Preliminaries

- From now on, we focus on (finite) structures over finite relational vocabularies
- σ-structure: (finite) structure over the vocabulary σ
- Every vocabulary implicitly contains $=$
- σ-formula: formula using extra-logical symbols from σ
- $\mathrm{STRUCT}[\sigma]$: set of all finite σ-structures
- All classes of structures we consider are assumed to be closed under isomorphism, that is, if $h\colon A \to B$ is an isomorphism between $\mathcal{A}$ and $\mathcal{B}$, then $\mathcal{A}$ belongs to a class of structures $\mathcal{K}$ if and only if $\mathcal{B} \in \mathcal{K}$

L. Libkin
Elements of Finite Model Theory, Springer, 2004

K. Doets
Introduction to Ehrenfeucht's Game, 2002

Definability of relations

Definition

Let $\mathcal{A}$ be a σ-structure. A σ-formula ϕ with $n > 0$ free variables $x_1, \ldots, x_n$ *defines* the n-ary relation

$$R_\phi^{\mathcal{A}} = \{ (a_1, \ldots, a_n) \mid \mathcal{A} \models \phi\{{}^{x_1}/a_1, \ldots, {}^{x_n}/a_n\} \}.$$

- A relation S is *definable* in a σ-structure $\mathcal{A}$ if there is a σ-formula ϕ such that $R_\phi^{\mathcal{A}} = S$
- This form of definability is relative to a given structure

Example

The successor relation $S = \{ (n, n+1) \mid n \in \mathbb{N} \}$ is definable in $(\mathbb{N}, <)$

- Let $\phi(x, y) \stackrel{\text{def}}{=} (x < y) \wedge \neg \exists z \, (x < z \wedge z < y)$
- Then $S = R_{\phi(x,y)}^{(\mathbb{N},<)}$

Definability in $(\mathbb{N}, <)$

Lemma

A set is definable in $(\mathbb{N}, <)$ iff it is finite or co-finite.

$(\Leftarrow)$

- $\pi_n(x)$ ("x has exactly n predecessors") is definable
    - $\pi_0(x) \stackrel{\text{def}}{=} \neg \exists y \, (y < x)$, $\pi_1(x) \stackrel{\text{def}}{=} \exists z \, (\pi_0(z) \wedge s(z, x))$ (where s is a $<$-formula defining the successor), ...
- $A \subseteq_{\text{fin}} \mathbb{N}$ can be defined by $\phi_A(x) \stackrel{\text{def}}{=} \bigvee_{n \in A} \pi_n(x)$
- $\mathbb{N} \setminus A$ can be defined by $\neg \phi_A(x)$

Definability in $(\mathbb{N}, <)$ (cont.)

$(\Rightarrow)$

- Let $\phi(x)$ define a set which is neither finite nor co-finite
- $(\mathbb{N}, <) \models \forall x \exists y\, (x < y \wedge \phi(y)) \wedge \forall x \exists y\, (x < y \wedge \neg\phi(y))$
- But $\mathbb{N} \equiv \mathbb{N} \triangleleft \mathbb{Z}$
- Hence, there is a in the $\mathbb{Z}$-part of $\mathbb{N} \triangleleft \mathbb{Z}$ that satisfies $\phi(x)$ and there is b in the $\mathbb{Z}$-part of $\mathbb{N} \triangleleft \mathbb{Z}$ that satisfies $\neg\phi(x)$
- But there is an automorphism of $\mathbb{N} \triangleleft \mathbb{Z}$ mapping a onto b
- A contradiction arises

Corollary

The set of even/odd natural numbers is not FO-definable in $(\mathbb{N}, <)$

Definability of σ-structures

Definition

A σ-sentence ψ *defines* the class $\mathcal{C}_\psi$ of models in which it is true, that is,

$$\mathcal{C}_\psi = \{\mathcal{A} \mid \mathcal{A} \in \mathsf{STRUCT}[\sigma] \wedge \mathcal{A} \models \psi\}.$$

- A given class of structures $\mathcal{D}$ is *definable* if there is a σ-sentence ψ such that $\mathcal{D} = \mathcal{C}_\psi$

Definition

A σ-sentence ψ *defines the class* $\mathcal{P}$ *relative to a class* $\mathcal{K}$ of σ-structures if, and only if,

$$\mathcal{K} \cap \mathcal{C}_\psi = \mathcal{P}.$$

Queries

Definition

For $m > 0$, an m-*ary query* on a class $\mathcal{K} \subseteq \mathrm{STRUCT}[\sigma]$ is a mapping Q that associates any σ-structure $\mathcal{A} \in \mathcal{K}$ with an m-ary relation over its universe A, such that Q is closed under isomorphism, that is, if $h\colon A \to B$ is an isomorphism between $\mathcal{A}$ and $\mathcal{B}$ and Q is an m-ary query, then $(a_1, \ldots, a_m) \in Q(\mathcal{A})$ if and only if $(h(a_1), \ldots, h(a_m)) \in Q(\mathcal{B})$.

