Model Checking the Logic of Allen's Relations *Meets* and *Started-by* is P^{NP}-Complete

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Model checking

- Model checking: the desired properties of a system are checked against a model of the system
    - the model is a (finite) state-transition graph
    - system properties are specified by a temporal logic (e.g., LTL, CTL, CTL*, ...)
- Distinctive features of model checking:
    - exhaustive verification of all the possible behaviours
    - fully automatic process
    - a counterexample is produced for a violated property

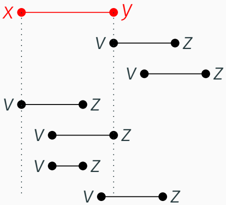
Point-based vs. interval-based model checking

- Model checking is usually point-based:
  - properties express requirements over points (snapshots) of a computation (states of the state-transition system)
  - they are specified by means of point-based temporal logics such as LTL and CTL
- Interval-based model checking:
  - Interval-based properties express conditions on computation stretches: accomplishments, actions with duration, and temporal aggregations
  - they are specified by means of interval temporal logics, e.g., HS

The logic HS

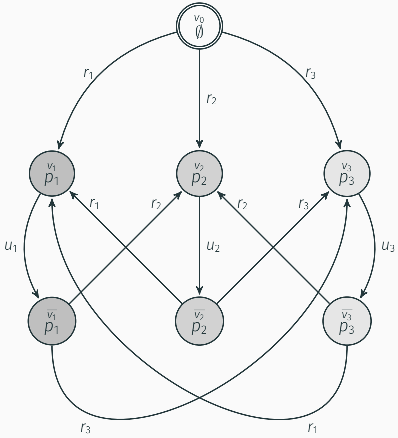
HS features a modality for any Allen ordering relation between pairs of intervals (except for equality)

Allen rel.	HS	Definition	Example
meets	$\langle A \rangle$	$[x, y]\mathcal{R}_A[v, z] \iff y = v$	
before	$\langle L \rangle$	$[x, y]\mathcal{R}_L[v, z] \iff y < v$	
started-by	$\langle B \rangle$	$[x, y]\mathcal{R}_B[v, z] \iff x = v \wedge z < y$	
finished-by	$\langle E \rangle$	$[x, y]\mathcal{R}_E[v, z] \iff y = z \wedge x < v$	
contains	$\langle D \rangle$	$[x, y]\mathcal{R}_D[v, z] \iff x < v \wedge z < y$	
overlaps	$\langle O \rangle$	$[x, y]\mathcal{R}_O[v, z] \iff x < v < y < z$	



All modalities can be expressed by means of $\langle A \rangle$, $\langle B \rangle$, $\langle E \rangle$ and their transposed modalities only

Kripke structures



An example of Kripke structure

- HS formulas are interpreted over (finite) state-transition systems whose states are labelled with sets of proposition letters (Kripke structures)
- An interval is a track (finite path) in a Kripke structure

HS semantics and model checking

Truth of a formula ψ over a track ρ of a Kripke structure $\mathcal{K}$:

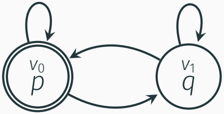
- $\mathcal{K}, \rho \models p$ iff $p \in \bigcap_{w \in \mathrm{states}(\rho)} \mu(w)$, for any letter $p \in \mathcal{AP}$ (homogeneity assumption);
- negation, disjunction, and conjunction are standard;
- $\mathcal{K}, \rho \models \langle \mathrm{A} \rangle\, \psi$ iff there is a track ρ' s.t. $\mathrm{lst}(\rho) = \mathrm{fst}(\rho')$ and $\mathcal{K}, \rho' \models \psi$;
- $\mathcal{K}, \rho \models \langle \mathrm{B} \rangle\, \psi$ iff there is a prefix ρ' of ρ s.t. $\mathcal{K}, \rho' \models \psi$;
- $\mathcal{K}, \rho \models \langle \mathrm{E} \rangle\, \psi$ iff there is a suffix ρ' of ρ s.t. $\mathcal{K}, \rho' \models \psi$;
- the semantic clauses for $\langle \overline{\mathrm{A}} \rangle$, $\langle \overline{\mathrm{B}} \rangle$, and $\langle \overline{\mathrm{E}} \rangle$ are similar

