Gaifman's theorem (for sentences)

- r-local formula (around $\vec{x}$): has "bounded" quantifiers:

$$\exists y\,(\delta(\vec{x}, y) \leqslant r \wedge \phi)$$
$$\forall y\,(\delta(\vec{x}, y) \leqslant r \rightarrow \phi)$$

  where ϕ is either quantifier-free or r-local (around $\vec{x}$)
- $\delta(\vec{x}, y) \leqslant r$ is FO-definable
- existentially r-local sentence:

$$\exists x_1 \cdots \exists x_s \Big(\bigwedge_{1 \leqslant i < j \leqslant s} \delta(x_i, x_j) > 2r \wedge \bigwedge_{1 \leqslant i \leqslant s} \phi_r^{(i)}(x_i) \Big)$$

  where $\phi_r^{(i)}$ are r-local formulas around x_i

Theorem (Gaifman's theorem)

Every first-order sentence is logically equivalent to a boolean combination of existentially local sentences.

Gaifman's theorem: preliminary technicalities

To prove the theorem, we need the following results

Lemma

Let Φ be a set of first-order σ-sentences. If all the σ-structures that agree on Φ are elementarily equivalent, then any first-order σ-sentence is equivalent to a boolean combination of sentences of Φ.

Lemma (Relativization lemma)

Let σ be a purely relational vocabulary. For every $r \in \mathbb{N}$ and σ-formula $\phi(\vec{x}, \vec{y})$, with $|\vec{x}| = n$ and $|\vec{y}| = p$, there is an r-local formula $\phi^{S_r(\vec{x})}(\vec{x}, \vec{y})$ such that, for any σ-structure $\mathcal{A}$, $\vec{a} \in A^n$ and $\vec{b} \in \left(S_r^{\mathcal{A}}(\vec{a})\right)^p$,

$$\mathcal{A} \models \phi^{S_r(\vec{x})} \{\vec{x}/\vec{a}, \vec{y}/\vec{b}\} \qquad \Longleftrightarrow \qquad \mathcal{N}_r^{\mathcal{A}}(\vec{a}) \models \phi \{\vec{x}/\vec{a}, \vec{y}/\vec{b}\}.$$

For the proofs, see Ebbinghaus and Flum's *Finite Model Theory*

Gaifman's theorem: the idea of the proof

- We only prove the theorem for sentences, but it can be extended to arbitrary formulas
- We show that $\mathcal{A} \equiv \mathcal{B}$ iff they agree on all existentially local sentences
- The theorem then follows from the first lemma of the previous slide

- One direction is trivial
- We prove the non-trivial direction by showing that **II** has a winning strategy in an m-round EF-game for every m

Gaifman's theorem: invariant for the proof

- Let $f\colon \mathbb{N} \to \mathbb{N}$ be some function (to be defined)
  - constraints on f will emerge during the proof
  - in particular, it will turn out that f will be monotonically non-increasing
- Let $a_1, \ldots, a_i$ and $b_1, \ldots, b_i$ be the elements chosen in the first i rounds
- After i rounds, **II** will maintain

$$\mathcal{N}^{\mathcal{A}}_{7^{m-i}-1}(a_1 \cdots a_i) \equiv_{f(i)} \mathcal{N}^{\mathcal{B}}_{7^{m-i}-1}(b_1 \cdots b_i)$$

- This invariant ensures that $a_1 \cdots a_i \mapsto b_1 \cdots b_i$ is a partial isomorphism
- **Notation:** $r_i \stackrel{\text{def}}{=} 7^{m-i} - 1$
- So, the invariant after i rounds is

$$\mathcal{N}^{\mathcal{A}}_{r_i}(a_1 \cdots a_i) \equiv_{f(i)} \mathcal{N}^{\mathcal{B}}_{r_i}(b_1 \cdots b_i)$$