Definition

A *Boolean query* on $\mathcal{K}$ is a mapping (closed under isomorphism) which assigns a value in $\{\textit{true}, \textit{false}\}$ to any given σ-structure $\mathcal{A} \in \mathcal{K}$.

- Uniform definability over a class of structures
- Binary query $\neq$ Boolean query
- A Boolean query is a statement of a property of a class
  - E.g., connectivity of graphs

Queries: examples

Example

Let $\mathcal{G}$ be the class of finite graphs and let $G = (V, E) \in \mathcal{G}$ be a finite graph. The following are queries on $\mathcal{G}$:

❶ "transitive closure of a graph" (binary query):

$$\mathrm{TC}(G) = \{ (s,t) \in V \times V \mid \text{there is a path from } s \text{ to } t \};$$

❷ "elements of degree m" (unary query):

$$D_m(G) = \{ v \in V \mid v \text{ has degree } m \};$$

❸ "connectivity" (Boolean query):

$$\mathrm{CONN}(G) = \begin{cases} \textit{true} & G \text{ is connected;} \\ \textit{false} & \text{otherwise.} \end{cases}$$

Definability of queries

- Let $m > 0$
- Let $\mathfrak{L}$ be a logic
- Let $\mathcal{K}$ be a class of σ-structures

Definition
An m-ary query Q on $\mathcal{K}$ is $\mathfrak{L}$-*definable* if there is a σ-formula ϕ of $\mathfrak{L}$ with m free variables such that for every $\mathcal{A} \in \mathcal{K}$

$$Q(\mathcal{A}) = R_\phi^{\mathcal{A}}$$

Definition
A Boolean query Q on $\mathcal{K}$ is $\mathfrak{L}$-*definable* if there is a σ-sentence ψ of $\mathfrak{L}$ such that

$$\{\mathcal{A} \mid \mathcal{A} \in \mathcal{K} \land Q(\mathcal{A}) = \textit{true}\} = \mathcal{C}_\psi \cap \mathcal{K}$$

First-order logic is too strong

- Any finite structure can be defined by a single sentence (up to isomorphism)

Example
Given a finite graph $G = (V, E)$, with $|V| = n$,

$$\exists x_1 \cdots \exists x_n \Big(\bigwedge_{i \neq j} \neg(x_i = x_j) \land (\forall y \bigvee_i (x_i = y))$$
$$\land (\bigwedge_{(v_i, v_j) \in E} E(x_i, x_j)) \land \bigwedge_{(v_i, v_j) \notin E} \neg E(x_i, x_j) \Big)$$

defines G.

- Every class of finite structures can be characterized by a set of sentences (up to isomorphism)
- Elementary equivalence is the same as isomorphism in the finite

First-logic is too strong (cont.)

Lemma

Let σ be a finite vocabulary. Every class $\mathcal{K}$ of finite σ-structures is definable by a set of σ-sentences.

- For every fixed $n > 0$, there is a only *finite* number of pairwise non-isomorphic σ-structures with n elements (because σ is finite)
- Let $\{\mathcal{A}_1, \ldots, \mathcal{A}_k\}$ be a maximal set of such structures
- Let $\phi_{\mathcal{A}_i}$ be the sentence that defines $\mathcal{A}_i$
- Let $\phi_{=n}$ be a σ-sentence that expresses the property "there are exactly n elements" in the domain
- Let $\psi_n \stackrel{\text{def}}{=} \phi_{=n} \to (\phi_{\mathcal{A}_1} \vee \cdots \vee \phi_{\mathcal{A}_k})$
- Then, $\mathcal{K}$ is precisely the class of models of $\{\, \psi_n \mid n > 0 \,\}$

First-order logic is too weak

- Natural properties cannot be expressed (such as, for instance, "the domain has even cardinality")
- "Weak" does not necessarily mean "bad"

