A Guided Tour through Interval Temporal Logics
Lecture 4: Interval logics: undecidability.

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Gargnano, August 20-25, 2012

Undecidability in interval logics: the initial shock

Theorem. The validity in HS, in the non-strict semantics, over any class of ordered structures containing at least one with an infinitely ascending sequence is r.e.-hard.

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Proof idea: reduction from the non-halting problem for Turing machines to testing satisfiability in HS.

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Proof idea: reduction from the problem of existence of a computation of a given non-deterministic Turing machine that enters the start state infinitely often to testing satisfiability in HS.

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Truth at the beginning/end of the current interval:

$$[[BP]]\phi := [B](\pi \rightarrow \phi); \quad [[EP]]\phi := [E](\pi \rightarrow \phi)$$

Reduction from the halting problem - 1

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Every cell on the tape represented by an interval satisfying

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Every ID is a sequence of cells, represented by an interval satisfying

$$\texttt{ID} := \langle B\rangle \texttt{cell}(*) \wedge \langle E\rangle \texttt{cell}(*) \wedge \langle D\rangle cell \wedge \neg\langle D\rangle \texttt{cell}(*)$$

Reduction from the halting problem - 2

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Starting/Final configurations:

$\texttt{startID} := \texttt{ID} \wedge \langle D \rangle (\texttt{cell}((q_0, 0)) \vee \texttt{cell}((q_0, 1)) \vee \texttt{cell}((q_0, B)))$

$\texttt{finalID} := \texttt{ID} \wedge \langle D \rangle (\texttt{cell}((q_f, 0)) \vee \texttt{cell}((q_f, 1)) \vee \texttt{cell}((q_f, B)))$

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Describing corr properly is the most ingenious and expressiveness demanding part of the reduction.

Reduction from the halting problem - 3

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Some of the conditions that must be imposed on corr:

- an interval over which corr is true starts and ends with a cell: $\texttt{cellRule} := \texttt{corr} \rightarrow (\langle B\rangle \texttt{cell} \wedge \langle E\rangle \texttt{cell})$
- one corr interval may not properly contain another one: $\texttt{notContainscorr} := \texttt{corr} \rightarrow$ $(\neg\langle B\rangle \texttt{corr} \wedge \neg\langle D\rangle \texttt{corr} \wedge \neg\langle E\rangle \texttt{corr})$

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For the satisfiability of NoHalt, any interval structure with an infinite ascending chain suffices.

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On Dedekind complete structures one can also express the property of a computation to visit infinitely often its starting state.

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Lodaya shows that the construction can be carried out on the ordinal ω^2, and even on $\omega + 1$.

More recent undecidability results

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In 2011, Marcinkowski and Michaliszyn showed undecidability of D over the classes of finite and discrete linear orderings.

J. Marcinkowski and J. Michaliszyn, *The Ultimate Undecidability Result for the Halpern-Shoham Logic*, LICS 2011

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Thus, other ideas and more refined encodings of tiling problems were needed for undecidability results on fragments of HS.

The Octant Tiling Problem

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This is the problem of establishing whether a given finite set of tile types $\mathcal{T} = \{t_1, \ldots, t_k\}$ can tile the 2nd octant of the integer plane:

$$\mathcal{O} = \{(i, j) : i, j \in \mathbb{N} \wedge 0 \leq i \leq j\},$$

while respecting the color constraints.

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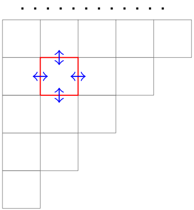

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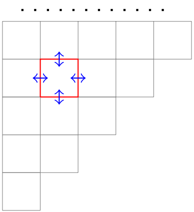

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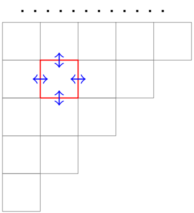

Proposition. The Octant Tiling Problem is undecidable.

Proof: by reduction from the tiling problem for $\mathbb{N} \times \mathbb{N}$, using König's Lemma.

Undecidability of interval logics via tiling: generic construction - 1

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Given a finite set of tiles, we consider a signature containing, inter alia, special propositional letters u, tile, ld, $\mathtt{t}_1, \ldots, \mathtt{t}_k$, cbb, cbe, ceb, corr, and possibly others.

