The monadic theory of one successor

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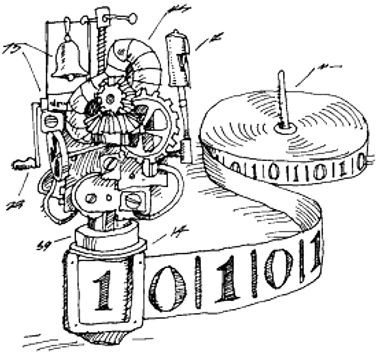

While FO talks of elements of $\mathbb{N}$, MSO talks of **subsets of $\mathbb{N}$**.

These subsets can be encoded by **infinite words**:

$$\text{Even} = \{ 0, 2, 4, 6, 8, \ldots \} \subseteq \mathbb{N}$$

$$\text{Square} = \{ 0, 1, 4, 9, \ldots \} \subseteq \mathbb{N}$$

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Accordingly, a language $L \subseteq \mathbb{B}^{\omega}$ encodes a **set of subsets of $\mathbb{N}$**.

Definition

A **Büchi automaton** is a tuple $\mathcal{A} = (Q, \Sigma, \Delta, I, F)$, where

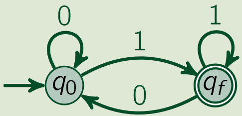
- Q is a finite set of control states
- Σ is a finite alphabet for transition labels
- $\Delta \subseteq Q \times \Sigma \times Q$ is a finite set of transition rules
- $I \subseteq Q$ is a set of initial states
- $F \subseteq Q$ is a set of final states

$\mathcal{A}$ **accepts** a word $w \in \Sigma^\omega$ if it admits a run ρ on w such that

$$\inf(\rho) \cap F \neq \emptyset$$

where $\inf(\rho) = \left\{ q \in Q \;\middle|\; \forall i.\ \exists j \geq i.\ \rho(j) = q \right\}$

Example



$\mathscr{L}(\mathcal{A}) = (0^\star 1)^\omega$

Decidability of S1S (Büchi '60)

One can decide if a given sentence ψ of **MSO[+1]** holds over $\mathbb{N}$.

1. Replace first-order variables x with set variables X satisfying

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2. By induction on all subformulas $\varphi(X_1, \ldots, X_m)$ of ψ, construct Büchi automata $\mathcal{A}_\varphi$ over $\Sigma_m = \mathbb{B}^m$ such that

$$\mathscr{L}(\mathcal{A}_\varphi) \;=\; \left\{ [X_1] \otimes \cdots \otimes [X_m] \in \Sigma_m^\omega \;\middle|\; (\mathbb{N}, +1) \models \varphi(X_1, \ldots, X_m) \right\}$$

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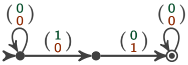
$$\varphi(X, Y) = (Y = X + 1)$$



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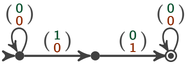

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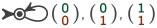

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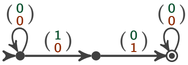

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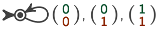

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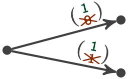

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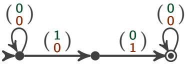

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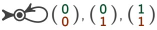

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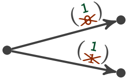

A Büchi automaton $\mathcal{A}$ induces a function $f : \Sigma^\star \to \{-\infty, 0, 1\}^{Q^2}$ such that

$$f(w) = \underset{p}{\rightarrow}\left[\begin{array}{c} \downarrow q \\ \boldsymbol{\iota_{p,q}} \end{array}\right]$$

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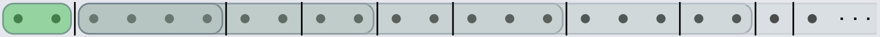

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Proof: $\supseteq$ straightforward, $\subseteq$ follows from Ramsey's Theorem

Every word can be factorized into $w_1, w_2, w_3, \ldots$ with $f(w_2) = f(w_2) = \ldots$

A Büchi automaton $\mathcal{A}$ induces a function $f : \Sigma^\star \to \{-\infty, 0, 1\}^{Q^2}$ such that

$$f(w) \;=\; \underset{p}{\rightarrow}\begin{bmatrix} & \downarrow q & \\ & \boldsymbol{\iota_{p,q}} & \\ & & \end{bmatrix}$$

