A Guided Tour through Interval Temporal Logics
Lecture 3: Languages and expressiveness.

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Outline of Lecture 3

- Comparing the expressiveness of fragments of HS (bisimulation and bisimulation games on interval structures)
- A complete classification of the expressiveness of HS fragments on the class of all linear orders
- Standard translations for interval temporal logics
- Expressive completeness results

Definability and inter-definability equations

A modal operator $\langle X\rangle$ of HS is definable in an HS-fragment $\mathcal{F}$, denoted $\langle X\rangle \lhd \mathcal{F}$, if $\langle X\rangle p \equiv \psi$ for some formula $\psi = \psi(p) \in \mathcal{F}$, for any fixed propositional variable p.

In such a case, the equivalence $\langle X\rangle p \equiv \psi$ is called an inter-definability equation for $\langle X\rangle$ in $\mathcal{F}$.

Notation.

For each modal operator $\langle X\rangle$, we denote by R_X the corresponding Allen's relation.

With every subset $\mathcal{X} = \{\langle X_1\rangle, \ldots, \langle X_k\rangle\}$ of HS modalities we associate the fragment $\mathcal{F}_{\mathcal{X}}$ of HS, denoted $\mathrm{X}_1\mathrm{X}_2\ldots\mathrm{X}_\mathrm{k}$, with formulas only featuring modalities from $\mathcal{X}$.

As an example, $\mathrm{B\overline{B}}$ denotes the fragment involving the modalities $\langle B\rangle$ and $\langle \overline{B}\rangle$ only.

Bisimulation - 1

To show undefinability of a given modality in a certain interval logic, one can use bisimulation and invariance of modal formulas with respect to bisimulations.

Let $\mathcal{F}$ be the considered interval logic. An $\mathcal{F}$-bisimulation between two interval models $M = \langle \mathbb{I}(\mathbf{D}), V \rangle$ and $M' = \langle \mathbb{I}(\mathbf{D}'), V' \rangle$ over the set of proposition letters $\mathcal{AP}$ is a relation $Z \subseteq \mathbb{I}(\mathbf{D}) \times \mathbb{I}(\mathbf{D}')$ satisfying the following properties:

- local condition: pairs of Z-related intervals satisfy the same proposition letters over $\mathcal{AP}$
- forward condition: if $([i,j],[i',j']) \in Z$ and $([i,j],[h,k]) \in R_X$ for some $\langle X \rangle \in \mathcal{F}$, then there exists $[h',k']$ such that $([i',j'],[h',k']) \in R_X$ and $([h,k],[h',k']) \in Z$

Bisimulation - 2

- backward condition: if $([i,j],[i',j']) \in Z$ and $([i',j'],[h',k']) \in R_X$ for some $\langle X \rangle \in \mathcal{F}$, then there exists $[h,k]$ such that $([i,j],\ [h,k]) \in R_X$ and $([h,k],\ [h',k']) \in Z$.

Since any $\mathcal{F}$-bisimulation preserves the truth of *all* formulas in $\mathcal{F}$, to prove that an operator $\langle X \rangle$ is not definable in $\mathcal{F}$,

it suffices to construct a pair of interval models M and M' and an $\mathcal{F}$-bisimulation between them such that

$M, [i,j] \Vdash \langle X \rangle p$ and

$M', [i',j'] \nVdash \langle X \rangle p$,

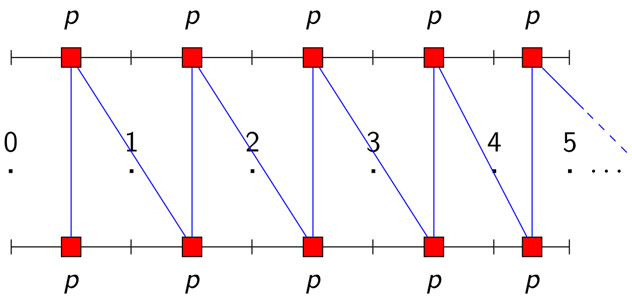
for a pair of $\mathcal{F}$-bisimilar intervals $[i,j] \in M$ and $[i',j'] \in M'$.

On the relationship between A and L over $\mathbb{N}$ - 1

Theorem. The modality $\langle \overline{L} \rangle$ is not definable in A over $\mathbb{N}$.
Proof. Let us consider the pair of interval models $M = \langle \mathbb{I}^-(\mathbb{N}), V \rangle$ and $M' = \langle \mathbb{I}^-(\mathbb{N}), V' \rangle$, over the set of proposition letters $\mathcal{AP} = \{p\}$, where $V(p) = V'(p) = \{[i, i+1] : i \geq 0\}$. Moreover, let $Z \subseteq \mathbb{I}^-(\mathbb{N}) \times \mathbb{I}^-(\mathbb{N})$ be the set

$$\{([i,j],[i,j]) : 0 \leq i < j\} \cup \{([i,j],[i+1,j+1]) : 0 \leq i < j\}$$

(Part of) the relation Z can be depicted as follows:



On the relationship between A and L over $\mathbb{N}$ - 2

Z is an A-bisimulation.

