A Guided Tour through Interval Temporal Logics
Lecture 2: Interval structures, relations, and logics.

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Intervals and interval structures in partial orders
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We will use \(\mathbb{I}(D)\) to denote either of these.
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• *(weakly) discrete* if every point with a successor/predecessor has an immediate successor/predecessor, that is,

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- **unbounded above** (resp., **below**) if every point has a successor (resp., predecessor); **unbounded** if unbounded above and below;

- **Dedekind complete** if every non-empty and bounded above set of points has a least upper bound.
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We will also consider the single interval structures on $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ with their usual orders.
Binary relations between intervals

There are 13 binary relations between two intervals on a linear order: those below and their inverses (the so-called Allen’s relations).
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- equals:
- ends:
- during:
- begins:
- overlaps:
- meets:
- before:
Sub-interval relations
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- \([s_0, s_1]\) is a sub-interval of \([d_0, d_1]\) if \(d_0 \leq s_0\) and \(s_1 \leq d_1\).

This relation of sub-interval will be denoted by \(\sqsubseteq\).
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Given a partial order \( \langle D, \prec \rangle \) and intervals \([s_0, s_1]\) and \([d_0, d_1]\) in it:

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  This relation of sub-interval will be denoted by \(\sqsubseteq\);

- \([s_0, s_1]\) is a \textbf{proper sub-interval} of \([d_0, d_1]\), denoted \([s_0, s_1] \sqsubset [d_0, d_1]\), if \([s_0, s_1] \sqsubseteq [d_0, d_1]\) and \([s_0, s_1] \neq [d_0, d_1]\).
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Given a partial order $\langle D, < \rangle$ and intervals $[s_0, s_1]$ and $[d_0, d_1]$ in it:

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- $[s_0, s_1]$ is a proper sub-interval of $[d_0, d_1]$, denoted $[s_0, s_1] \subset [d_0, d_1]$, if $[s_0, s_1] \sqsubseteq [d_0, d_1]$ and $[s_0, s_1] \neq [d_0, d_1]$.
- $[s_0, s_1]$ is a strict sub-interval of $[d_0, d_1]$ (Allen’s relation during), denoted $[s_0, s_1] \prec [d_0, d_1]$, if $d_0 < s_0$ and $s_1 < d_1$. 
Ternary relations between intervals
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k \\
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i.e., \( C_{ijk} \) if \( i \) meets \( j \), \( i \) begins \( k \), and \( j \) ends \( k \).
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The relation \textit{chop} has 5 associated ‘residual’ relations, e.g.:
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More generally, let $\mathcal{R} = \{R_1, \ldots, R_k\}$ be a family of interval relations, hereafter called an interval relational type.
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An interval $\mathcal{R}$-structure is a relational interval structure of the type $\langle \mathbb{I}(D), R_1, \ldots, R_k \rangle$. 
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An interval \( \mathcal{R} \)-structure is a relational interval structure of the type \( \langle \mathbb{I}(D), R_1, \ldots, R_k \rangle \).

An interval \( \mathcal{R} \)-frame is any abstract relational structure of the type \( \langle I, R_1, \ldots, R_k \rangle \), where \( I \) is a non-empty set and \( R_1, \ldots, R_k \) are relations on \( I \) corresponding to \( R_1, \ldots, R_k \).
Example: Begin-End structures and frames
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Interval BE-structure: $\langle \mathbb{I}(D), B, E \rangle$, where $\mathbb{I}(D)$ is a linear interval structure and $B, E$ are the binary relations ‘begins’ and ‘ends’ in $\mathbb{I}(D)$
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- \( iBj \) holds if and only the interval \( i \) is a proper beginning of the interval \( j \), i.e., \( i = [d_0, d_1] \) and \( j = [d_0, d_2] \) for some \( d_0, d_1, d_2 \in D \) such that \( d_0 \leq d_1 < d_2 \).
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Strict interval BE-structure: $\langle \mathbb{I}^-(D), B, E \rangle$.

BE-frame: a relational structure $F = \langle I, B, E \rangle$ where $I$ is a non-empty set and $B, E$ are binary relations on $I$. 
Abstract (first-order) characterizations and representation theorems for interval frames
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Abstract (first-order) characterizations
and representation theorems for interval frames

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A first-order isomorphism characterization of the type $\mathcal{R}$ is a set of
sentences $\Gamma$ in the first-order language respective to $\mathcal{R}$ such that
any interval $\mathcal{R}$-frame satisfies all sentences in $\Gamma$ iff it is isomorphic
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Likewise, a first-order **embedding characterization** of the type \( \mathcal{R} \) is a set of sentences \( \Gamma \) in the first-order language respective to \( \mathcal{R} \) such that any interval \( \mathcal{R} \)-frame satisfies all sentences in \( \Gamma \) iff it is isomorphically embeddable into an interval \( \mathcal{R} \)-structure.
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An important general problem is to establish abstract (first-order) characterizations of various interval relational types.
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Such results are known as representation theorems.
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**TR:** *Transitivity of* $B$ *and* $E$.