$\iota_{p,q} = -\infty$ if there is no path from p to q

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Complementation: $$\mathscr{L}(\mathcal{A})^C = \bigcup_{\mathscr{L}(\mathcal{A}) \not\supseteq f^{-1}(M) \cdot f^{-1}(N)^\omega} f^{-1}(M) \cdot f^{-1}(N)^\omega$$

A converse translation: from automata to logic

One can translate every Büchi automaton $\mathcal{A}$ into a formula $\psi_{\mathcal{A}} = \exists \bar{X}.\ \varphi(\bar{X})$, where φ is a first-order formula, such that

$$\mathscr{L}(\mathcal{A}) = \left\{ w \in \Sigma^\omega \;\middle|\; w \models \psi_{\mathcal{A}} \right\}$$

Encode an accepting run of $\mathcal{A}$ into monadic variables $\bar{X}$:

$$\begin{aligned}
w \in \mathscr{L}(\mathcal{A}) \quad & \text{iff} \quad \exists\ \rho \textit{ accepting run of } \mathcal{A} \textit{ on } w \\
& \text{iff} \quad \exists (X_t)_{t \in \Delta}.\ \textit{exactly one transition on each position} \\
& \qquad\qquad\qquad \wedge\ \textit{transitions respect symbols of } w \\
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Corollary (collapse of quantifier hierarchy)

$$\text{MSO}[<] \quad = \quad \text{Büchi automata} \quad = \quad \exists\,\text{MSO}[+1]$$

Application example 1 (interpretation of the integers)

One can **logically define $\mathbb{Z}$ inside $\mathbb{N}$**:

$$\begin{aligned} \varphi_{\leq_\mathbb{Z}}(x, y) \quad &= \quad \big(\mathrm{Even}(x) \;\wedge\; \mathrm{Even}(y) \;\wedge\; x \leq_\mathbb{N} y\big) \\ &\vee \quad \big(\mathrm{Odd}(x) \;\wedge\; \mathrm{Odd}(y) \;\wedge\; y \leq_\mathbb{N} x\big) \\ &\vee \quad \big(\mathrm{Odd}(x) \;\wedge\; \mathrm{Even}(y)\big) \end{aligned}$$

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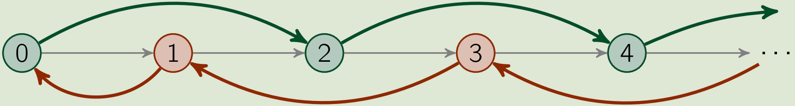

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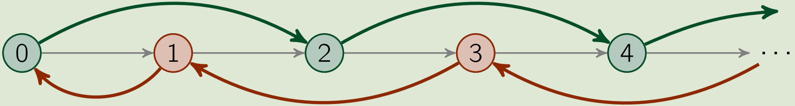

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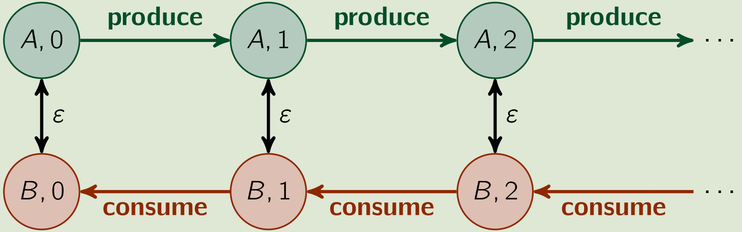
Corollary

The **MSO[≤]** theory of $\mathbb{Z}$ is decidable.

Application example 2 (interpretation of a counter system)



Any property of the above system expressed by an MSO formula

$$\psi \;=\; \ldots \quad \forall X \quad \ldots \left(y \xleftarrow{\text{consume}} z \right) \ldots \left(\quad y \;\updownarrow_{\varepsilon}\; z \quad \right) \ldots$$

can be translated into an **equi-satisfiable** formula over $(\mathbb{N}, +1)$

$$\hat{\psi} \;=\; \ldots \forall X_1, X_2 \ldots \left(\; y_2 + 1 = z_2 \; \right) \ldots \left(y_1 = z_2 \vee y_2 = z_1 \right) \ldots$$

and then checked for validity.

