Temporal Representation and Reasoning in Interval Temporal Logics

Part II: decision procedures and tableau systems

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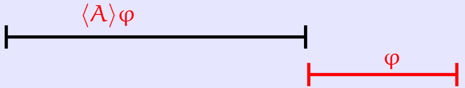
The interval temporal logic $A\overline{A}$

Formulas of the logic are recursively defined by the following grammar:

$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle A\rangle\varphi \mid \langle\overline{A}\rangle\varphi$ ($[A] = \neg\langle A\rangle\neg$ as usual; same for $[\overline{A}]$)



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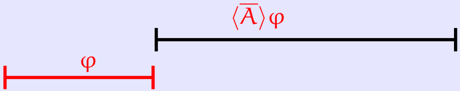


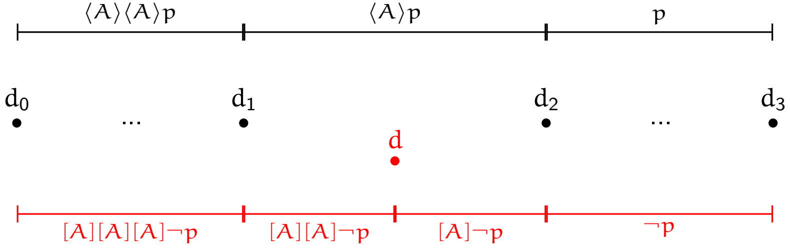
$A\overline{A}$ expressiveness - the formula $\mathrm{ImmediateSucc}$

It can be shown that $A\overline{A}$ is expressive enough to distinguish between satisfiability over the class of all linear orders and the class of dense (resp., discrete) ones.

Let $\mathrm{ImmediateSucc}$ be the $A\overline{A}$ formula:

$$\langle A\rangle\langle A\rangle p \wedge [A][A][A]\neg p$$

$\mathrm{ImmediateSucc}$ is satisfiable over the class of all (resp., discrete) linear orders, but it is not satisfiable over dense linear orders.

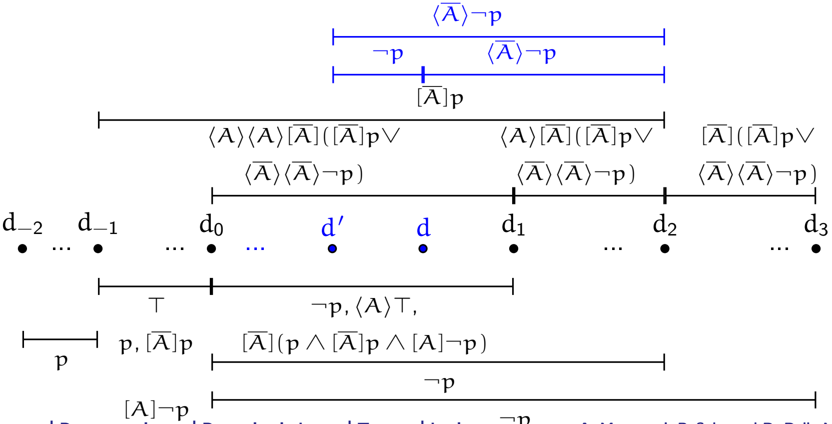


$A\overline{A}$ expressiveness - the formula NoImmediateSucc

Let $\mathrm{NoImmediateSucc}$ be the $A\overline{A}$ formula:

$$\langle\overline{A}\rangle\top \wedge [\overline{A}](p \wedge [\overline{A}]p \wedge [A]\neg p) \wedge \langle A\rangle\langle A\rangle[\overline{A}]([\overline{A}]p \vee \langle\overline{A}\rangle\langle\overline{A}\rangle\neg p)$$

$\mathrm{NoImmediateSucc}$ is satisfiable over the class of all (resp., dense) linear orders, but it is not satisfiable over discrete linear orders.



