Verification of infinite state systems

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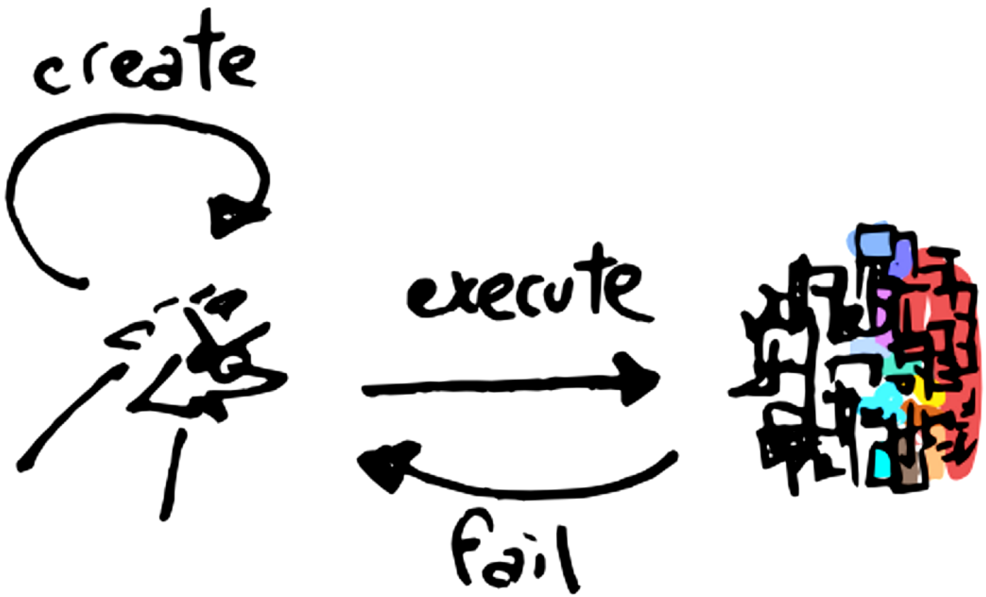

Outline of the course

1. **Warm-up**

   (transition systems, automata, logics)

2. **First-order theories**

   (undecidability, Presburger logic, automatic structures)

3. **The monadic theory of one successor**

   (contraction and composition methods, factorization forests)

4. **The monadic theory of two successors**

   (Rabin's complementation, application examples)

5. **The transformational approach**

   (interpretations, context-free and prefix-rewriting graphs, unfoldings, Caucal hierarchy, recursive program schemes)

6. **Reachability via saturation**

   (pushdown systems, VAS / Petri nets, lossy counter machines)

Goal

Automatic verification of **properties** of **systems**.

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- which properties?

