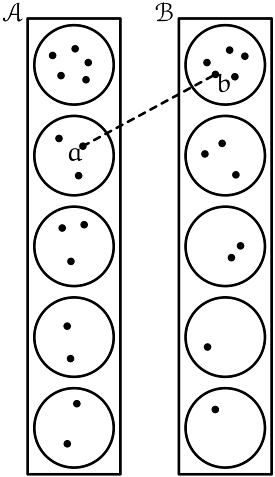
- **I** selects q_t distinct classes of size t ("global" moves)
- Then, he plays one more "global" move in a class of size t to which **II** cannot reply "appropriately"
- Then, he plays $\leqslant t$ rounds in the same class ("local" moves)

Example

- 2-locally safe, but not 3-locally safe

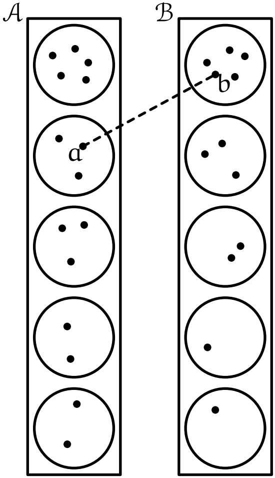
t	$q_t^{(\mathcal{A},a)}$	$q_t^{(\mathcal{B},b)}$
1	0	2
2	2	1
3	1	1
4	0	0
5	1	0

- $\Delta^{(\mathcal{A},a)}_{(\mathcal{B},b)} = \{1, 2, 5\}$



Example (cont.)

- $\Delta^{(\mathcal{A},a)}_{(\mathcal{B},b)} = \{1, 2, 5\}$
- $q_1 = 0$, $q_2 = 1$, $q_5 = 0$
- $1 \in \Delta^{(\mathcal{A},a)}_{(\mathcal{B},b)} \Rightarrow$ I can reach a not 1-locally safe configuration in $q_1 + 1 = 1$ round
- $2 \in \Delta^{(\mathcal{A},a)}_{(\mathcal{B},b)} \Rightarrow$ I can reach a not 2-locally safe configuration in $q_2 + 1 = 2$ rounds
- $5 \in \Delta^{(\mathcal{A},a)}_{(\mathcal{B},b)} \Rightarrow$ I can reach a not 5-locally safe configuration in $q_5 + 1 = 1$ round



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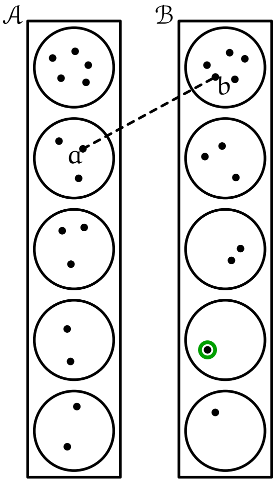

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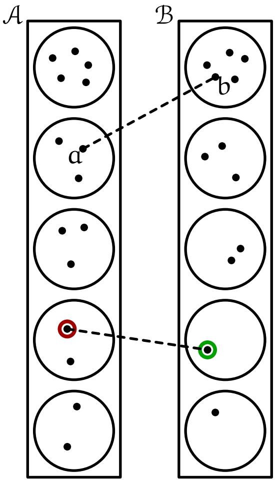

Example (cont.)

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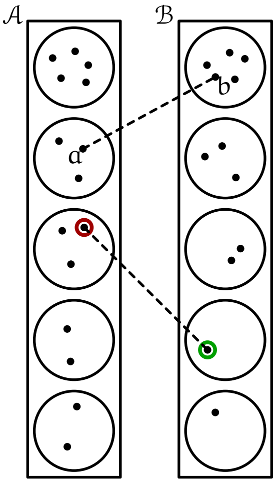

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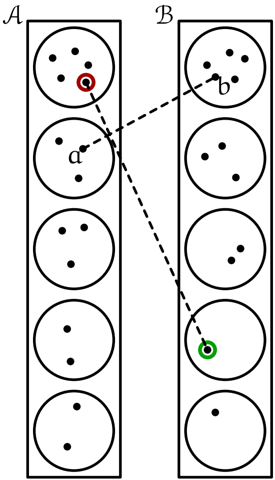

Example (cont.)

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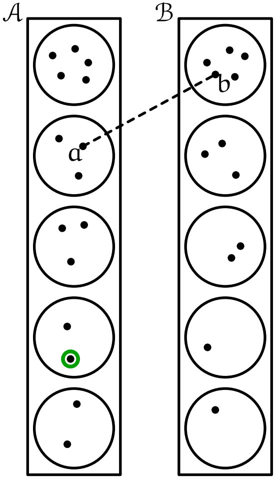

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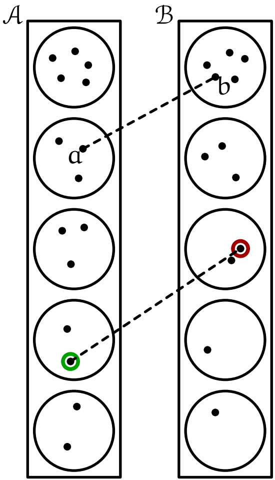

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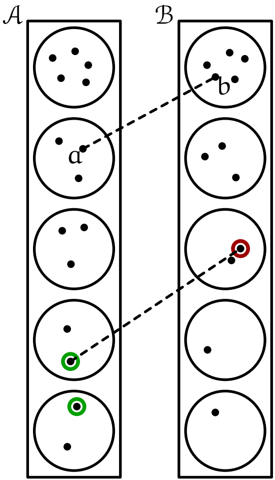

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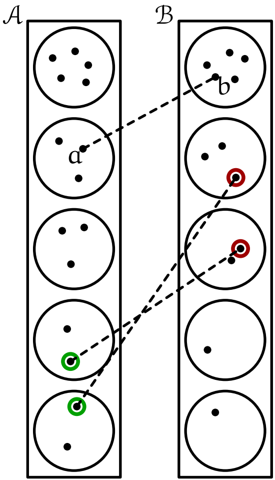

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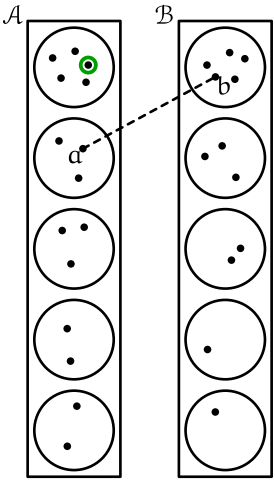

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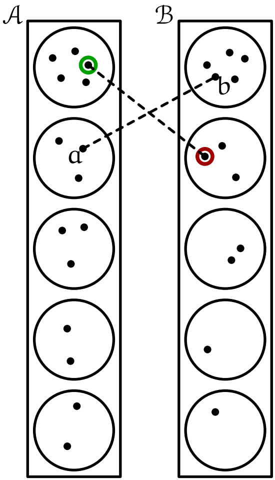


Equivalence relations: "large disparity"

- $q_{\geqslant t}^{(\mathcal{A},\vec{a})}$: number of free classes of size $\geqslant t$
- Let $\Gamma_{(\mathcal{B},\vec{b})}^{(\mathcal{A},\vec{a})} = \{\, t \mid q_{\geqslant t}^{(\mathcal{A},\vec{a})} \neq q_{\geqslant t}^{(\mathcal{B},\vec{b})} \,\}$
- Let $q_{\geqslant t} = \min\{q_{\geqslant t}^{(\mathcal{A},\vec{a})}, q_{\geqslant t}^{(\mathcal{B},\vec{b})}\}$

Lemma

Given $(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ *and* $t \in \Gamma_{(\mathcal{B},\vec{b})}^{(\mathcal{A},\vec{a})}$, ***I*** *can reach a position that is not* $(t-1)$*-locally safe after* $q_{\geqslant t} + 1$ *rounds.*

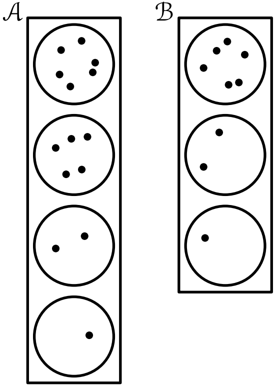
Corollary

I *has a winning strategy in at most* $q_{\geqslant t} + t$ *rounds, with* $t \in \Gamma_{(\mathcal{B},\vec{b})}^{(\mathcal{A},\vec{a})}$.

- **I** selects $q_{\geqslant t}$ distinct free classes of size $\geqslant t$ ("global" moves)
- Then, only one structure remains with a free class of size $\geqslant t$
- **I** plays t rounds in that class ("local" moves)

Example

- Initially, empty configuration
- Let $t = 3$
- Then $q_{\geqslant t} = 1$
- let **I** pick a free class with $\geqslant t$ elements
- **II** replies accordingly
- Now there is a free class of size $\geqslant t$ only in $\mathcal{A}$
- **II** replies with a "small" class
- **I** starts to play locally
- **II** must reply locally
- **I** wins
- $q_{\geqslant t} + t$ rounds needed



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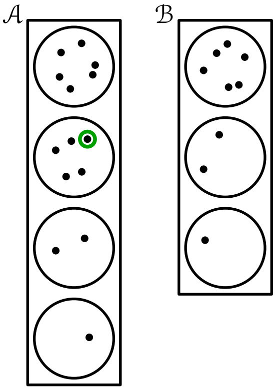

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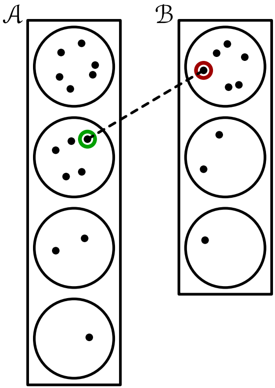

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- II must reply locally
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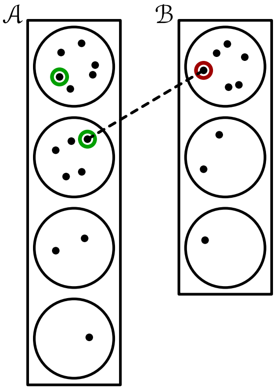

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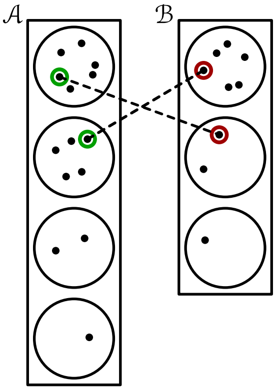

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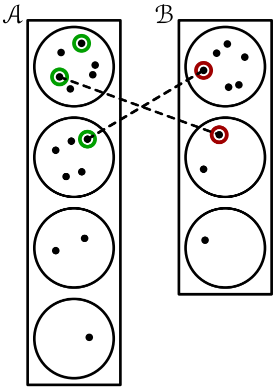

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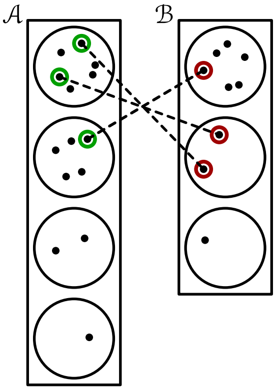

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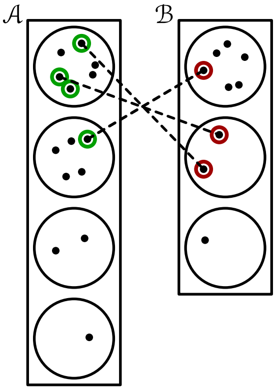


Equivalence relations: characterization

Definition

Given $(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ and $m \in \mathbb{N}$, $(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ is m-globally safe iff

- $q_t > m - t - 1$ for all $t \in \Delta^{(\mathcal{A}, \vec{a})}_{(\mathcal{B}, \vec{b})}$
- $q_{\geqslant t} > m - t$ for all $t \in \Gamma^{(\mathcal{A}, \vec{a})}_{(\mathcal{B}, \vec{b})}$

Theorem

***II** wins $\mathcal{G}_m(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ iff $(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ is m-locally safe and m-globally safe.*

Corollary

The remoteness of $\mathcal{G}(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ is the minimum between the minimum m such that $(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ is not m-locally safe and

$$\min \left\{ \min\{ t + q_{\geqslant t} \mid t \in \Gamma^{(\mathcal{A}, \vec{a})}_{(\mathcal{B}, \vec{b})} \}, 1 + \min\{ t + q_t \mid t \in \Delta^{(\mathcal{A}, \vec{a})}_{(\mathcal{B}, \vec{b})} \} \right\}.$$

The remoteness can be computed in $O(|A| + |B|)$ time and space.

