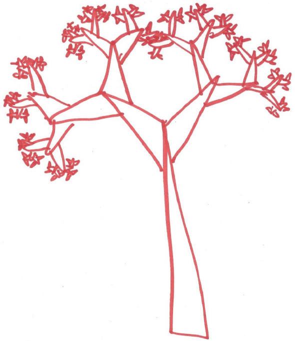
The monadic theory of two successors

Gabriele Puppis

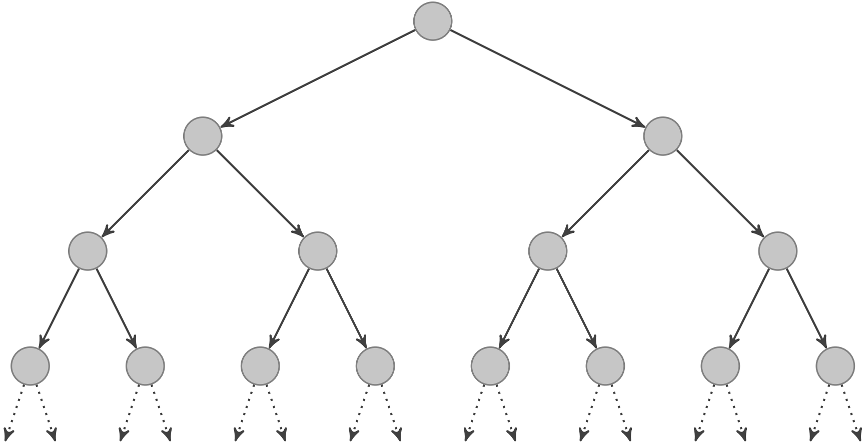
LaBRI / CNRS



Decidability of S2S (Rabin'69)

One can decide if a sentence of $\mathsf{MSO}[E_1, E_2]$ holds over the binary tree.

Same approach as with $(\mathbb{N}, +1)$: transform formulas to automata



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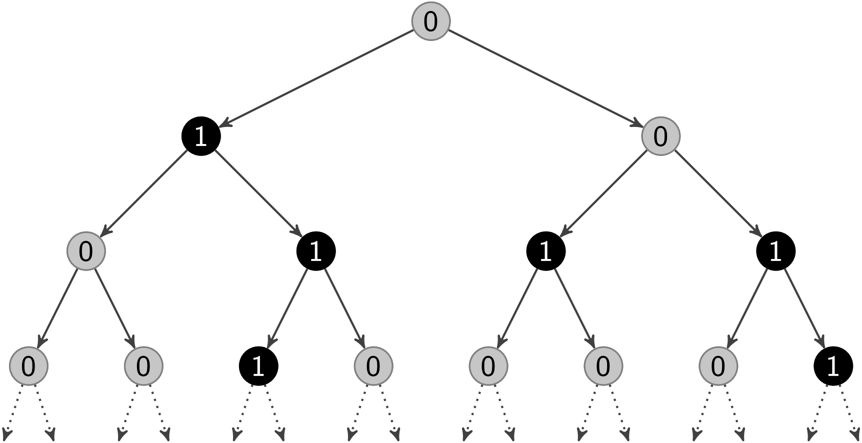

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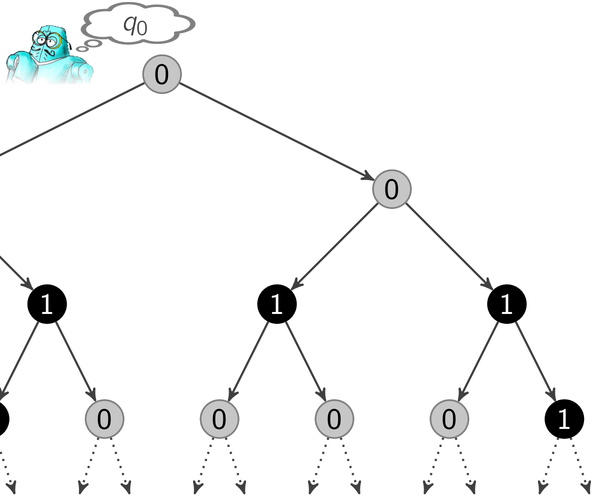