Checking that it satisfies the local condition is immediate.

As for the forward condition, we must distinguish two cases.

First, we must consider any pair of the form $([i,j],[i,j])$. In such a case, the $\langle A \rangle$-move from $[i,j]$ to $[j,k]$ in M can be simulated by the very same $\langle A \rangle$-move from $[i,j]$ to $[j,k]$ in M'. Notice that p-intervals come into play when $j = i + 1$ or $k = j + 1$ (or both).

The second case is that of pairs of the form $([i,j],[i+1,j+1])$. In such a case, the $\langle A \rangle$-move from $[i,j]$ to $[j,k]$ in M can be simulated by the $\langle A \rangle$-move from $[i+1,j+1]$ to $[j+1,k+1]$ in M'. As in the previous case, p-intervals come into play when $j = i + 1$ or $k = j + 1$ (or both).

Satisfaction of the backward condition can be checked in a very similar way.

On the relationship between A and L over $\mathbb{N}$ - 3

To conclude the proof, it suffices to show that Z does not preserve the relation induced by the modality $\langle \overline{L} \rangle$. To this end, consider the pair $([1,2],[2,3]) \in Z$.

We have that $M', [2,3] \models \langle \overline{L} \rangle p$, while $M, [1,2] \not\models \langle \overline{L} \rangle p$. □

Corollary. The modality $\langle \overline{A} \rangle$ is not definable in A over $\mathbb{N}$.

Proof. As the modal operator $\langle \overline{L} \rangle$ is definable in any interval logic featuring the modal operator $\langle \overline{A} \rangle$ (for any fixed proposition letter p, it holds that $\langle \overline{L} \rangle p \equiv \langle \overline{A} \rangle \langle \overline{A} \rangle p$), the thesis immediately follows from the above theorem. □

It is worth pointing out that the bisimulation exploited in the proof of the above theorem still works if we replace $\mathbb{N}$ by $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$.

Exercise. To show that the modality $\langle \overline{A} \rangle$ is not definable in A$\overline{\text{L}}$ over $\mathbb{N}$.

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We illustrate such games by an example.

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Bisimulation games for $A\overline{A}^{+}$ and $A\overline{A}^{-}$ are defined likewise.

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Proposition. Let $\mathcal{P}$ be a finite set of propositional letters. For all $k \geq 0$, Player II has a winning strategy in the k-round $A\overline{A}$-bisimulation game on M_0 and M_1, with initial configuration $([a_0, b_0], [a_1, b_1])$, iff $[a_0, b_0]$ and $[a_1, b_1]$ satisfy the same $A\overline{A}$-formulas over $\mathcal{P}$ with modal operator depth at most k.

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Corollary. Player II has a winning strategy in the infinite $A\overline{A}$-bisimulation game on M_0 and M_1, with initial configuration $([a_0, b_0], [a_1, b_1])$, iff $[a_0, b_0]$ and $[a_1, b_1]$ satisfy the same $A\overline{A}$-formulas over $\mathcal{P}$.

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Classification of HS fragments w.r.to expressiveness

We compare and classify the expressiveness of all fragments of HS on the class of all interval structures over linear orders (with strict semantics). Formally, let $\mathcal{F}_1$ and $\mathcal{F}_2$ be any pair of such fragments. We say that:

- $\mathcal{F}_2$ is at least as expressive as $\mathcal{F}_1$, denoted $\mathcal{F}_1 \preceq \mathcal{F}_2$, if every operator $\langle X \rangle \in \mathcal{F}_1$ is definable in $\mathcal{F}_2$ ($\langle X \rangle \lhd \ F_2$).
- $\mathcal{F}_1$ is strictly less expressive than $\mathcal{F}_2$, denoted $\mathcal{F}_1 \prec \mathcal{F}_2$, if $\mathcal{F}_1 \preceq \mathcal{F}_2$ but not $\mathcal{F}_2 \preceq \mathcal{F}_1$.
- $\mathcal{F}_1$ and $\mathcal{F}_2$ are equally expressive (or expressively equivalent), denoted $\mathcal{F}_1 \equiv \mathcal{F}_2$, if $\mathcal{F}_1 \preceq \mathcal{F}_2$ and $\mathcal{F}_2 \preceq \mathcal{F}_1$.
- $\mathcal{F}_1$ and $\mathcal{F}_2$ are expressively incomparable, denoted $\mathcal{F}_1 \not\equiv \mathcal{F}_2$, if neither $\mathcal{F}_1 \preceq \mathcal{F}_2$ nor $\mathcal{F}_2 \preceq \mathcal{F}_1$.