Sketch of the proof

Theorem

II *wins* $\mathcal{G}_m(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ *iff* $(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ *is* m*-locally safe and* m*-globally safe.*

- If a position is m-locally safe and **I** play a local move, then **II** can reach a position $(m-1)$-locally safe
- If a position is m-globally safe, then **II** can reach a position $(m-1)$-globally safe
    - The only tricky case is when **I** chooses an element in a free class of size $t \in \Delta^{(\mathcal{A}, \vec{a})}_{(\mathcal{B}, \vec{b})}$ or $t \in \Gamma^{(\mathcal{A}, \vec{a})}_{(\mathcal{B}, \vec{b})}$
    - But, m-global safety allows **II** to reply properly

The result easily extends to structures colored homogeneously, i.e., if $E(x, y)$ then $x \in P \Leftrightarrow y \in P$, for all $x, y \in A$ and unary predicate P

Equivalence structures with one color

Definition

Structures $\mathcal{A} = (A, E, P)$, where E is an equivalence relation on A and P is a unary predicate.

Definition

- Let $P[a_i]$ be the set of elements $a_j \in [a_i]$ such that $P(a_j)$ holds ("a_j is colored")
- Let $\bar{P}[a_i]$ be the set of elements $a_j \in [a_i]$ such that $\neg P(a_j)$ holds ("a_j is non-colored")
- Type of an equivalence class X of $\mathcal{A}$: $tp(X) = (i, j)$, where i is the number of elements e in X such that $P(e)$ holds and j is the number of elements e in X such that $\neg P(e)$ holds

Colored and non-colored equivalences

Remind: for $m, n, t \in \mathbb{N}$, $m =_t n$ iff $m = n$ or both $m, n \geqslant t$

Definition

Two types (i, j) and (i', j') are *colored* n*-equivalent*, denoted by $(i, j) \equiv_n^C (i', j')$ if, and only if,

1. $i =_n i'$
2. $j =_{n-1} j'$

Definition

Two types (i, j) and (i', j') are *non-colored* n*-equivalent*, denoted by $(i, j) \equiv_n^N (i', j')$ if, and only if,

1. $i =_{n-1} i'$
2. $j =_n j'$

Lemma

If either $(i, j) \equiv_n^C (i', j')$ *or* $(i, j) \equiv_n^N (i', j')$*, then* $(i, j) \equiv_{n-1}^C (i', j')$ *and* $(i, j) \equiv_{n-1}^N (i', j')$

Counting up to colored equivalence

- For structures $\mathcal{A}$ and $\mathcal{B}$, type (i, j), and $k \geqslant 1$,

$$C^{\mathcal{A}}_{(i,j),k} \stackrel{\text{def}}{=} \{ X \mid X \text{ is an equivalence class of } \mathcal{A} \text{ and } \mathrm{tp}(X) \equiv^{C}_{k} (i, j) \}$$

- Let $q^{\mathcal{A},C}_{(i,j),k} \stackrel{\text{def}}{=} |C^{\mathcal{A}}_{(i,j),k}|$
- Let $q^{C}_{(i,j),k} \stackrel{\text{def}}{=} \min(q^{\mathcal{A},C}_{(i,j),k}, q^{\mathcal{B},C}_{(i,j),k})$
- Let $\mathcal{A}^{C}((i, j), k)$ be the structure obtained from $\mathcal{A}$ by removing $q^{C}_{(i,j),k}$ equivalence classes in $C^{\mathcal{A}}_{(i,j),k}$
- $N^{\mathcal{A}}_{(i,j),k}$, $q^{\mathcal{A},N}_{(i,j),k}$, $q^{N}_{(i,j),k}$ and $\mathcal{A}^{N}((i, j), k)$ are defined similarly w.r.t. $\equiv^{N}_{k}$

Colored and non-colored disparity

Definition

We say that a *colored disparity* occurs in a game $\mathcal{G}_n(\mathcal{A}, \mathcal{B})$ if there exists a type (i, j) and $n > k \geqslant 0$ such that the following holds:

1. $k = q^C_{(i,j),n-k}$
2. In one of $\mathcal{A}^C((i, j), n - k)$ and $\mathcal{B}^C((i, j), n - k)$, there is an equivalence class whose type is colored $(n - k)$-equivalent to (i, j), and no such equivalence class exists in the other structure

Non-colored disparity is defined in a similar way.

Theorem

II *has a winning strategy in* $\mathcal{G}_n(\mathcal{A}, \mathcal{B})$ *if and only if neither colored disparity nor non-colored disparity occurs.*

Proof's idea

($\Rightarrow$)

- Assume that colored disparity occurs for some (i, j) and k
- W.l.o.g, suppose that in $\mathcal{A}^C((i, j), n - k)$ there is an equivalence class whose type is colored $(n - k)$-equivalent to (i, j), and no such class exists in $\mathcal{B}^C((i, j), n - k)$
- First, **I** chooses $k = q^C_{(i,j),n-k}$ mutually non-equivalent elements in $C^{\mathcal{A}}_{(i,j),n-k}$
- Then, **I** selects a colored element in a class X with $tp(X) \equiv^C_{n-k} (i, j)$ and he plays the rest of the game inside it ($n - k - 1$ rounds suffice to spot the difference)

Proof's idea (contn'd)

$(\Leftarrow)$

- We describe a winning strategy for **II** that maintains the invariant below.

  Let $(a_1, b_1), \ldots, (a_k, b_k)$ be the result of the played k-round game and, for $1 \leqslant l \leqslant k$, let (i_l, j_l) and (i'_l, j'_l) be respectively the types of a_l and b_l.

  We show that the following invariant is preserved:

  1. for $1 \leqslant l \leqslant k$, a_l is colored iff b_l is colored
  2. for $1 \leqslant l, m \leqslant k$, $E(a_l, a_m)$ iff $E(b_l, b_m)$
  3. for $1 \leqslant l \leqslant k$, $(i_l, j_l) \equiv^{C}_{n-l} (i'_l, j'_l)$ and $(i_l, j_l) \equiv^{N}_{n-l} (i'_l, j'_l)$
  4. $\mathcal{G}_{n-k}$ has neither colored disparity nor non-colored disparity

Proof's idea (contn'd)

- Assume that **I** selects an element $a_{k+1} \in A$.
  - If $E(a_{k+1}, a_l)$, for some $1 \leqslant l \leqslant k$, then **II** chooses a new $b_{k+1} \in B$ such that $E(b_{k+1}, b_l)$ and b_{k+1} is colored iff a_{k+1} is colored (the existence of such a b_{k+1} is guaranteed by item (3) above
  - If a_{k+1} is a colored (resp. non-colored) element belonging to a class X, which differs from $[a_1], \ldots, [a_k]$, then item (4) above guarantees that there exists a colored (resp, non-colored) element $b_{k+1} \in B$ belonging to a class Y such that Y differs from $[b_1], \ldots, [b_k]$ and $tp(X) \equiv^C_{n-k} tp(Y)$ (resp., $tp(X) \equiv^N_{n-k} tp(Y)$)

Proof's idea (contn'd)

- It can easily shown that both item (1) and item (2) hold for $(a_1, b_1), \ldots, (a_{k+1}, b_{k+1})$

- As for item (3), let (i_{k+1}, j_{k+1}) and (i'_{k+1}, j'_{k+1}) be the types of $[a_{k+1}]$ and $[b_{k+1}]$, respectively. The above strategy guarantees that one of $(i_{k+1}, j_{k+1}) \equiv^C_{n-k} (i'_{k+1}, j'_{k+1})$ and $(i_{k+1}, j_{k+1}) \equiv^N_{n-k} (i'_{k+1}, j'_{k+1})$ holds and thus $(i_{k+1}, j_{k+1}) \equiv^C_{n-k-1} (i'_{k+1}, j'_{k+1})$ and $(i_{k+1}, j_{k+1}) \equiv^N_{n-k-1} (i'_{k+1}, j'_{k+1})$.

- The fact that item (4) is preserved as well can be easily proved by contradiction.

Hence, by item (1) and item (2), the strategy is a winning strategy for **II**.

Embedded equivalence structures: local strategy

Definition

Structures $\mathcal{A} = (A, E_1, \ldots, E_h)$, where each E_i is an equivalence relation on A and $E_i \subseteq E_j$ for $i < j$.

- We consider the case $h = 2$
- Let $\mathcal{A} = (A, E_1, E_2)$ and $\mathcal{B} = (B, E_1, E_2)$

Definition

A local game on $(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ is a game played only within non-free equivalence classes, i.e., classes containing some $a_i \in \vec{a}$ or $b_i \in \vec{b}$.

Definition

$(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ is t-locally safe iff **II** has a winning strategy in the t-round local game on $(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$.