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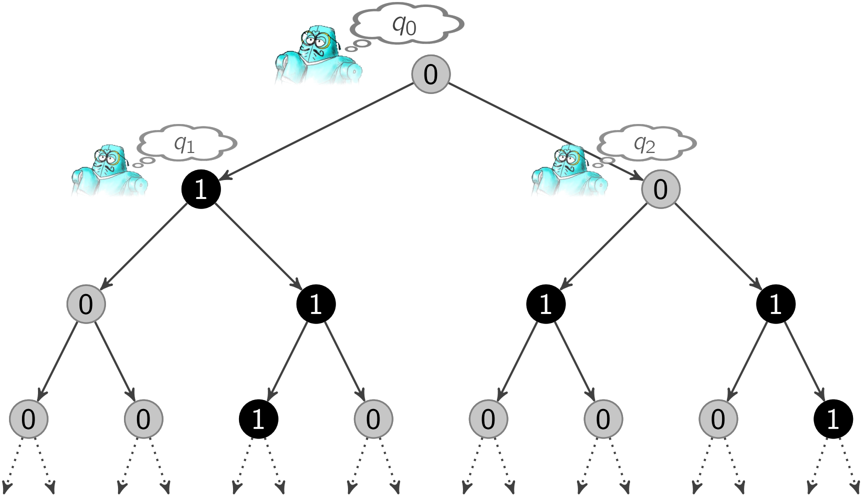

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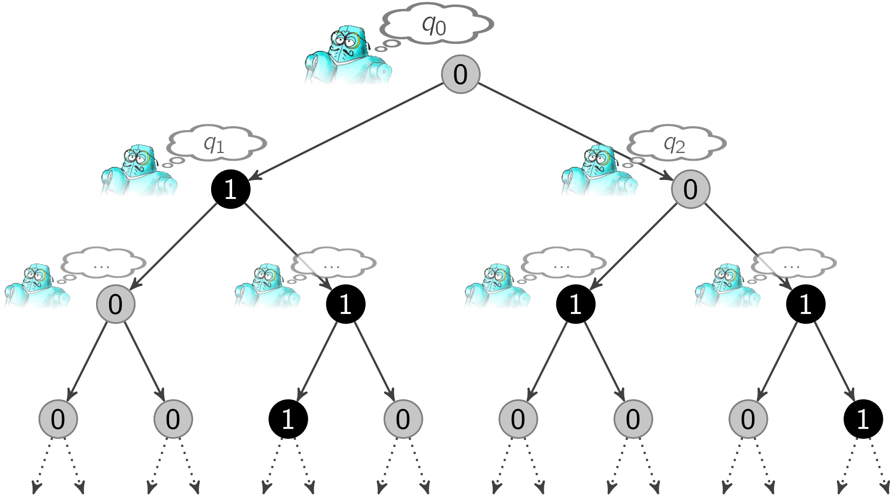