Model Checking

$\mathcal{K} \models \psi \iff$ for all *initial* tracks ρ of $\mathcal{K}$, it holds that $\mathcal{K}, \rho \models \psi$

Possibly infinitely many tracks!

BE-descriptors



BE_2-descriptor for the track $\rho = v_0 v_1 v_0^4 v_1$
(only the part for prefixes is shown)

- $(v_0, \{v_0, v_1\}, v_1)$
  - $(v_0, \{v_0, v_1\}, v_0)$
    - $(v_0, \{v_0, v_1\}, v_0)$
    - $(v_0, \{v_1\}, v_0)$
    - $(v_0, \emptyset, v_1)$
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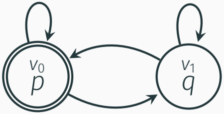

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- FACT 1: For any Kripke structure $\mathcal{K}$ the number of different descriptors of bounded depth k is finite
- FACT 2: Two tracks ρ and ρ' of a Kripke structure $\mathcal{K}$ described by the same BE_k-descriptor are k-equivalent

Decidability of HS model checking

Theorem

The model checking problem for full HS on Kripke structures is decidable (non-elementary *algorithm)*

Reference

A. Molinari, A. Montanari, A. Murano, G. Perelli, and A. Peron. Checking interval properties of computations.

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Theorem

The model checking problem for BE *on Kripke structures is EXPSPACE-hard*

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L. Bozzelli, A. Molinari, A. Montanari, A. Peron, and P. Sala. Interval Temporal Logic MC: the Border Between Good and Bad HS Fragments.

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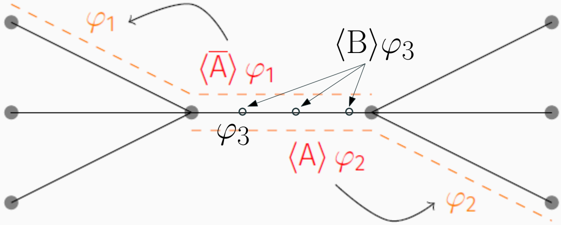
The fragment $A\bar{A}B$



- Branching semantics of $\langle A \rangle / \langle \bar{A} \rangle$
- MC for $A\bar{A}B$ is complete for $\mathsf{P}^{\mathsf{NP}} = \Delta_2^p$

P^{NP} MC algorithm for $\mathsf{A}\overline{\mathsf{A}}\mathsf{B}$ formulas

Algorithm 1 $\mathrm{MC}(\mathcal{K}, \psi, \textsc{direction})$

for all $\langle \mathrm{A} \rangle\, \phi \in \mathrm{ModSubf}_{\mathrm{A}\overline{\mathrm{A}}}(\psi)$ **do**
    $\mathrm{MC}(\mathcal{K}, \phi, \textsc{forward})$
for all $\langle \overline{\mathrm{A}} \rangle\, \phi \in \mathrm{ModSubf}_{\mathrm{A}\overline{\mathrm{A}}}(\psi)$ **do**
    $\mathrm{MC}(\mathcal{K}, \phi, \textsc{backward})$
for all $v \in \mathrm{states}(\mathcal{K})$ **do**
    if $\textsc{direction}$ is $\textsc{forward}$ **then**
        $V_{\mathrm{A}}(\psi, v) \leftarrow \mathit{Success}(\mathtt{Oracle}(\mathcal{K}, \psi, v, \textsc{forward}, V_{\mathrm{A}} \cup V_{\overline{\mathrm{A}}}))$
    else if $\textsc{direction}$ is $\textsc{backward}$ **then**
        $V_{\overline{\mathrm{A}}}(\psi, v) \leftarrow \mathit{Success}(\mathtt{Oracle}(\mathcal{K}, \psi, v, \textsc{backward}, V_{\mathrm{A}} \cup V_{\overline{\mathrm{A}}}))$

