Ehrenfeucht-Fraïssé Games: Applications and Complexity

Angelo Montanari   Nicola Vitacolonna

Department of Mathematics and Computer Science
University of Udine, Italy

ESSLLI 2010 CPH

Outline

- Introduction to EF-games
- Inexpressivity results for first-order logic
- Normal forms for first-order logic
- Algorithms and complexity for specific classes of structures
- General complexity bounds

Background on finite model theory

Books

- H.-D. Ebbinghaus and J. Flum
  Finite Model Theory
  Springer, 2nd edition, 2005
- L. Libkin
  Elements of Finite Model Theory
  Springer, 2004

Why finite model theory?

- Connections with computation
  - **Verification**
    finite structures can be coded as words and thus can be objects of computations; moreover, finite structures can be used to describe finite runs of machines
  - **Database theory**
    the relational model identifies a database with a finite relational structure (formulas of a formal language can be viewed as programs to evaluate their meaning in a structure and, vice versa, one can express queries of a certain computational complexity in a given formal language)
    Genuinely finite queries, e.g.,
    - Has the relation R even cardinality?
  - **Computational complexity**
    logical description of complexity classes (e.g., the problem $P = NP$ amounts to the question whether two fixed-point logics have the same expressive power in finite structures)

Most theorems fail, one method survives

We focus our attention on first-order (FO) logic

- Results of model theory often do not apply to the finite
  - Gödel's completeness theorem
  - Compactness theorem
  - Löwenheim-Skolem theorem
  - Definability and interpolation results
  - etc.
- Ehrenfeucht-Fraïssé games are an exception

An application of the compactness theorem

Theorem (Compactness Theorem)

(i) *If ψ is a consequence of Φ, then ψ is a consequence of a finite subset of Φ.*

(ii) *If every finite subset of Φ is satisfiable, then Φ is satisfiable.*

- Connectivity is not FO-definable over the class of all graphs $\mathcal{G} = (G, E)$
  - The proof is via compactness
  - Assume ϕ defines connectivity
  - ψ_n: "there is no path of length $n + 1$ from c_1 to c_2"
  - Let $T = \{\psi_n \mid n > 0\} \cup \{c_1 \neq c_2, \neg E(c_1, c_2), \phi\}$
  - Every finite subset of T is satisfiable, but T is not

Compactness fails in the finite

- γ_n: "there are at least n distinct elements"
  - $\gamma_n \stackrel{\text{def}}{=} \exists x_1 \cdots \exists x_n \bigwedge_{1 \leqslant i < j \leqslant n} (x_i \neq x_j)$
- $\Gamma = \{\gamma_n \mid n > 0\}$
- **General case**: every finite subset of Γ is satisfiable and thus (compactness theorem) Γ is satisfiable, that is, it has an (infinite) model
- **Finite structures**: every finite subset of Γ is satisfiable (it has a finite model), but Γ has no finite model
- Is connectivity definable over all finite graphs? We cannot exploit the compactness theorem to answer the question

Isomorphic and elementarily equivalent structures

Definition (Isomorphic structures)

Two structures $\mathcal{A}$, $\mathcal{B}$, over the same finite vocabulary τ, are isomorphic ($\mathcal{A} \cong \mathcal{B}$) if there is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$, that is, a bijection $\pi : \mathcal{A} \mapsto \mathcal{B}$ preserving relations and constants.

Theorem

Every finite structure can be characterized in FO logic up to isomorphism, that is, for every finite structure $\mathcal{A}$ there exists a FO sentence $\varphi_{\mathcal{A}}$ such that, for every $\mathcal{B}$, we have

$$\mathcal{B} \models \varphi_{\mathcal{A}} \textit{ iff } \mathcal{A} \cong \mathcal{B}.$$

Definition (Elementarily equivalent structures)

Two structures $\mathcal{A}$, $\mathcal{B}$ are elementarily equivalent ($\mathcal{A} \equiv \mathcal{B}$) if they satisfy the same FO sentences.

Notation

- Vocabulary: finite set of relation symbols including $=$ (for the sake of simplicity, we restrict ourselves to a purely relational vocabulary; however, all results extend to vocabularies that have constant symbols).
- $\mathcal{A}$ and $\mathcal{B}$ structures on the same vocabulary
- $\vec{a} = a_1, \ldots, a_k \in \mathrm{dom}(\mathcal{A})$
- $\vec{b} = b_1, \ldots, b_k \in \mathrm{dom}(\mathcal{B})$
- $(\mathcal{A}, \vec{a})$: expansion of structure $\mathcal{A}$ by k elements from its universe
- $(\mathcal{B}, \vec{b})$: expansion of structure $\mathcal{B}$ by k elements from its universe
- Configuration: $(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$, with $|\vec{a}| = |\vec{b}|$
  - It represents the relation $\{(a_i, b_i) \mid 1 \leqslant i \leqslant |\vec{a}|\}$

A weakening of elementary equivalence: m-equivalent structures

Quantifier rank $qr(\phi)$ of a FO-formula ϕ = maximum number of nested quantifiers in ϕ:

- if ϕ is atomic then $qr(\phi) = 0$;
- $qr(\neg\phi_1) = qr(\phi_1)$; $qr(\phi_1 \vee \phi_2) = \max(qr(\phi_1), qr(\phi_2))$;
- $qr(\exists x\, \phi_1) = qr(\phi_1) + 1$.