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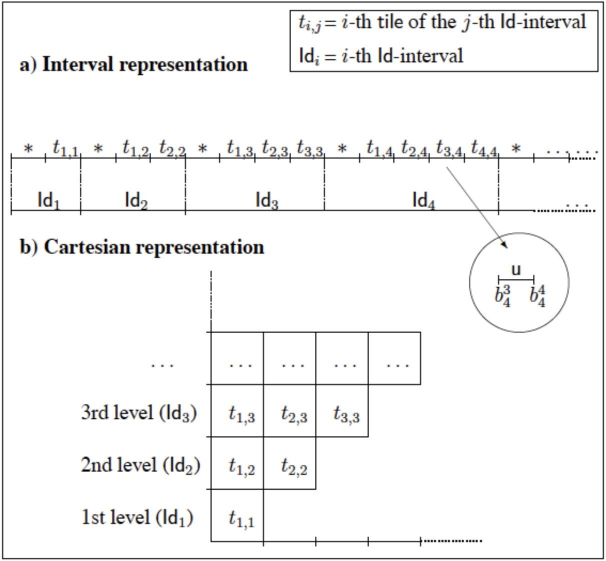
Then, ID-intervals are introduced to represent the layers of tiles.

Undecidability of interval logics via tiling: generic construction - 2



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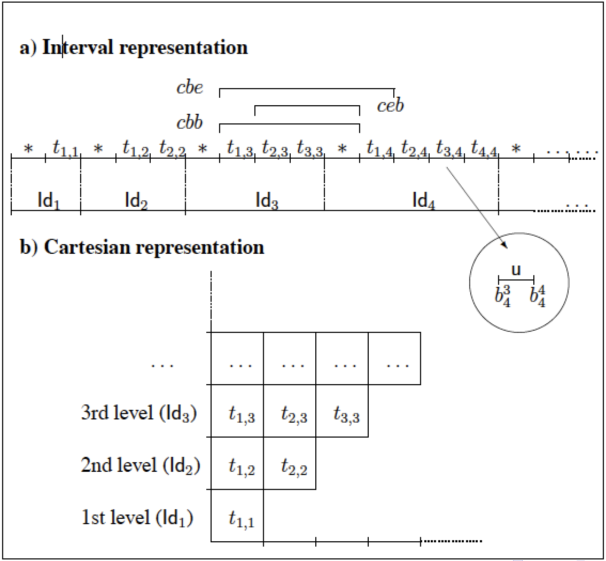
For that, we use several auxiliary propositional letters to refine and implement the idea of `corr`: cbb for matching the beginning point of a tile to the beginning point of the corresponding tile above; cbe, for matching beginning point with ending point above, and ceb for matching ending point with a beginning point above.

Undecidability of interval logics via tiling: generic construction - 4



Undecidability of interval logics via tiling: generic construction - 5

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$$[F](\langle A\rangle \text{tile} \rightarrow \bigvee_{up(t_i)=down(t_j)} (\langle A\rangle \mathtt{t_i} \wedge \langle A\rangle(\text{cbb} \wedge \langle A\rangle \mathtt{t_j})))$$

Undecidability of the logic O over discrete linear orderings

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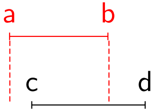
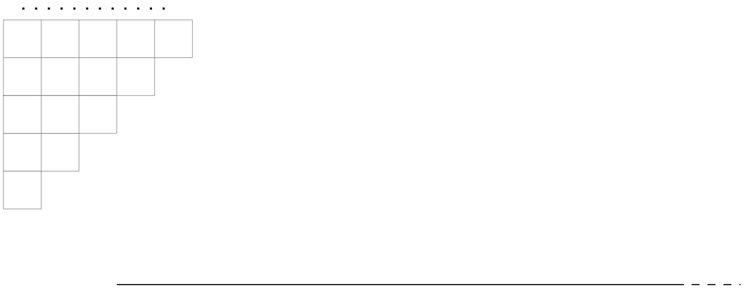


Encoding the Octant



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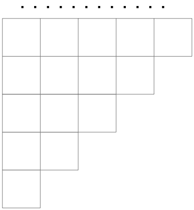

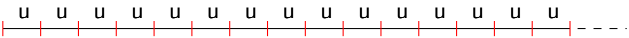

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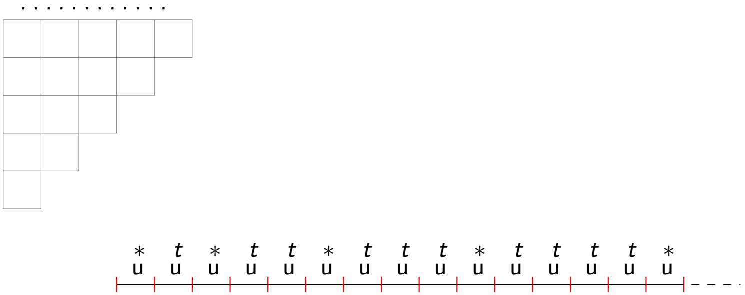