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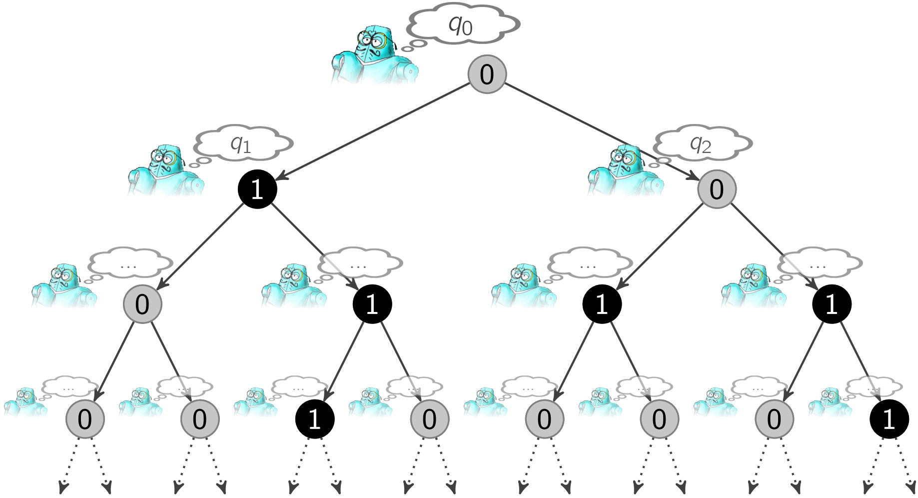


Definition

A **tree automaton** is a tuple $\mathcal{A} = (Q, \Sigma, \Delta, I, \Omega)$, where

- Q is a finite set of control states
- Σ is a finite alphabet for transition labels
- $\Delta \subseteq Q \times \Sigma \times Q^2$ is a finite set of transition rules
- $I \subseteq Q$ is a set of initial states
- $\Omega \subseteq Q^\omega$ defines the acceptance condition by specifying the allowed sequences of states along **all paths** in the tree

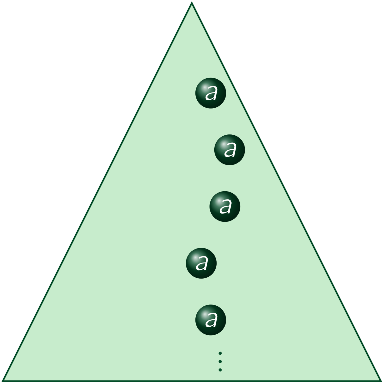
  (more details in the next slide...)

E.g. in a Büchi tree automaton one has

$$\Omega = \left\{ \rho \in Q^\omega \mid \inf(\rho) \cap F \neq \varnothing \right\}$$

Need of **non-determinism** and **stronger acceptance conditions**

$$\exists X.\ X \text{ is a path } \wedge\ \exists^{\infty} y \in X.\ a(y)$$



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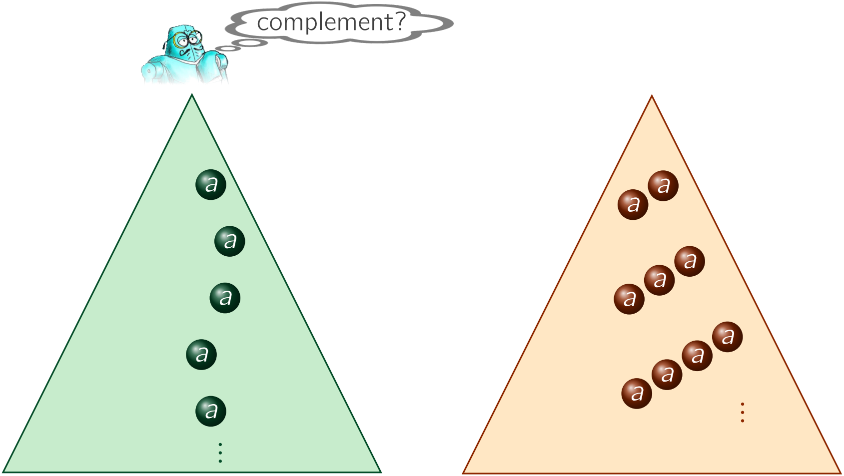