safety	*"something bad never happens"*	reachability
liveness	*"something good eventually happens"*	reachability
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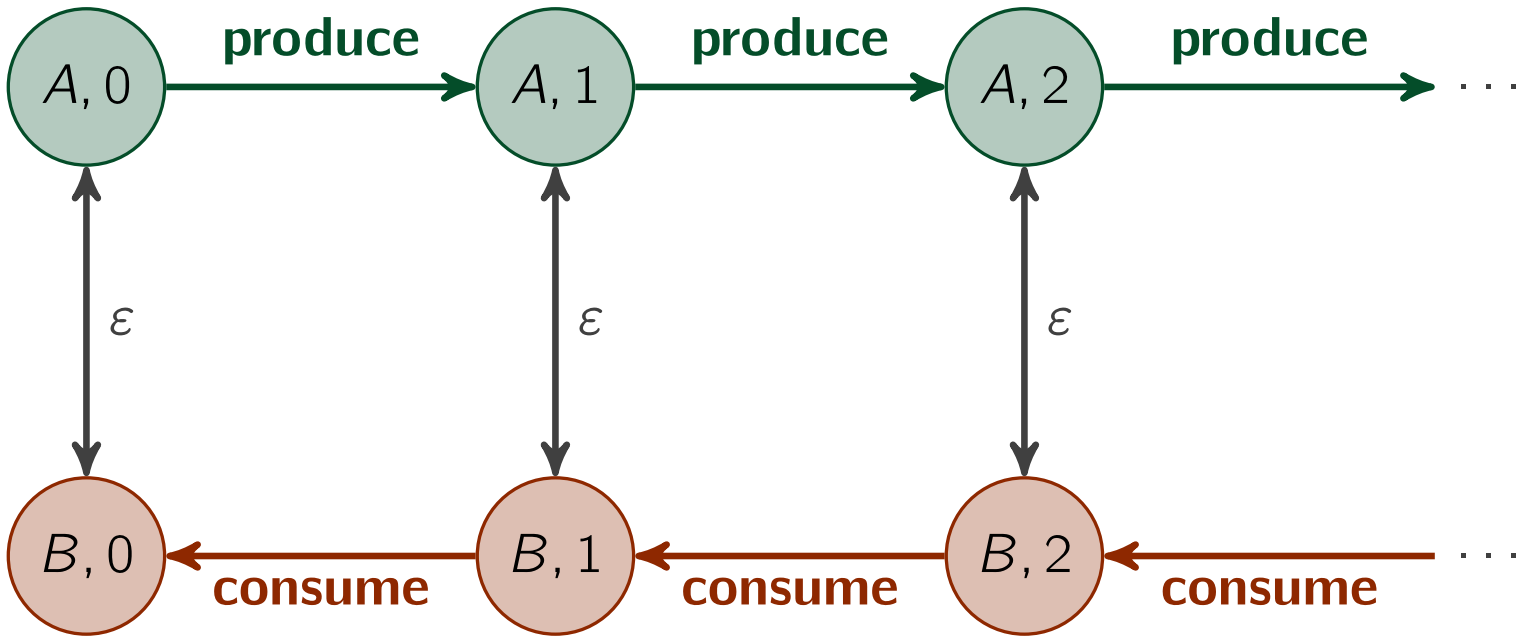
- which systems?

reactive	*"transitions enabled on the basis of input"*
infinite	*stacks* (or *recursion*) *variables* *queues* *lists*

A : repeat forever
do atomically
produce
$count := count + 1$

$\|$

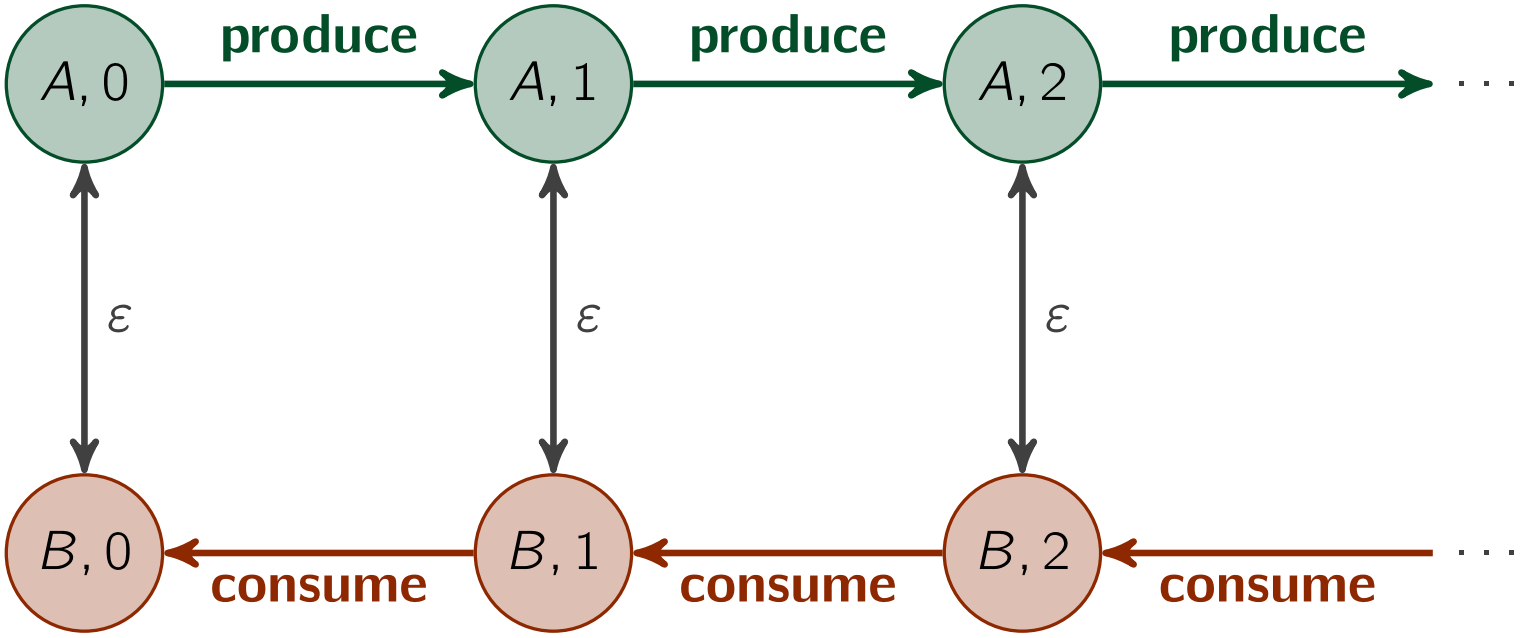
B : repeat forever
do atomically
if $count > 0$ then
consume
$count := count - 1$

