A Guided Tour through Interval Temporal Logics
Lecture 2: Interval structures, relations, and logics.

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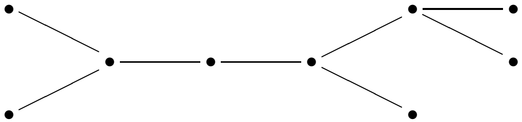

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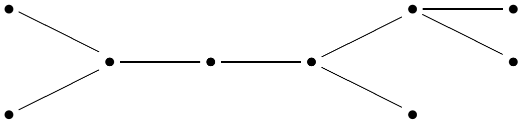

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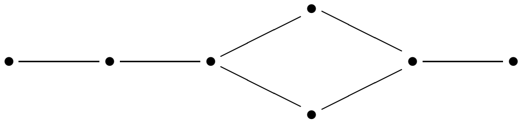

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We will also consider the single interval structures on $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$ with their usual orders.

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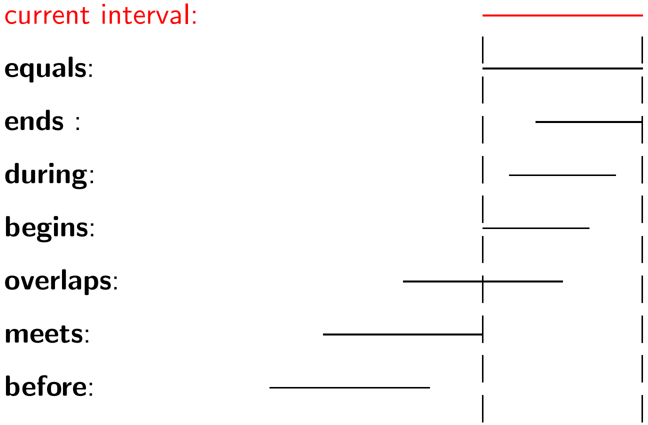

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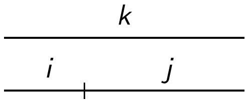

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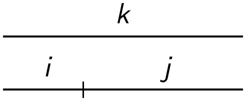

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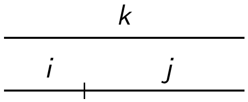

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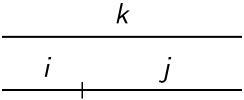

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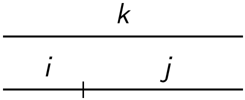

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An interval $\mathcal{R}$-frame is any abstract relational structure of the type $\langle \mathbf{I}, R_1, \ldots, R_k \rangle$, where $\mathbf{I}$ is a non-empty set and $R_1, \ldots, R_k$ are relations on $\mathbf{I}$ corresponding to $\mathrm{R}_1, \ldots, \mathrm{R}_\mathrm{k}$.

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BE-frame: a relational structure $\mathbf{F} = \langle \mathbf{I}, B, E \rangle$ where $\mathbf{I}$ is a non-empty set and B, E are binary relations on $\mathbf{I}$.

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Such results are known as representation theorems.

Representation theorem for interval BE-frames - 1

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- UD: *Unique directedness* of intervals:

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NO: *No overlap* of B and E : $\neg \exists x \exists y (xBy \wedge xEy)$.

Representation theorem for interval BE-frames - 2

Representation theorem for interval BE-frames

A BE-frame is an interval BE-frame iff it is isomorphic to an interval BE-structure.

Y. Venema, *Expressiveness and completeness*, Research Report LP-1988-02, ILLC Publications, University of Amsterdam, 1988

Interval neighborhood structures

Interval neighborhood structures

Interval neighborhood structure: $\langle \mathbb{I}(\mathbf{D}), \mathrm{R}, \mathrm{L} \rangle$, where $\mathbb{I}(\mathbf{D})$ is a linear interval structure and R, L are the binary relations '*right neighbor*' and '*left neighbor*' in $\mathbb{I}(\mathbf{D})$

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- $\mathbf{i}\mathrm{R}\mathbf{j}$ holds if and only the interval $\mathbf{j}$ is a right neighbor of the interval $\mathbf{i}$, i.e. $\mathbf{i} = [d_0, d_1]$ and $\mathbf{j} = [d_1, d_2]$ for some $d_0, d_1, d_2 \in \mathbb{D}$ such that $d_0 \leq d_1 \leq d_2$.

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Thus, interval neighborhood structures correspond to the interval relation '*meet*' and its inverse.

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For every sequence $S_1, ..., S_k \in \{R, L\}$, we denote the composition of the relations $S_1, ..., S_k$ by $S_1...S_k$.

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3. Every connected, open, strict and normal interval neighborhood frame is isomorphic to a strict unbounded interval neighborhood structure.

V. Goranko, A. Montanari, and G. Sciavicco, *On Propositional Interval Neighborhood Temporal Logics*, Journal of Universal Computer Science, 9(9):1137–1167, 2003

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  P. Ladkin, *The Logic of Time Representation*, PhD thesis, University of California, Berkeley, 1987

There are still various unexplored representation problems

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- Respectively, a large variety of relational interval structures and frames.
- Representation theorems characterize up to isomorphism (or isomorphic embedding) the class of concrete relational interval structures of a given type.
- Several representation theorems have been obtained, but many interesting cases are still unexplored.

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Thus, the formulas of HS are:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \psi \mid \langle B \rangle\phi \mid \langle E \rangle\phi \mid \langle \overline{B} \rangle\phi \mid \langle \overline{E} \rangle\phi.$$

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$\mathbb{I}(\mathbb{D})$ will mean either $\mathbb{I}(\mathbb{D})^+$ or $\mathbb{I}(\mathbb{D})^-$, and $\mathbf{M}$ will denote a strict or a non-strict interval model.

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It is definable as either $[B]\bot$ or $[E]\bot$, so it is only needed in weaker fragments of HS.

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What happens in the strict semantics?

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What happens in the strict semantics? The modalities over the neighborhood relations must be added

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Each of these, considered over various classes of interval structures: all, dense, (weakly) discrete, finite, etc., with strict or non-strict semantics.

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D is quite expressive, e.g.: for non-trivial combinatorial relationships between width and depth of an interval, of the type:

$$\bigwedge_{i=1}^{d(n)} \langle D \rangle \left(p_i \wedge \bigwedge_{j \neq i} \langle D \rangle \neg p_j \right) \rightarrow \langle D \rangle^n \top$$

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Also, for special properties of the models, e.g.: the formula

$$\langle D \rangle \langle D \rangle \top \wedge [D](\langle D \rangle \top \rightarrow \langle D \rangle \langle D \rangle \top \wedge \langle D \rangle [D] \bot)$$

for proper subinterval relation has no discrete or dense models in the strict semantics, but is satisfiable in the Cantor space over $\mathbb{R}$.

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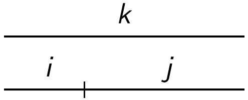

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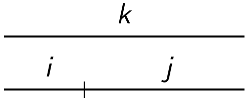

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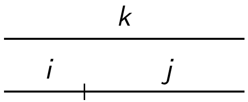

The logic CDT contains binary modalities associated with the relation *chop* and its residuals.

Syntax of CDT:

$$\phi ::= \pi \mid p \mid \neg\phi \mid \phi \wedge \psi \mid \phi C \psi \mid \phi D \psi \mid \phi T \psi.$$

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D can be read as *Done*, *T* as *To do*.

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On the other hand, none of C, D, T is expressible in HS (CDT is strictly more expressive than HS).

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