Relativized Hintikka formulas

- Let $\mathcal{H}^{f(i)}_{\vec{a}}(\vec{x})$ denote the $f(i)$-Hintikka formula for $\mathcal{N}^{\mathcal{A}}_{r_i}(\vec{a})$
- By the definition of Hintikka formulas,

$$\mathcal{N}^{\mathcal{A}}_{r_i}(\vec{a}) \models \mathcal{H}^{f(i)}_{\vec{a}}(x)\{\vec{x}/\vec{a}\}$$

- Let

$$\hat{\mathcal{H}}^{i}_{\vec{a}}(\vec{x}) \stackrel{\text{def}}{=} \left(\mathcal{H}^{f(i)}_{\vec{a}}(\vec{x})\right)^{S_{r_i}(\vec{a})}$$

  that is, $\hat{\mathcal{H}}^{i}_{\vec{a}}(\vec{x})$ is the relativized version (as in the relativization lemma) of $\mathcal{H}^{f(i)}_{\vec{a}}(x)$ with respect to $S_{r_i}(\vec{a})$
- By the relativization lemma,

$$\mathcal{A} \models \hat{\mathcal{H}}^{i}_{\vec{a}}(\vec{x})\{\vec{x}/\vec{a}\}$$

- By the relativization lemma and Ehrenfeucht theorem, for every structure $\mathcal{B}$ and $\vec{b} \in B^n$

$$\mathcal{B} \models \hat{\mathcal{H}}^{i}_{\vec{a}}(\vec{x})\{\vec{x}/\vec{b}\} \iff \mathcal{N}^{\mathcal{B}}_{r_i}(\vec{b}) \equiv_{f(i)} \mathcal{N}^{\mathcal{A}}_{r_i}(\vec{a})$$

The first round

- W.l.o.g, assume that **I** plays in $\mathcal{A}$
- Let **I** choose $a_1 \in A$
- $\mathcal{N}^{\mathcal{A}}_{r_1}(a_1)$ is characterized, up to f(1)-equivalence, by $\mathcal{H}^{f(1)}_{a_1}(x)$
- Since the relativized version $\hat{\mathcal{H}}^1_{a_1}(x)$ is r_1-local, then $\exists x\, \hat{\mathcal{H}}^1_{a_1}(x)$ is an existentially local sentence that holds in $\mathcal{A}$
- By the hypothesis, $\exists x\, \hat{\mathcal{H}}^1_{a_1}(x)$ must hold in $\mathcal{B}$ as well
- That is, there is $b_1 \in B$ such that $\mathcal{B} \models \hat{\mathcal{H}}^1_{a_1}(x)\{{}^{x}/{}_{b_1}\}$
- Hence, $\mathcal{N}^{\mathcal{B}}_{r_1}(b_1) \equiv_{f(1)} \mathcal{N}^{\mathcal{A}}_{r_1}(a_1)$
- But this is exactly the invariant after one round
- Therefore, b_1 is a suitable reply for **II**

The inductive step: first case

- Suppose that after i rounds, with $i < m$, the invariant holds
- Let **I** choose $a_{i+1} \in A$
- Two possibilities: either $\delta(a_{i+1}, \vec{a}) \leqslant 2r_{i+1} + 1$ or not
- Suppose that $\delta(a_{i+1}, \vec{a}) \leqslant 2r_{i+1} + 1$
- Recall that $r_{i+1} = 7^{m-(i+1)} - 1$
- Then, the $S^{\mathcal{A}}_{r_{i+1}}(a_{i+1}) \subseteq S^{\mathcal{A}}_{r_i}(\vec{a})$
- Therefore,
$$\mathcal{N}^{\mathcal{A}}_{r_i}(\vec{a}) \models \Theta\{{}^{\vec{x}}/{}_{\vec{a}}\}$$
where Θ is
$$\exists x_{i+1} \left(\delta(x_{i+1}, \vec{x}) \leqslant 2r_{i+1} + 1 \wedge \hat{\mathcal{H}}^{i+1}_{\vec{a}, a_{i+1}}(\vec{x}, x_{i+1})\right)$$

The inductive step: first case (cont.)