Decidability of $A\overline{A}$ over the class of all linear orders

How to check an $A\overline{A}$ formula φ for satisfiability?

Outline of the proof:

- FROM the existence of an interval model for φ
- TO the existence of a (possibly infinite) φ-labeled interval structure (STANDARD)
- TO the existence of a finite pseudo-model for φ (DIFFICULT)
- TO the existence of a tableau for φ with a blocked branch (EASY)

D. Bresolin, A. Montanari, P. Sala, and G. Sciavicco. Optimal tableau systems for propositional neighborhood logic over all, dense, and discrete linear orders. TABLEAUX 2011

Basic machinery

closure of φ: the set $\mathrm{CL}(\varphi)$ of all subformulae of φ and of their negations

temporal formulae of φ: the set $\mathrm{TF}(\varphi) \subseteq \mathrm{CL}(\varphi)$ of subformulae of the forms $\langle A \rangle \psi$, $[A]\psi$, $\langle \overline{A} \rangle \psi$, and $[\overline{A}]\psi$

maximal set of requests for φ: a subset of $\mathrm{TF}(\varphi)$ such that for every $\langle A \rangle \psi \in \mathrm{TF}(\varphi)$, $\langle A \rangle \psi \in S$ iff $\neg \langle A \rangle \psi \notin S$ (the same for $\langle \overline{A} \rangle \psi$)

φ-atom: a set $A \subseteq \mathrm{CL}(\varphi)$ such that (i) for every $\psi \in \mathrm{CL}(\varphi)$, $\psi \in A$ iff $\neg \psi \notin A$, and (ii) for every $\psi_1 \vee \psi_2 \in \mathrm{CL}(\varphi)$, $\psi_1 \vee \psi_2 \in A$ iff $\psi_1 \in A$ or $\psi_2 \in A$.

We denote by A_φ the set of all φ-atoms.

Interval models and φ-labeled interval structures

Let D be a set of points, $\mathbb{D} = \langle D, < \rangle$ be a linear order on it, and $\mathbb{I}(\mathbb{D})$ be the set of all intervals over $\mathbb{D}$

Interval model: a pair $\mathbf{M} = \langle \mathbb{D}, \mathcal{V} \rangle$, where $\mathbb{D} = \langle D, < \rangle$ and $\mathcal{V} : \mathbb{I}(\mathbb{D}) \mapsto 2^{AP}$

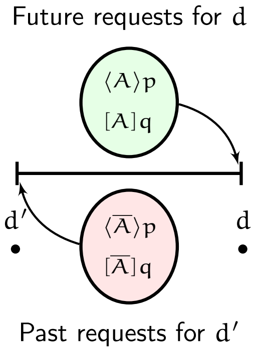
φ-labeled interval structure (φ-LIS): a pair $\mathbf{L} = \langle \mathbb{D}, \mathcal{L} \rangle$, where $\mathcal{L} : \mathbb{I}(\mathbb{D}) \mapsto \mathcal{A}_\varphi$ is such that, for every pair $[d_i, d_j], [d_j, d_k] \in \mathbb{I}(\mathbb{D})$ and every $[A]\psi \in \mathsf{CL}(\varphi)$, if $[A]\psi \in \mathcal{L}([d_i, d_j])$, then $\psi \in \mathcal{L}([d_j, d_k])$ (the same for $[\overline{A}]\psi$)

φ-LIS represent *candidate models* (they satisfy local conditions and universal temporal conditions). We must guarantee that existential temporal conditions are satisfied as well: fulfilling φ-LIS

Theorem. φ is satisfiable iff there exists a fulfilling φ-LIS $\mathbf{L} = \langle \mathbb{D}, \mathcal{L} \rangle$ with $\varphi \in \mathcal{L}([d_i, d_j])$ for some $[d_i, d_j] \in \mathbb{I}(\mathbb{D})$

How to make the notion of fulfilling φ-LIS effective?