- $\mathrm{ModSubf}_{\mathrm{A}\overline{\mathrm{A}}}(\psi)$: $\mathrm{A}\overline{\mathrm{A}}$-modal-subformulas of ψ

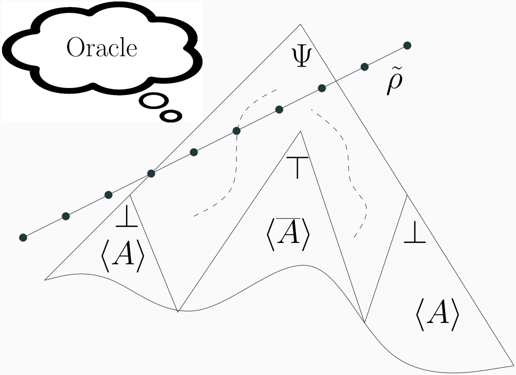
P^{NP} MC algorithm for $\mathsf{A\overline{A}B}$ formulas

Algorithm 2 MC($\mathcal{K}, \psi$, DIRECTION)

for all $\langle A \rangle \phi \in \mathrm{ModSubf}_{A\overline{A}}(\psi)$ **do**
  MC($\mathcal{K}, \phi$, FORWARD)
for all $\langle \overline{A} \rangle \phi \in \mathrm{ModSubf}_{A\overline{A}}(\psi)$ **do**
  MC($\mathcal{K}, \phi$, BACKWARD)
for all $v \in \mathrm{states}(\mathcal{K})$ **do**
  if DIRECTION is FORWARD **then**
    $V_A(\psi, v) \leftarrow \mathit{Success}(\texttt{Oracle}(\mathcal{K}, \psi, v, \text{FORWARD}, V_A \cup V_{\overline{A}}))$
  else if DIRECTION is BACKWARD **then**
    $V_{\overline{A}}(\psi, v) \leftarrow \mathit{Success}(\texttt{Oracle}(\mathcal{K}, \psi, v, \text{BACKWARD}, V_A \cup V_{\overline{A}}))$

- $\texttt{Oracle}(\mathcal{K}, \psi, v, \text{DIRECTION}, V_A \cup V_{\overline{A}})$ is called for all $v \in \mathrm{states}(\mathcal{K})$
- $V_A(\phi, v) = \top \iff \exists$ a track $\rho \in \mathrm{Trk}_{\mathcal{K}}$ starting from v s.t. $\mathcal{K}, \rho \models \phi$
- DIRECTION = FORWARD / BACKWARD (for $\langle A \rangle$ / $\langle \overline{A} \rangle$)

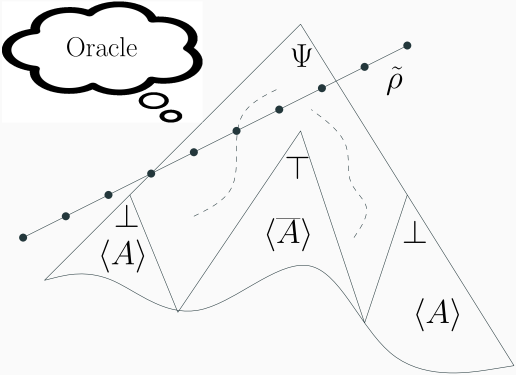
NP oracle



The oracle:

- generates $\tilde{\rho}$ by non-deterministically visiting the unravelling of the Kripke structure.
- performs a bottom-up deterministic verification of Ψ against $\tilde{\rho}$ (for all the subformulas / for all the prefixes).