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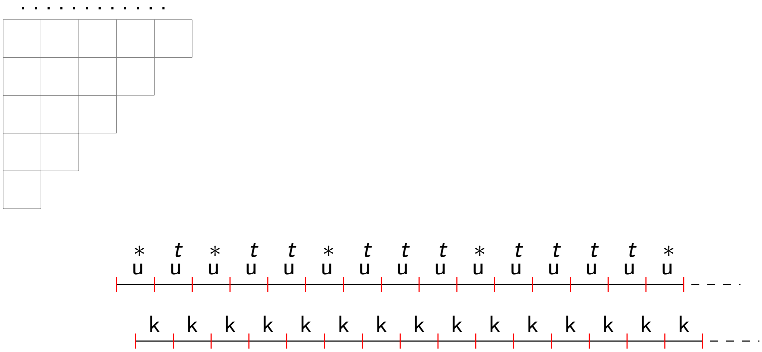

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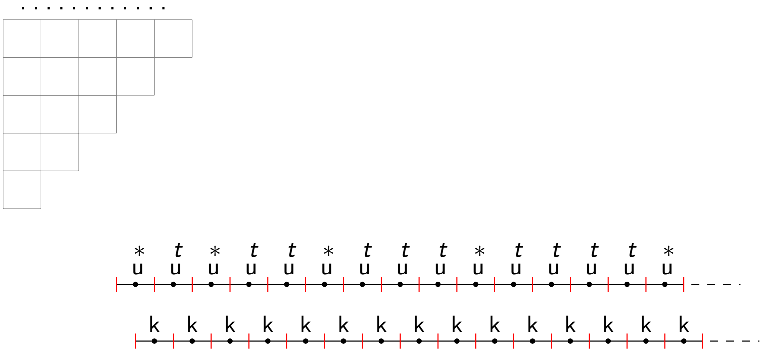
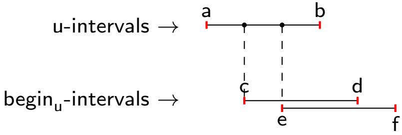


Encoding the Octant: u- and k-intervals of length 2



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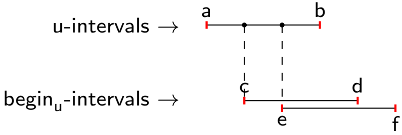

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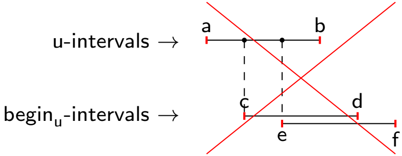

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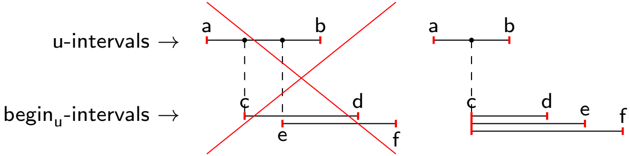

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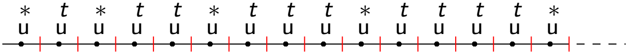
Encoding the Octant: the Above-Neighbor Relation

. . . .

t_1^5	t_2^5	t_3^5	t_4^5	t_5^5
t_1^4	t_2^4	t_3^4	t_4^4	
t_1^3	t_2^3	t_3^3		
t_1^2	t_2^2			
t_1^1				



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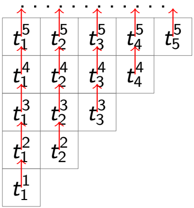

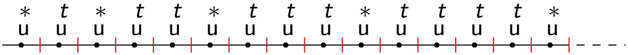

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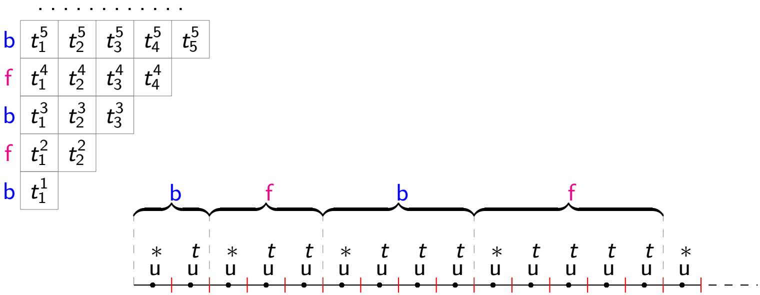

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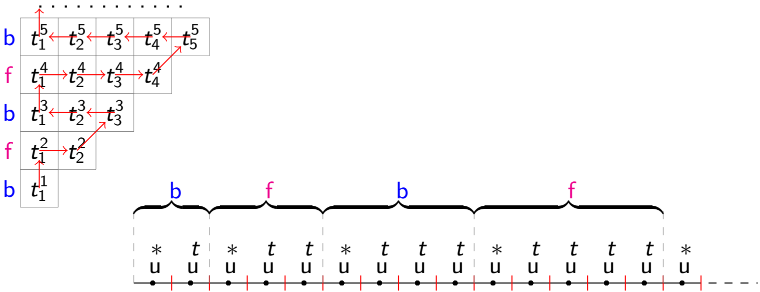