Example

$\phi = \forall x\, (P(x) \to \exists y\, Q(x, y) \vee \exists y\, R(y))$ has $qr(\phi) = 2$.

Definition (m-equivalent structures)

Two structures $\mathcal{A}$ and $\mathcal{B}$ are m-equivalent, denoted $\mathcal{A} \equiv_m \mathcal{B}$, with $m \geqslant 0$, if they satisfy the same FO sentences of quantifier rank up to m.

m-equivalence can be easily generalized to expanded structures: $(\mathcal{A}, \vec{a}) \equiv_m (\mathcal{B}, \vec{b})$ if they satisfy the same FO formulas of quantifier rank m with at most $|\vec{a}|$ free variables

A weakening of isomorphism: m-isomorphic structures

Definition (partial isomorphisms)

$(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ is a partial isomorphism if it is an isomorphism of the substructures induced by $\vec{a}$ and $\vec{b}$, respectively.

Let $I_1, \ldots, I_m$ be sets of partial isomorphisms such that, for every k, I_k contains partial isomorphisms which allow k-fold extensions.

Definition (m-isomorphic structures)

Two pairs $(\mathcal{A}, \vec{a})$ and $(\mathcal{B}, \vec{b})$ are m-isomorphic, denoted $(\mathcal{A}, \vec{a}) \cong_m (\mathcal{B}, \vec{b})$, if there are nonempty sets $I_0, I_1, \ldots, I_m$ of partial isomorphisms, each of them extending the partial isomorphism $(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$, such that, for all $k = 1, \ldots, m$,

- (forth property) $\forall p \in I_k \forall a \in A \exists b \in B (p \cup \{(a, b)\} \in I_{k-1})$
- (back property) $\forall p \in I_k \forall b \in B \exists a \in A (p \cup \{(a, b)\} \in I_{k-1})$

Theorem (Fraïssé, 1954)

For $m \geqslant 0$, $(\mathcal{A}, \vec{a}) \equiv_m (\mathcal{B}, \vec{b})$ *iff* $(\mathcal{A}, \vec{a}) \cong_m (\mathcal{B}, \vec{b})$.

Combinatorial Games

Ehrenfeucht-Fraïssé games are (logical) combinatorial games.

- Combinatorial games:
  - Two opponents
  - Alternate moves
  - No chance
  - No hidden information
  - No loops
  - The player who cannot move loses[1]

E. R. Berlekamp, J. H. Conway, and R. K. Guy
Winning Ways for your mathematical plays
A K Peters LTD, 2nd edition, 2001

[1] In Combinatorial Game Theory (CGT), this is called *normal play* (the opposite rule: "the player who cannot move wins" is called *misère play*, and it gives rise to quite a different theory)

Ehrenfeucht-Fraïssé games (EF-games)

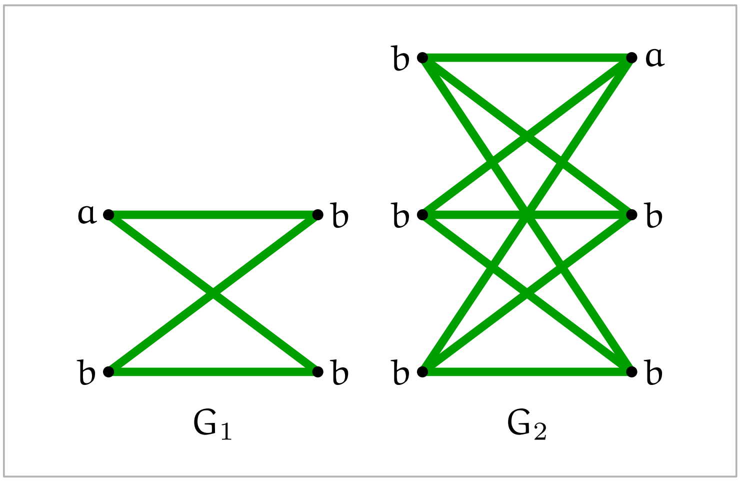
- (Logical) combinatorial games
- The playground: two relational structures $\mathcal{A}$ and $\mathcal{B}$ (over the same finite vocabulary)
- Two players: **I** (Spoiler) and **II** (Duplicator)
- Perfect information
- Move by **I** : select a structure and pick an element in it
- Move by **II** : pick an element in the opposite structure
- Round: a move by **I** followed by a move by **II**
- Game: sequence of rounds
- **II** tries to imitate **I**
- A player who cannot move loses