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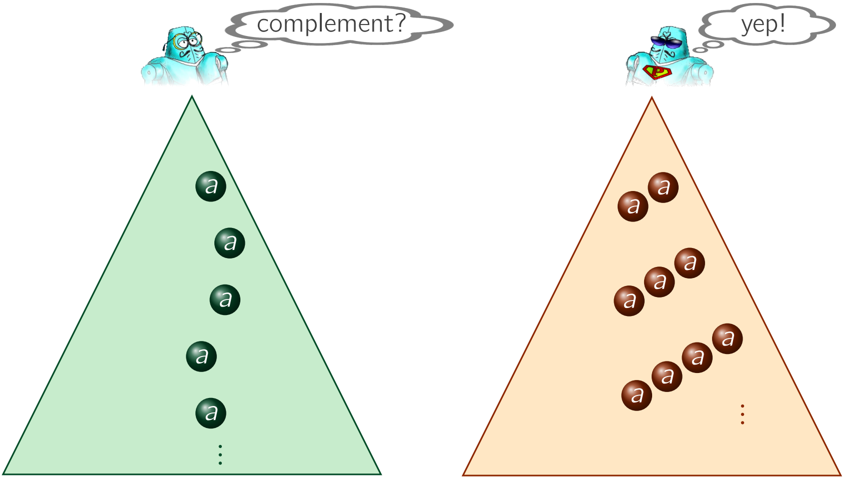

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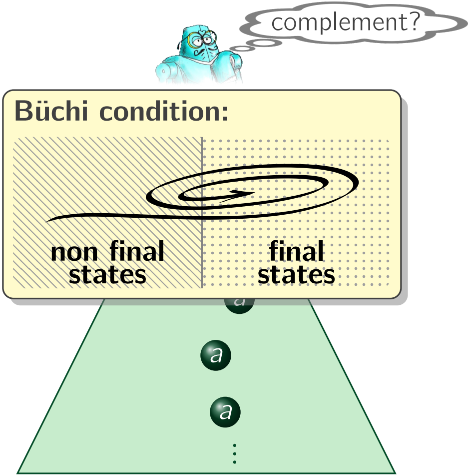
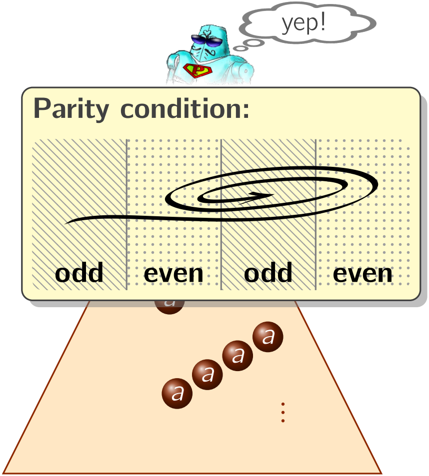




Decidability of S2S (Rabin'69)

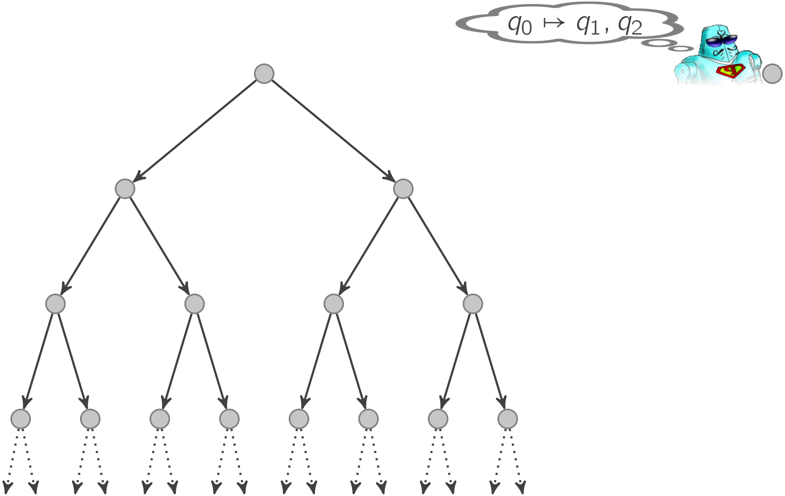
One can decide if a sentence of $\mathsf{MSO}[E_1, E_2]$ holds over the binary tree.