- t-round local games are characterized as in “flat” equivalence games

Embedded equivalence structures: global strategy

Definition

- Type of an E_2-class X of $\mathcal{A}$: $\mathrm{tp}(X) = (q_1, \ldots, q_t)$, if the largest E_1-equivalence class in X has size t and, for all $1 \leqslant i \leqslant t$, q_i is the number of E_1-classes of size i in X
- $\mathrm{tp}(X) \equiv_t \mathrm{tp}(Y)$ iff **II** wins $\mathcal{G}_t((X, E_1 \restriction X), (Y, E_1 \restriction Y))$
- (Free) t-multiplicity of type σ in $(\mathcal{A}, \vec{a})$:

$$q_{\sigma,t}^{(\mathcal{A},\vec{a})} \stackrel{\text{def}}{=} \left|\{\, Y \mid Y \text{ is a free } E_2\text{-class of } (\mathcal{A}, \vec{a}) \wedge \mathrm{tp}(Y) \equiv_t \sigma \,\}\right|$$

- $\Delta_{(\mathcal{B},\vec{b})}^{(\mathcal{A},\vec{a})} = \{\, (\sigma, t) \mid q_{\sigma,t}^{(\mathcal{A},\vec{a})} \neq q_{\sigma,t}^{(\mathcal{B},\vec{b})} \}$

Lemma

Given $(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ *and* $(\sigma, t) \in \Delta_{(\mathcal{B},\vec{b})}^{(\mathcal{A},\vec{a})}$, ***I*** *has a winning strategy in* $\min\{\, q_{\sigma,t}^{(\mathcal{A},\vec{a})}, q_{\sigma,t}^{(\mathcal{B},\vec{b})} \,\} + 1 + t$ *rounds.*

- A complete characterization can be given

Trees with height h

Definition

A tree $\mathcal{T}$ is a pair $(T, \preceq)$ where

1. $\preceq$ is a partial ordering with a unique minimum
2. for all $x \in T$, $\{y \mid y \preceq x\}$ is finite and linearly ordered
3. maximal elements are leaves
4. Level of a node: distance from the root
5. Height of $\mathcal{T}$: number of levels -1

- $\mathcal{K}_h$: class of trees of height h
- $x \preceq y$ iff x is an ancestor of y
- The idea of Khoussainov and Liu's paper is to map $\mathcal{K}_h$ into the class of embedded equivalence relations of height h
- Sounds nice!
- Unfortunately, it does not work (without a level predicate)

Mapping trees onto embedded equivalences

- $T' \stackrel{\text{def}}{=} T \cup \{\, a_x \mid x \text{ is a leaf of } \mathcal{T} \,\}$
- E_1: minimal equivalence containing $\{\, (x, a_x) \mid x \text{ is a leaf of } \mathcal{T} \,\}$
- E_{i+1}: minimal equivalence containing $E_i \cup (T_1 \times T_1) \cup \cdots \cup (T_k \times T_k)$, where $T_1, \ldots, T_k$ are the subtrees rooted at nodes of level $h - i + 1$
- $E_i \subseteq E_{i+1}$ (E_i is finer than E_{i+1})
- Embedded equivalence structure induced by $\mathcal{T}$:

$$\mathcal{A}(\mathcal{T}) \stackrel{\text{def}}{=} (T', E_1, \ldots, E_h)$$

Claim

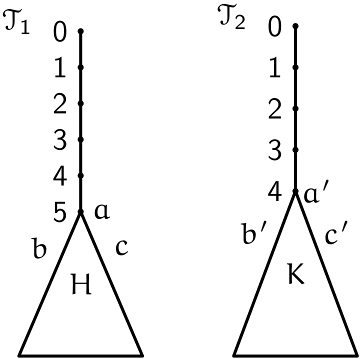
1. $\mathcal{T}_1 \cong \mathcal{T}_2$ *iff* $\mathcal{A}(\mathcal{T}_1) \cong \mathcal{A}(\mathcal{T}_2)$ *(ok!)*
2. ***II*** *wins* $\mathcal{G}_m(\mathcal{T}_1, \mathcal{T}_2)$ *iff* ***II*** *wins* $\mathcal{G}_m(\mathcal{A}(\mathcal{T}_1), \mathcal{A}(\mathcal{T}_2))$ *(wrong!)*

Why it does not work

Claim (wrong)

***II** wins* $\mathcal{G}_m(\mathcal{T}_1, \mathcal{T}_2)$ *iff* ***II** wins* $\mathcal{G}_m(\mathcal{A}(\mathcal{T}_1), \mathcal{A}(\mathcal{T}_2))$.

- Observe that $x \preceq y$ iff x has level t, y has level $s \geqslant t$ and $E_{h-t+1}(x, y)$
- Every winning strategy for **II** in $\mathcal{G}_m(\mathcal{A}(\mathcal{T}_1), \mathcal{A}(\mathcal{T}_2))$ must map elements at level k in $\mathcal{A}(\mathcal{T}_1)$ to elements at level k in $\mathcal{A}(\mathcal{T}_2)$
- How to fix the correspondence? Enrich the tree structure with a level predicate



Labelled successor structures (LSS)

- Let Σ be a finite alphabet
- Let $u \in \Sigma^*$ be a word on Σ
- Let $u[i]$ be the ith letter of u

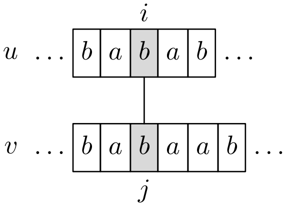
Definition

A (labelled) successor structure is a pair $(u, \overrightarrow{i})$, where the elements of $\overrightarrow{i}$ are distinguished indices of u.
Successor structures $(u, \overrightarrow{i})$ interpret FO-formulas $\phi(\overrightarrow{x})$ in the vocabulary $(=, s, (P_a)_{a \in \Sigma})$ according to the following rules:

$$
\begin{aligned}
(u, \overrightarrow{i}) &\models x_h = x_l && \text{if } i_h = i_l; \\
(u, \overrightarrow{i}) &\models s(x_h, x_l) && \text{if } i_l = i_h + 1; \\
(u, \overrightarrow{i}) &\models P_a(x_h) && \text{if } u[i_h] = a.
\end{aligned}
$$

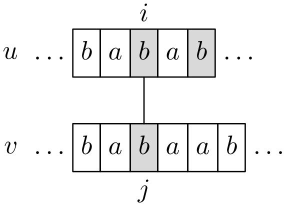
Warm up example

$$(u, i) \qquad (v, j) \qquad u, v \in \{a, b\}^*$$



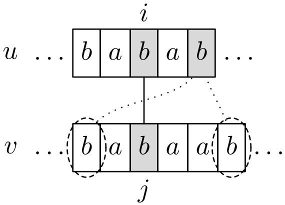
Warm up example

$$(u, i) \qquad (v, j) \qquad u, v \in \{a, b\}^*$$



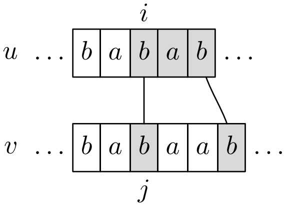
Warm up example

$(u, i) \qquad (v, j) \qquad u, v \in \{a, b\}^*$

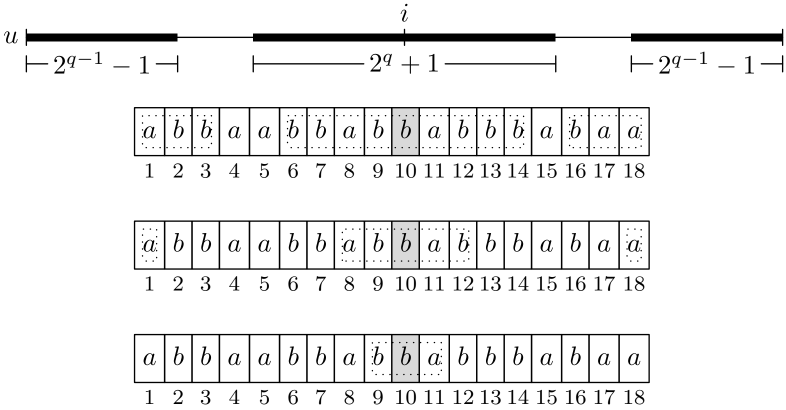


Warm up example

$(u, i) \qquad (v, j) \qquad u, v \in \{a, b\}^*$



Local strategy



- The number of remaining rounds q determines a q-entailing region (dashed lines) where **II**'s strategy is "rigid":
  - the prefix and suffix of length $2^{q-1}-1$
  - the factors of radius 2^{q-1} centered at distinguished indices
- q-free region: the complement of the entailing region

Local conditions

$$\eta_d(i,j) = \begin{cases} j - i & \text{if } |i-j| \leqslant d; \\ \infty & \text{otherwise.} \end{cases}$$

Definition

A configuration $(u, \vec{i}, v, \vec{j})$ is t-locally safe iff, for all $i_h, i_l \in \vec{i}$,

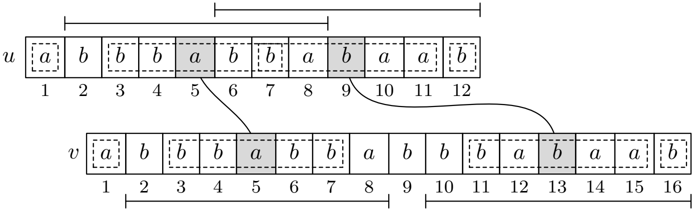
❶ $\eta_{2^t}(i_h, i_l) = \eta_{2^t}(j_h, j_l)$

❷ $\mathcal{N}^u_{2^t-1}(i_h) = \mathcal{N}^v_{2^t-1}(j_h)$

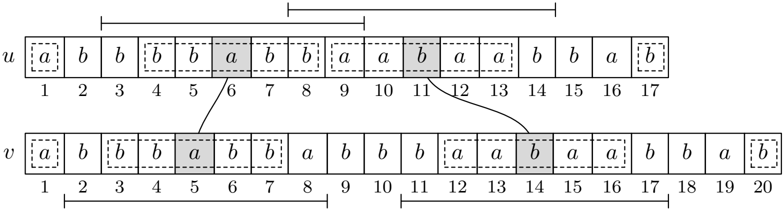
- If a configuration is not t-locally safe, **I** has a "local" winning strategy in t rounds
- **II** can turn a t-locally safe configuration into a $(t-1)$-locally safe configuration if **I** plays "locally"

Local safety: an example

Not 2-locally safe:



2-locally safe:

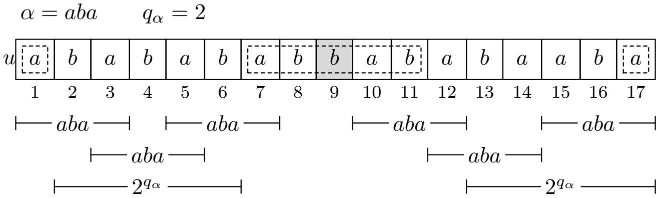


Free factors

Definition

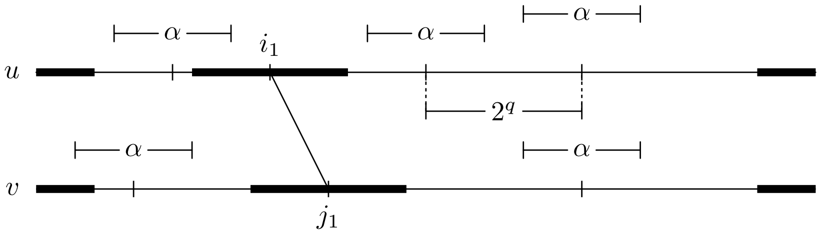
- Let α be a word of length $2^t - 1$
- An occurrence of α centered at index k in $(u, \vec{i})$ is free iff $|k - \vec{i}| > 2^{t-1}$, that is, k falls inside the t-free region of $(u, \vec{i})$
- (Free) multiplicity of α in $(u, \vec{i})$: number of free occurrences of α in $(u, \vec{i})$
- Scattering of α in $(u, \vec{i})$: cardinality of a maximal 2^t-scattered subset of the free occurrences of α in $(u, \vec{i})$
- (A set $X \in \mathbb{N}$ is d-scattered iff $|x - y| > d$ for all $x, y \in X$)