> *"[...] weak expressive power can also be a good thing, as it implies transfer of properties across different situations. In non-standard arithmetic, one computes in the structure $\mathbb{N} \triangleleft \mathbb{Z}$ using the infinite numbers to simplify calculations, and then transfers the outcome back to $\mathbb{N}$, provided it is a first-order statement about $<$."*

(van Benthem's course on logical games, Ch. 2, *Model Comparison Games*)

Example (Transfer of properties)

Assume that **II** has a winning strategy in $\mathcal{G}_3((A, R), (B, R'))$ and R is dense. Then, R' is also dense.

Definability and EF-games

- Let $\mathcal{K}$ be a class of σ-structures
- Let Q be a Boolean query on $\mathcal{K}$
- The following are equivalent (corollary of Ehrenfeucht theorem):
  - Q is FO-definable on $\mathcal{K}$
  - There is $m \in \mathbb{N}$ such that, for every $\mathcal{A}, \mathcal{B} \in \mathcal{K}$ such that $\mathcal{A}$ has property Q and $\mathcal{B}$ does not, **I** has a winning strategy in $\mathcal{G}_m(\mathcal{A}, \mathcal{B})$
- How to prove an inexpressivity result?
- For every $m \in \mathbb{N}$, find $\mathcal{A}, \mathcal{B} \in \mathcal{K}$ such that
  1. $\mathcal{A}$ has property Q
  2. $\mathcal{B}$ does not have property Q
  3. **II** has a winning strategy in $\mathcal{G}_m(\mathcal{A}, \mathcal{B})$
- Soundness: the method above proves that Q is not definable
- Completeness: if Q is not definable, the method above can (in principle) be used to prove it

Definability and EF-games (cont.)

- Let $\mathcal{K}$ be a class of σ-structures
- Let Q be an m-ary query on $\mathcal{K}$
- The following are equivalent (corollary of Ehrenfeucht theorem):
  - Q is FO-definable on $\mathcal{K}$
  - There is $m \in \mathbb{N}$ such that, for every $\mathcal{A}, \mathcal{B} \in \mathcal{K}$ and m-tuples $\vec{a}, \vec{b}$ such that $\vec{a} \in Q(\mathcal{A})$ and $\vec{b} \notin Q(\mathcal{B})$, **I** has a winning strategy in $\mathcal{G}_m(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$
- How to prove an inexpressivity result?
- For every $m \in \mathbb{N}$, find $\mathcal{A}, \mathcal{B} \in \mathcal{K}$ and m-tuples $\vec{a}, \vec{b}$ such that
  1. $\vec{a} \in Q(\mathcal{A})$
  2. $\vec{b} \notin Q(\mathcal{B})$
  3. **II** has a winning strategy in $\mathcal{G}_m(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$
- Soundness: the method above proves that Q is not definable
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Games on sets

- Let A and B be two sets
- Empty vocabulary (only equality)

Lemma

If $|A|, |B| \geqslant m$ *then* $A \equiv_m B$.

- Assume that, after i rounds ($i < m$), the mapping $A \to B$

$$(a_1, \ldots, a_i) \mapsto (b_1, \ldots, b_i)$$

  is a partial isomorphism
- At round $i+1$, assume that **I** picks a_{i+1}
- W.l.o.g, $a_{i+1} \notin \{a_1, \ldots, a_i\}$
- **II** responds with $b_{i+1} \in B \setminus \{b_1, \ldots, b_i\}$
- $B \setminus \{b_1, \ldots, b_i\}$ is non-empty by hypothesis
- Dual reasoning for **I** playing in $\mathcal{B}$

Games on sets: remarks

- Even Cardinality:

$$\mathsf{EC}(\mathcal{A}) = \begin{cases} \textit{true} & A \text{ has even cardinality} \\ \textit{false} & \text{otherwise} \end{cases}$$

Lemma

EC is not FO-definable over sets.

- $\{a_1, \ldots, a_m\} \equiv_m \{b_1, \ldots, b_{m+1}\}$ (see previous slide)
- Finiteness:

$$\mathsf{FIN}(\mathcal{A}) = \begin{cases} \textit{true} & A \text{ has finite cardinality} \\ \textit{false} & \text{otherwise} \end{cases}$$

Lemma

FIN is not FO-definable over sets.