By induction on all subformulas $\varphi(X_1, \ldots, X_m)$,
construct **tree automata** $\mathcal{A}_\varphi$ such that

$$\mathscr{L}(\mathcal{A}_\varphi) \;=\; \left\{ t \in \Sigma_m^{\{1,2\}^*} \;\middle|\; t \models \varphi \right\}$$

logical disjunction	$\vee$	$\mapsto$	union
existential quantification	$\exists$	$\mapsto$	projection
logical negation	$\neg$	$\mapsto$	complementation

Complementation via **parity games**: Automaton vs Pathfinder



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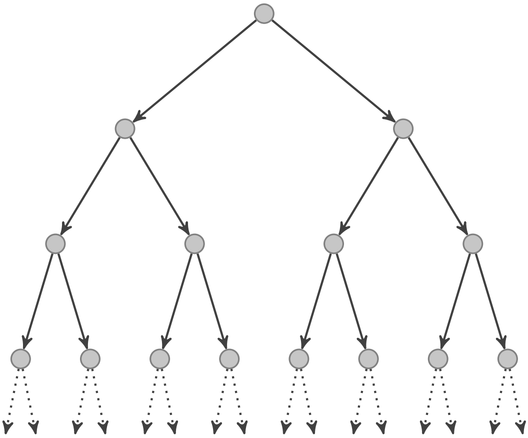