Winning strategies

- A play from $(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ proceeds by extending the initial configuration with the pair of elements chosen by the two players, e.g.,
  - if **I** picks c in $\mathcal{A}$
  - and **II** replies with d in $\mathcal{B}$
  - then the new configuration is $(\mathcal{A}, \vec{a}, c, \mathcal{B}, \vec{b}, d)$
- Ending condition: a player repeats a move or the configuration is not a partial isomorphism

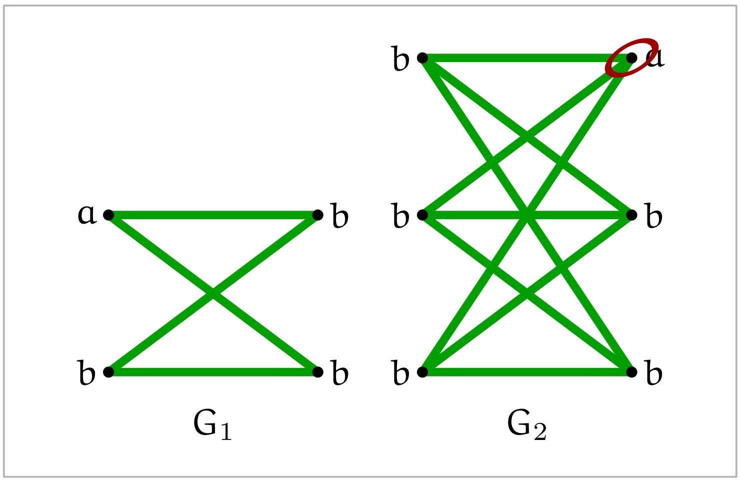
Definition
II has a winning strategy from $(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ if every configuration of the game until an ending configuration is reached is a partial isomorphism, no matter how **I** plays.

An example on graphs



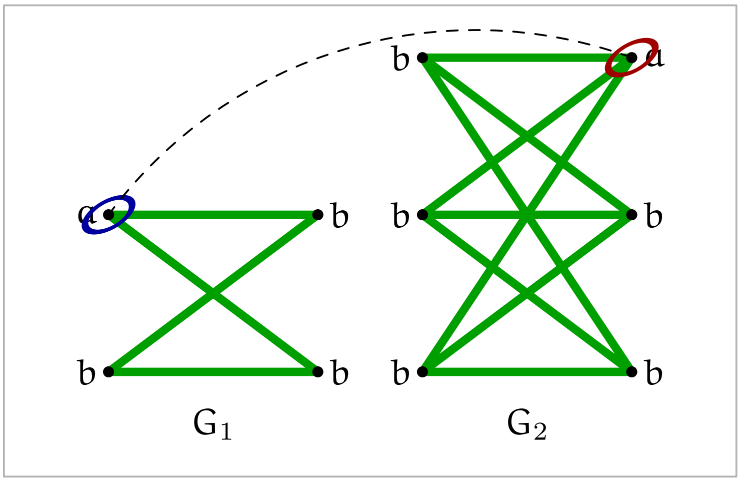
- **II** must respect the adjacency relation...
- ...and pick nodes with the same label as **I** does

An example on graphs



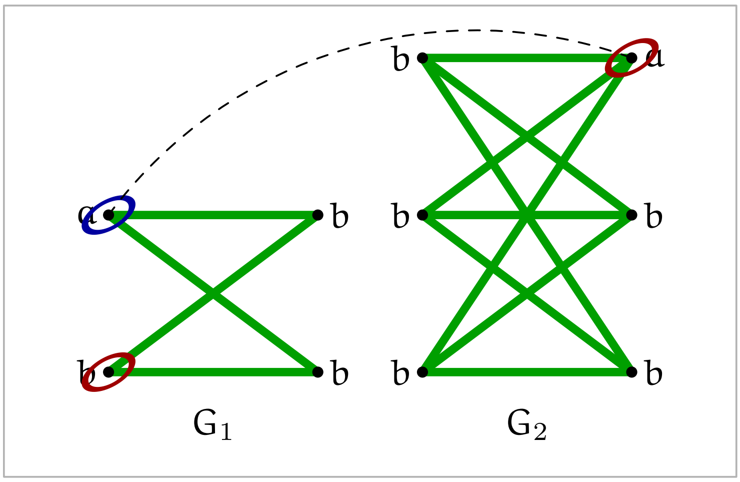
- **II** must respect the adjacency relation...
- ...and pick nodes with the same label as **I** does