NP oracle



- "polynomial-size model-track property":
  if ρ is a track of $\mathcal{K}$, ϕ is an A$\overline{\text{A}}$B formula, and $\mathcal{K}, \rho \models \phi \Rightarrow$ $\exists \rho'$ such that $|\rho'| \leq |W| \cdot (2|\phi| + 1)^2$ and $\mathcal{K}, \rho' \models \phi$.

P^{NP} MC algorithm for $\mathsf{A\overline{A}B}$ formulas

Theorem

Let $\mathcal{K}$ be a finite Kripke structure, w_0 be its initial state, and ψ an $\mathsf{A\overline{A}B}$ formula. If $\mathtt{MC}(\mathcal{K}, \neg\psi, \textit{FORWARD})$ is executed, then

$$V_{\mathsf{A}}(\neg\psi, w_0) = \bot \iff \mathcal{K} \models \psi.$$

Corollary

The model checking problem for $\mathsf{A\overline{A}B}$ formulas over finite Kripke structures is in P^{NP}.

P^{NP}-hardness of MC for AB formulas

Definition (SNSAT: a P^{NP}-complete problem)

An instance $\mathcal{I}$ of SNSAT:

- a set of Boolean variables $X = \{x_1, \cdots, x_n\}$
- a set of Boolean formulas $\{F_1(Z_1), F_2(x_1, Z_2), \cdots, F_n(x_1, \cdots, x_{n-1}, Z_n)\}$ (where Z_i are private variables)

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$v_{\mathcal{I}}$ is the valuation of the variables in X defined as:

$$v_{\mathcal{I}}(x_i) = \top \iff F_i(v_{\mathcal{I}}(x_1), \cdots, v_{\mathcal{I}}(x_{i-1}), Z_i) \text{ is satisfiable.}$$

SNSAT: to decide whether $v_{\mathcal{I}}(x_n) = \top$.

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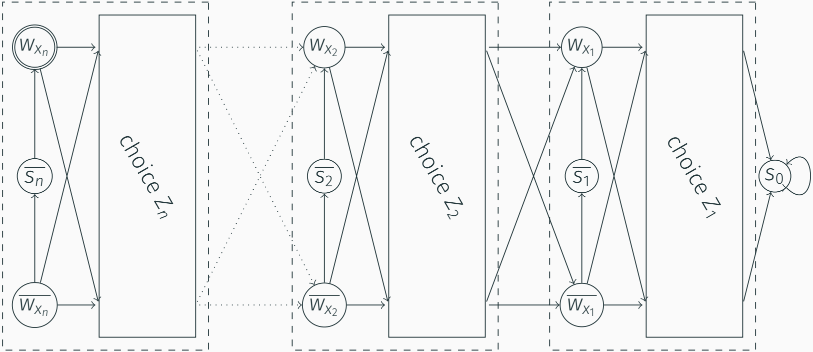
Given $\mathcal{I}$, we build a Kripke structure $\mathcal{K}_{\mathcal{I}}$ and an AB formula $\Phi_{\mathcal{I}}$ s.t.

$$v_{\mathcal{I}}(x_n) = \top \iff \mathcal{K}_{\mathcal{I}} \models \Phi_{\mathcal{I}}.$$

P^{NP}-hardness of MC for AB formulas



$\psi_n =$ (1) $\bigwedge_i x_i \Rightarrow F_i(x_1, \cdots, x_{i-1}, Z_i)$ is true
(2) $\bigwedge_i \neg x_i \Rightarrow F_i(x_1, \cdots, x_{i-1}, Z_i)$ is unsat for any choice of Z_i
(3) the track reaches the last state s_0

P^{NP}-hardness of MC for AB formulas

Theorem

$$v_{\mathcal{I}}(x_n) = \top \iff \mathcal{K}_{\mathcal{I}} \models [B]\bot \to \psi_n.$$

Corollary

The model checking problem for AB *formulas over finite Kripke structures is* P^{NP}*-hard (under* LOGSPACE *reductions).*

Therefore AB, $A\overline{A}B$, $\overline{A}E$, $A\overline{A}E$ are P^{NP}-complete.