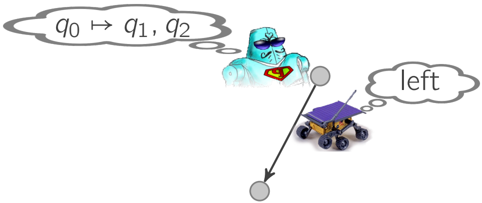

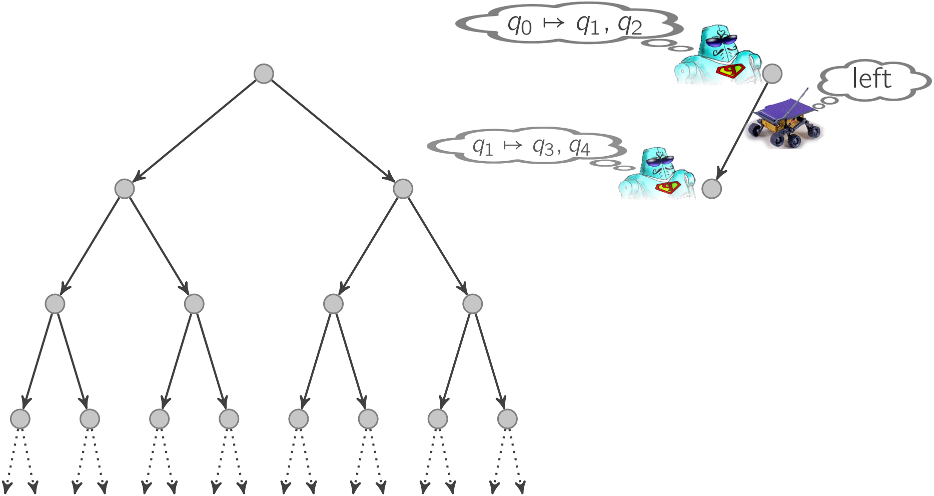

$q_0 \mapsto q_1, q_2$
left
$q_1 \mapsto q_3, q_4$

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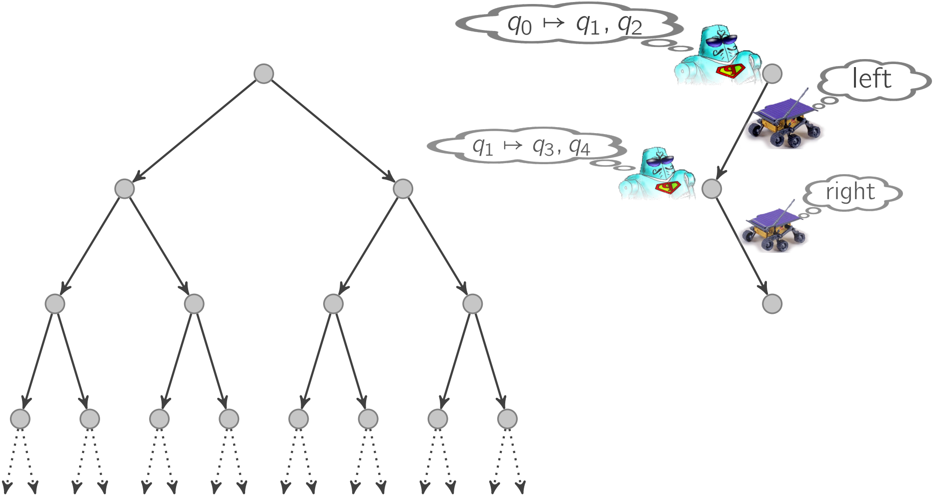

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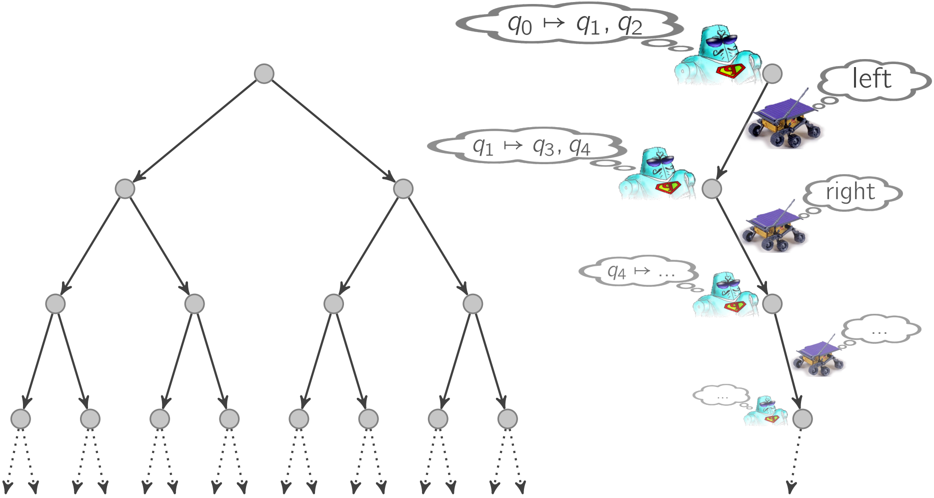

Positional determinacy of parity games

Either Automaton wins with a **positional strategy**, or Pathfinder does.

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When a tree is rejected, Pathfinder's winning strategy can be converted to a strategy for Automaton over a game with new transition rules

$t \notin \mathscr{L}(\mathcal{A})$

iff Pathfinder has a winning positional strategy

$\sigma : \{1,2\}^\star \times \Delta \to \{1,2\}$ (or, equally, $\sigma : \{1,2\}^\star \to \{1,2\}^\Delta$)

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iff $\exists\ t \otimes \sigma : \{1,2\}^\star \to \Sigma \times \{1,2\}^\Delta$ expansion of t such that
$\forall$ infinite path $\pi \in \{1,2\}^\omega$
$\forall$ seq. of transitions $\tau \in \Delta^\omega$
if π is consistent with $(t \otimes \sigma)|\pi$ and τ,
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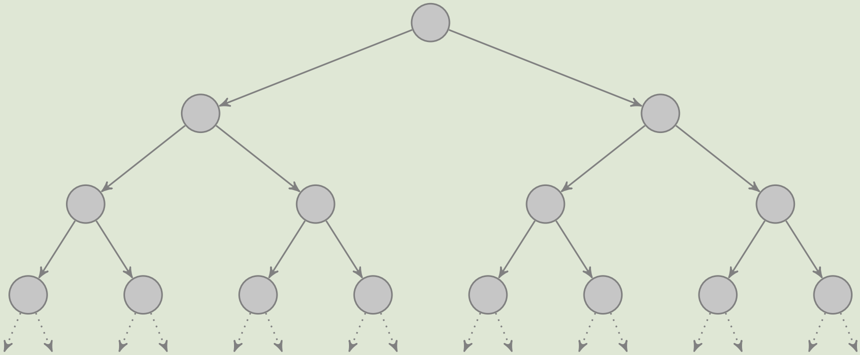
iff $t \in \mathscr{L}(\mathcal{A}^C)$