Given a φ-LIS $\mathbf{L}$ and $d \in D$, we define the sets of future and past requests for d ($\mathrm{REQ}_f^{\mathbf{L}}(d)$ and $\mathrm{REQ}_p^{\mathbf{L}}(d)$, respectively)



We say that a future request $\langle A \rangle \psi$ is fulfilled for d in $\mathbf{L}$ if there exists $\overline{d} \in D$ such that $\psi \in \mathcal{L}([d, \overline{d}])$ (the same for past requests). We say that d is fulfilled in $\mathbf{L}$ if all its future and past requests ($\mathrm{REQ}^{\mathbf{L}}(d)$) are fulfilled.

The key notion of interval tuple

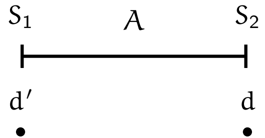
Let φ be an $A\overline{A}$ formula, $\mathcal{A}$ be a φ-atom, and $S_1, S_2 \subseteq \mathsf{TF}(\varphi)$ be two maximal sets of requests. We say that the triplet $\langle S_1, \mathcal{A}, S_2\rangle$ is an interval tuple if

(i) for every $[A]\psi \in S_1$, $\psi \in \mathcal{A}$;

(ii) for every $\langle A\rangle\psi \in \mathsf{TF}(\varphi)$, $\langle A\rangle\psi \in \mathcal{A}$ iff $\langle A\rangle\psi \in S_2$;

(iii) for every $\psi \in \mathcal{A}$ such that $\langle \overline{A}\rangle\psi \in \mathsf{TF}(\varphi)$, $\langle \overline{A}\rangle\psi \in S_1$.

The same for past operators.



Let $\mathbf{L}$ be a φ-LIS for φ and $d, d' \in \mathbb{D}$. It can be easily shown that $\langle \mathsf{REQ}^{\mathbf{L}}(d), \mathcal{L}([d, d']), \mathsf{REQ}^{\mathbf{L}}(d')\rangle$ is an interval tuple

From fulfilling φ-LISs to pseudo-models

Let $\mathbf{L}$ be a φ-LIS and $\langle R, A, R' \rangle$ be an interval tuple. If there exists $[d, d']$ such that $\mathcal{L}([d, d']) = A$, $\mathsf{REQ}^{\mathbf{L}}(d) = R$, and $\mathsf{REQ}^{\mathbf{L}}(d') = R'$, we say that $\langle R, A, R' \rangle$ occurs in $\mathbf{L}$ (at $[d, d']$). Moreover, if $\langle R, A, R' \rangle$ occurs in $\mathbf{L}$ at $[d, d']$ and both d and d' are fulfilled in $\mathbf{L}$, we say that $\langle R, A, R' \rangle$ is fulfilled in $\mathbf{L}$ (via $[d, d']$).

Given a finite φ-LIS $\mathbf{L}$ for φ, we say that $\mathbf{L}$ is a pseudo-model for φ if every interval tuple $\langle R, A, R' \rangle$ that occurs in $\mathbf{L}$ is fulfilled.

Being $\mathbf{L}$ is a pseudo-model for φ does not guarantee $\mathbf{L}$ to be fulfilling, since $\mathbf{L}$ can feature multiple occurrences of the same interval tuple, associated with different intervals.

However, it is possible to prove that any pseudo-model can be turned into a fulfilling LIS (for φ).

Decidability

Lemma 1. Given a pseudo-model $\mathbf{L}$ for φ, there exists a fulfilling LIS $\mathbf{L}'$ that satisfies φ.

Lemma 2. Given a formula φ and a fulfilling LIS $\mathbf{L}$ that satisfies it, there exists a pseudo-model $\mathbf{L}'$ for φ, with $|\mathrm{D}'| \leqslant 2 \cdot |\varphi| \cdot 2^{3 \cdot |\varphi| + 1}$.