An example on graphs



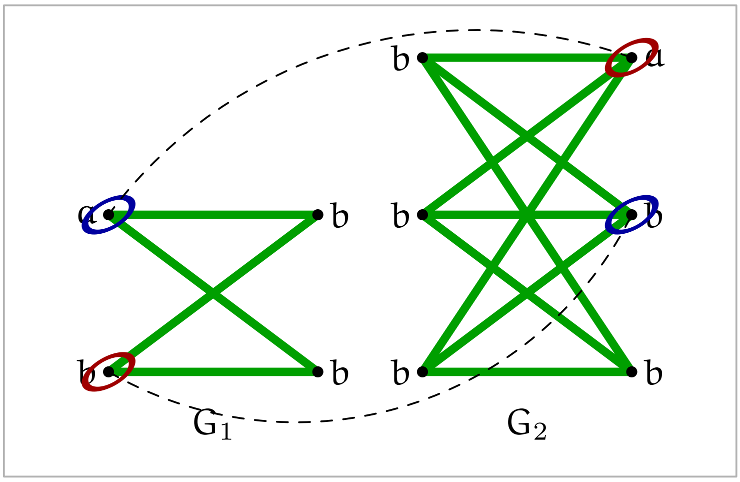
- **II** must respect the adjacency relation...
- ...and pick nodes with the same label as **I** does

An example on graphs



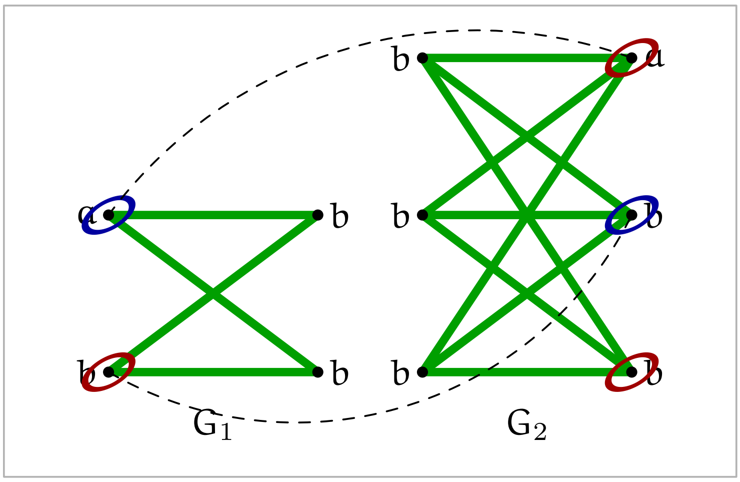
- **II** must respect the adjacency relation. . .
- . . . and pick nodes with the same label as **I** does

An example on graphs



- **II** must respect the adjacency relation. . .
- . . . and pick nodes with the same label as **I** does

An example on graphs



- **II** must respect the adjacency relation...
- ...and pick nodes with the same label as **I** does

Bounded and unbounded games

How long does a game last?

- Bounded game: $\mathcal{G}_m(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ ($\mathcal{G}_m(\mathcal{A}, \mathcal{B})$ if $k = 0$)
  - the number of rounds is fixed: the game ends after m rounds have been played
- Unbounded game: $\mathcal{G}(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ ($\mathcal{G}(\mathcal{A}, \mathcal{B})$ if $k = 0$)
  - the game goes on as long as either a player repeats a move or the current configuration in not partial isomorphism
- **II** wins if and only if the ending configuration is a partial isomorphism

Unbounded games turn out to be useful to compare (finite) structures (comparison games): the remoteness (duration) of an unbounded game as a measure of structure similarity (the notion of remoteness will be formalized later).

Main result

First-order EF-games capture m-equivalence

Theorem (Ehrenfeucht, 1961)

***II** has a winning strategy in* $\mathcal{G}_m(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ *iff* $(\mathcal{A}, \vec{a}) \equiv_m (\mathcal{B}, \vec{b})$.

Remarks.

- If two structures A and B are m-equivalent for every natural number m, then they are elementarily equivalent
- In finite structures, A and B are elementarily equivalent if and only if they are isomorphic (in general, this is not the case: consider, for instance, $\mathbb{N}$ and the ordered sum $\mathbb{N} \triangleleft \mathbb{Z}$)

Definition (EF-problem)

The EF-problem is the problem of determining whether **II** has a winning strategy in $\mathcal{G}_m(\mathcal{A}, \mathcal{B})$, given $\mathcal{A}$, $\mathcal{B}$ and an integer m.

Correspondence between games and formulas

EF-games have a natural logical counterpart which is based on the following simple properties of **II** winning strategies.