- $\{a_1, \ldots, a_m\} \equiv_m \mathbb{N}$
- Does $A \equiv_m B$ imply $|A|, |B| \geqslant m$?
- Of course not! $A \equiv_m B \Leftrightarrow |A|, |B| \geqslant m \vee |A| = |B|$

Games on sets: remarks

- Even Cardinality:

$$\mathsf{EC}(\mathcal{A}) = \begin{cases} \textit{true} & A \text{ has even cardinality} \\ \textit{false} & \text{otherwise} \end{cases}$$

Lemma
EC is not FO-definable over sets.

- $\{a_1, \dots, a_m\} \equiv_m \{b_1, \dots, b_{m+1}\}$ (see previous slide)

- Finiteness:

$$\mathsf{FIN}(\mathcal{A}) = \begin{cases} \textit{true} & A \text{ has finite cardinality} \\ \textit{false} & \text{otherwise} \end{cases}$$

Lemma
FIN is not FO-definable over sets.

- $\{a_1, \dots, a_m\} \equiv_m \mathbb{N}$
- Does $A \equiv_m B$ imply $|A|, |B| \geqslant m$?
- Of course not! $A \equiv_m B \Leftrightarrow |A|, |B| \geqslant m \vee |A| = |B|$

Games on linear orderings

- $\mathcal{L}_n = (\{1, \dots, n\}, <)$
- $\mathcal{L}_n^{<k} = (\{1, \dots, k\}, <)$ and
  $\mathcal{L}_n^{>k} = (\{k+1, \dots, n\}, <) \cong (\{1, \dots, n-k\}, <)$
- For $m, n, t \in \mathbb{N}$, $m =_t n$ iff $m = n$ or $m, n > t$

Lemma
If $n, p \geqslant 2^m - 1$ then $\mathcal{L}_n \equiv_m \mathcal{L}_p$.

- Direct proof by induction on m maintaining the following invariant: if $(a_1 \dots, a_k) \mapsto (b_1, \dots, b_k)$ is the mapping after k rounds then, for every $1 \leqslant i, j \leqslant k$ and $t = 2^{m-k} - 1$,
  1. $a_i < a_j$ iff $b_i < b_j$
  2. $|a_i - a_j| =_t |b_i - b_j|$, $a_i =_t b_i$ and $n - a_i =_t p - b_i$
- Proof using the congruence of linear orderings:
  - $\mathcal{L}_n \equiv_{m+1} \mathcal{L}_p$ if, and only if, for every $i \in L_n$ there is $j \in L_p$ such that

    $$\mathcal{L}_n^{<i} \equiv_m \mathcal{L}_p^{<j} \wedge \mathcal{L}_n^{>i} \equiv_m \mathcal{L}_p^{>j}$$

    and for every $j \in L_p$ there is $i \in L_n$ such that

    $$\mathcal{L}_n^{<i} \equiv_m \mathcal{L}_p^{<j} \wedge \mathcal{L}_n^{>i} \equiv_m \mathcal{L}_p^{>j}$$

Games on linear orderings: remarks

Lemma
EC is not FO-definable on the class of (finite) linear orderings.

- For every m, $\mathcal{L}_{2^m-1} \equiv_m \mathcal{L}_{2^m}$

Lemma
FIN is not FO-definable on the class of linear orderings.

- For all m, $\mathcal{L}_{2^m-1} \equiv_m \mathbb{N} \triangleleft \mathbb{N}^R$ ($\mathbb{N}^R$ is like a reversed copy of $\mathbb{N}$)
- Note that the class of finite linear orderings is axiomatizable (i.e., a sentence is true on the class of linear orderings iff it is a logical consequence of the following axioms)
  - transitivity
  - trichotomy (exactly one among $a < b$, $b < a$ and $a = b$ holds)
  - existence of endpoints
  - discreteness (existence of successor/predecessor)

A game-theoretic proof of undefinability in $(\mathbb{N}, <)$

Lemma
The set of even natural numbers is not FO-definable in $(\mathbb{N}, <)$

- "The set of even natural numbers" is a unary query
- Let $\mathcal{A}$ be $(\mathbb{N}, <)$ and let $\mathcal{B}$ be $(\mathbb{N}, <)$
- Fix m
- Let a be any even number in $\mathcal{A} \geqslant 2^m$
- Let b be any odd number in $\mathcal{B} \geqslant 2^m - 1$
- Then, **II** has a winning strategy in $\mathcal{G}_m(\mathcal{A}, a, \mathcal{B}, b)$
  - $(\{0, \ldots, a-1\}, <) \equiv_m (\{0, \ldots, b-1\}, <)$
  - $(\{a+1, \ldots\}, <) \equiv (\{b+1, \ldots\}, <)$ (they are both $\equiv (\mathbb{N}, <)$)
  - Just compose the strategies

Undefinability on graphs

Lemma

The class of all (finite or infinite) connected graphs is not FO-definable.