Theorem. The satisfiability problem for $A\overline{A}$ over the class of all linear orders is decidable.

The decidability proof for $A\overline{A}$ over all linear orders can be tailored to the cases of dense linear orders and (weakly) discrete linear orders.

The dense case

In the case of dense models, we force each point in a pseudo-model for φ to satisfy the following covering condition.

Let $\mathbf{L} = \langle \mathbb{D}, \mathcal{L} \rangle$ be a pseudo-model for an $A\overline{A}$ formula φ and $d \in D$. We say that d is *covered* if either d is not unique or (d is unique and) both its immediate predecessor (if any) and successor (if any) are not unique.

Such a condition guarantees us that we can always insert a point in between any pair of consecutive points, thus producing a dense model for φ.

The dense case: the idea

We want to build a dense model for φ by duplicating *non-unique* points only.

The idea behind the notion of *covering* is the following:
the closest points to a non-unique point d in a pseudo-model must not be unique, and thus we can duplicate them instead of d.

The discrete case

In the discrete case, we force each point in a pseudo-model for φ to satisfy the following safety property.

Let $\mathbf{L} = \langle \mathbb{D}, \mathcal{L} \rangle$ be a pseudo-model for an $A\overline{A}$ formula φ and $d \in D$. We say that d is *safe* if either d is not unique or (d is unique and) both its immediate predecessor (if any) and successor (if any) are fulfilled.

Such a condition guarantees us that all points added during the construction of the fulfilling LIS get their (definitive) immediate successor and immediate predecessor in at most one step.

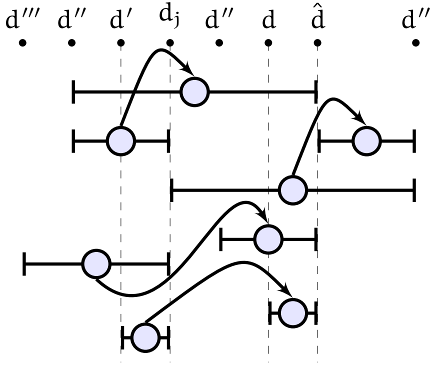
How does the proof of Lemma 1 work?

Basic idea: we show how to obtain a fulfilling LIS $\mathbf{L}'$ starting from a pseudo-model $\mathbf{L}$ as the limit of a possibly infinite sequence of pseudo-models $\mathbf{L}_0 (= \mathbf{L}), \mathbf{L}_1, \mathbf{L}_2, \ldots$ by fixing defects of points in the current pseudo-model (that is, existential temporal formulae whose requests are not fulfilled) in a principled way.

Points that must be checked for fulfillment are managed by a queue (this guarantees us that all defects are sooner or later fixed).

Initially, the queue consists of all and only those points $\mathrm{d} \in \mathrm{D}$ such that d is not fulfilled in the given pseudo-model $\mathbf{L}$.

Proof of Lemma 1: fulfilling requests



d is the node on the top of the queue.

d' is the maximum node with $REQ^{\mathbf{L}}(d') = REQ^{\mathbf{L}}(d)$ which is fulfilled.

d_j is the node such that $\psi \in \mathcal{L}_i(d', d_j)$ and ψ is not fulfilled for d.

It can be proved that $Past^{\mathbf{L}_i}(d) = Past^{\mathbf{L}_i}(d')$.

$\hat{d}$ is the point introduced at step $i + 1$. We have $REQ^{\mathbf{L}}(\hat{d}) = REQ^{\mathbf{L}}(d_j)$ and $\mathcal{L}_{i+1}(d, \hat{d}) = \mathcal{L}_i(d', d_j)$.

A tableau system for $A\overline{A}$ over all linear orders

Basic notions.

A tableau for φ: a special decorated tree $\mathcal{T}$.

We associate a finite linear order $\mathbb{D}_B = \langle D_B, < \rangle$ and a request function $\mathcal{R}eq_B : D_B \mapsto \mathrm{REQ}_\varphi$ with every branch B of $\mathcal{T}$.