Given two structures $\mathcal{A}$ and $\mathcal{B}$, a tuple $\vec{a}$ of elements of A and a tuple $\vec{b}$ of elements of B, with $|\vec{a}| = |\vec{b}|$, and $m \geqslant 0$, we have that:

- **II** wins $\mathcal{G}_0(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ iff $(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ is a partial isomorphism
- for every $m > 0$, **II** wins $\mathcal{G}_m(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$ iff
  - for all $a \in A$, there exists $b \in B$ such that **II** wins $\mathcal{G}_{m-1}(\mathcal{A}, \vec{a}, a, \mathcal{B}, \vec{b}, b)$
  - for all $b \in B$, there exists $a \in A$ such that **II** win $\mathcal{G}_{m-1}(\mathcal{A}, \vec{a}, a, \mathcal{B}, \vec{b}, b)$

From games to formulas: Hintikka formulas

Definition (Hintikka formulas)

Given a structure $\mathcal{A}$, a tuple $\vec{a}$ of elements of A, with $|\vec{a}| = k$, and a tuple $\vec{x}$ of variables $x_1, \ldots, x_k$, let

$$\varphi^0_{(\mathcal{A},\vec{a})}(\vec{x}) \stackrel{\text{def}}{=} \bigwedge_{\substack{\varphi(\vec{x}) \text{ atomic} \\ (\mathcal{A},\vec{a}) \models \varphi(\vec{x})}} \varphi(\vec{x}) \;\wedge \bigwedge_{\substack{\varphi(\vec{x}) \text{ atomic} \\ (\mathcal{A},\vec{a}) \models \neg\varphi(\vec{x})}} \neg\varphi(\vec{x})$$

and, for $m \geqslant 0$,

$$\varphi^{m+1}_{(\mathcal{A},\vec{a})}(\vec{x}) \stackrel{\text{def}}{=} \bigwedge_{a \in A} \exists x_{k+1}\, \varphi^m_{(\mathcal{A},\vec{a},a)}(\vec{x}, x_{k+1}) \wedge$$
$$\forall x_{k+1} \bigvee_{a \in A} \varphi^m_{(\mathcal{A},\vec{a},a)}(\vec{x}, x_{k+1}).$$

For each m, $\varphi^m_{(\mathcal{A},\vec{a})}(\vec{x})$ is called the m-*Hintikka formula*.

From games to formulas: Hintikka formulas (cont.)

The Hintikka formula $\varphi^0_{(\mathcal{A},\vec{a})}(\vec{x})$ describes the isomorphism type of the substructure of $\mathcal{A}$ induced by $\vec{a}$.

In general, $\varphi^m_{(\mathcal{A},\vec{a})}(\vec{x})$ describes to which isomorphism types the tuple $\vec{a}$ can be extended in m steps by adding one element in each step. Since the vocabulary is finite, the above conjunctions and disjunctions are finite even if the structure is infinite.

Theorem (Ehrenfeucht, 1961 - cont.)

For any given $(\mathcal{A}, \vec{a})$, $(\mathcal{B}, \vec{b})$, *and* $m \geqslant 0$, *we have*

$$(\mathcal{B}, \vec{b}) \models \varphi^m_{(\mathcal{A},\vec{a})}(\vec{x}) \iff (\mathcal{A}, \vec{a}) \equiv_m (\mathcal{B}, \vec{b}) \iff$$

II *has a winning strategy in* $\mathcal{G}_m(\mathcal{A}, \vec{a}, \mathcal{B}, \vec{b})$.

Distributive normal form

Hintikka formulas are the basis of a normal form for FO formulas:

- the class of structures which satisfies a given FO formula $\varphi(\vec{x})$ of quantifier rank m must be a union of $\equiv_m$-classes
- each $\equiv_m$-class is defined by a Hintikka formula
- hence, $\varphi(\vec{x})$ is logically equivalent to the (finite) disjunction of those Hintikka formulas which define these $\equiv_m$-classes (distributive normal form for FO logic)

FO definability

A winning strategy for **I** in $\mathcal{G}_m(\mathcal{A}, \mathcal{B})$ can be converted into a FO sentence of quantifier rank at most m that is true in exactly one of A and B (the Hintikka formula $\varphi^m_{(\mathcal{A},\vec{a})}(\vec{x})$ or the Hintikka formula $\varphi^m_{(\mathcal{B},\vec{b})}(\vec{x})$).

A characterization of FO-definable (FO-axiomatizable) classes

- A class $\mathcal{K}$ of structures (on the same finite vocabulary) is FO-definable if and only if there is $m \in \mathbb{N}$ such that **I** has a winning strategy whenever $\mathcal{A} \in \mathcal{K}$ and $\mathcal{B} \notin \mathcal{K}$.

The same characterization holds in the finite case (classes of finite structures) – the same argument applies.

FO undefinability

FO-undefinable classes of structures

- A class $\mathcal{K}$ of structures is not FO-definable if and only if, for all $m \in \mathbb{N}$, there are $\mathcal{A} \in \mathcal{K}$ and $\mathcal{B} \notin \mathcal{K}$ such that **II** has a winning strategy in $\mathcal{G}_m(\mathcal{A}, \mathcal{B})$.