- First constraint on f: impose $f(i) \geqslant qr(\Theta)$
- Then, by the invariant and the inductive hypothesis,

$$\mathcal{N}^{\mathcal{B}}_{r_i}(\vec{b}) \models \Theta\{\vec{x}/\vec{b}\}$$

- That is, there is $b_{i+1} \in B$ (with $b_{i+1} \leqslant 2r_{i+1} + 1$) such that (after applying the relativization lemma)

$$\mathcal{N}^{\mathcal{B}}_{r_{i+1}}(\vec{b}, b_{i+1}) \models \mathcal{H}^{f(i+1)}_{\vec{a}, a_{i+1}}(\vec{x}, x_{i+1})\{\vec{x}/\vec{b}, x_{i+1}/b_{i+1}\}$$

- Which implies

$$\mathcal{N}^{\mathcal{A}}_{r_{i+1}}(\vec{a}, a_{i+1}) \equiv_{f(i+1)} \mathcal{N}^{\mathcal{B}}_{r_{i+1}}(\vec{b}, b_{i+1})$$

- Therefore, the invariant is preserved after round $i + 1$

Inductive step: second case

- Suppose that $\delta(a_{i+1}, \vec{a}) > 2r_{i+1} + 1$
- Then, $\mathcal{N}^{\mathcal{A}}_{r_{i+1}}(\vec{a})$ and $\mathcal{N}^{\mathcal{A}}_{r_{i+1}}(a_{i+1})$ are not adjacent (i.e., there is no tuple in any relation of $\mathcal{A}$ which connects the two neighbourhoods)
- Hence, the disjoint union of $\mathcal{N}^{\mathcal{A}}_{r_{i+1}}(\vec{a})$ and $\mathcal{N}^{\mathcal{A}}_{r_{i+1}}(a_{i+1})$ is isomorphic to $\mathcal{N}^{\mathcal{A}}_{r_{i+1}}(\vec{a}, a_{i+1})$ (this will be used later)
- We will show that **II** is able to find an element b_{i+1} with the same $f(i + 1)$-isomorphism type as a_{i+1} and such that $\delta(b_{i+1}, \vec{b}) > 2r_{i+1} + 1$
- Since $\mathcal{A}$ and $\mathcal{B}$ agree on all existentially local sentences, the existence of a suitable b_{i+1} will be guaranteed by the properties they express on the *scattering* of small neighbourhoods

Maximal sets of scattered neighbourhoods

- For $n \in \mathbb{N}$, let

$$\theta(x_1, \ldots, x_n) \stackrel{\text{def}}{=} \bigwedge_{1 \leqslant j < k \leqslant n} \delta(x_j, x_k) > 4r_{i+1} + 2 \wedge \bigwedge_{1 \leqslant j \leqslant n} \hat{\mathcal{H}}^{i+1}_{a_{i+1}}(x_j)$$

- "$\{x_1, \ldots, x_n\}$ is a $(4r_{i+1} + 2)$-scattered set of n elements whose r_{i+1}-neighbourhoods have the same $f(i+1)$-isomorphism type as a_{i+1}"
- Let t be the cardinality of a maximal subset of elements of A with the above property, that is, let t be such that

$$\mathcal{A} \models \exists x_1 \cdots \exists x_t \, \theta(x_1, \ldots, x_t)$$

but

$$\mathcal{A} \not\models \exists x_1 \cdots \exists x_{t+1} \, \theta(x_1, \ldots, x_{t+1})$$

- Note that $\exists \vec{x} \, \theta(\vec{x})$ is an existentially local sentence
- Hence, $\mathcal{B}$ agrees with $\mathcal{A}$ on the above sentences

Maximal sets of scattered neighbourhoods (cont.)