Multiplicity and scattering: an example



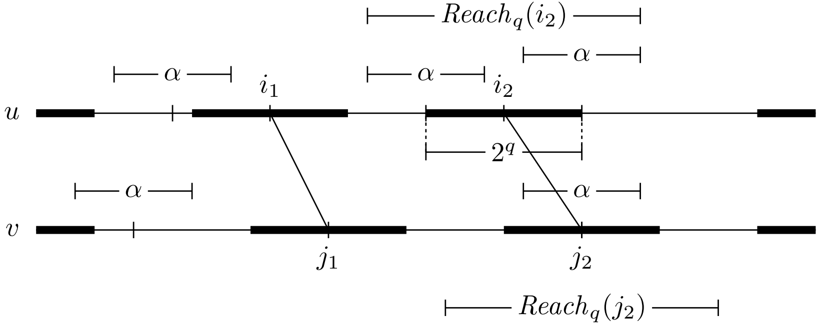
- Let $\alpha = aba$ ($t = 2$)
- Centers of free occurrences of aba in $(u, 9)$: $\{2, 4, 6, 13, 16\}$
- Multiplicity: 5
- Scattering: 2 ($\{2, 4, 6\}$, $\{13, 16\}$)
- Note that the scattering is the cardinality of a coarsest partition of the free occurrences in which each class contains elements at mutual distance $\leqslant 2^t$

An instance of the global strategy



- Let $|\alpha| = 2^q - 1$
- Let the thick lines denote the q-entailing region
- α has the same scattering (2) but different multiplicity (resp, 3 and 2) in u and v

An instance of the global strategy (cont.)



- **I** chooses i_2, causing two α-occurrences to fall inside the 2^q-entailing region
- **II** replies with j_2, "capturing" only one α-occurence
- In the new configuration, **II** has obtained that the multiplicity and scattering of α are the same
- But the position is necessarily not q-locally safe

Global safety

- Let $p_\alpha^{(u,\vec{i})}$ denote the free multiplicity
- Let $q_\alpha^{(u,\vec{i})}$ denote the scattering
- Let $\Delta_{(v,\vec{j})}^{(u,\vec{i})} = \{ \alpha \mid p_\alpha^{(u,\vec{i})} \neq p_\alpha^{(v,\vec{j})} \vee q_\alpha^{(u,\vec{i})} \neq q_\alpha^{(v,\vec{j})} \}$
- $\Delta_{(v,\vec{j})}^{(u,\vec{i})}$ is the set of words that **I** can potentially exploit in order to win
- All words $\alpha \in \Delta_{(v,\vec{j})}^{(u,\vec{i})}$ have length $2^t - 1$ for some t
- Let $q_\alpha = \min\{ q_\alpha^{(u,\vec{i})}, q_\alpha^{(v,\vec{j})} \}$

Definition

A configuration $(u, \vec{i}, v, \vec{j})$ is m-globally safe iff $q_\alpha > m - \log_2(|\alpha| + 1)$ for all words $\alpha \in \Delta_{(v,\vec{j})}^{(u,\vec{i})}$.

Intuition: there are enough scattered free occurrences of α in both structures for each α that **I** might use to win

LSS: Characterization

Lemma

Given $(u, \vec{i}, v, \vec{j})$ *and* $\alpha \in \Delta^{(u,\vec{i})}_{(v,\vec{j})}$, ***I*** *can reach a position not* $(t_\alpha - 1)$*-locally safe after* $q_\alpha + 1$ *rounds, with* $t_\alpha = \log_2(|\alpha| + 1)$.

Corollary

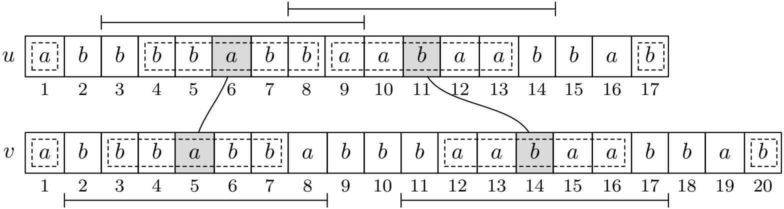
I *has a winning strategy in* $\leqslant q_\alpha + t_\alpha$ *rounds, with* $\alpha \in \Delta^{(u,\vec{i})}_{(v,\vec{j})}$.

- **I** plays q_α rounds with the goal of decreasing the scattering of α by 1 in each round
- After that, if the position is not t_α-locally safe, **I** plays locally
- Otherwise, the scattering of α must be 0 in u and positive in v (or vice versa)
- **I** plays to decrease the scattering of α where it is positive
- After that, the position is certainly not $(t_\alpha - 1)$-locally safe

Theorem

II *has a winning strategy in* $\mathcal{G} = \mathcal{G}_m(u, \vec{i}, v, \vec{j})$ *iff* $\mathcal{G}$ *is* m*-locally safe and* m*-globally safe.*

Example 1



	α	$p_\alpha^{(u,6,11)}$	$p_\alpha^{(v,5,14)}$	$q_\alpha^{(u,6,11)}$	$q_\alpha^{(v,5,14)}$
$q = 1$	a	4	5	4	5
	b	7	9	4	5
$q = 2$	abb	2	3	2	3
	bab	1	2	1	2
	bba	1	2	1	2
	bbb	1	1	1	1

It is also 2-globally safe!

Example 2

Who wins the 2-round game between
$u = abbfbfbbaydddddaba$ and
$v = abaydddabbbbbbfbfbba$?

	α	p^u_α	p^v_α	q^u_α	q^v_α
$q = 1$	a	4	4	3	3
	b	7	9	4	5
	d	5	4	2	2
	f	2	2	1	1
	y	1	1	1	1
$q = 2$	aba, abb, ayd, bay, bba, bbf, dab, dda, fbb, fbf, ydd	1	1	1	1
	bbb	1	5	1	2
	bfb	2	2	1	1
	ddd	3	2	1	1

Definability and m-equivalence

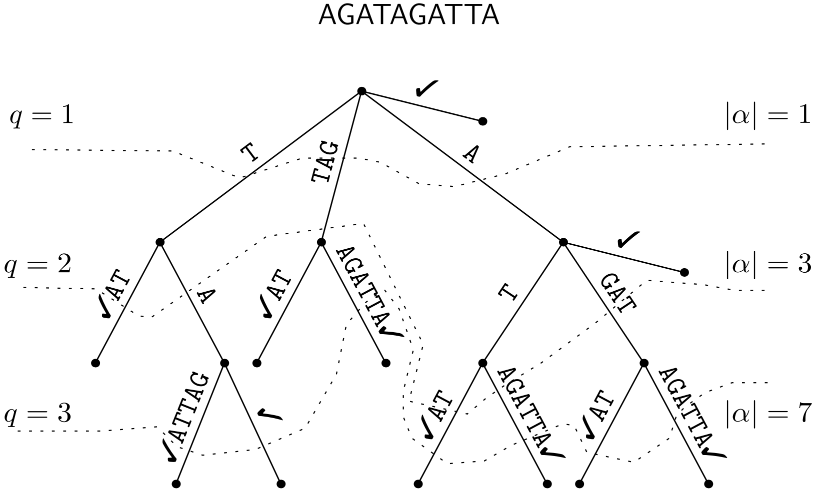
$\mathcal{L}$	Definable class	m-equivalence
FO(s)	threshold locally testable	Previous theorem

- From FO(s) to FO($<$):

$$\mathrm{FO}(<_p), \text{ where } x <_p y \Leftrightarrow 0 < y - x \leqslant p.$$

$\mathcal{L}$	Definable class	m-equivalence
FO($<$)	$*$-free	$<_p$, with $p \to \infty$

Suffix trees



Features of suffix trees

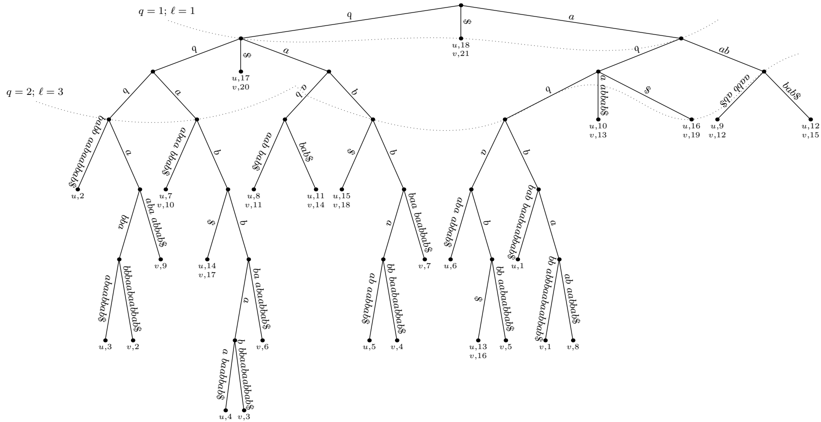
"Miryad virtues" (Apostolico)

- Space $\propto$ sequence size
- Build time $\propto$ sequence size
- Fast motif search
- Fast repeat detection
- Longest common prefix queries
- etc...

One defect

- No approximate queries

Testing $\equiv_m$ with generalized suffix trees



- Let $n = |u| + |v|$
- Remoteness of $\mathcal{G}(u, v)$: $O(n \log n)$ time and space
- **I**'s optimal moves: $O(n^2 \log n)$ time, $O(n \log n)$ space
- **II**'s optimal moves: $O(n)$ time and space (if the remoteness is known)

An emerging pattern

Let $\mathcal{A}$ and $\mathcal{B}$ arbitrary structures.

Definition

A t-round local game on $(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ is a game played on $\mathcal{N}^{\mathcal{A}}_{2^t-1}(\vec{a})$ and $\mathcal{N}^{\mathcal{B}}_{2^t-1}(\vec{b})$ such that, at round $t-k+1$, with $1 \leqslant k \leqslant t$, **I** must choose an element at distance at most 2^{k-1} from $\vec{a}$ or from $\vec{b}$.

Definition

A configuration $(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ is t-locally safe if **II** has a winning strategy in the t-round local game on $(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$.

- We write $(\mathcal{A}, \vec{a}) \equiv^{\text{loc}}_t (\mathcal{B}, \vec{b})$
- **II** can play t rounds provided that **I** plays "near" distinguished elements (nearer and nearer after each round)

How to count neighbourhoods?

- The analysis of equivalence structures shows that we need to count up to isomorphism and up to $\equiv_t^{\mathsf{loc}}$-equivalence (in equivalence structures, neighbourhoods coincide with equivalence classes; two equivalence classes are isomorphic iff they have the same number of elements and they are $\equiv_t^{\mathsf{loc}}$-equivalent iff they both have at least t elements)
- The analysis of labelled successor structures shows that we need to count both the (free) multiplicity and the scattering of neighbourhoods (for equivalence structures, the two notions collapse into one)

Conjecture

Counting the multiplicity and scattering of "small" neighbourhoods up to isomorphism and up to $\equiv_t^{loc}$-equivalence is enough for characterizing the "global" winning strategy for arbitrary structures.