The fragment $\overline{\mathsf{A}}\mathsf{B}$

- AB allows one to impose specific constraints on the branches departing from a state occurring in a given path... $\langle B \rangle \langle A \rangle \theta$
  $\Rightarrow$ P^{NP}-hardness of AB.

- $\overline{\mathsf{A}}\mathsf{B}$...

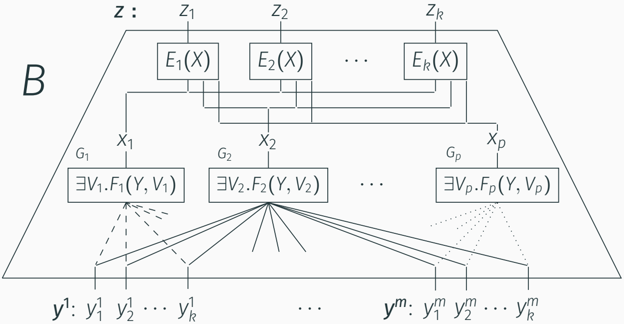
The fragment $\overline{\mathrm{A}}\mathrm{B}$

- AB allows one to impose specific constraints on the branches departing from a state occurring in a given path... $\langle \mathrm{B} \rangle \langle \mathrm{A} \rangle \theta$
  $\Rightarrow$ P^{NP}-hardness of AB.

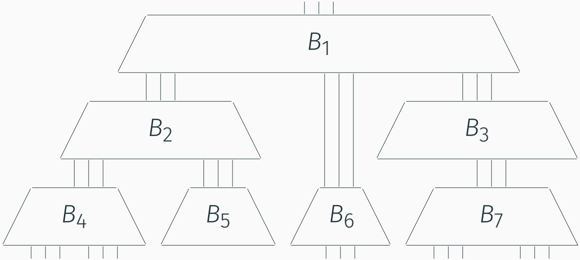
- $\overline{\mathrm{A}}\mathrm{B}$... can't express constraints of this form: pairing $\langle \overline{\mathrm{A}} \rangle$ and $\langle \mathrm{B} \rangle$ does not give any advantage in terms of expressiveness
  $\Rightarrow$ MC for $\overline{\mathrm{A}}\mathrm{B}$ in $\mathrm{P}^{\mathrm{NP}[O(\log^2 n)]}$.

The fragment $\overline{\mathrm{A}}\mathrm{B}$

Membership to $\mathsf{P}^{\mathsf{NP}[O(\log^2 n)]}$ is proved by means of Boolean circuits with SAT oracles



The fragment $\overline{\mathrm{A}}\mathrm{B}$: tree of blocks



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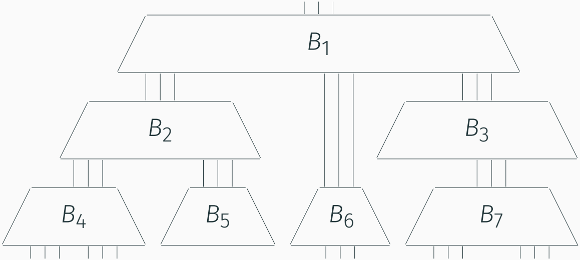


Theorem

TB(SAT) is P^{NP}*-complete.*
TB(SAT)$_{1\times M}$ *(i.e., F_i can use* only one bit from each input vector *of B) is* $\mathsf{P}^{\mathsf{NP}[O(\log^2 n)]}$*-complete.*