Every node n in B is labeled with a pair $\langle [d_i, d_j], A_n \rangle$ such that the triple $\langle \mathcal{R}eq_B(d_i), A_n, \mathcal{R}eq_B(d_j) \rangle$ is an interval tuple.

The initial tableau for φ consists of a single node (a single branch B) labeled with a pair $\langle [d_0, d_1], A \rangle$, where $\mathbb{D}_B = \{d_0 < d_1\}$ and $\varphi \in A$.

Fulfilling conditions

Given a point $d \in D_B$ and a formula $\langle A \rangle \psi \in \mathrm{REQ}_B(d)$, we say that $\langle A \rangle \psi$ is fulfilled in B for d if there exists a node $n' \in B$ such that n' is labeled with $\langle [d, d'], A_{n'} \rangle$ and $\psi \in A_{n'}$ (same for the past).

Given a point $d \in D_B$, we say that d is fulfilled in B if every $\langle A \rangle \psi$ (resp., $\langle \overline{A} \rangle \psi$) in $\mathcal{R}eq_B(d)$ is fulfilled in B for d.

Let $\mathcal{T}$ be a tableau and B be a branch of $\mathcal{T}$, with $\mathbb{D}_B = \{d_0 < \ldots < d_k\}$.

We denote by $B \cdot n$ the expansion of B with an immediate successor node n and by $B \cdot n_1 | \ldots | n_h$ the expansion of B with h immediate successor nodes $n_1, \ldots, n_h$.

Expansion rules

To expand B, we apply one of the following expansion rules:
$\langle A\rangle$-rule: there exist $d_j \in D_B$ and $\langle A\rangle\psi \in \text{REQ}_B(d_j)$ such that $\langle A\rangle\psi$ is not fulfilled in B for d_j.

- There is not an interval tuple $\langle \mathcal{R}eq_B(d_j),\ \mathcal{A}, \mathcal{S}\rangle$, with $\psi \in \mathcal{A}$. We *close* B.
- Let $\langle \mathcal{R}eq_B(d_j), \mathcal{A}, \mathcal{S}\rangle$ be such an interval tuple. We take a new point d and we expand B with $h = k - j + 1$ immediate successor nodes $n_1, \ldots, n_h$ such that, for every $1 \leqslant l \leqslant h$, $\mathbb{D}_{B \cdot n_l} = \mathbb{D}_B \cup \{d_{j+l-1} < d < d_{j+l}\}$, $n_l = \langle [d_j, d], \mathcal{A}\rangle$, with $\text{REQ}_{B \cdot n_l}(d) = \mathcal{S}$, and $\text{REQ}_{B \cdot n_l}(d') = \text{REQ}_B(d')$ for every $d' \in D_B$.

$\langle \overline{A}\rangle$-rule: symmetric to the $\langle A\rangle$-rule.
Fill-in rule:

- There are d_i, d_j, with $d_i < d_j$, such that no node in B is decorated with $[d_i, d_j]$, but there is an interval tuple $\langle \text{REQ}_B(d_i), \mathcal{A}, \text{REQ}_B(d_j)\rangle$. We expand B with a node $n = \langle [d_i, d_j], \mathcal{A}\rangle$.
- Such an interval tuple does not exist. We *close* B.

The notion of blocked branch

A node $n = \langle [d_i, d_j], A \rangle$ in a branch B is active if for every predecessor $n' = \langle [d, d'], A' \rangle$ of n in B, the interval tuples $\langle \mathcal{R}eq_B(d_i), A, \mathcal{R}eq_B(d_j) \rangle$ and $\langle \mathcal{R}eq_B(d), A', \mathcal{R}eq_B(d') \rangle$ are different.

A point $d \in D_B$ is active if there is an active node n in B such that $n = \langle [d, d'], A \rangle$ or $n = \langle [d', d], A \rangle$, for some $d' \in D_B$ and some atom A.