Strings with a bounded ordering

- Let Σ be a finite alphabet
- Let $u \in \Sigma^*$ be a word on Σ
- For a given p, let $<_p$ be a bounded ordering on $\mathbb{N}$, that is, $i <_p j$ iff $0 < j - i < p$
- Every word u induces a relational structure

$$\mathcal{U} = (\{1, \ldots, |u|\}, =, <_p, (P_a)_{a \in \Sigma})$$

such that, for $\vec{i} = i_1, \ldots, i_k$,

$$\begin{aligned}
(\mathcal{U}, \vec{i}) &\models x_h = x_l && \text{if } i_h = i_l; \\
(\mathcal{U}, \vec{i}) &\models x_h <_p x_l && \text{if } i_l <_p i_h; \\
(\mathcal{U}, \vec{i}) &\models P_a(x_h) && \text{if } u[i_h] = a.
\end{aligned}$$

A few conventions

- In what follows, we assume that p is fixed (we omit it in the notation)
- We write $(u, \vec{i})$ for $(\mathcal{U}, \vec{i})$
- To be able to treat prefixes and suffixes of word u uniformly, we assume that u is extended infinitely on both sides with a symbol $\$ \notin \Sigma$:
$$\cdots \$\$\$\, u\, \$\$\$ \cdots$$
- Besides, we assume that in every $(u, \vec{i})$ positions $-p$ and $|u| + p + 1$ are always distinguished elements
- Therefore, every partial isomorphism between words u and v maps $-p$ to $-p$ and $|u| + p + 1$ to $|v| + p + 1$

Note: For the time being, we ignore the unary predicates and focus only on $<_p$

Games on strings with a bounded ordering

Let η_d be the difference up to a threshold between two positions:

$$\eta_d(i,j) = \begin{cases} j - i & \text{if } |i-j| \leqslant d \\ \infty & \text{otherwise} \end{cases}$$

Let π_d be the oriented distance up to a threshold in the Gaifman graph of $\mathcal{U}$:

$$\pi_d(i,j) = sgn(j-i) \cdot \left\lceil \frac{|\eta_d(i,j)|}{p} \right\rceil$$

Example: For $p = 2$,

- $\pi_4(1,2) = \pi_4(1,3) = 1$ and $\pi_4(2,1) = \pi_4(3,1) = -1$
- $\pi_4(1,4) = \pi_4(1,5) = 2$ and $\pi_4(4,1) = \pi_4(5,1) = -2$
- $\pi_4(1,6) = \pi_4(6,1) = \infty$
- $\pi_4(-p,1) = p$

Remarks:

- For every p and $d > p$, $\pi_d(-p,1) = \lceil (1+p)/p \rceil = 2$
- Similarly, $\pi_d(|u|, |u| + p + 1) = 2$

Local strategy: first approximation

- For a given q, the q-entailing region of $(u, \vec{i})$ is defined as for LSSs (using the distance in the Gaifman graph)
- That is, a position k is in the q-entailing region of $(u, \vec{i})$ iff the number of "$<_p$-steps" between k and some $i \in \vec{i}$ is $\leqslant 2^q$
- Equivalently, $|k - i| \leqslant p \cdot 2^q$ for some $i \in \vec{i}$ (where i may be one of the two "spurious" indexes)

Definition

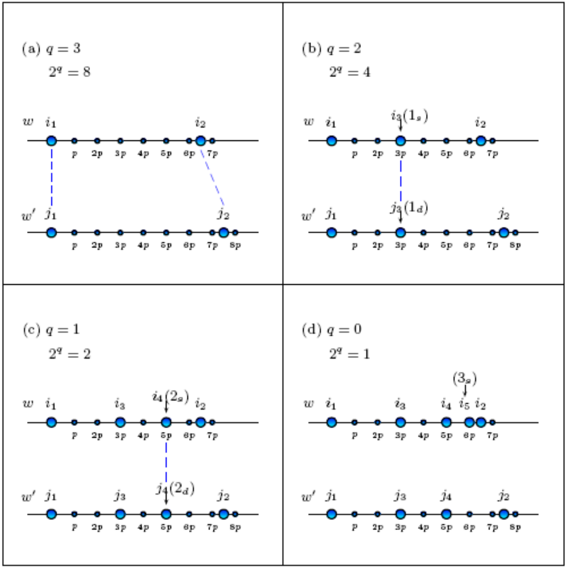
A configuration $(u, \vec{i}, v, \vec{j})$ is t-step safe iff $\pi_{p \cdot 2^t}(i_h, i_l) = \pi_{p \cdot 2^t}(j_h, j_l)$ for all $i_h, i_l \in \vec{i}$

Lemma

If $(u, \vec{i}, v, \vec{j})$ is not t-step safe, ***I*** *wins in t rounds.*

- **I** can spot a difference in the number of "$<_p$-steps" between two pairs of distinguished elements
- **II** can turn a t-step safe configuration into a $(t-1)$-step safe configuration if **I** plays in the t-entailing region

Example



Very local strategy

- t-step safety is essentially a necessary condition for (generalized) successor structures
- But $<_p$ shares some features of $<$, too
- In particular, if i_h and i_l are "very close" to each other, that is, $|i_h - i_l| \leqslant 2^q - 1$ for some q, then **I** wins in q rounds if $i_l - i_h \neq j_l - j_h$
- This is the same property as for $<$ on linear orderings
- In fact, it does not depend on p

Definition

A configuration $(u, \vec{i}, v, \vec{j})$ is η_d-safe iff $\eta_d(i_h, i_l) = \eta_d(j_h, j_l)$ for all $i_h, i_l \in \vec{i}$

Lemma

If $(u, \vec{i}, v, \vec{j})$ *is not* η_{2^t-1}*-safe,* ***I*** *wins in* t *rounds.*

Rigid and elastic intervals

- Let $q > 1$
- Let $r_0 = 2^{q-1} - 1$
- For $k > 0$, let

$$r_k = 1 + \sum_{r=\lceil \log_2 k \rceil}^{q-2} (2^r - 1) = 2^{q-1} - 2^{\lceil \log_2 k \rceil} - q + \lceil \log_2 k \rceil + 2$$

Definition

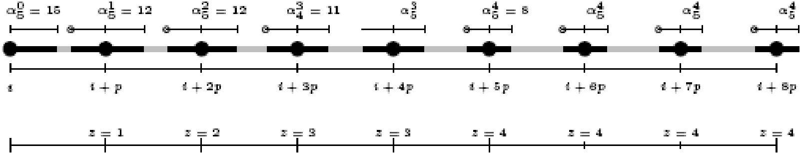
Let i be a distinguished index in $(u, \vec{i})$ and let $q > 1$. The 0th q-rigid interval induced by i is $[i - r_0, i + r_0]$. For $0 < k \leqslant 2^{q-2}$, the kth right q-rigid interval induced by i is

$$(i + kp - r_k, i + kp + r_k]$$

The kth right q-elastic interval is the interval between the $(k-1)$th and the kth q-rigid interval. Left intervals are defined similarly.

Rigid and elastic intervals: intuition

Rigid (black) and elastic (gray) right intervals for $q = 5$ induced by i:



- If **I** chooses i inside a rigid interval induced by some distinguished element i_h then **II** must reply with j such that $j - j_h = i - i_h$ (i.e., at the same relative position)
- If **I** plays in the kth elastic interval instead, **II** must reply in the corresponding kth elastic interval in the opposite structure, but not necessarily at exactly the same relative position
- Elastic intervals exist only when p is large wrt q

Local strategy: characterization

Definition

A configuration $(u, \vec{i}, v, \vec{j})$ is t-distance safe iff, for all $i_h, i_l \in \vec{i}$:

1. $(u, \vec{i}, v, \vec{j})$ is t-step safe
2. whenever i_h is in some t-rigid interval induced by i_l or j_h is in some t-rigid interval induced by j_l then $i_h - i_l = j_h - j_l$

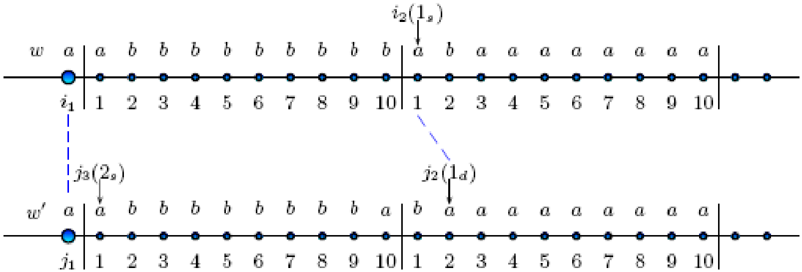
- Condition 1 captures the requirement that **II** must reply within a specific elastic interval when **I** chooses an element inside an elastic interval
- Condition 2 adds the stronger constraint for rigid intervals
- Condition 2 subsumes η_{2^t-1}-safety

Theorem

If $(u, \vec{i}, v, \vec{j})$ *is not* t*-distance safe,* ***I*** *wins in* t *rounds.*

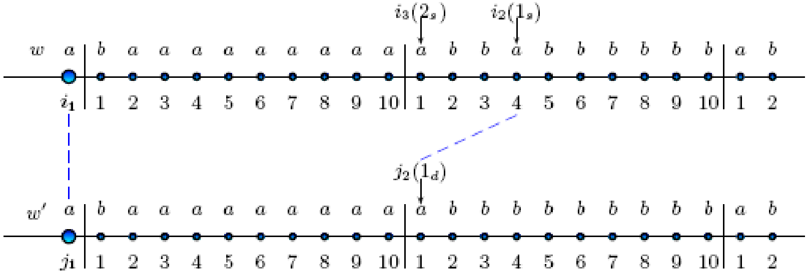
Vice versa, if $(u, \vec{i}, v, \vec{j})$ is t-distance safe and **I** plays in the t-entailing region, then **II** can reach a configuration that is $(t-1)$-distance safe

What if we add labels?



- Starting configuration is (w, i_1, w', j_1) and $p = 10$
- We play a 2-round game
- **I** plays i_2, which is inside the first 2-rigid interval induced by i_1
- **II** is forced to reply in a way that does not respect 2-distance safety
- **I** wins by playing j_3, because $j_1 \leqslant_p j_3 \not\leqslant_p j_2$ and there no i_3 in w such that $i_1 \leqslant_p i_3 \not\leqslant_p i_2$

What if we add labels?