- Compactness argument (first lesson)

Lemma

CONN is not FO-definable on the class of finite graphs.

- Given m, let $r = 3^{m+1}$
- Let $d > 2r + 1$
- Let $\mathcal{G}_1$ consist of a cycle of length $2d$
- Let $\mathcal{G}_2$ consist of two disjoint cycles of length d
- Every r-neighbourhood is a path of length $2r$
- By the Sphere theorem, $\mathcal{G}_1 \equiv_m \mathcal{G}_2$

Games on trees

- The class of finite trees is not FO-definable over the class of finite graphs
    - compare a path with a cycle
    - E.g., see Libkin 2004

K. Doets
On n-Equivalence of Binary Trees
Notre Dame Journal of Formal Logic, 1987

This note presents a simple characterization of the class of all trees which are n-elementary equivalent with B_m: the binary tree with one root all of whose branches have length m (for each pair of positive integers n and m). [...] Section 2 introduces the class $Q(n)$ of binary trees and proves that every tree in it is n-equivalent with B_m whenever $m \geqslant 2^n - 1$. Section 3 shows that, conversely, each n-equivalent of a B_m with $m > 2^n - 1$ belongs to $Q(n)$. Finally, all n-equivalents of B_m for $m < 2^n - 1$ are isomorphic to B_m.

Notions of locality

- Hanf's theorem (seen) and Gaifman's theorem (to be seen) are results about the *locality* of FO
- They have inspired a slightly different methodology to prove inexpressivity results

How to prove that query Q is not definable in logic $\mathfrak{L}$?

1. Provide a definition of locality for queries
2. Prove that every $\mathcal{L}$-definable query is local according to the given definition
3. Prove that Q is not local according to the same definition

Hanf-locality

- Recall that $\mathcal{A} \leftrightarrows_r \mathcal{B}$ means that there is a bijection $f \colon A \to B$ such that $\mathcal{N}_r^{\mathcal{A}}(a) \cong \mathcal{N}_r^{\mathcal{B}}(f(a))$ for every $a \in A$

Definition

A Boolean query Q on a class $\mathcal{K}$ of σ-structures is *Hanf-local* if, and only if, there is $r \in \mathbb{N}$ such that, for every $\mathcal{A}, \mathcal{B} \in \mathcal{K}$,

$$\text{if } \mathcal{A} \leftrightarrows_r \mathcal{B} \text{ then } (Q(\mathcal{A}) \Leftrightarrow Q(\mathcal{B}))$$

Example

CONN is not Hanf-local

- By contradiction, let CONN be Hanf-local for a given r
- Let $d > 2r + 1$
- Let $\mathcal{G}_1$ consist of a cycle of length $2d$
- Let $\mathcal{G}_2$ consist of two disjoint cycles of length d
- Let f an arbitrary bijection between $\mathcal{G}_1$ and $\mathcal{G}_2$
- Every r-neighbourhood is a path of length $2r$
- Then, $\mathcal{A} \leftrightarrows_r \mathcal{B}$; but, $\mathcal{G}_1$ is connected and $\mathcal{G}_2$ is not

Gaifman-locality

Definition

An m-ary query Q on a class $\mathcal{K}$ of σ-structures is *Gaifman-local* if, and only if, there is $r \in \mathbb{N}$ such that, for every $\mathcal{A} \in \mathcal{K}$ and m-tuples $\vec{a}_1, \vec{a}_2 \in A^m$,

$$\text{if } \mathcal{N}_r^{\mathcal{A}}(\vec{a}_1) \cong \mathcal{N}_r^{\mathcal{A}}(\vec{a}_2) \text{ then } (\vec{a}_1 \in Q(\mathcal{A}) \Leftrightarrow \vec{a}_2 \in Q(\mathcal{A})).$$