Let B be a non-closed branch. B is complete if for every $d_i, d_j \in D_B$, with $d_i < d_j$, there is a node n in B labeled with $n = \langle [d_i, d_j], A \rangle$, for some atom A.

If B is complete, then the pair $\langle \mathbb{D}_B, \mathcal{L}_B \rangle$ such that, for every $[d_i, d_j] \in \mathbb{I}(\mathbb{D}_B)$, $\mathcal{L}_B([d_i, d_j]) = A$ if and only if there is a node n in B labeled with $\langle [d_i, d_j], A \rangle$, is a LIS.

Let B be a non-closed branch. B is blocked if B is complete and, for every active point $d \in B$, d is fulfilled in B.

Expansion strategy

We start from an initial tableau for φ and we apply the expansion rules to all the non-blocked and non-closed branches B.

The expansion strategy is the following one:

1. Apply the *Fill-in rule* until it generates no new nodes in B.

2. If there exist an active point $d \in D_B$ and a formula $\langle A\rangle\psi \in \mathcal{R}eq_B(d)$ such that $\langle A\rangle\psi$ is not fulfilled in B for d, then apply the $\langle A\rangle$*-rule* on d. Go back to step 1.

3. If there exist an active point $d \in D_B$ and a formula $\langle\overline{A}\rangle\psi \in \mathcal{R}eq_B(d)$ such that $\langle\overline{A}\rangle\psi$ is not fulfilled in B for d, then apply the $\langle\overline{A}\rangle$*-rule* on d. Go back to step 1.

A tableau $\mathcal{T}$ for φ is final if and only if every branch B of $\mathcal{T}$ is closed or blocked.

Termination, soundness, and completeness

Termination.
Let $\mathcal{T}$ be a final tableau for φ and $\mathcal{B}$ be a branch of $\mathcal{T}$. We have that $|\mathcal{B}| \leqslant (2 \cdot |\varphi| \cdot 2^{3 \cdot |\varphi|+1}) \cdot (2 \cdot |\varphi| \cdot 2^{3 \cdot |\varphi|+1} - 1)/2$.

Soundness.
Let $\mathcal{T}$ be a final tableau for φ. If $\mathcal{T}$ features one blocked branch, then φ is satisfiable over all linear orders.

Completeness.
Let φ be an $\mathsf{A}\overline{\mathsf{A}}$ formula which is satisfiable over the class of all linear orders. Then there exists a final tableau for it with at least one blocked branch.

The tableau system can be tailored to the dense and discrete cases.

The dense case

By adding the following rule, we obtain a procedure for the dense case.

> Dense rule: If there exist two consecutive points d_i, d_{i+1} in $\mathbb{D}_B$ which are not covered, we proceed as follows. If there is not an interval tuple $\langle \mathcal{R}eq_B(d_i), \mathcal{A}, \mathcal{S} \rangle$ for some $\mathcal{S} \in \text{REQ}_\varphi$ and some atom $\mathcal{A} \in \mathcal{A}_\varphi$, we *close* the branch B.
> Otherwise, let $\langle \mathcal{R}eq_B(d_i), \mathcal{A}, \mathcal{S} \rangle$ be such an interval tuple. We expand B with a node n, labeled with $\langle [d_i, d], \mathcal{A} \rangle$, such that $\text{REQ}_{B \cdot n}(d) = \mathcal{S}$ and $\mathbb{D}_{B \cdot n} = \mathbb{D}_B \cup \{d_i < d < d_{i+1}\}$.

The discrete case

By adding the following rules, we obtain a procedure for the discrete case.