Application example 3 (expanded theories)

Recall inductive invariant: $\mathscr{L}(\mathcal{A}_\varphi) = \left\{ [\bar{X}] \in \Sigma_m^\omega \;\middle|\; \mathbb{N} \vDash \varphi(\bar{X}) \right\}$

and recall f-equivalence on factors of infinite words:

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We have

$$(\mathbb{N}, +1, \mathrm{Squares}) \models \varphi$$

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since $\boldsymbol{f(w_n) = f(w_{n+\ell})}$ for some $\ell > 0$ and all $n > n_0$

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Contraction method (Elgot & Rabin '66)

Let P be a subset of $\mathbb{N}$ that is (effectively) **profinitely periodic**

i.e. $[P] = w_0 \; w_1 \; w_2 \; \ldots$ and for every *semigroup morphism* f
the series $f(w_0) \; f(w_1) \; f(w_2) \; \ldots$ is (effectively) periodic.

Then one can decide whether $(\mathbb{N}, +1, P) \vDash \psi$.

Examples of effectively profinitely periodic subsets

- Squares $= \{n^2 \mid n \in \mathbb{N}\}$
- Powers $= \{2^n \mid n \in \mathbb{N}\}$
- Factorials $= \{n! \mid n \in \mathbb{N}\}$
- Fibonacci $= \{0, 1, 2, 3, 5, 8, 11, \ldots\}$
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Composition method (Shelah '75)

m = number of free variables $\quad$ k = number of nested quantifiers

Definition

Let $\boldsymbol{g_m^k}$ map words w over $\boldsymbol{\Sigma_m} = \{0,1\}^{\boldsymbol{m}}$ to **logical types**:

$$g_m^k(w) \;=\; \left\{ \boldsymbol{\varphi(X_1, \ldots, X_m)} \textit{ with } k \textit{ nested quantifiers such that } \boldsymbol{w \vDash \varphi} \right\}$$

Example: $\quad g_2^0\left(\binom{0}{1}\binom{0}{1}\binom{1}{1}\binom{1}{1}\right) \;=\; \left\{ X_1 \subseteq X_2, \;\ldots \right\}$

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Up to logical equivalence (e.g. $\neg\wedge = \vee\neg$, $\neg\forall = \exists\neg$, $\exists\vee = \vee\exists$)

$$\begin{aligned} g_m^0(w) &= \{ \ X_i \subseteq X_j \ \mid \ \forall n.\ w(n)[i] = 1 \rightarrow w(n)[j] = 1 \ \} \\ &\cup \{ \neg X_i \subseteq X_j \ \mid \ \exists n.\ w(n)[i] = 1 \ \wedge \ w(n)[j] = 0 \ \} \end{aligned}$$

$$\begin{aligned} g_m^{k+1}(w) &= \left\{ \ \exists Y.\, \varphi(X_1, \dots, X_m, Y) \ \middle| \ \begin{array}{l} Y \subseteq \mathrm{dom}(w) \\ \varphi \in g_{m+1}^k(w \otimes Y) \end{array} \right\} \\ &\cup \left\{ \neg\exists Y.\, \varphi(X_1, \dots, X_m, Y) \ \middle| \ \begin{array}{l} Y \subseteq \mathrm{dom}(w) \\ \varphi \notin g_{m+1}^k(w \otimes Y) \end{array} \right\} \end{aligned}$$

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Like f, the type function g_m^k is **compositional**, namely

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Example

$$\exists x.\, \exists y.\; x < y \quad \mapsto \quad (\exists x_1.\, \exists y_1.\; x_1 < y_1) \;\vee\; (\exists x_2.\, \exists y_2.\; x_2 < y_2) \;\vee\; (\exists x_1.\, \exists y_2.\; \text{true}) \;\vee\; (\exists x_2.\, \exists y_1.\; \text{false})$$

$$\exists X.\, \exists Y.\; X \subseteq Y \quad \mapsto \quad (\exists X_1.\, \exists Y_1.\; X_1 \subseteq Y_1) \;\wedge\; (\exists X_2.\, \exists Y_2.\; X_2 \subseteq Y_2)$$

Transform φ into $\bigvee_{i=1\ldots n} (\varphi_{i,1} \;\wedge\; \varphi_{i,2})$ in such a way that

$$\boldsymbol{w_1 \cdot w_2 \vDash \varphi} \qquad \text{iff} \qquad \boldsymbol{w_1 \vDash \varphi_{i,1}} \text{ and } \boldsymbol{w_2 \vDash \varphi_{i,2}} \text{ for some } i = 1\ldots n$$

Accordingly, constuct $g_m^k(w_1 \cdot w_2)$ on the basis of $g_m^k(w_1)$ and $g_m^k(w_2)$.