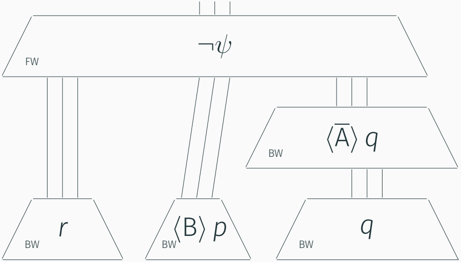
Reference

P. Schnoebelen. Oracle circuits for branching-time model checking.

In *ICALP*, pages 790–801, 2003

The fragment $\overline{\mathsf{A}}\mathsf{B}$: from a formula to a tree of blocks

$$\psi = ((\langle\overline{\mathsf{A}}\rangle\, r \wedge \langle\overline{\mathsf{A}}\rangle\langle\overline{\mathsf{A}}\rangle\, q) \rightarrow \langle\overline{\mathsf{A}}\rangle\langle\mathsf{B}\rangle\, p)$$



Every formula F_i of a block B:

1. is a translation of the oracle algorithm;
2. is built starting from the $\overline{\mathsf{A}}\mathsf{B}$ formula associated with B

The fragment $\overline{\mathsf{A}}\mathsf{B}$: complexity

Theorem

$B_{root}(z_1) = \bot \iff \mathcal{K} \models \psi.$

Corollary

The model checking problem for $\overline{\mathsf{A}}\mathsf{B}$ *formulas, over finite Kripke structures, is in* $\mathsf{P}^{\mathsf{NP}[O(\log^2 n)]}$.

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$\mathsf{P}^{\mathsf{NP}[O(\log n)]}$-hardness follows immediately from that of $\overline{\mathsf{A}}$

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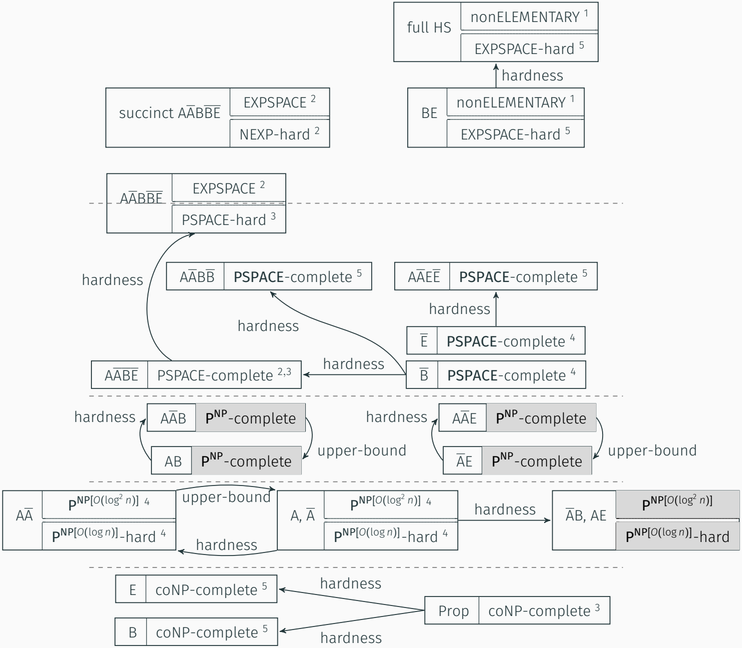
In *KR*, pages 473–483, 2016

Complexity picture



Current/future work

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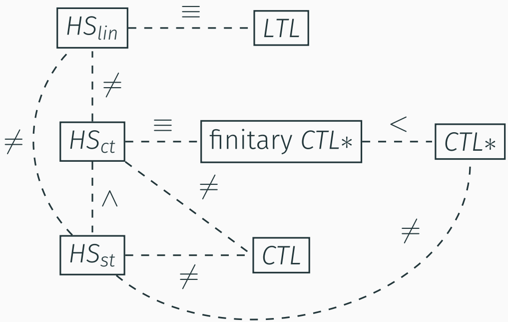
- Application: Planning as Model Checking in Interval Temporal Logic - *IN PROGRESS*

Expressiveness comparison



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