Application example 1 (interpretation of the ternary tree)

Consider the binary tree t_2



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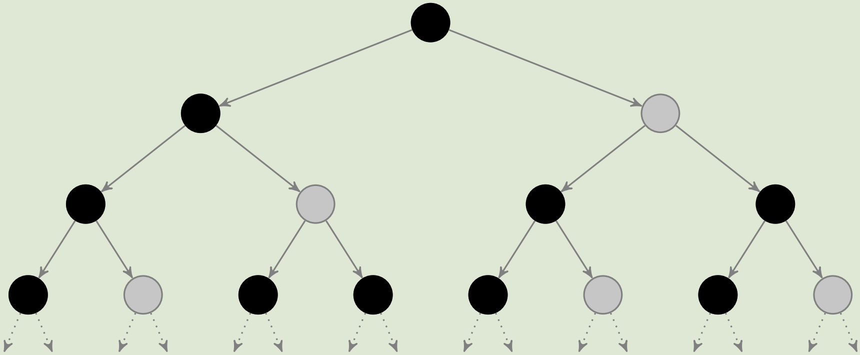

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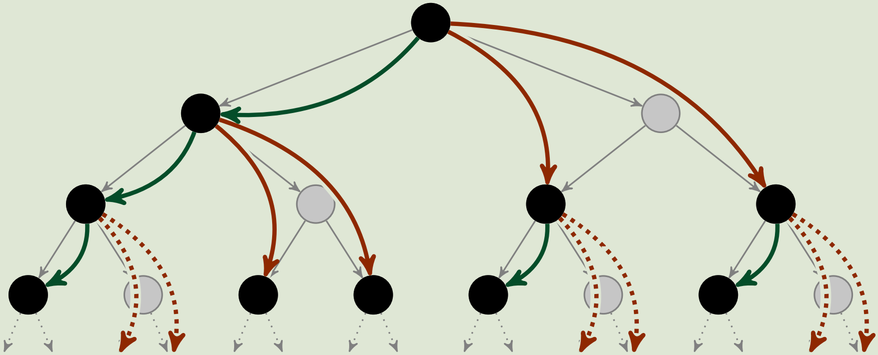

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2. define the three successor relations of t_3 by the formulas

   $$\varphi_1(x,y) \;=\; (x,y) \in E_1 \qquad \varphi_{2/3}(x,y) \;=\; (x,y) \in (E_2 \circ E_{1/2})$$

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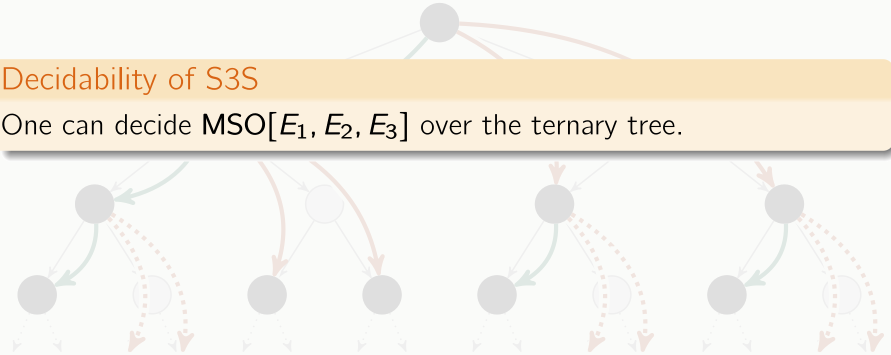