- Starting configuration is (w, i_1, w', j_1) and $p = 10$
- We play a 2-round game
- **I** chooses i_2, which is inside a 2-elastic interval induced by i_1
- But the corresponding elastic interval in w' contains only b's
- Hence, **II** is forced to play elsewhere (j_2), violating 2-distance safety
- **I** wins by choosing i_3, so that $i_1 \not<_p i_3$ and $i_3 <_p i_2$
- There is no j_3 in w' such that $j_1 \not<_p j_3$ and $j_3 <_p j_2$

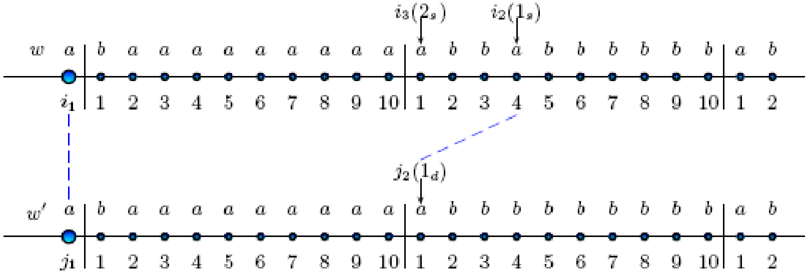
q-colors

- To characterize local strategies in the presence of labels, we need to describe the "type" of each element in terms of how such element is labelled and what there is around it
- We call this description the "q-color" of an element
- The 0-color of index i is $w[i]$
- The $(q+1)$-color of i is a triple $(w[i], \sigma, \tau)$ where each σ is a list that describes the q-colors in the interval $(i, i+p \cdot 2^q]$, which can be built from left to right as follows:
- if position k is q-rigid wrt i, add the q-color of k
- if positions $k, \ldots, k+n$ are q-elastic wrt i, add the set of q-colors of $k, \ldots, k+n$
- τ is built similarly for the interval $[i-p \cdot 2^q, i)$

Lemma

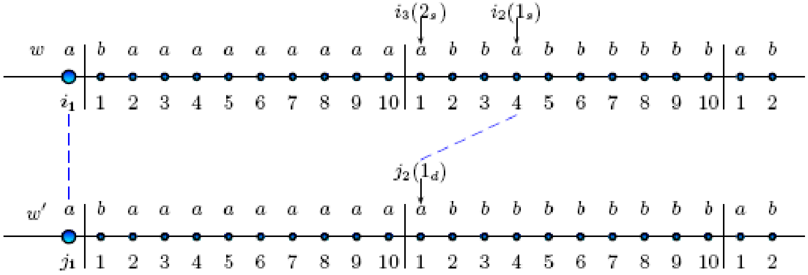
*Given a configuration $(u, \vec{i}, v, \vec{j})$, if i_h and j_h do not have the same q-color for some h, then **I** wins in q rounds*

Example (revised)



- Starting configuration is (w, i_1, w', j_1) and $p = 10$
- We play a 2-round game
- The 0-color of i_1 is a, and it is the same as the 0-color of j_1
- The 1-color of i_1 is $(a, \sigma, \emptyset)$, where $\sigma = \langle \{a, b\} \rangle$, and it is the same as the 1-color of j_1

Example (revised)



- To define the 2-color of i_1 we need to know:
- the 1-color of position 1 (because 1 is 2-rigid wrt i_1)
- the set of 1-colors occurring in positions from 2 to 9 (because they are 2-elastic wrt i_1)
- the 1-color of position 10 (because 10 is 2-rigid wrt i_1)
- the 1-color of position 11 (because 10 is 2-rigid wrt i_1)
- Go on till position $p \cdot 2^q = 40 \ldots$

Games on labelled bounded orderings: a characterization

Theorem
$(u, \vec{i}) \equiv_q^{loc} (v, \vec{j})$ *if and only if* $(u, \vec{i}, v, \vec{j})$ *is* q*-distance safe and* i_h *and* j_h *have the same* q*-color for all* h*.*

- By counting the free multiplicity and scattering of suitable substructures, it is possible to provide a characterization of the full game (in which **I** can play non-local moves), in a similar way as we have done for LSSs
- The remoteness can be computed in $O(p^2 n^3 \log n)$ where $n = \min(|u|, |v|)$

Introduction to EF-games

Inexpressivity results for first-order logic

Normal forms for first-order logic

Algorithms and complexity for specific classes of structures

General complexity bounds

Complexity theory: PSPACE-completeness

In computational complexity theory, a decision problem is PSPACE-complete if

- it can be solved using an amount of memory that is polynomial in the input length (polynomial space) and
- every other problem that can be solved in polynomial space can be transformed to it in polynomial time

The problems that are PSPACE-complete can be thought of as the hardest problems in PSPACE, because a solution to any one such problem could easily be used to solve any other problem in PSPACE

Complexity theory: PSPACE-completeness (contn'd)

The PSPACE-complete problems are widely suspected to be outside of the well-known complexity classes P and NP, but that is not known

It is known that they lie outside of the class NC (a class of problems with highly efficient parallel algorithms):

problems in NC can be solved in an amount of space polynomial in the logarithm of the input size, and the class of problems solvable in such a small amount of space is strictly contained in PSPACE by the space hierarchy theorem

The complexity class NC

In complexity theory, the class NC (for Nick's Class) is the set of decision problems decidable in polylogarithmic time on a parallel computer with a polynomial number of processors:

a problem is in NC if there exist constants c and k such that it can be solved in time $O(\log^c n)$ using $O(n^k)$ parallel processors

Cook coined the name "Nick's class" after Nick Pippenger, who had done extensive research on circuits with polylogarithmic depth and polynomial size

The complexity class NC (contn'd)

Just as the class P can be thought of as the tractable problems (Cobham's thesis), so NC can be thought of as the problems that can be efficiently solved on a parallel computer

NC is a subset of P because polylogarithmic parallel computations can be simulated by polynomial-time sequential ones

It is unknown whether $\mathrm{NC} = \mathrm{P}$, but most researchers suspect this to be false, meaning that there are probably some tractable problems that are ”inherently sequential” and cannot significantly be sped up by using parallelism

(just as the class NP-complete can be thought of as "probably intractable", so the class P-complete, when using NC reductions, can be thought of as "probably not parallelizable" or "probably inherently sequential")

Examples of PSPACE-complete problems

Some well-known examples of PSPACE-complete problems:

- **universality problem for regular expressions**: given a regular expression R, to determine whether it generates every string over its alphabet
- **the word problem for deterministic context-sensitive grammars**: given a set of grammatical transformations which can increase, but cannot decrease, the length of a sentence, to determine if a given sentence could be produced by these transformations (since PSPACE is closed under nondeterminism, non-deterministic context-sensitive grammars are in PSPACE as well)
- **Quantified Boolean Formulas**

Complexity of the EF-Problem

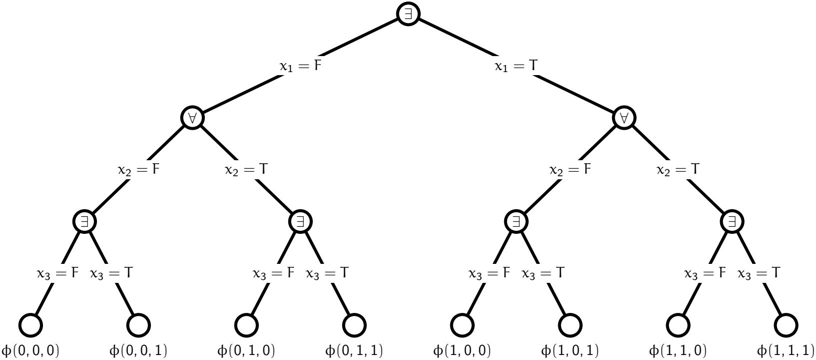
As for the the EF-Problem,

- It is easy to prove that the problem is in PSPACE
- The difficult part is proving hardness for PSPACE
- The problem is in fact PSPACE-complete
- It is proved by reducing QBF (Quantified Boolean Formulas) to the problem of determining whether **II** has a winning strategy
- QBF formulas have the form

$$\exists x_1 \forall x_2 \exists x_3 \cdots Q x_k \, (C_1 \wedge \cdots \wedge C_n)$$

where each C_j is a disjunction of literals

QBF is in PSPACE



- Exhaustive search of the evaluation tree
- For each node, only one bit of information (true/false)
- $\forall$-nodes are true iff both children are true
- $\exists$-nodes are true iff at least one child is true
- Space proportional to the tree height (recursion depth)

QBF is in PSPACE

On input ϕ:

- If ϕ has no quantifiers, then evaluate ϕ and accept iff it is true.
- If $\phi = \exists x\, \phi$, then recursively evaluate $\phi'[x = 0]$ and $\phi'[x = 1]$ and accept iff either computation accepts.
- If $\phi = \forall x\, \phi'$, then recursively evaluate $\phi'[x = 0]$ and $\phi'[x = 1]$ and accept iff both computations accept.
- Recursion depth = number of variables of ϕ and each level stores values of formula for one variable, so total space used for recursion is linear. Evaluating ϕ at each level also requires linear space, but this can be shared between calls.

The EF-problem is PSPACE-complete

Theorem (Pezzoli)

The EF-problem for finite structures over any fixed signature that contains at least one binary and one ternary relation is PSPACE-complete.

- The proof for hardness goes by reducing QBF to the EF-problem
- Given a QBF formula ϕ of the form

$$\exists x_1 \forall x_2 \cdots \exists x_{2r-1} \forall x_{2r} \, (C_1 \wedge \cdots \wedge C_n),$$

  we build two structures $\mathcal{A}$ and $\mathcal{B}$ over $\Sigma = \{E, H\}$, where E is binary and H is ternary, such that **I** wins $\mathcal{G}_{2r+1}(\mathcal{A}, \mathcal{B})$ iff ϕ is satisfiable

Sketch of the proof

- **I**'s moves correspond to existential quantifiers
- **II**'s moves correspond to universal quantifiers
- Structures $\mathcal{A}$ and $\mathcal{B}$ consist of r blocks
- Each block is made of a certain number of subgraphs, called "gadgets", which are of three types: J, L, and I
- Some elements of the domains are labelled by truth values or pairs of truth values
- Some elements in the last block (block r) are labelled by clauses of ϕ
- A pair of consecutive rounds i, $i+1$ is played within block $\lceil i/2 \rceil$ and corresponds to instantiate a pair of consecutive variables $\exists x_i \forall x_{i+1}$

Sketch of the proof (cont.)

- At round i, **I** assigns the truth value T (resp., F) to variable x_i by choosing an element in block $\lceil i/2 \rceil$ of one of the structures (say, $\mathcal{A}$) "labelled" by T (resp., F)
- **II** is forced to reply by choosing an element "labelled" by a pair of truth values TT or TF (resp., FT or FF) in $\mathcal{B}$, which corresponds to recording **I**'s assignment (the first truth value) and to assign a truth value to variable x_{i+1} (the second truth value)
- At round $i+1$, **I** chooses an "unlabelled" element in $\mathcal{B}$
- **II** is forced to reply by recording the truth value of x_{i+1} in $\mathcal{A}$ by choosing an element "labelled" the same as the second truth value chosen at round i

Sketch of the proof (cont.)