Example

Let $\mathcal{L}_k \overset{\text{def}}{=} (\{1, \ldots, k\}, <)$. It is possible to show that

$$n, p \geqslant 2^m - 1 \Rightarrow \textbf{II} \text{ wins } \mathcal{G}_m(\mathcal{L}_n, \mathcal{L}_p)$$

"The class of linear orderings of even cardinality is not FO-definable": given m, choose $\tilde{n} = 2^m$ and $\tilde{p} = 2^m + 1$; **II** wins $\mathcal{G}_m(\mathcal{L}_{\tilde{n}}, \mathcal{L}_{\tilde{p}})$ (i.e., $\mathcal{L}_{\tilde{n}} \equiv_m \mathcal{L}_{\tilde{p}}$).

Other applications will be given later (inexpressivity results for FO logic).

From differentiating formulas to games

- Let $\mathcal{A}$ and $\mathcal{B}$ be fixed
- Let ϕ be a formula with quantifier rank m
- Let $\mathcal{A} \models \phi$ but $\mathcal{B} \not\models \phi$
- Repeat m times:

1. If $\phi = \forall x_1\, \psi$, let $\phi \leftarrow \neg\phi$ and swap $\mathcal{A}$ and $\mathcal{B}$
   - So, ϕ holds in $\mathcal{A}$ but not in $\mathcal{B}$ and its first quantifier is $\exists$
2. Let $\psi \leftarrow \psi\{x_1/\bar{c}_1\}$, with $\bar{c}_1$ a fresh constant symbol
3. Let **I** pick a_1 in $\mathcal{A}$ such that $(\mathcal{A}, a_1) \models \psi[\bar{c}_1/a_1]$ (since $\mathcal{A} \models \phi$, such an a_1 must exist)
4. Whatever b_1 **II** chooses in $\mathcal{B}$, $(\mathcal{B}, b_1) \not\models \psi[\bar{c}_1/b_1]$
5. Let $\mathcal{A} \leftarrow (\mathcal{A}, a_1)$, $\mathcal{B} \leftarrow (\mathcal{B}, b_1)$ and $\phi \leftarrow \psi$

- Switching between models is encoded in ϕ as quantifier alternations (step 1)

Example

Consider the formula for density:
$\phi = \forall x_1 \forall x_2 \exists x_3 \, (x_1 < x_2 \to x_1 < x_3 < x_2)$,
which holds in $(\mathbb{Q}, <)$ but not in $(\mathbb{Z}, <)$.

(step 1) $\phi \leftarrow \exists x_1 \exists x_2 \forall x_3 \, (x_1 < x_2 \land \neg(x_1 < x_3 < x_2))$

(step 2) $\psi \leftarrow \exists x_2 \forall x_3 \, (x_1 < x_2 \land \neg(x_1 < x_3 < x_2))\{x_1/\bar{c}_1\} = \exists x_2 \forall x_3 \, (\bar{c}_1 < x_2 \land \neg(\bar{c}_1 < x_3 < x_2))$

(step 3) **I** chooses z in $(\mathbb{Z}, <)$ such that
$(\mathbb{Z}, <, z) \models \psi \, [\bar{c}_1/z]$

(step 4) **II** replies q in $(\mathbb{Q}, <)$ such that
$(\mathbb{Q}, <, q) \not\models \psi \, [\bar{c}_1/q]$

(step 2) $\psi \leftarrow \forall x_3 \, (\bar{c}_1 < x_2 \land \neg(\bar{c}_1 < x_3 < x_2))\{x_2/\bar{c}_2\} = \forall x_3 \, (\bar{c}_1 < \bar{c}_2 \land \neg(\bar{c}_1 < x_3 < \bar{c}_2))$

Example (cont.)

(step 3) **I** chooses $z + 1$ in $(\mathbb{Z}, <, z)$ such that
$(\mathbb{Z}, <, z, z+1) \models \psi \, [\bar{c}_1/z, \bar{c}_2/z+1]$

(step 4) **II** replies with $q' > q$ in $(\mathbb{Q}, <, q)$ (otherwise it loses immediately) such that
$(\mathbb{Q}, <, q, q') \not\models \psi \, [\bar{c}_1/q, \bar{c}_2/q']$

(step 1) $\phi \leftarrow \exists x_3 \, (\bar{c}_1 < \bar{c}_2 \to (\bar{c}_1 < x_3 < \bar{c}_2))$

(step 2) $\psi \leftarrow \bar{c}_1 < \bar{c}_2 \to (\bar{c}_1 < x_3 < \bar{c}_2)\{x_3/\bar{c}_3\} = \bar{c}_1 < \bar{c}_2 \to (\bar{c}_1 < \bar{c}_3 < \bar{c}_2)$

(step 3) **I** chooses $q + \frac{q'-q}{2}$ in $(\mathbb{Q}, <, q)$ such that
$(\mathbb{Q}, <, q, q', q + \frac{q'-q}{2}) \models \bar{c}_1 < \bar{c}_2 \to \bar{c}_1 < \bar{c}_3 < \bar{c}_2 \, [\bar{c}_1/q, \bar{c}_2/q', \bar{c}_3/q+(\frac{q'-q}{2})]$

Example (cont.)