- Let

$$\Lambda(n) \stackrel{\text{def}}{=} \exists x_1 \cdots \exists x_s \, \Big(\bigwedge_{1 \leqslant j \leqslant s} \delta(x_j, \vec{a}) \leqslant 2r_{i+1} + 1 \wedge \\ \wedge \, \theta(x_1, \ldots, x_s) \Big)$$

- "There is a $(4r_{i+1} + 2)$-scattered set of n elements of $S^{\mathcal{A}}_{2r_{i+1}+1}(\vec{a})$ whose r_{i+1}-neighbourhoods have the same $f(i+1)$-isomorphism type as a_{i+1}"
- Let s be the cardinality of a maximal subset of elements with the above property, that is, let s be such that

$$\mathcal{N}^{\mathcal{A}}_{r_i}(\vec{a}) \models \Lambda(s)$$

but

$$\mathcal{N}^{\mathcal{A}}_{r_i}(\vec{a}) \not\models \Lambda(s+1)$$

- Note that $\Lambda(n)$ is not an existentially local sentence

Inductive step: second case (cont.)

- There are no two $(4r_{i+1}+2)$-scattered points in a sphere of radius $2r_{i+1}+1$
- It is possible to choose at most one element for each $S^{\mathcal{A}}_{2r_{i+1}+1}(a_j)$, with $1 \leqslant j \leqslant i$
- Therefore, $s \leqslant i$
- Clearly, $s \leqslant t$, too (t may be ∞)
- By hypothesis,

$$\mathcal{B} \models \exists x_1 \cdots \exists x_t\, \theta(x_1, \ldots, x_t)$$

and

$$\mathcal{B} \not\models \exists x_1 \cdots \exists x_{t+1}\, \theta(x_1, \ldots, x_{t+1})$$

- Second constraint on f: impose $f(i) \geqslant \mathrm{qr}(\Lambda(s))$
- Then, by the invariant,

$$\mathcal{N}^{\mathcal{B}}_{r_i}(\vec{b}) \models \Lambda(s)$$

and

$$\mathcal{N}^{\mathcal{B}}_{r_i}(\vec{b}) \not\models \Lambda(s+1)$$

First sub-case: $s = t$

- Suppose that $s = t$
- Then, *any* element $e \in A$ with the same $f(i+1)$-Hintikka type of a_{i+1} is such that

$$\begin{aligned}\delta(e, \vec{a}) &\leqslant (2r_{i+1}+1) + (4r_{i+1}+2) = 6r_{i+1}+3 < \\ &< 7r_{i+1}+3 = 7 \cdot 7^{m-(i+1)} - 4 < 7^{m-i} - 1 = r_i\end{aligned}$$

- In particular, the above implies that $S^{\mathcal{A}}_{r_{i+1}}(e) \subseteq S^{\mathcal{A}}_{r_i}(\vec{a})$
- This holds for a_{i+1}, too
- Therefore:

$$\mathcal{N}^{\mathcal{A}}_{r_i}(\vec{a}) \models \Pi\{\vec{x}/\vec{a}\}$$

where Π is

$$\exists z\, (2r_{i+1}+1 < \delta(\vec{x}, z) \leqslant 6r_{i+1}+3 \wedge \hat{\mathcal{H}}^{i+1}_{a_{i+1}}(z) \wedge \hat{\mathcal{H}}^{i+1}_{\vec{a}}(\vec{x}))$$

- Third constraint on f: impose $f(i) \geqslant \mathrm{qr}(\Pi)$
- Then, by the invariant,

$$\mathcal{N}^{\mathcal{B}}_{r_i}(\vec{b}) \models \Pi\{\vec{x}/\vec{b}\}$$

First sub-case: $s = t$ (cont.)