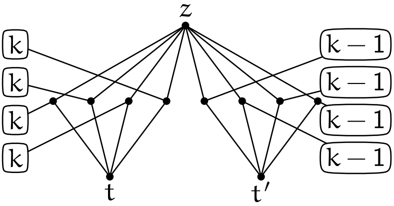
E.g., the pair of rounds may go like this:

round i	round $i+1$	
$s : T(x_i)$	$d : F(x_{i+1})$	$\mathcal{A}$
$d : TF(x_i x_{i+1})$	$s : r$	$\mathcal{B}$

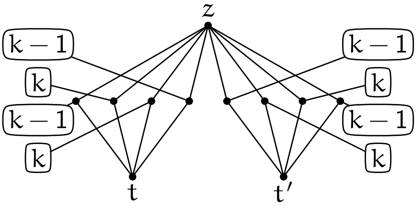
- The "labelling" is encoded by a ternary relation H such that $H(u, v, w)$ holds iff
  - u and v are adjacent in the same block
  - w is in the last block and is labelled by clause C_j
  - Clause C_j is made true by the truth values that label u and/or v

Gadgets J_k, L_k

Circled node are special neighbours



Gadget J_k

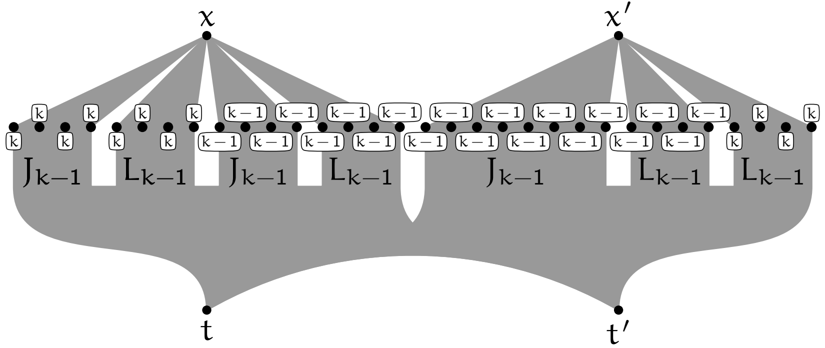


Gadget L_k

- four nodes in the middle have k special neighbours and target t
- four nodes in the middle have $k-1$ special neighbours and target t'

- four nodes in the middle have k special neighbours (two with target t and two with target t')
- four nodes in the middle have $k-1$ special neighbours (two with target t and two with target t')

Gadget I_k



- x and x' are linked to 16 nodes each (*nodes in the middle*)
- Each node in the middle is the source of a gadget J_{k-1} or L_{k-1}
- All gadgets share the same two targets t and t'
- Each node in the middle has either k or $k-1$ special neighbours
- I_k is symmetric if I_k's special neighbours are removed

Forcing pairs

Lemma (Forcing lemma)

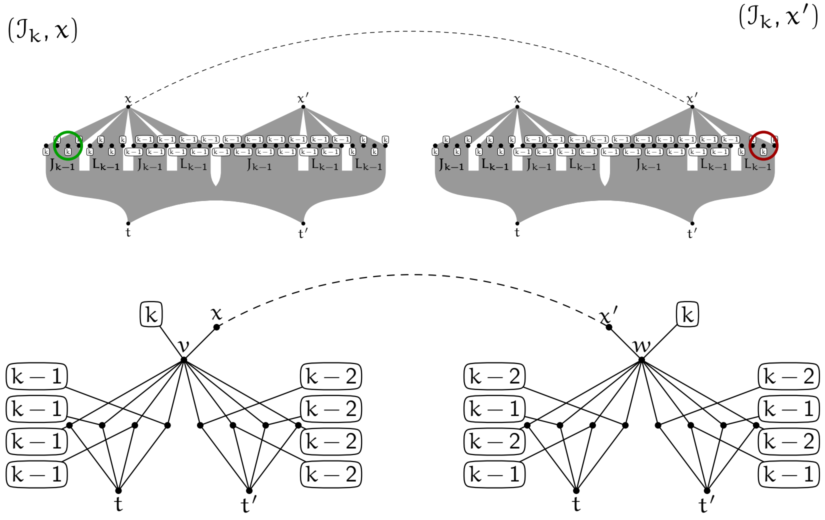
In the $(k+1)$*-moves EF-game on* (I_k, x, I_k, x')*,* ***I*** *can force the pair* (t, t') *(that is,* ***I*** *can play* t*, resp.,* t'*, and* ***II*** *must reply by* t'*, resp.,* t*, not to lose the game), but* ***II*** *has a winning strategy in the* k*-moves EF-game that allows him to reply* t *to* t *and* t' *to* t'*.*

- **Notation:** let kxG denote a node adjacent to x, with k special neighbours, and which is the source of a gadget of type G_{k-1}, with $G \in \{J, L\}$
- In the $(k+1)$-moves game **I** starts by playing $v = kxJ$
- **II** must answer with $w = kx'L$
    - otherwise, **I** wins by moving into the special neighbours
- **I** chooses $w(k-1)t'$ in L_{k-1} (a vertex in the middle of L_{k-1}, connected to w and t', and with $k-1$ special neighbours)
- **II** must answer $v(k-1)t$ in J_{k-1}

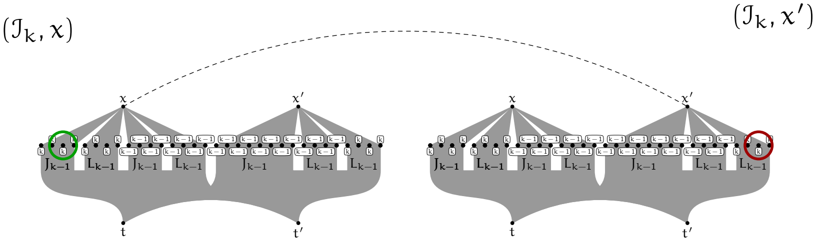
Remark

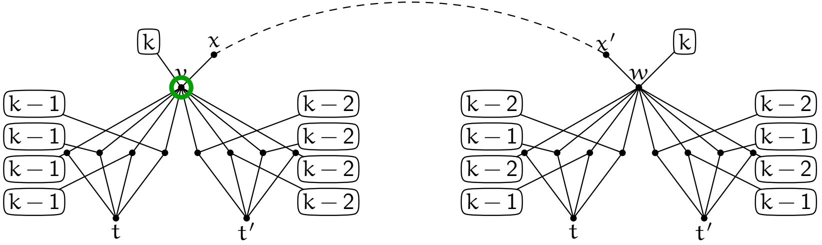
The above lemma says nothing about who has a winning strategy.

Forcing pairs (cont.)

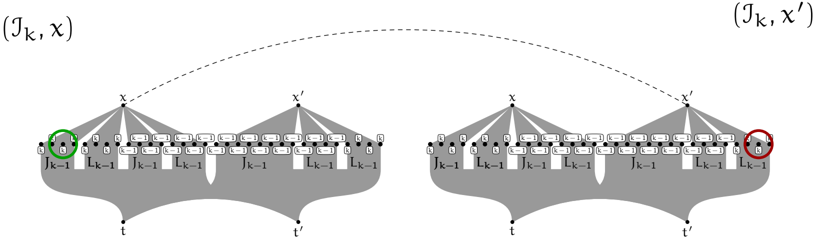


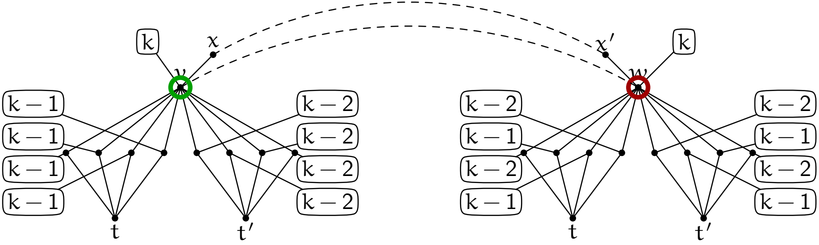
Forcing pairs (cont.)



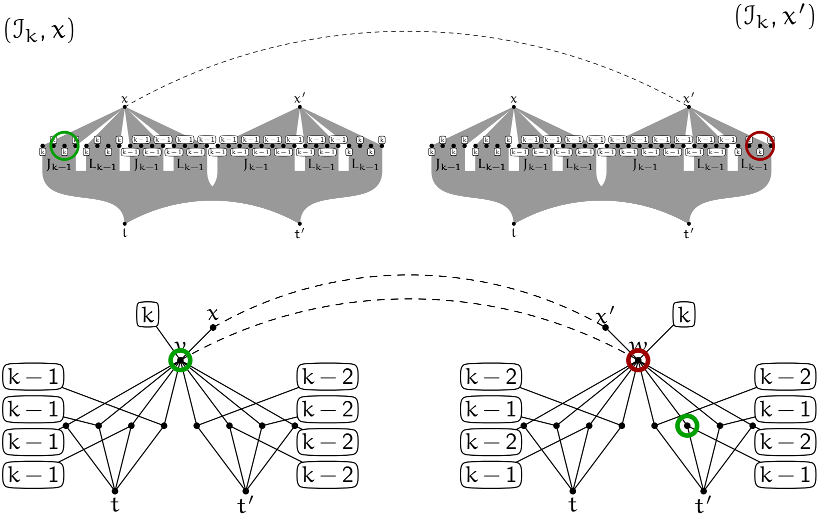


Forcing pairs (cont.)

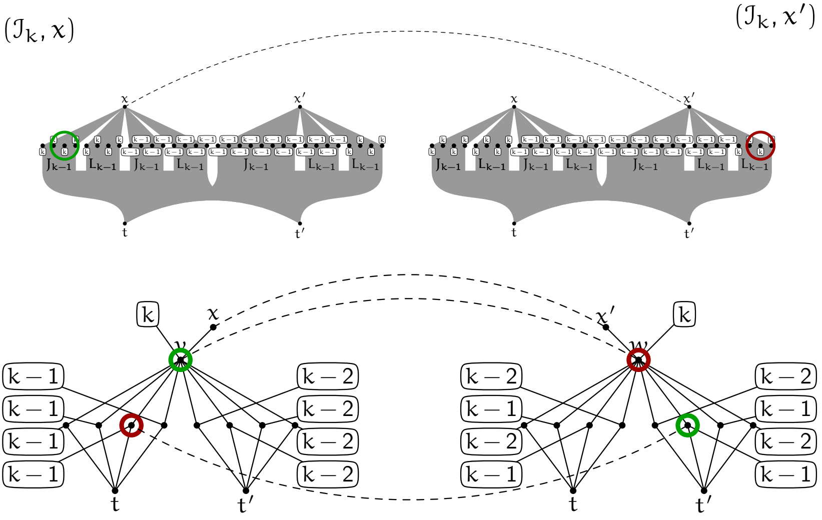




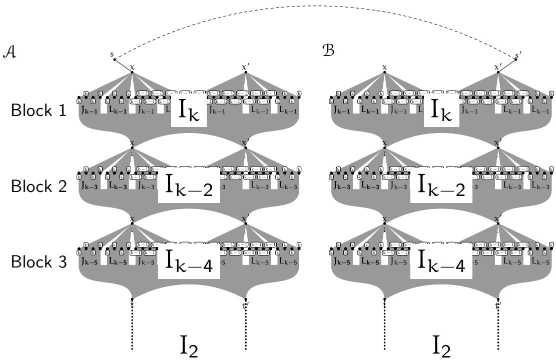
Forcing pairs (cont.)