In a similar way, one can "compute" **types of ω-products**:

$$g_m^k(w_1 \cdot w_2 \cdot w_3 \cdot \ldots) \; = \; g_m^k(w_1) \; \odot \; g_m^k(w_2) \; \odot \; g_m^k(w_3) \; \odot \; \ldots$$

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One can decide whether a formula ψ of $\mathrm{MSO}[\Sigma,<]$ holds over $\mathbb{N}$.

1. Start by computing types of singleton words a, for all $a \in \Sigma$
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Ramsey’s Theorem implies completeness of the above procedure!

From Ramsey's Theorem to **Factorization Forests** (Simon '90)

Fix a semigroup morphism $f : \Sigma^\star \to (S, \odot)$ (e.g. logical types)
and recall that every infinite word has a Ramseyan factorization, i.e.

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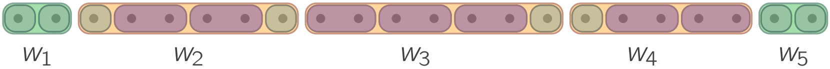

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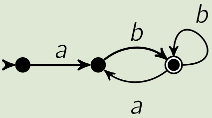
We can recursively factorize factors until we get single characters

- only **a few nested factorizations** (linear in $|S|$, independent of $|w|$)
- each factor is divided into **a few non-idempotent sub-factors**

An example of factorization forest

Consider the automaton recognizing $(a\,b^+)^+$ and the induced function f:



$$f(a) = \begin{bmatrix} -\infty & 0 & -\infty \\ -\infty & -\infty & -\infty \\ -\infty & 1 & -\infty \end{bmatrix} \qquad f(b) = \begin{bmatrix} -\infty & -\infty & -\infty \\ -\infty & -\infty & 1 \\ -\infty & -\infty & 1 \end{bmatrix}$$

$$f(a\,b) = \begin{bmatrix} -\infty & -\infty & 1 \\ -\infty & -\infty & -\infty \\ -\infty & -\infty & 1 \end{bmatrix} \qquad f(a\,b\,a) = \begin{bmatrix} -\infty & 1 & -\infty \\ -\infty & 1 & -\infty \\ -\infty & -\infty & -\infty \end{bmatrix}$$

$$w = a \quad b \;\; b \ldots b \quad a \quad b \;\; b \ldots b \quad \ldots \quad a \quad b \;\; b \ldots b$$

An example of factorization forest

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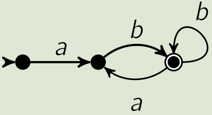

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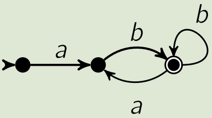

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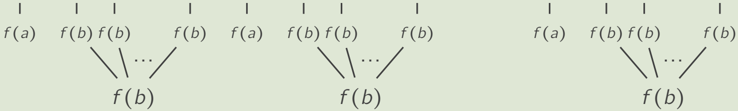

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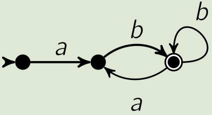

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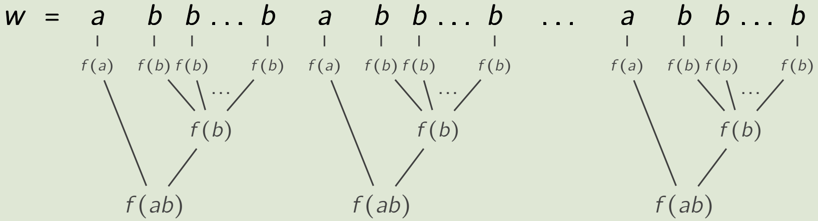