- So, there is $b_{i+1} \in B$ such that

$$2r_{i+1} + 1 < \delta(\vec{x}, z) \leqslant 6r_{i+1} + 3$$

  and

$$\mathcal{N}^{\mathcal{A}}_{r_{i+1}}(a_{i+1}) \equiv_{f(i+1)} \mathcal{N}^{\mathcal{B}}_{r_{i+1}}(b_{i+1})$$

- Last constraint on f: impose $f(i) \geqslant f(i+1)$
- Then, the invariant implies

$$\mathcal{N}^{\mathcal{A}}_{r_{i+1}}(\vec{a}) \equiv_{f(i+1)} \mathcal{N}^{\mathcal{B}}_{r_{i+1}}(\vec{b})$$

- As the neighbourhood around a_{i+1} (resp., b_{i+1}) is not adjacent to the neighbourhood around $\vec{a}$ (resp., $\vec{b}$) we may take their disjoint union and conclude that

$$\mathcal{N}^{\mathcal{A}}_{r_{i+1}}(\vec{a}, a_{i+1}) \equiv_{f(i+1)} \mathcal{N}^{\mathcal{B}}_{r_{i+1}}(\vec{b}, b_{i+1})$$

Second sub-case: $s < t$

- Suppose that $s < t$
- Remember that $s < t$ holds in $\mathcal{B}$, too
- Then, there is an element $b_{i+1} \in B$ such that
  1. $\mathcal{N}^{\mathcal{B}}_{r_{i+1}}(b_{i+1})$ is not adjacent to $\mathcal{N}^{\mathcal{B}}_{r_{i+1}}(\vec{b})$, and
  2. $\mathcal{B}, \{x/b_{i+1}\} \models \hat{\mathcal{H}}^{i+1}_{a_{i+1}}(x)$
- Again, by applying the relativization lemma and the Ehrenfeucht theorem,

$$\mathcal{N}^{\mathcal{A}}_{r_{i+1}}(a_{i+1}) \equiv_{f(i+1)} \mathcal{N}^{\mathcal{B}}_{r_{i+1}}(b_{i+1})$$

- The thesis is then obtained as in the preceding case

Remarks on Gaifman's theorem

- First-order logic can only talk of *scattered small substructures*
- First-order logic can only express *local properties*
- Gaifman's normal form is effective
- Gaifman's proof uses EF-games to prove the invariant

$$\mathcal{N}^{\mathcal{A}}_{7^{m-i}-1}(a_1 \cdots a_i) \equiv_{f(i)} \mathcal{N}^{\mathcal{B}}_{7^{m-i}-1}(b_1 \cdots b_i)$$

- r-local formulas with $r \leqslant 7^{\mathrm{qr}(\phi)}$
- $f(i)$-equivalence instead of isomorphism as in the Sphere theorem
- Notion of *scattered* substructures
- No counting up to a threshold as in the Sphere theorem

Schwentick and Bartelmann's normal form

Theorem

Every first-order formula is logically equivalent to a formula of the form $\exists x_1 \cdots \exists x_n \forall y\, \phi$, where ϕ is r-local around y for some r.

- Consider a differentiating formula ψ
- By the theorem, $\psi \Leftrightarrow \exists x_1 \cdots \exists x_n \forall y\, \phi$, with $\mathrm{qr}(\phi) = k$
- In game-theoretic terms, there is winning strategy for **I** such that **I** plays n rounds by choosing elements in the same structure (the one that satifies ψ)
- Then, **I** plays a round in the opposite structure
- Finally, **I** plays k "local" rounds

T. Schwentick and K. Barthelmann
Local Normal Forms for First-Order Logic with Applications to Games and Automata
Discrete Mathematics and Theoretical Computer Science, 1999

Shrinking games

- Similar to Schwentick's extension theorem, but it works in the opposite direction, by shrinking the playground according to a sequence of "scattering parameters"
- The authors use Ehrenfeucht–Fraïssé type games with a shrinking horizon between structures to obtain a spectrum of normal form theorems of the Gaifman type
- They improve the bound in the proof of Gaifman's theorem from $7^{qr(\phi)}$ to $4^{qr(\phi)}$ and they provide bounds for other normal form theorems