Example

Transitive Closure (TC) is not Gaifman-local

- E.g., consider $(\mathbb{Z}, \mathit{succ})$
- Given r, take a, b such that $b - a > 2r + 1$
- Then, $\mathcal{N}_r^{\mathcal{A}}(a) \cong \mathcal{N}_r^{\mathcal{A}}(b)$
- Since the neighbourhoods are not adjacent, then $\mathcal{N}_r^{\mathcal{A}}(a, b) \cong \mathcal{N}_r^{\mathcal{A}}(b, a)$
- However, $(a, b) \in \text{TC}$ but $(b, a) \notin \text{TC}$

Locality and first-order logic

- Hanf-locality can be applied only when $|A| = |B|$
- (The version of) Hanf-locality (generalized to m-queries) implies Gaifman-locality
- Every FO-definable query is Hanf-local
- Hence, every FO-definable query is Gaifman-local
- Exponential lower bounds on the locality ranks (the minimum integer such that the locality property holds) can be proved

Explicit definability

- Let σ be a purely relational vocabulary
- Let R be a relation symbol in σ
- Let σ_0 be $\sigma \setminus \{R\}$
- Let T be a set of σ-sentences (closed under entailment)

Definition

T *explicitly defines* R iff there is a σ_0-formula ϕ such that

$$T \models \forall \vec{x}\,(R(\vec{x}) \leftrightarrow \phi(\vec{x})).$$

Or, for a complete logic (as FO),

$$T \vdash \forall \vec{x}\,(R(\vec{x}) \leftrightarrow \phi(\vec{x})).$$

- Equivalently, ϕ *explicitly defines* R *relative to* T
- Syntactic notion of definability
- Obviously implies that any two models of T that agree on the interpretation of σ_0 must also agree on the interpretation of R

Explicit definability in FO: an example

- Let $\sigma = \{<, s\}$ and $\sigma_0 = \{<\}$
- Let T be the theory of linear orderings plus the following:
  1. $\forall x \forall y \forall y'\,((s(x,y) \wedge s(x,y')) \rightarrow y = y')$
  2. $\forall x \forall y\,(s(x,y) \rightarrow x < y)$
  3. $\forall x \forall y\,(x < y \rightarrow \exists y'\,(y' \leqslant y \wedge s(x,y')))$
- Then, s is explicitly definable relative to T:

$$T \models \forall x \forall y\,\big(s(x,y) \leftrightarrow \phi(x,y)\big)$$

where

$$\phi(x,y) \equiv x < y \wedge \neg \exists w\,(x < w < y)$$

Explicit definability in FO: an example (cont.)

($\rightarrow$) $s(x, y)$ holds by hypothesis

- By (2), $s(x, y)$ implies $x < y$
- We need to prove that $s(x, y)$ implies $\neg\exists w\,(x < w < y)$
- For the sake of contradiction, assume that w exists such that $x < w < y$
- Then, by (3), $s(x, w')$ holds for some $w' \leqslant w$
- But then, by (1), $y = w' \leqslant w$, which contradicts $w < y$

($\leftarrow$) $x < y \wedge \neg\exists w\,(x < w < y)$ holds by hypothesis

- By (3), there is $y' \leqslant y$ such that $s(x, y')$ holds
- By (2), $x < y' \leqslant y$
- By hypothesis, no w exists such that $x < w < y$
- Hence, $y' = y$ and $s(x, y)$ holds

Implicit definability

- Let σ be a purely relational vocabulary
- Let R be a relation symbol in σ
- Let σ_0 be $\sigma \setminus \{R\}$
- Let T be a set of σ-sentences (closed under entailment)
- Let S be a fresh relation symbol with the same arity as R
- Let T' be like T with occurrences of R replaced by S

Definition

T *implicitly defines* R iff any σ_0-structure has *at most one expansion* to a model of T, i.e.,

$$T \cup T' \models \forall \vec{x}\,(R(\vec{x}) \leftrightarrow S(\vec{x})).$$

- I.e., every pair of models of T that agree on the interpretation of σ_0 also agree on the interpretation of R
- R can be characterized uniquely
- Semantic notion of definability

Beth theorem

- Explicit definability entails implicit definability
- What about the converse?

Definition (Beth Property)

A logic has the Beth property iff for every relation symbol $R \in \sigma$ and for every set of σ-sentences T, if T implicitly defines R then T explicitly defines R.

Theorem (Beth theorem)

First-order logic has the Beth property.