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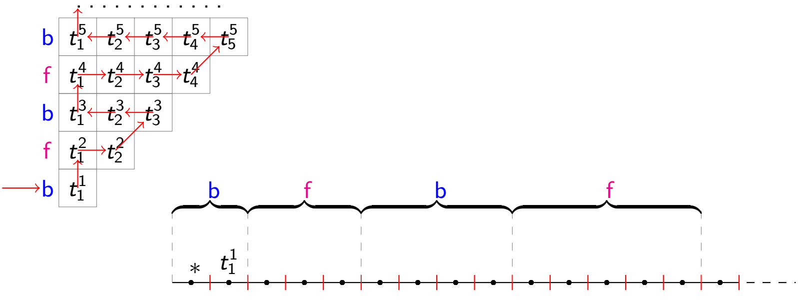

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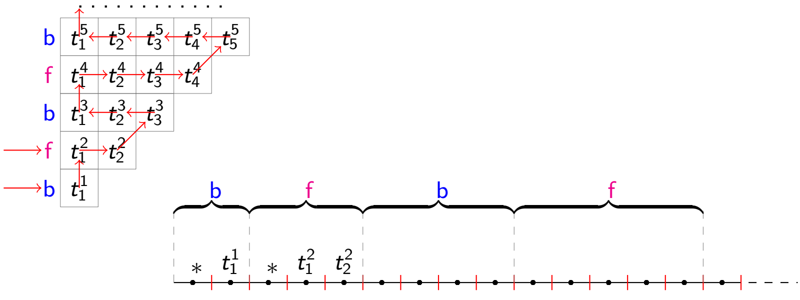

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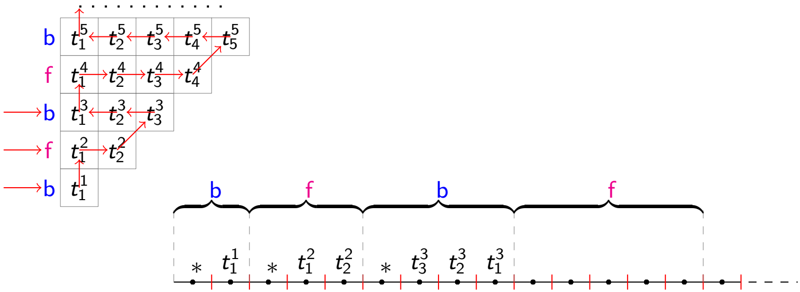

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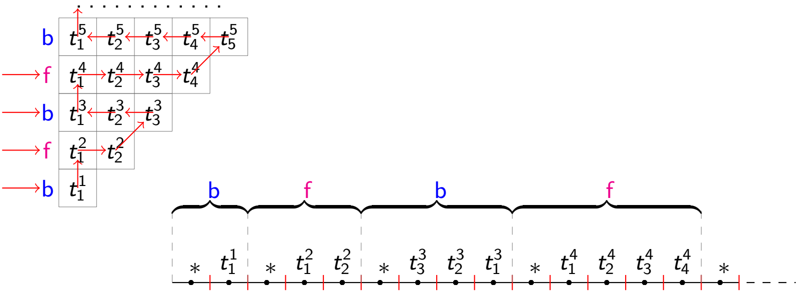

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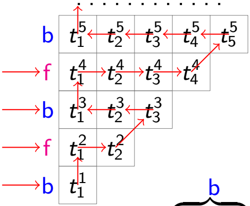
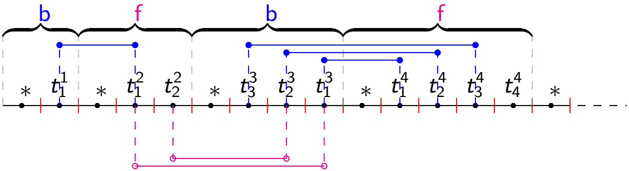


$$\text{up\_rel}^{\text{b}} \rightarrow \neg\langle O\rangle \text{up\_rel}^{\text{b}}$$

$$\text{up\_rel}^{\text{f}} \rightarrow \neg\langle O\rangle \text{up\_rel}^{\text{f}}$$



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Theorem. The satisfiability problem for the logic O (resp., $\overline{\text{O}}$) is undecidable over any class of discrete linear orderings that contains at least one linear ordering with an infinite ascending (resp., descending) sequence.

D. Bresolin, D. Della Monica, V. Goranko, A. Montanari, and G. Sciavicco, *Undecidability of the Logic of Overlap Relation over Discrete Linear Orderings*, Electronic Notes in Theoretical Computer Science (Proceedings of the 6th Workshop on Methods for Modalities - M4M 6, 2009), 262:65–81, 2010

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