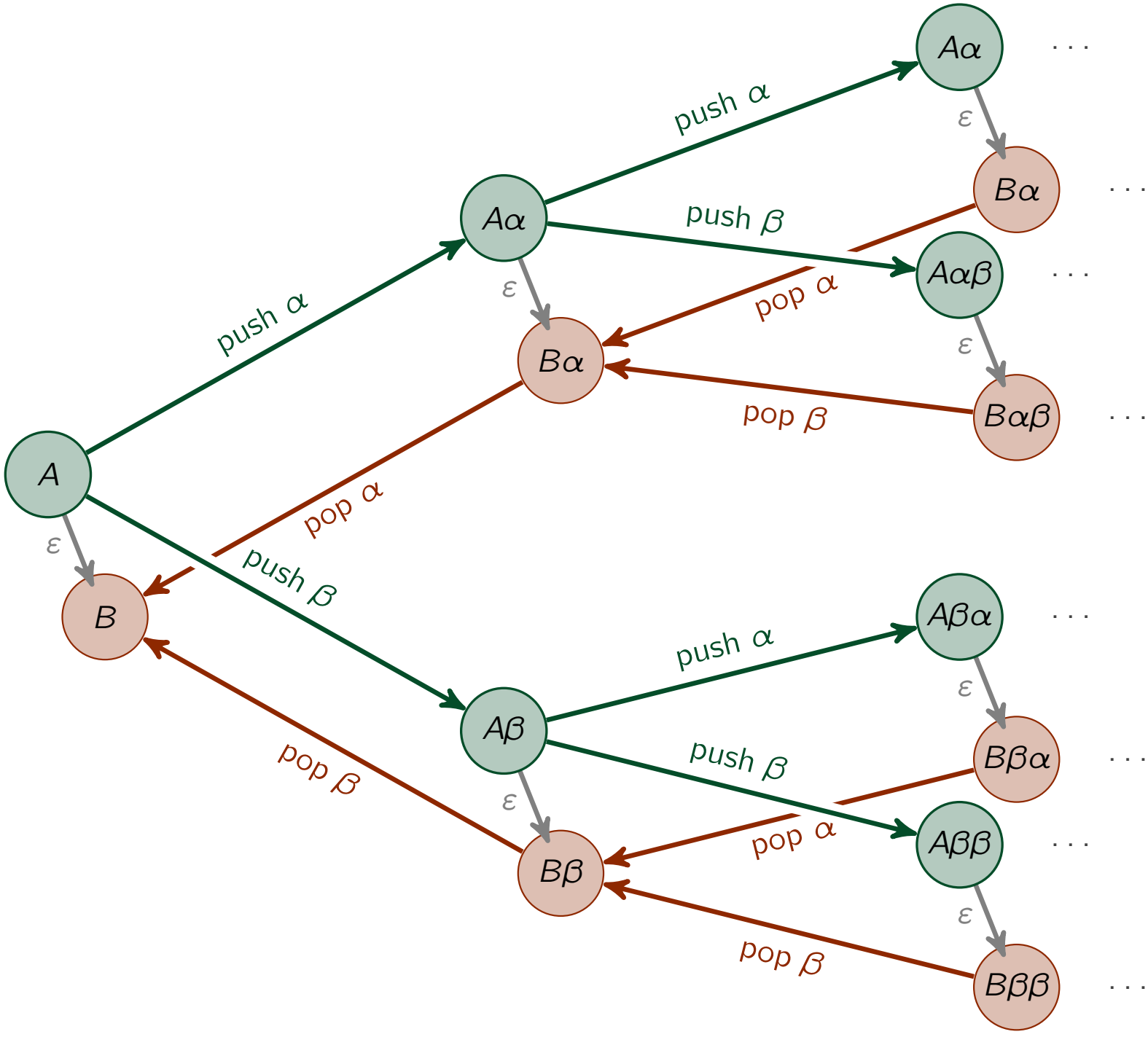


Definition

A **transition system** is a graph $G = \left((V_a)_{a \in \Sigma}, (E_b)_{b \in \Delta}\right)$ where

- vertices are associated with labels from a finite alphabet Σ
- edges are associated with labels from a finite alphabet Δ




A
B
Aα
Bα
Aαβ
Bαβ
Aβ
Bβ
Aβα
Bβα
Aββ
Bββ
push α
pop α
push β
pop β
ε
...

Automata = finite transition systems

 but mostly used as representations of **languages**

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Definition

A **finite state automaton** is a tuple $\mathcal{A} = (Q, \Sigma, \Delta, I, F)$, where

- Q is a finite set of control states
- Σ is a finite alphabet for transition labels
- $\Delta \subseteq Q \times \Sigma \times Q$ is a finite set of transition rules
- $I \subseteq Q$ is a set of initial states
- $F \subseteq Q$ is a set of final states

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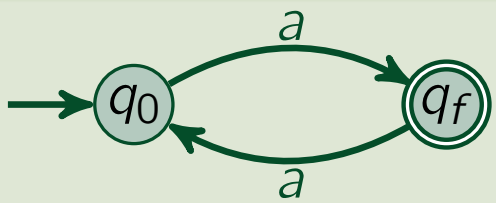
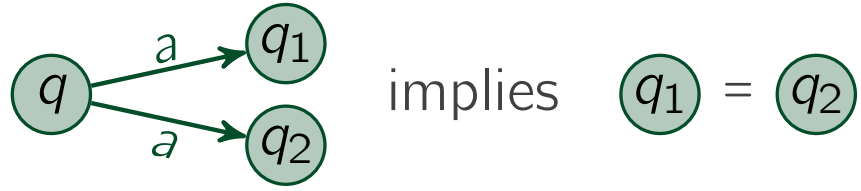
Example



$$\mathscr{L}(\mathcal{A}) = a\,(a\,a)^\star$$

Different types of automata:

- **deterministic**

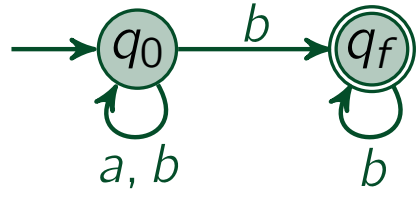


- **with ε-transitions**

- **complete**

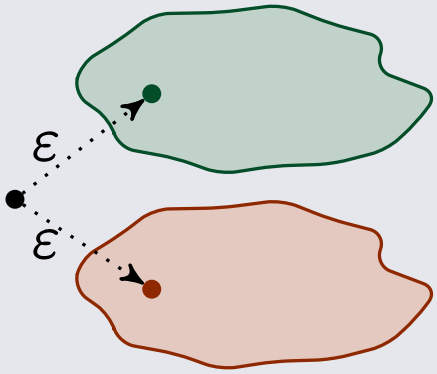
$\forall a.\ \forall q.\ \exists q'.$

- **Büchi/parity conditions**

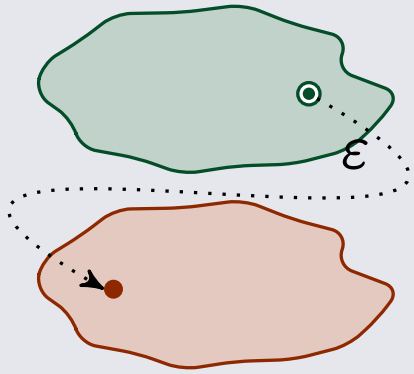


Language theoretic operations on automata

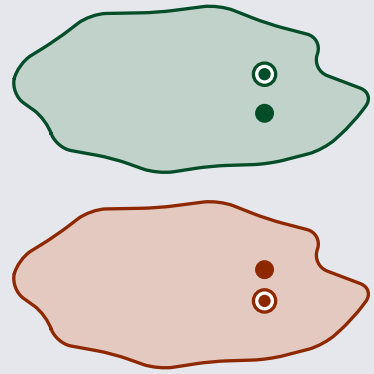
union



concatenation



complementation



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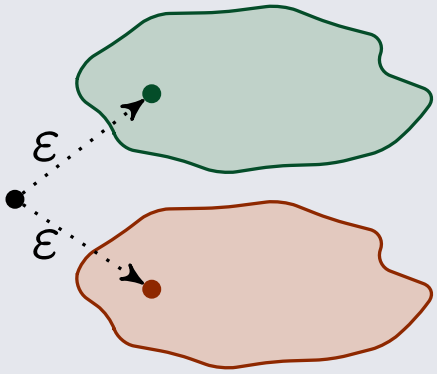

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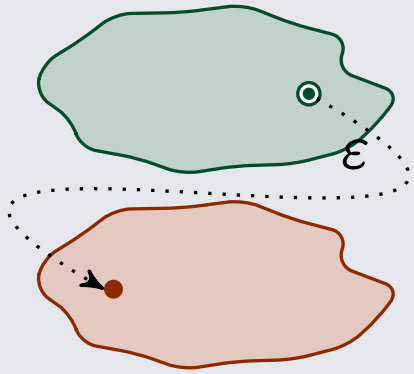
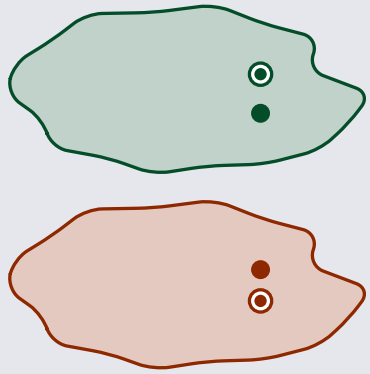


complementation



Language theoretic operations on automata

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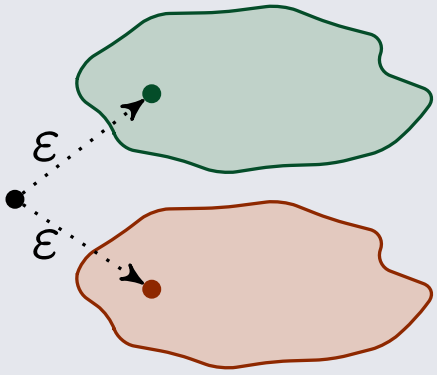