(step 4) Of course, whatever z' **II** chooses, we have $(\mathbb{Z}, <, z, z+1, z') \not\models \bar{c}_1 < \bar{c}_2 \rightarrow \bar{c}_1 < \bar{c}_3 < \bar{c}_2 \, [\bar{c}_1/z, \bar{c}_2/z+1, \bar{c}_3/z']$

(game over) The resulting mapping from $\mathbb{Q}$ to $\mathbb{Z}$:

$$\begin{array}{rcl} q & \mapsto & z \\ q' & \mapsto & z+1 \\ q + \dfrac{q'-q}{2} & \mapsto & z' \end{array}$$

is not a partial isomorphism, so **I** wins

Applications of EF-games

EF-games have been exploited to prove some basic results about (the expressive power of) FO logic:

- Hanf's theorem
- Sphere lemma
- Gaifman's theorem

EF-games have been extensively used to prove negative expressivity results (sufficient conditions that guarantee a winning strategy for **II** suffice)

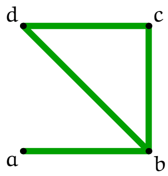
Gaifman's theorem and normal forms for FO logic

Gaifman graph

- Gaifman graph $G(\mathcal{A})$ of a structure $\mathcal{A}$: undirected graph $(\text{dom}(\mathcal{A}), E)$ where $(a, b) \in E$ iff a and b occur in the same tuple of some relation of $\mathcal{A}$
- If $\mathcal{A}$ itself is a (directed) graph, then $G(\mathcal{A})$ is (the undirected version of) $\mathcal{A}$ itself, plus all self-loops
- The *degree* of a node a is the number of nodes $b(\neq a)$ such that $(a, b) \in E$ (the degree of G is the maximum of the degrees of its nodes)
- $\delta(a, b)$: length of the shortest path between a and b in $G(\mathcal{A})$ (if there is not such a path, $\delta(a, b) = \infty$)

Example

$\mathcal{A} = (\{a, b, c, d\}, R, S)$, $R = \{(a, b)\}$, $S = \{(b, c, d)\}$
$\delta(a, c) = \delta(a, d) = 2$



r-sphere and r-neighborhood

Definition (r-sphere)

Let $\mathcal{A}$ be a structure with domain A, $a \in A$, and $r \in \mathbb{N}$. The r-sphere of a (in $\mathcal{A}$), denoted $S_r^{\mathcal{A}}(a)$, is defined as follows:

$$S_r^{\mathcal{A}}(a) \stackrel{\text{def}}{=} \{b \in A \mid \delta(a, b) \leqslant r\}.$$

The notion of r-sphere can be extended to a vector $\vec{a} = a_1 \dots a_s$ (r-sphere $S_r^{\mathcal{A}}(\vec{a})$):

$$S_r^{\mathcal{A}}(\vec{a}) \stackrel{\text{def}}{=} \{b \in A \mid \delta(\vec{a}, b) \leqslant r\} = S_r^{\mathcal{A}}(a_1) \cup \dots \cup S_r^{\mathcal{A}}(a_s).$$

Definition (r-neighborhood)

The r-neighborhood $\mathcal{N}_r^{\mathcal{A}}(\vec{a})$ is the substructure of $\mathcal{A}$ induced by $S_r^{\mathcal{A}}(\vec{a})$.

If we restrict ourselves to graphs of degree $\leqslant d$ for some fixed d, there are, for any $r > 0$, only finitely many possible isomorphism types of r-spheres.

Hanf's theorem

- $\mathcal{A} \leftrightarrows_r \mathcal{B}$: there is a bijection $f\colon A \to B$ such that $\mathcal{N}_r^{\mathcal{A}}(a) \cong \mathcal{N}_r^{\mathcal{B}}(f(a))$ for every $a \in A$

The relation $\mathcal{A} \leftrightarrows_r \mathcal{B}$ states that *locally* $\mathcal{A}$ and $\mathcal{B}$ look the same.

Theorem (Hanf, 1965)

Let $\mathcal{A}$ and $\mathcal{B}$ be two structures such that, for any $r \in \mathbb{N}$, each r-sphere in $\mathcal{A}$ or $\mathcal{B}$ contains finitely many elements. Then, $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent if $\mathcal{A} \leftrightarrows_r \mathcal{B}$ for every $r \in \mathbb{N}$.