Forcing pairs (cont.)

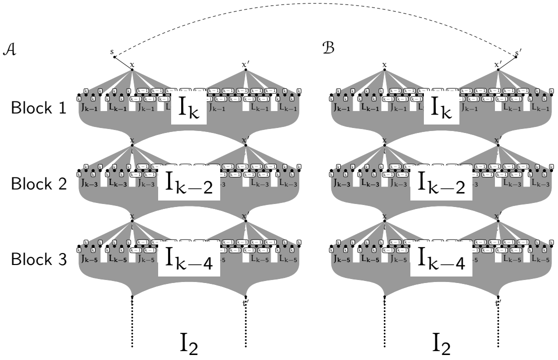


The whole structure



- The game will proceed by choosing two vertices in each block, from top to bottom, according to the strategy of the forcing lemma
- The two vertices in each block are the source of a gadget J_h or L_h and a vertex in the middle in the same gadget
- The pairs of vertices connecting two blocks are never chosen by the players

The whole structure (cont.)

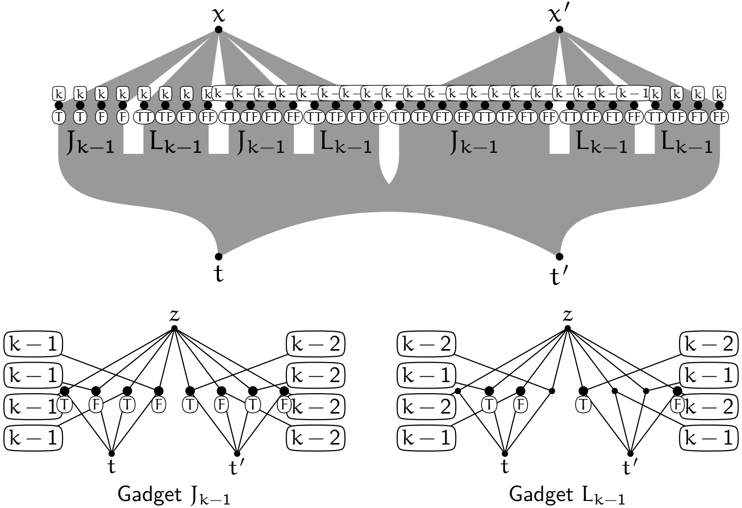


- Up to now, $\mathcal{A}$ and $\mathcal{B}$ are $(2r + 1)$-equivalent
- A "meta-labelling" of the vertices is introduced
- The last block of each structure is slightly changed
- The "meta-labelling" induces a ternary relation H
- H relates a winning strategy for **I** to the satisfiability of a formula ϕ

The truth-value labelling

- Same labelling no matter what ϕ is
- Just a convenience for defining H
- There are no unary predicates in the vocabulary
- Of the four vertices kxJ, two are labelled T and the other two F
- For each group of four vertices kxL, $(k-1)xJ$, $(k-1)xL$, $(k-1)x'J$ (two groups of four vertices),$(k-1)x'L$ and $kx'L$, one is labelled TT, one TF, one FT, one FF
- Of the four vertices in the middle of any gadget J_{k-1} with $k-1$ special neighbours, or $k-2$ special neighbours, two are labelled T and two F
- In gadget L_{k-1} the two vertices $(k-1)zt'$ and the two vertices $(k-2)zt$ are not labelled
- Of the two remaining vertices $(k-1)zt$ and the two $(k-2)zt'$, one is labelled T and the other F

The truth-value labelling (cont.)



- Only the sources of the gadgets J_{k-1} and L_{k-1}, the vertices in the middle of J_{k-1} and half of the vertices in the middle of L_{k-1} are labelled by (pairs of) truth values

Labelling vertices by clauses

- The last block is labelled in a way that depends on ϕ
- In the last block, t and t' are replaced by two sets of elements labelled by clauses of ϕ
- t' is replaced by $2r + 1$ vertices labelled C_1, $2r + 1$ vertices labelled C_2, ..., $2r + 1$ vertices labelled C_n
- t is replaced by $2r + 1$ vertices labelled C_1, ..., $2r + 1$ vertices labelled C_n, *plus an unlabelled vertex*
- The new vertices are not mutually adjacent, but they are adjacent to all the vertices previously connected to t' or t, respectively
- The labelling of vertices with (pairs of) truth values and clauses is used to define the ternary relation H

The ternary relation H

Definition (Ternary relation H)

$H(u, v, w)$ holds if, and only iff, u and v are consecutive in the same block $\lceil i/2 \rceil$, w is in the last block, w is labelled by a clause C and one of the following holds:

- u is labelled $a \in \{T, F\}$, v is labelled $b \in \{T, F\}$, or
- u is labelled ab, with $a, b \in \{T, F\}$, v is not labelled, or
- u is labelled ac, v is labelled b, with $a, b, c \in \{T, F\}$,

and assigning a to x_i and b to x_{i+1} makes C true

Lawful strategies

- **I** starts playing in $\mathcal{A}$
- Then, **I** will play in $\mathcal{A}$ at every odd round and in $\mathcal{B}$ at every even round
- Besides, **I** plays on the "left" of $\mathcal{A}$ in odd rounds and on the "right" of $\mathcal{B}$ in even rounds
- At each odd round, **II** is forced to record **I**'s choice in $\mathcal{B}$, i.e., if **I** picks an element labelled T in $\mathcal{A}$ then **II** must reply with TT or TF, but not with FF or FT (otherwise, she is bound to lose in less than $2r + 1$ rounds)
- Similarly, **II** is forced to record its choice in $\mathcal{A}$ at the next round, i.e., if she has chosen TF in $\mathcal{B}$ then she will pick an element labelled by F in $\mathcal{A}$
- If **II** fails to play like that, at some following round **I** may pick an element labelled by a clause C that appears in some triple of H, but **II** would not be able to do so in the opposite structure

What if **II** does not record **I**'s choices?

Example

$$\phi \stackrel{\text{def}}{=} \exists x_1 \forall x_2 \exists x_3 \forall x_4 \left((\bar{x}_3 \vee x_2) \wedge \bar{x}_1 \wedge (x_1 \vee \bar{x}_3) \wedge (\bar{x}_3 \vee x_4) \right)$$

Suppose that during a game the following labelling is determined:

round 1	round 2	round 3	round 4	
$s : F(x_1)$	$d : F(x_2)$	$s : F(x_3)$	$d : F(x_4)$	$\mathcal{A}$
$d : FF(x_1x_2)$	$s : r$	$d : TF(x_3x_4)$	$s : r'$	$\mathcal{B}$

- **II** does not record the move made by **I** at round 3
- At round 5, **I** jumps to an element labelled by clause $\bar{x}_3 \vee x_4$ in $\mathcal{A}$, which determines a triple in H
- **II**, however, cannot find a corresponding element in $\mathcal{B}$ (no clause is satisfied when x_4 is false, but x_3 is true)

What if **II** does not record **I**'s choices?

Example

$$\phi \stackrel{\text{def}}{=} \exists x_1 \forall x_2 \exists x_3 \forall x_4 \left((\bar{x}_3 \vee x_2) \wedge \bar{x}_1 \wedge (x_2 \vee x_3) \wedge (x_3 \vee x_4) \right)$$

Suppose that during a game the following labelling is determined:

round 1	round 2	
$s : F(x_1)$	$d : F(x_2)$	$\mathcal{A}$
$d : TF(x_1 x_2)$	$s : r$	$\mathcal{B}$

- **II** does not record the move made by **I** at round 1
- At round 3, **I** may choose an element labelled by $\bar{x}_1$ in $\mathcal{A}$, which determines a triple in H
- **II**, however, cannot find a corresponding element in $\mathcal{B}$ (no clause is satisfied when x_1 is true and x_2 is false)

How **I** wins if ϕ is satisfiable

- Suppose that ϕ is satisfiable
- Assuming that **I** follows a lawful strategy and **II** correctly records the truth values, the choices of the players will determine the same truth assignment for the variables of ϕ, both in $\mathcal{A}$ and $\mathcal{B}$
- At the last round, **I** chooses the only vertex w not labelled by any clause at the bottom of $\mathcal{A}$
- But, by the forcing lemma, **II** is bound to choose a vertex w' at the bottom of $\mathcal{B}$ labelled by some clause C, or to choose a vertex not adjacent to the choice **I** has made in $\mathcal{B}$ in the previous round
- In the latter case, **II** loses immediately
- In the former case, since **I** has played in such a way to build a satisfying assignment and **II** has recorded such assignment in $\mathcal{B}$, the last choice by **II** will determine a triple (u', v', w') of $H^{\mathcal{B}}$, for some previously chosen vertices u' and v'
- But $(u, v, w) \notin H^{\mathcal{A}}$ for corresponding u, v in $\mathcal{A}$

Complexity results for pebble games

- Pebble games are a variant of EF-games in which each player has a limited number of pebbles and re-uses them
- They correspond to formulas with a bounded number of variables

Theorem

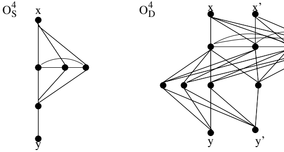
*Given a positive integer k and structures $\mathcal{A}$ and $\mathcal{B}$ the problem of determining whether **II** has a winning strategy in the existential k-pebble game on $\mathcal{A}$ and $\mathcal{B}$ is EXPTIME-complete.*

Corollary

All algorithms for determining whether k-strong consistency can be established are inherently exponential.

P. G. Kolaitis, J. Panttaja
On the Complexity of Existential Pebble Games
CSL 2003

The proof of EXPTIME-completeness is not that easy...



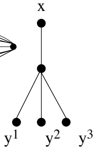


Fig. 3. Single Input One-Way Switch O^4

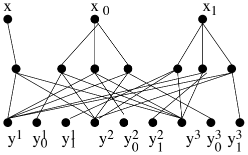
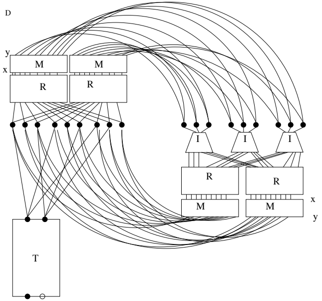
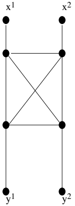


Fig. 6. I^3 Gadget





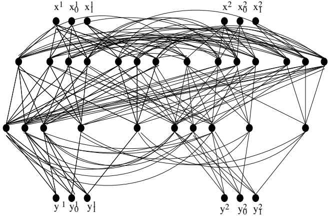


Fig. 7. A subgraph of M^4

Fig. 10. This is component decomposition of the Duplicator's graph for the reduction

Conclusions

- EF-games not explored much algorithmically
  - What is the complexity of the EF-problem for (labelled) arbitrary trees?
  - What is complexity of the EF-problem for signature containing only a binary relations E (i.e., graphs)?
  - The question for the complexity of first-order equivalence for finite structures, that is, isomorphism, is open (strictly related to the graph isomorphism problem)
- Simpler proofs?
- May notions from Combinatorial Game Theory help?
  - Berlekamp's et al. *Winning Ways*