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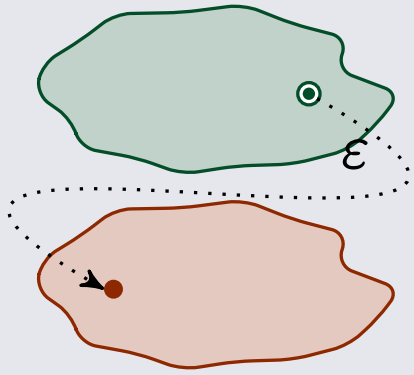
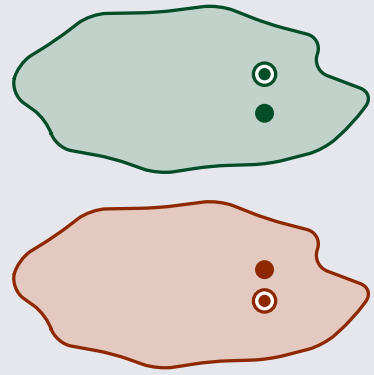
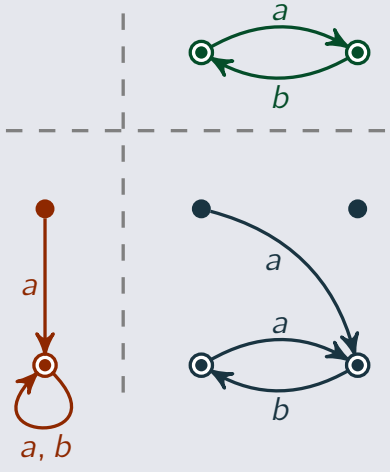