- A model-theoretic notion of definability coincides with a proof-theoretic notion of definability
- Good balance between syntax and semantics
- Unfortunately, FO interpreted over finite structures does not have the Beth property

Beth theorem fails in the finite

- Let $\sigma = \{<, P\}$ with P unary predicate
- Let T be the theory of linear orderings plus
    1. $\exists x\,(P(x) \wedge \forall y\,(x \leqslant y))$
    2. $\forall x \forall y\,(s(x,y) \to (P(x) \leftrightarrow \neg P(y)))$, where $s(x,y)$ is a shorthand for "y is an immediate successor of x"

Lemma

T *implicitly defines* P *on finite models*

- Let $\mathcal{M}$ be a finite model of T
- $M = \{m_1, \ldots, m_k\}$ with $m_1 < m_2 < \cdots < m_k$
- According to T, $m_i \in P^{\mathcal{M}}$ iff i is odd
- Hence, the interpretation of P is uniquely determined
- Note that, on infinite models, this does not need to be the case
    - E.g., on $\mathbb{R}^+$ with $<$ interpreted as usual, any P containing 0 yields a model of T

Beth theorem fails in the finite (cont.)

Lemma

There is no explicit definition for P *relative to* T.

- For the sake of contradiction, suppose that a $\{<\}$-formula $\phi(x)$ that defines P exists ($\phi(x)$ means "x has odd index")
- Let k be the quantifier rank of ϕ
- Consider the formula

$$\psi \equiv \exists x\,(\phi(x) \wedge \forall y\,(y \leqslant x))$$

- ψ must be true in every finite model of T iff its cardinality is odd
- ψ has quantifier rank $k+1$
- We know that $\mathcal{L}_{2^{k+1}-1} \equiv_{k+1} \mathcal{L}_{2^{k+1}}$
- Hence, $\mathcal{L}_{2^{k+1}-1} \models \psi$ and $\mathcal{L}_{2^{k+1}} \not\models \psi$ is a contradiction

Explicit and implicit definability of queries

Definition

A m-ary query Q is *explicitly definable* iff there is a σ-formula $\phi(x_1, \ldots, x_m)$ such that, for every $\mathcal{A}$, $R_\phi^\mathcal{A} = Q(\mathcal{A})$

Definition

Let P be an m-ary relation symbol not occurring in σ. An m-ary query Q is *implicitly definable* iff there is a $(\sigma \cup \{P\})$-sentence ψ such that every σ-structure $\mathcal{A}$ has a unique expansion to a $(\sigma \cup \{P\})$-structure that satisfies ψ, namely $(\mathcal{A}, Q(\mathcal{A}))$

- Let Q be "the set of even elements of a finite linear ordering"
- Q is implicitly definable
  - See two slides before
  - $\psi \equiv \exists x\,(P(x) \wedge \forall y\,(x \leqslant y)) \wedge \forall x \forall y\,(s(x,y) \rightarrow (P(x) \leftrightarrow \neg P(y))$
- There is no explicit definition for Q
  - See the previous slide

Explicit and implicit definability: exercise

- Let $\sigma = \{s\}$ (interpreted as the successor relation)
- Let Q be binary query: "the transitive closure of s" ($<$ relation)

❶ Using EF-games, prove that Q is not explicitly definable over the class of successor structures

❷ Is Q implicitly definable?

A library of sufficient conditions

- Sufficient conditions allow us to prove negative expressivity results

R. Fagin and L. J. Stockmeyer and M. Y. Vardi
On monadic NP vs monadic co-NP
Information and Computation, 1995

T. Schwentick
On winning Ehrenfeucht games and monadic NP
Annals of Pure and Applied Logic, 1996

S. Arora and R. Fagin
On winning strategies in Ehrenfeucht-Fraïssé games
Theoretical Computer Science, 1997

Arora and Fagin's condition

- "Approximately" isomorphic neighborhoods
- Still based on a multiplicity argument
- Neighborhoods must be tree-like structures

Definition (simplified for directed graphs)

- The $(m, 0)$-color of an element a is its label plus a description of whether it is a constant and whether it has a self-loop
- the $(m, r+1)$-color of a is its (m, r)-color plus a list of triples, one for each possible (m, r)-color τ:
  1. the number of elements b with (m, r)-color τ such that $E(a, b)$ but not $E(b, a)$, counted up to m
  2. the number of elements b with (m, r)-color τ such that $E(b, a)$ but not $E(a, b)$, counted up to m
  3. the number of elements b with (m, r)-color τ such that $E(a, b)$ and $E(b, a)$, counted up to m

Arora and Fagin's condition (cont.)