- Hanf's result does not hold if the Gaifman graph of (at least) one structure has infinite degree, e.g., the usual ordering relation on natural numbers

From the infinite case to the finite one

- Hanf's theorem is of interest only for infinite structures: as we already pointed out, two finite structures are elementarily equivalent if and only if they are isomorphic

- A weakened version of Hanf's theorem, called sphere theorem, provides a sufficient condition for m-equivalence (instead of a sufficient condition for elementary equivalence) and it turns out to be of interest for finite structures

- The proofs of both Hanf's theorem and sphere theorem use Fraïssé's theorem

Sphere theorem

- $\mathcal{A} \leftrightarrows_r^t \mathcal{B}$: isomorphic r-neighborhoods occur the same number of times in both structures (that is, they have the same multiplicity) or they occur more than t times in both structures

Theorem (Sphere theorem)

Given $\mathcal{A}$ and $\mathcal{B}$ with degree at most d and $m \in \mathbb{N}$, if $\mathcal{A} \leftrightarrows_r^t \mathcal{B}$ for $r = 3^{m+1}$ and $t = m \cdot d^{3^{m+1}}$, then $\mathcal{A} \equiv_m \mathcal{B}$.

- For all m there are r and t such that $\leftrightarrows_r^t$ is finer than $\equiv_m$ with respect to the class of structures with degree $\leqslant d$
- Strong hypotheses (it is a sufficient condition)
  - isomorphic neighborhoods
  - uniform threshold for all neighborhood sizes
  - scattering of neighborhoods is not taken into account

Sphere theorem: a proof

Thanks to Fraïssé's theorem, it suffices to show that $(\mathcal{A}, \vec{a}) \cong_m (\mathcal{B}, \vec{b})$.

The required sequence of sets $I_0, \dots, I_m$ of partial isomorphisms is defined as follows: $p = \{(a_1, b_1), \dots, (a_{m-k}, b_{m-k})\} \in I_k$ iff

$$\mathcal{N}^{\mathcal{A}}_{3^k}(a_1, \dots, a_{m-k}) \cong \mathcal{N}^{\mathcal{B}}_{3^k}(b_1, \dots, b_{m-k})$$

To prove the forth property (a similar argument holds for the back property), we assume that such a condition holds for p and we show that, for every possible choice of $a(= a_{m-(k-1)}) \in A$, we can find $b(= b_{m-(k-1)}) \in B$ such that:

$$\mathcal{N}^{\mathcal{A}}_{3^{k-1}}(a_1, \dots, a_{m-(k-1)}) \cong \mathcal{N}^{\mathcal{B}}_{3^{k-1}}(b_1, \dots, b_{m-(k-1)})$$

Sphere theorem: a proof (cont.)

We must distinguish two cases:

- if $a \in S^{\mathcal{A}}_{2/3 \cdot 3^k}(a_i)$ for some a_i, then we may choose a corresponding b from $S^{\mathcal{B}}_{2/3 \cdot 3^k}(b_i)$ ($S^{\mathcal{A}}_{3^{k-1}}(a)$ is contained in $S^{\mathcal{A}}_{3^k}(a_i)$ and $S^{\mathcal{B}}_{3^{k-1}}(b)$ is contained in $S^{\mathcal{B}}_{3^k}(b_i)$, and thus $\mathcal{N}^{\mathcal{A}}_{3^{k-1}}(a) \cong \mathcal{N}^{\mathcal{B}}_{3^{k-1}}(b)$);
- otherwise, $S^{\mathcal{A}}_{3^{k-1}}(a)$ (of some isomorphism type σ) is disjoint from $S^{\mathcal{A}}_{3^{k-1}}(a_i)$, for $i = 1, \ldots, m-k$. From $\mathcal{A} \leftrightarrows^t_r \mathcal{B}$, with $r = 3^{m+1}$ and $t = m \cdot d^{3^{m+1}}$, it follows that the number of occurrences of spheres of type σ in $\mathcal{B}$ is large enough to guarantee that we may find one which is disjoint from $S^{\mathcal{B}}_{3^{k-1}}(b_i)$, for $i = 1, \ldots, m-k$.

By sphere lemma and distributive normal form, any FO formula is equivalent (over graphs of degree $\leqslant d$) to a Boolean combination of statements of the form "there exist $\geqslant k$ occurrences of spheres of types σ": FO logic can only express local properties of graphs.

References for Hanf's and Sphere theorems

W. Hanf
Model-Theoretic Methods in the Study of Elementary Logic
The Theory of Model, 1965

W. Thomas
On logics, tilings, and automata
Proc. 18th ICALP, LNCS 510, 1991

W. Thomas
On the Ehrenfeucht-Fraïssé game in Theoretical Computer Science
Proc. 4th TAPSOFT, LNCS 668, 1993

R. Fagin, L. J. Stockmeyer, and M. Y. Vardi
On monadic NP vs monadic co-NP
Information and Computation, 1995