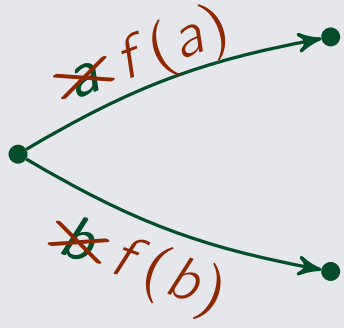
complementation



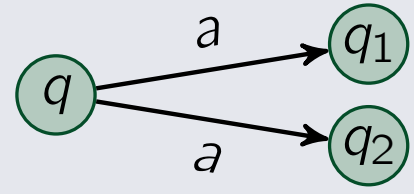
intersection

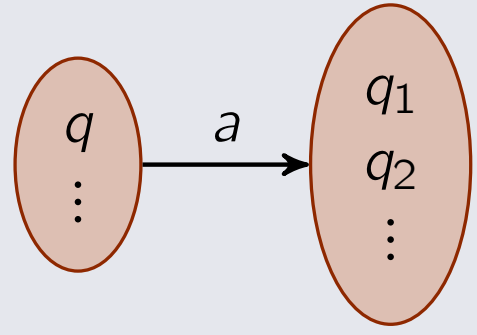


projection



subset construction

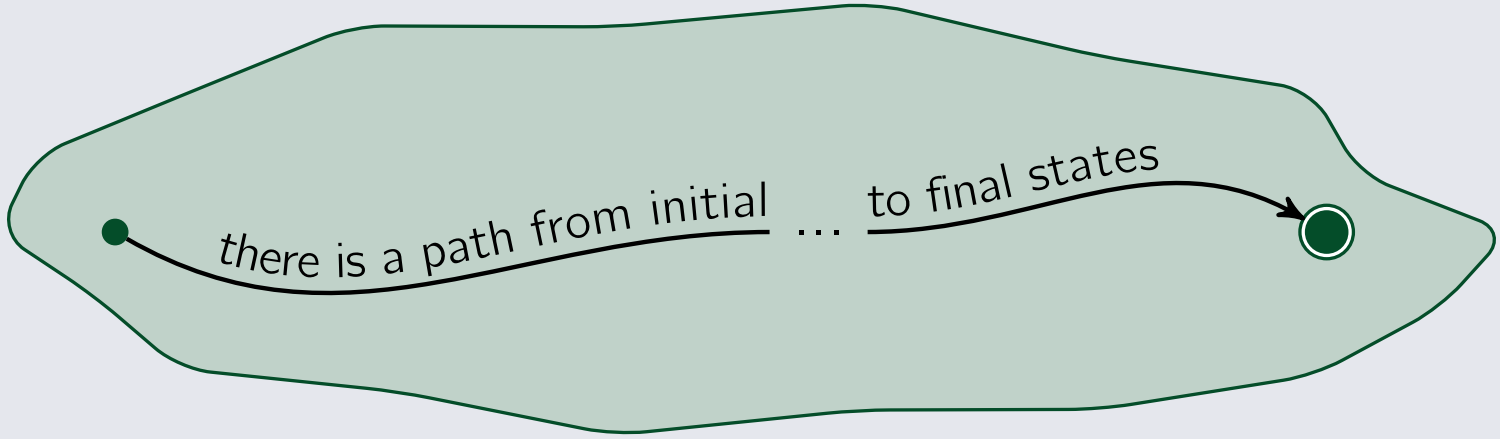




Problems on automata

- **Non-emptiness**

  $\mathscr{L}(\mathcal{A}) \neq \varnothing$ iff



- **Universality**

  $\mathscr{L}(\mathcal{A}) = \Sigma^\star$ iff $\mathscr{L}(\mathcal{A}^\mathsf{C}) = \varnothing$

- **Containment**

  $\mathscr{L}(\mathcal{A}) \subseteq \mathscr{L}(\mathcal{B})$ iff $\mathscr{L}(\mathcal{A}) \cap \mathscr{L}(\mathcal{B}^\mathsf{C}) = \varnothing$

☞ These are simple graph search problems!

Logics for specification of properties

- **Propositional logic**

$b \vee \neg b$

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$$a(x_0) \land \forall x.\, \big(a(x) \to b(x)\big) \to b(x_0)$$

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- **Monadic second-order logic**

$$\exists Z.\ \forall x.\ \exists y.\ (Z(y) \land y = x + 1)$$

Examples of sentences and formulas

$$\psi_{\text{dense}} \quad = \quad \forall x, y.\ \exists z.\ (x < y \rightarrow x < z \ \wedge \ z < y)$$

$$\psi_{\text{connected}} \quad = \quad \forall Z.\ \big(\exists x, y.\ Z(x) \ \wedge \ \neg Z(y)\big) \ \rightarrow \ \big(\exists x, y.\ Z(x) \ \wedge \ \neg Z(y) \ \wedge \ E(x, y)\big)$$

$$\psi_{\text{path}}(x, y) \quad = \quad \forall Z.\ Z(x) \ \wedge \ \forall x', y'.\ \big(Z(x') \wedge E(x', y') \rightarrow Z(y')\big) \ \rightarrow \ Z(y)$$

Examples of sentences and formulas