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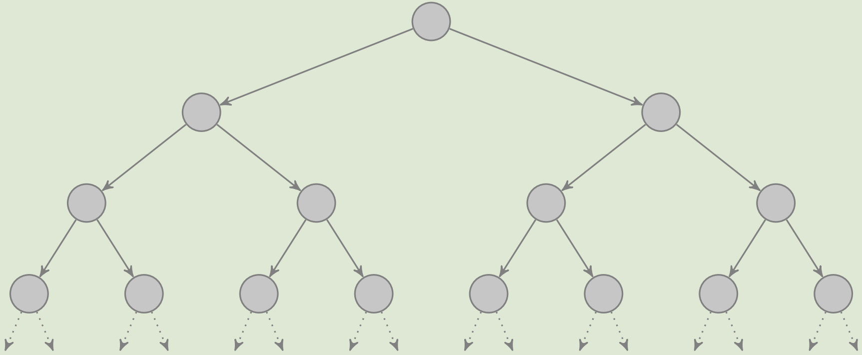
Application example 2 (interpretation of the rationals)

Consider again the binary tree t_2



Inside t_2 one can logically define the **the rationals** $\mathbb{Q}$

- the *dense order* on $\mathbb{Q}$ can be seen as the *infix order* of t_2

$$\begin{aligned}\varphi_{\le}(x,y) \quad = \quad \exists z. \quad & \big((z,x) \in E_1 \circ (E_1 \cup E_2)^\star \ \lor \ z = x\big) \\ \land \quad & \big((z,y) \in E_2 \circ (E_1 \cup E_2)^\star \ \lor \ z = y\big)\end{aligned}$$

Application example 2 (interpretation of the rationals)

Consider again the binary tree t_2

Monadic theories of linear orders (Shelah '75)

One can decide **MSO[≤]** over $\mathbb{Q}$ and
over the class of all countable linear orders.

One **cannot** decide **MSO[≤]** over $\mathbb{R}$.

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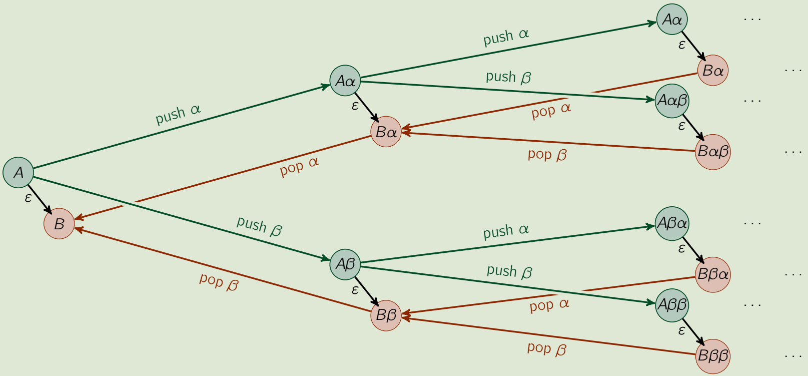
Application example 3 (interpretation of a pushdown system)



Any property of the above system expressed by an MSO formula

$$\psi \;=\; \dots \quad \forall X \quad \dots \Big(\; y \xrightarrow{\textbf{push } \beta} z \;\Big) \dots \Big(\; y \downarrow_{\varepsilon} z \;\Big) \dots \Big(\; z \xleftarrow{\textbf{pop } \beta} y \;\Big) \dots$$

can be translated into an **equi-satisfiable** formula over the binary tree

$$\hat{\psi} \;=\; \dots \forall X_1, X_2 \dots \Big((y_1, z_1) \in E_2 \Big) \dots \Big(y_1 = z_2 \Big) \dots \Big((y_2, z_2) \in E_2 \Big) \dots$$