H. J. Keisler and W. B. Lotfallah
Shrinking games and local formulas
Annals of Pure and Applied Logic, 2004

Shrinking games

- Let $\vec{s} = s_0, s_1, \ldots$ a possibly infinite sequence of natural numbers, called scattering parameters
- The sequence of local radii associated with $\vec{s}$ is defined as follows:

$$r_0 = 1$$
$$r_{n+1} = 2r_n + s_n$$

- A set C is s-scattered if $\delta(a, b) > s$ for all distinct $a, b \in C$
- A sequence $\vec{s}$ shrinks rapidly if $2r_j \leqslant s_j$ for all j
- Given $\vec{s} = s_0, s_1, \ldots$ that shrinks rapidly, if C is s_j-scattered then the r_j-neighborhood around any $c \in C$ does not contain any other element of C

Shrinking games: local rounds

Let $\vec{s} = s_0, s_1, \ldots$ be a sequence that shrinks rapidly

Definition ($\vec{s}$-shrinking game)

Given $\mathcal{A}$ and $\mathcal{B}$ and $m \in \mathbb{N}$, the m-round $\vec{s}$-shrinking game is as follows:

- **I** chooses $1 \leqslant i < m$ and plays either a local or a scattered round
- a local round is played as follows (assuming that **I** plays in $\mathcal{A}$):
  1. **I** chooses $a \in \mathcal{N}^{\mathcal{A}}_{r_i+s_i}(\vec{a})$
  2. **II** replies with $b \in \mathcal{N}^{\mathcal{B}}_{r_i+s_i}(\vec{b})$

Shrinking games: scattered rounds

- a scattered round is played as follows:
  1. **I** chooses a non-empty set of s_i-scattered elements $C \subseteq \mathcal{N}^{\mathcal{A}}_{r_i}(\vec{a})$ such that **II** has a winning strategy in each i round $(s_0, \ldots, s_{i-1})$-shrinking game from $(\mathcal{A}, c, \mathcal{A}, d)$ for $c, d \in C$ (if $|\vec{a}| = 0$ then **I** chooses $m - i$ elements in A)
  2. **II** replies with a non empty set of s_i-scattered elements $D \subseteq \mathcal{N}^{\mathcal{B}}_{r_i}(\vec{b})$ such that $|C| = |D|$
  3. **I** chooses $d \in D$
  4. **II** chooses $c \in C$
  5. the position is extended with (c, d) and i rounds are left
- The ending and winning conditions are as in standard EF-game

Theorem

*Let $m \in \mathbb{N}$ and let $\vec{s} = s_0, s_1, \ldots$ be a sequence that shrinks rapidly. If **II** has a winning strategy in the m-round $\vec{s}$-shrinking game for $\mathcal{A}$ and $\mathcal{B}$ then **II** has a winning strategy in $\mathcal{G}_m(\mathcal{A}, \mathcal{B})$.*

Normal forms of the Gaifman type

Definition

An s-scattered r-local sentence of width k is a sentence of the form

$$\exists x_1 \cdots x_k \big(\bigwedge_{1 \leqslant i < j \leqslant k} \delta(x_i, x_j) > s \wedge \bigwedge_{1 \leqslant i \leqslant k} \phi(x_i) \big)$$

where $\phi(x)$ is r-local.

Theorem

Fix a scattering sequence $\vec{s}$ shrinking rapidly. Then each FO-sentence ψ with qr$(\psi) \leqslant n$ is logically equivalent to a finite Boolean combination of sentences each of which is s_m-scattered and $(r_m - 1)$-local of width at most $n - m$, for some $m < n$.

Corollary

ψ is logically equivalent to a finite Boolean combination of sentences each of which is $2 \cdot 4^m$-scattered and $(4^m - 1)$-local of width at most $n - m$, for some $m < n$.