$$\psi_{\text{dense}} = \underbrace{\forall x, y.\ \exists z.\ (x < y \to x < z \ \wedge\ z < y)}_{\text{FO}[<] \text{ over } \mathbb{Q}}$$

$$\psi_{\text{connected}} = \underbrace{\forall Z.\ \big(\exists x, y.\ Z(x) \ \wedge\ \neg Z(y)\big) \ \to\ \big(\exists x, y.\ Z(x) \ \wedge\ \neg Z(y) \ \wedge\ E(x, y)\big)}_{\text{MSO}[E] \text{ over } G=(V,E)}$$

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☞ The underlying **signature** and **domain** are important!

- MSO[+1] over $\mathbb{N}$ = MSO[<] over $\mathbb{N}$
- FO[0, 1, +] = Presburger arithmetic
- MSO over $\mathbb{N}$ = FO[$\subseteq$] over $2^{\mathbb{N}}$
- FO[$\in$] over models of Zermelo–Fraenkel set theory...

Other examples of MSO properties

$$\begin{aligned}\psi_{\text{3-colorability}} \;=\; & \exists X, Y, Z.\ \forall v.\ \big(X(v) \vee Y(v) \vee Z(v)\big) \\ & \forall u, v.\ E(u,v) \;\rightarrow\; \neg\big(X(u) \wedge X(v)\big) \wedge \\ & \qquad\qquad \neg\big(Y(u) \wedge Y(v)\big) \wedge \\ & \qquad\qquad \neg\big(Z(u) \wedge Z(v)\big)\end{aligned}$$

$$\psi_{\mathrm{K5}}(x_1, \ldots, x_5) \;=\; \bigwedge_{i \neq j} \big(x_i \neq x_j \;\wedge\; \psi_{\text{path}}(x_i, x_j)\big)$$

$$\psi_{\mathrm{K3,3}}(x_1, \ldots, x_3, y_1, \ldots, y_3) \;=\; \bigwedge_{i \neq j} (x_i \neq x_j \wedge y_i \neq y_j) \;\wedge\; \bigwedge_{i,j} \psi_{\text{path}}(x_i, y_j)$$

$$\begin{aligned}\psi_{\text{planar}} \;=\; & \neg\exists x_1, \ldots, x_5.\ \psi_{\mathrm{K5}}(x_1, \ldots, x_5) \;\wedge \\ & \neg\exists x_1, \ldots, x_3, y_1, \ldots, y_3.\ \psi_{\mathrm{K33}}(x_1, \ldots, x_3, y_1, \ldots, y_3)\end{aligned}$$