A definability $\langle X \rangle \lhd \ \mathcal{F}$ is optimal if $\langle X \rangle \not\lhd \mathcal{F}'$ for any fragment $\mathcal{F}'$ such that $\mathcal{F}' \prec \mathcal{F}$. A set of such definabilities is optimal if it consists of optimal definabilities.

The complete set of inter-definability equations

$\langle L\rangle p \equiv \langle A\rangle\langle A\rangle p$	$\langle L\rangle \lhd \mathsf{A}$
$\langle \overline{L}\rangle p \equiv \langle \overline{A}\rangle\langle \overline{A}\rangle p$	$\langle \overline{L}\rangle \lhd \overline{\mathsf{A}}$
$\langle O\rangle p \equiv \langle E\rangle\langle \overline{B}\rangle p$	$\langle O\rangle \lhd \overline{\mathsf{B}}\mathsf{E}$
$\langle \overline{O}\rangle p \equiv \langle B\rangle\langle \overline{E}\rangle p$	$\langle \overline{O}\rangle \lhd \mathsf{B}\overline{\mathsf{E}}$
$\langle D\rangle p \equiv \langle E\rangle\langle B\rangle p$	$\langle D\rangle \lhd \mathsf{BE}$
$\langle \overline{D}\rangle p \equiv \langle \overline{E}\rangle\langle \overline{B}\rangle p$	$\langle \overline{D}\rangle \lhd \overline{\mathsf{BE}}$
$\langle L\rangle p \equiv \langle \overline{B}\rangle[E]\langle \overline{B}\rangle\langle E\rangle p$	$\langle L\rangle \lhd \overline{\mathsf{B}}\mathsf{E}$
$\langle \overline{L}\rangle p \equiv \langle \overline{E}\rangle[B]\langle \overline{E}\rangle\langle B\rangle p$	$\langle \overline{L}\rangle \lhd \mathsf{B}\overline{\mathsf{E}}$

Theorem. The above set of inter-definability equations is sound, complete, and optimal.

Soundness is easy; completeness is hard.

The structure of the completeness proof

The completeness proof is organized as follows.
For each HS operator $\langle X \rangle$, we show that $\langle X \rangle$ is not definable in any fragment of HS that does not contain as definable (according to given table) all operators of some of the fragments in which $\langle X \rangle$ is definable (according to the given table).

Formally, for each HS operator $\langle X \rangle$, the proof consists of the following steps:

1. using the given table, find all fragments $\mathcal{F}_i$ such that $\langle X \rangle \lhd \mathcal{F}_i$;
2. identify the list $\mathcal{M}_1, \ldots, \mathcal{M}_m$ of all $\subseteq$-maximal fragments of HS that contain neither the operator $\langle X \rangle$ nor any of the fragments $\mathcal{F}_i$ identified by the previous step;
3. for each fragment $\mathcal{M}_i$, with $i \in \{1, \ldots, m\}$, provide a bisimulation for $\mathcal{M}_i$ which is not a bisimulation for X.

Some cases are easier

Lemma. The set of inter-definability equations for $\langle L \rangle$ and $\langle \overline{L} \rangle$ given in the table is complete.

Proof. According to the given table, $\langle L \rangle$ is definable in terms of A and $\overline{\mathrm{B}}\mathrm{E}$. Hence, the fragments $\mathrm{BEDO}\overline{\mathrm{ALEDO}}$ and $\mathrm{BDO}\overline{\mathrm{ALBEDO}}$ are the only $\subseteq$-maximal ones not featuring $\langle L \rangle$ and containing neither A nor $\overline{\mathrm{B}}\mathrm{E}$.

To prove the thesis, it suffices to exhibit a bisimulation for each one of these two fragments that does not preserve the relation induced by $\langle L \rangle$.

Thanks to soundness, $\mathrm{BEDO}\overline{\mathrm{ALEDO}}$ and $\mathrm{BDO}\overline{\mathrm{ALBEDO}}$ are expressively equivalent to $\mathrm{BEO}\overline{\mathrm{AED}}$ and $\mathrm{BDO}\overline{\mathrm{ABE}}$, respectively. Thus, we can refer to the latter ones instead of the former ones. We give the details for $\mathrm{BEO}\overline{\mathrm{AED}}$ and leave the case of $\mathrm{BDO}\overline{\mathrm{ABE}}$ as an exercise.