- Successor rule. If there is $d_j \in D_B$ such that d_j is unique in D_B, its immediate successor d_{j+1} in D_B is not fulfilled, there is a node n labeled by $\langle [d_j, d_{j+1}], A_n, free \rangle$, for some atom A_n, in B, and there is no node n' labeled by $\langle [d_j, d_{j+1}], A_{n'}, unit \rangle$, for some atom $A_{n'}$, in B, then we proceed as follows. We expand B with 2 immediate successor nodes n_1, n_2 such that $n_1 = \langle [d_j, d_{j+1}], A_n, unit \rangle$ and $n_2 = \langle [d_j, d], A', unit \rangle$, with $d_j < d < d_{j+1}$ and there is an interval tuple $\langle \mathcal{R}eq_B(d_j), A', S \rangle$, for some A' and S (the existence of such an interval tuple is guaranteed by the existence of a node n with label $\langle [d_j, d_{j+1}], A_n, free \rangle$). We have that $\mathbb{D}_{B \cdot n_1} = \mathbb{D}_B$ and $\mathbb{D}_{B \cdot n_2} = \mathbb{D}_B \cup \{d_j < d < d_{j+1}\}$. Moreover, $\mathrm{REQ}_{B \cdot n_2}(d) = S$ and $\mathrm{REQ}_{B \cdot n_2}(d') = \mathrm{REQ}_B(d')$ for every $d' \in D_B$.
- Predecessor rule Symmetric to the *Successor rule*

What about $A\overline{A}$ formulas interpreted over the reals?

When we consider a fragment of first-order logic over the reals, we must keep in mind that:

> PROPERTY: **Löwenheim-Skölem Theorem** states that if a first-order sentence φ has a model, then it has a countable one. Such a result allows us to restrict our attention to $A\overline{A}$ formulas which are satisfiable on both $\mathbb{Q}$ and $\mathbb{R}$, since the case of a formula which is satisfiable on $\mathbb{R}$ and unsatisfiable on $\mathbb{Q}$ is not possible;
>
> PROBLEM: when models built on real numbers come into play, we cannot rely anymore on an enumeration procedure for generating the models.

As we will see, we can use the property to solve the problem by introducing suitable sufficient and necessary conditions under which a model on $\mathbb{Q}$ can be turned on a model on $\mathbb{R}$.

Separability power of $A\overline{A}$: the formula NoReal

Let NoReal be the $A\overline{A}$ formula:

$$p \wedge \langle A\rangle\langle A\rangle q \wedge [G]((p \rightarrow \langle A\rangle p) \wedge (q \rightarrow \langle\overline{A}\rangle q)\wedge$$

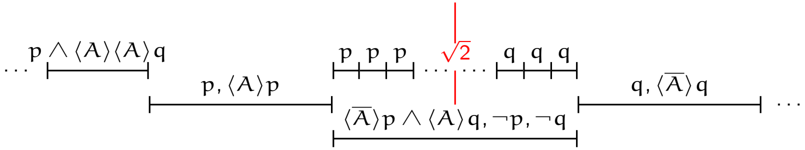
$$(p \rightarrow [A]([\overline{A}]p \wedge [\overline{A}][\overline{A}]p)) \wedge (q \rightarrow [\overline{A}]([A]q \wedge [A][A]q))\wedge$$

$$\neg(p \wedge q) \wedge (\neg p \wedge \neg q \rightarrow \langle\overline{A}\rangle p \wedge \langle A\rangle q)),$$

where $[G]$ is the *universal* operator defined as follows:

$$[G]\psi = \psi \wedge [\overline{A}][\overline{A}][A]\psi \wedge [\overline{A}][A][A]\psi \wedge [A][A][\overline{A}]\psi \wedge [A][\overline{A}][\overline{A}]\psi$$

NoReal is satisfiable over the class of dense linear orders (over the rationals), but it is not satisfiable over the real numbers



Forcing $\mathbb{Q}$ models to be safely turned into $\mathbb{R}$ ones

An $A\overline{A}$ formula φ is satisfiable over the reals iff there exist a model $\mathbf{L} = \langle \mathbb{Q}, \mathcal{L} \rangle$ for it and two functions:

- $\mathcal{F}_{inf} : \mathrm{REQ}^{\mathbf{L}} \to \mathbb{D}$ (infimum region function) such that for each $R \in \mathrm{REQ}^{\mathbf{L}}$ $\mathcal{F}_{inf}(R) = d \neq -\infty$ implies for each $d' < d$ $\mathrm{REQ}^{\mathbf{L}}(d') \neq R$ and if $\mathrm{REQ}^{\mathbf{L}}(d) \neq R$ we have that for each $\epsilon > 0$ there exists d'' with $\mathrm{REQ}^{\mathbf{L}}(d'') = R$ and $d'' - d < \epsilon$
- $\mathcal{F}_{sup} : \mathrm{REQ}^{\mathbf{L}} \to \mathbb{D}$ (supremum region function) symmetric to $\mathcal{F}_{inf}$

A tableau for $A\overline{A}$ over the reals (1)

The tableau system for dense linear orders (**Tableau System for General Linear Orders + Dense Rule**) can be extended to obtain a tableau system for $\mathbb{R}$ by adding the following expansion rules:

- inf-rule: Let $R \in range(REQ_B)$ be such that $\mathcal{F}^B_{inf}$ is undefined, d_i be the least point in D_B, with $REQ_B(d_i) = R$, and S be a set of requests. We expand B with $h = 2i + 3$ accumulation nodes $n_0, \ldots, n_{h-1}$, where $n_0 = \langle R, inf, -\infty \rangle$ and for each $0 \leqslant j \leqslant i$, $n_{2j+2} = \langle R, inf, d_j \rangle$, where $d_{j-1} < d < d_j$ ($d < d_0$ for $j = 0$) is a new point with $REQ_{B \cdot n_{2j+1}}(d) = S$;
- sup-rule: symmetric to the inf-rule;

A tableau for $A\overline{A}$ over the reals (cont'd)

- inf-chain rule: Let $d \in D_B$ be such that $\mathcal{R}^B_{inf} = \{R_0, \ldots, R_n\} (\neq \emptyset)$ and $1 \leqslant i \leqslant n$ be such that for each $d' \in D_B$, with $\mathrm{REQ}(d') = R$, $\mathrm{Future}^B(d') \neq \mathrm{Future}^B(d)$. If there is not an interval tuple of the form $\langle \mathrm{REQ}_B(d), \mathcal{A}, R_i \rangle$, for some $\mathcal{A}$, we close B.
  Otherwise, let $\langle \mathrm{REQ}_B(d), \mathcal{A}, R_i \rangle$ be such an interval tuple and $\overline{d}$ be the immediate successor of d in $\mathbb{D}_B$. We choose a new point $d < d' < \overline{d}$ and we expand B with a node $n = \langle [d, d'], \mathcal{A} \rangle$ such that $\mathrm{REQ}_{B \cdot n}(d') = R$;
- sup-chain rule: symmetric to the inf-chain rule.

A. Montanari, P.Sala. An optimal tableau system for the logic of temporal neighborhood over the reals, TIME 2012.

$A\overline{A}$ extensions undecidable over the reals

As we have already pointed out, Löwenheim-Skölem Theorem is a good starting point to prove decidability of fragments of first-order logic over the reals.

However, there are fragments of first-order logic that turn out to be decidable over $\mathbb{Q}$ and undecidable over $\mathbb{R}$.

Recently, we have discovered that this is the case with the interval temporal logic $\mathrm{A}\overline{\mathrm{A}}\mathrm{B}\overline{\mathrm{B}}$ which turns out to be decidable over $\mathbb{Q}$ and undecidable over $\mathbb{R}$.

A. Montanari, G. Puppis, and P. Sala, Decidability of the interval temporal logic $\mathrm{A}\overline{\mathrm{A}}\mathrm{B}\overline{\mathrm{B}}$ over the rationals, MFCS 2014.