A real example

let $Foo(g, h)$ =

$g(h)$

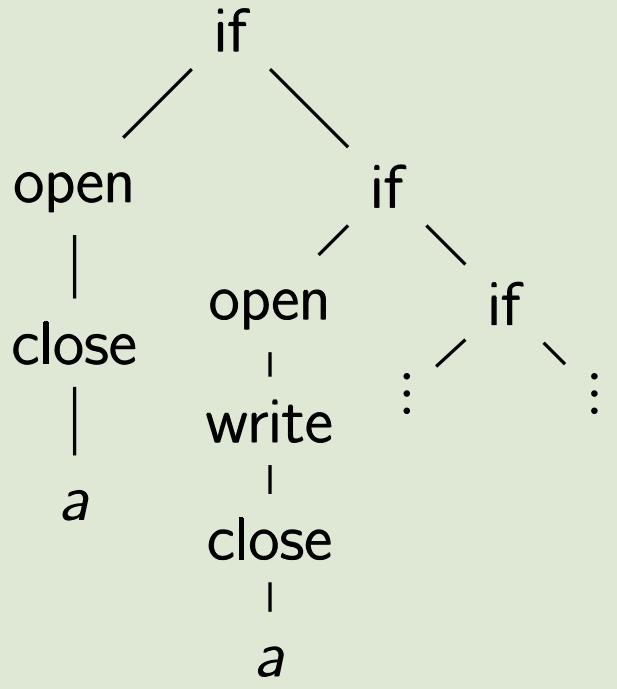
if [*user hits key*] then

$g \cdot \mathsf{close}(h)$

else

$Foo(g \cdot \mathsf{write}, h)$

$Foo(\mathsf{open}, a)$



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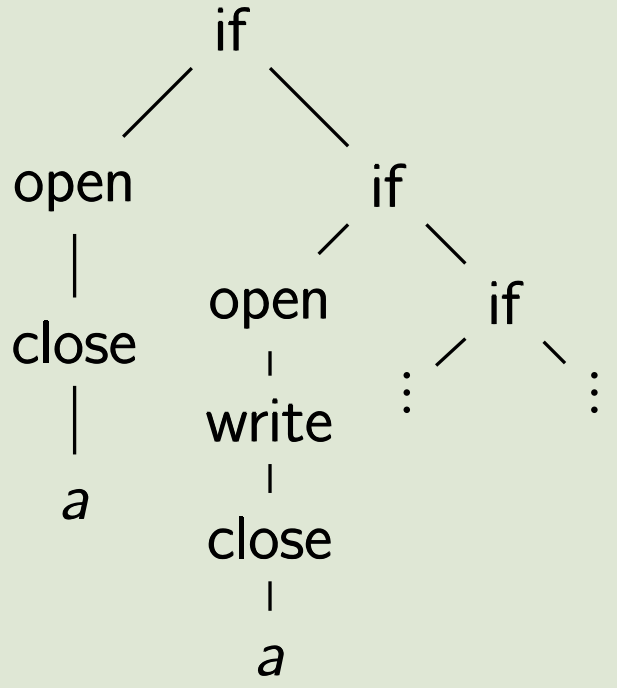

One may want to verify that all sequences of write operations occur between open and close operations:

$$\forall Z \text{ path}.\ \forall z \in Z.\ \ \mathsf{write}(z) \ \rightarrow\ \exists x, y \in Z.\ \ x < z < y \ \wedge\ \mathsf{open}(x) \ \wedge\ \mathsf{close}(y)$$

Next