Let the color of a directed edge be the ordered pair of colors of its nodes.

Theorem

Let $\mathcal{A} = (A, E)$ and $\mathcal{B} = (B, E)$ be two structures of degree at most d, and let $m \in \mathbb{N}$. If

- *there is a bijection $f\colon A \to B$ such that a and $f(a)$ have the same (m, r)-color, with $r = 3^{2m}$, for all $a \in A$,*
- *$\mathcal{A}$ and $\mathcal{B}$ do not have (undirected) cycles of length less than r,*
- *whenever $E^{\mathcal{A}}(a, b)$ holds but $E^{\mathcal{B}}(f(a), f(b))$ does not hold, or vice versa, then there are at least d^r edges in both structures having the same (m, r)-color as (a, b), (resp., $(f(a), f(b))$),*

*then **II** has a winning strategy in $\mathcal{G}_m(\mathcal{A}, \mathcal{B})$.*

Applications of Arora and Fagin's condition

- Directed reachability is not in monadic Σ_1^1 (a simpler proof of Ajtai and Fagin's result)
- Graph connectivity is not in monadic Σ_1^1
- Both results can be shown to hold even if the vocabulary is expanded with particular built-in relations of degree $n^{o(1)}$, where n is the size of the structure
- The requirement of the absence of small cycles can be relaxed at the expense of adding further hypotheses

Schwentick's extension theorem

Schwentick's work moves from the following question: Under which conditions can a "local" strategy be extended?

He develops a method that allows, under certain conditions, the extension of a winning strategy for **II** on some small parts of two finite structures to a global winning strategy.

- The structures must be isomorphic except for some small parts, for which local winning strategies exist by hypothesis
- The advantage is that there are no further constraints, either on the degree or on the internal characteristics of the substructures.

Schwentick's extension theorem (cont.)

- Let $\mathcal{C}$ and $\mathcal{D}$ be subtructures of $\mathcal{A}$ and $\mathcal{B}$, respectively
- Suppose that **II** has a winning strategy in $\mathcal{G}_m(\mathcal{C}, \mathcal{D})$ for some m
- **II** has a winning strategy in $\mathcal{G}_m(\mathcal{A}, \mathcal{B})$ if
  1. **II**'s strategy for $\mathcal{G}_m(\mathcal{C}, \mathcal{D})$ can be extended to a winning strategy in $\mathcal{G}_m(\mathcal{N}_{2^m}^{\mathcal{A}}(C), \mathcal{N}_{2^m}^{\mathcal{B}}(D))$, so that, at every round the two chosen elements have the same distance from C and D, respectively
  2. there is an isomorphism $\alpha\colon (A \setminus C) \to (B \setminus D)$ such that $\delta(x, C) = \delta(\alpha(x), D)$ for all $x \in \mathcal{N}_{2^m}^{\mathcal{A}}(C) \setminus C$

Proof's idea

- Divide the domains of the structures into three regions:
- inner area: $I = C \cup D$
- outer area: $O = (A \setminus \mathcal{N}_{2^m}^{\mathcal{A}}(C)) \cup (B \setminus \mathcal{N}_{2^m}^{\mathcal{B}}(D))$
- the area in between
- At each round, the inner or outer areas may grow, according to the played moves
- Separation invariant: after round i the distance from every element in the inner area and every element in the outer area is greater than 2^{m-i}
- So, the winning strategy for **II** is guaranteed by the isomorphism α in the outer area, and by the extended winning strategy in the inner area and the area in between

Extensions

- different distance functions can be used
- winning strategies for several pairs of substructures can be combined
- The separation invariant may be required for some relations, but not for others (e.g., linear ordering), by adding a kind of homogeneity condition that guarantees that elements in the inner and outer areas behave in the same way with respect to the relations that do not satisfy the separation invariant

Applications of Schwentick's extension theorem

- Connectivity of finite graphs is not expressible in monadic Σ_1^1 in the presence of built-in relations of degree $n^{o(1)}$ (the same result as Arora and Fagin's) or even in the presence of a built-in linear ordering
- Monadic Σ_1^1 with a built-in linear ordering is more expressive than monadic Σ_1^1 with a built-in successor relation