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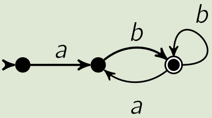

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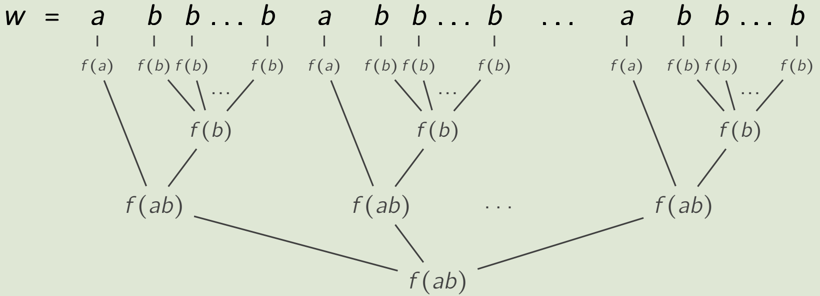

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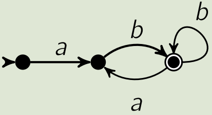

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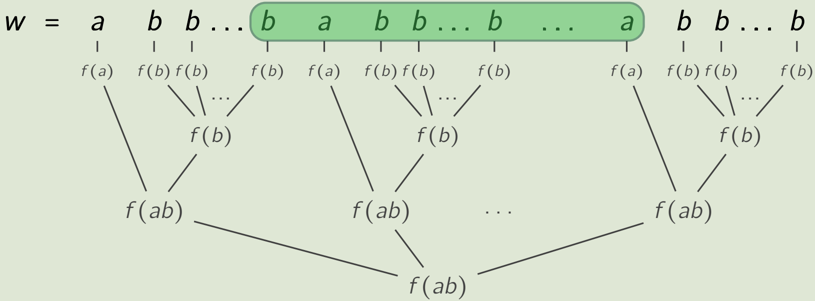

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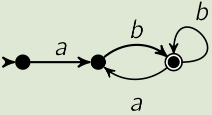

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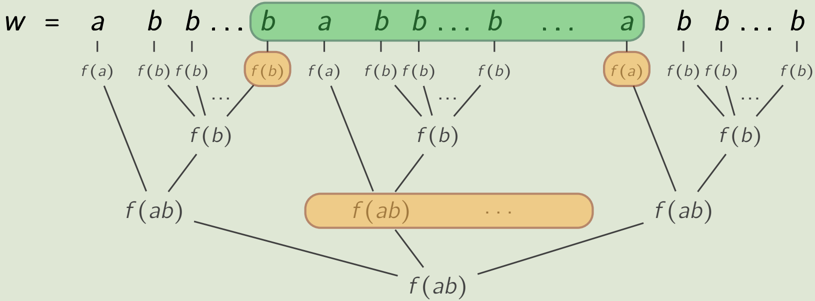

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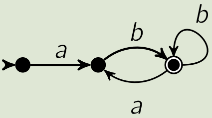

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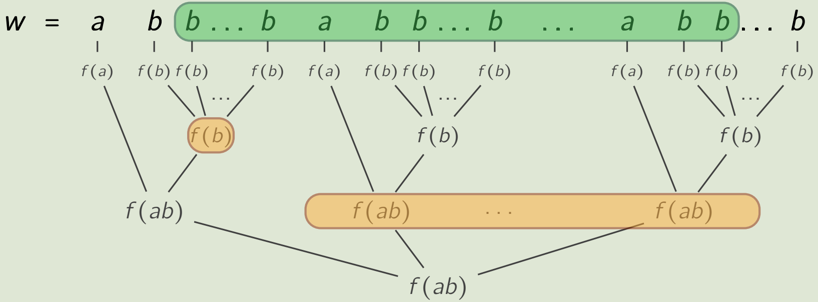

An application: infix matching

For a fixed morphism f (or automaton, or formula), one can receive a word w, construct its factorization forest, and then use it as an **index structure** to evaluate in **constant time** the f-image of any given infix of w.

Other applications:

- constant-delay enumeration of answers to a query
- number of nested Kleene $\star$ needed in a regular expression
- determinization of Büchi automata ($\rightarrow$ parity automata)
- convert semigroups to formulas