The fragment $\mathsf{BEO}\overline{\mathsf{AED}}$

Let $M_1 = \langle \mathbb{I}^-(\mathbb{N}), V_1 \rangle$ and $M_2 = \langle \mathbb{I}^-(\mathbb{N}), V_2 \rangle$ be two models and let V_1 and V_2 be such that $V_1(p) = \{[2,3]\}$ and $V_2(p) = \emptyset$, where p is the only propositional letter of the language.

Moreover, let Z be a relation between (intervals of) M_1 and M_2 defined as $Z = \{([0,1],[0,1])\}$.

It can be easily shown that Z is a $\mathsf{BEO}\overline{\mathsf{AED}}$-bisimulation.

The local property is trivially satisfied, since all Z-related intervals satisfy $\neg p$.

As for the forward and backward conditions, it suffices to notice that, starting from the interval $[0,1]$, it is not possible to reach any other interval using any of the modal operators of the fragment.

It can be easily checked that Z does not preserve the relation induced by the modality $\langle L \rangle$. Indeed, $([0,1],[0,1]) \in Z$ and $M_1, [0,1] \Vdash \langle L \rangle p$, but $M_2, [0,1] \Vdash \neg \langle L \rangle p$. Therefore, $\langle L \rangle$ is not definable in $\mathsf{BEDO}\overline{\mathsf{ALEDO}}$. □

Other cases are much more difficult

Lemma. The set of inter-definability equations for $\langle A \rangle$ and $\langle \overline{A} \rangle$ given in the table is complete.

To get the requested bisimulation, we exploit a well-known property of the set of real numbers $\mathbb{R}$: $\mathbb{R}$ (resp., $\mathbb{Q}$, $\overline{\mathbb{Q}} = \mathbb{R} \setminus \mathbb{Q}$) can be partitioned into a countable number of pairwise disjoint subsets, each one of which is dense in $\mathbb{R}$.

Formally, there are countably many nonempty sets $\mathbb{R}_i$ (resp., $\mathbb{Q}_i$, $\overline{\mathbb{Q}}_i$), with $i \in \mathbb{N}$, such that, for each $i \in \mathbb{N}$, $\mathbb{R}_i$ (resp., $\mathbb{Q}_i$, $\overline{\mathbb{Q}}_i$) is dense in $\mathbb{R}$, $\mathbb{R} = \bigcup_{i \in \mathbb{N}} \mathbb{R}_i$ (resp., $\mathbb{Q} = \bigcup_{i \in \mathbb{N}} \mathbb{Q}_i$, $\overline{\mathbb{Q}} = \bigcup_{i \in \mathbb{N}} \overline{\mathbb{Q}}_i$), and $\mathbb{R}_i \cap \mathbb{R}_j = \emptyset$, (resp., $\mathbb{Q}_i \cap \mathbb{Q}_j = \emptyset$, $\overline{\mathbb{Q}}_i \cap \overline{\mathbb{Q}}_j = \emptyset$), for each $i, j \in \mathbb{N}$ with $i \neq j$.

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Hereafter we omit the lower index 2.

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Let x_i, x_j, x_k be 3 distinct variables.

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Note that the only free variables in $ST_{x_i,x_j,x_k}(\varphi)$ are x_i, x_j.

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$$M, [b, e] \models \varphi \text{ iff } M \Vdash ST_{x_i, x_j, x_k}(\varphi)[x_i := b, x_j := e]$$

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Corollary. HS $\prec$ CDT $\preceq \mathrm{FO}^3[<]$.

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Corollary. CDT is expressively complete for $\mathrm{FO}^3[<]$.

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For technical convenience, we can assume that both variables x and y occur as (possibly vacuous) free variables in every formula $\alpha \in \mathrm{FO}^2[<]$, that is, $\alpha = \alpha(x, y)$.

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Theorem. For any $\mathsf{A}\overline{\mathsf{A}}^{\pi+}$-formula φ, non-strict interval model $M = \langle \mathbb{I}(\mathbf{D})^+, V \rangle$, and interval $[a, b]$ in M:

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Corollary. An $\mathsf{A}\overline{\mathsf{A}}^{\pi+}$-formula φ is satisfiable in a class of non-strict interval structures built over a class of linear orderings $\mathcal{C}$ iff $ST_{x,y}(\varphi)$ is satisfiable in the class of all $\mathrm{FO}^2[<]$-models expanding linear orderings from $\mathcal{C}$.

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$\tau[x,y](x=x) = \tau[x,y](y=y) = \top$.	$\tau[x,y](\neg\alpha) = \neg\tau[x,y](\alpha)$.
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D. Bresolin, V. Goranko, A. Montanari, and G. Sciavicco, *Propositional interval neighborhood logics: Expressiveness, decidability, and undecidable extensions*, Annals of Pure and Applied Logic, 161(3):289–304, 2009

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