**LL:** *Left linearity of* $B$ *and* $E$:

$$\forall x \forall y \forall z (xBz \land yBz \rightarrow xBy \lor x = y \lor yBx),$$

and likewise for $E$. 
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$$\forall x (\exists z (zBx) \rightarrow \exists y (yBx \land \neg \exists z (zBy))),$$ and likewise for $E$.

**PI:** *Proper intervals*:
$$\forall x (\exists z (zBx) \leftrightarrow \exists z (zEx)).$$
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\forall x (\exists z (zBx) \leftrightarrow \exists z (zEx)), \text{ and likewise for } E.
\]

**PI:** *Proper intervals*:
\[
\forall x \forall y \forall z (xBy \land xEz \rightarrow \exists ! u (zBu \land y Eu)),
\]
\[
\forall x \forall y \forall z (xBy \land zEx \rightarrow \exists ! u (zBu \land uEy)),
\]
\[
\forall x \forall y \forall z (xEy \land zBx \rightarrow \exists ! u (uBy \land zEu)).
\]
Representation theorem for interval BE-frames - 1

Interval BE-frame $\mathbf{F} = \langle I, B, E \rangle$ satisfying the following:

**TR:** Transitivity of $B$ and $E$:

$\forall x \forall y \forall z (x B z \land y B z \rightarrow x B y \lor x = y \lor y B x)$, and likewise for $E$.

**LL:** Left linearity of $B$ and $E$:

$\forall x (\exists z (z B x) \rightarrow \exists y (y B x \land \neg \exists z (z B y)))$, and likewise for $E$.

**AT:** Atomicity for $B$ and $E$:

$\forall x (\exists z (z B x) \leftrightarrow \exists z (z E x))$.

**PI:** Proper intervals:

$\forall x \forall y \forall z (x B y \land x E z \rightarrow \exists ! u (z B u \land y E u))$,

$\forall x \forall y \forall z (x B y \land z E x \rightarrow \exists ! u (z B u \land u E y))$,

$\forall x \forall y \forall z (x E y \land z B x \rightarrow \exists ! u (u B y \land z E u))$.

**UD:** Unique directedness of intervals:

$\forall x \forall y \forall z (x B y \land x E z \rightarrow \exists ! u (z B u \land y E u))$, 

$\forall x \forall y \forall z (x B y \land z E x \rightarrow \exists ! u (z B u \land u E y))$, 

$\forall x \forall y \forall z (x E y \land z B x \rightarrow \exists ! u (u B y \land z E u))$.

**NO:** No overlap of $B$ and $E$:

$\neg \exists x \exists y (x B y \land x E y)$.
Representation theorem for interval BE-frames

A BE-frame is an interval BE-frame iff it is isomorphic to an interval BE-structure.

Interval neighborhood structures
Interval neighborhood structures

Interval neighborhood structure: $\langle \mathbb{I}(D), R, L \rangle$, where $\mathbb{I}(D)$ is a linear interval structure and $R, L$ are the binary relations ‘right neighbor’ and ‘left neighbor’ in $\mathbb{I}(D)$.
Interval neighborhood structures: \( \langle \mathbb{I}(D), R, L \rangle \), where \( \mathbb{I}(D) \) is a linear interval structure and \( R, L \) are the binary relations ‘right neighbor’ and ‘left neighbor’ in \( \mathbb{I}(D) \), i.e.
Interval neighborhood structures

Interval neighborhood structure: \( \langle \mathbb{I}(D), R, L \rangle \), where \( \mathbb{I}(D) \) is a linear interval structure and \( R, L \) are the binary relations ‘right neighbor’ and ‘left neighbor’ in \( \mathbb{I}(D) \), i.e.:

- \( iRj \) holds if and only the interval \( j \) is a right neighbor of the interval \( i \), i.e. \( i = [d_0, d_1] \) and \( j = [d_1, d_2] \) for some \( d_0, d_1, d_2 \in D \) such that \( d_0 \leq d_1 \leq d_2 \).
Interval neighborhood structure: $\langle \mathbb{I}(D), R, L \rangle$, where $\mathbb{I}(D)$ is a linear interval structure and $R, L$ are the binary relations ‘right neighbor’ and ‘left neighbor’ in $\mathbb{I}(D)$, i.e.:

- $i R j$ holds if and only the interval $j$ is a right neighbor of the interval $i$, i.e. $i = [d_0, d_1]$ and $j = [d_1, d_2]$ for some $d_0, d_1, d_2 \in D$ such that $d_0 \leq d_1 \leq d_2$.
- $L$ is the inverse of $R$, i.e., $i L j$ iff $j R i$. 


Interval neighborhood structures

Interval neighborhood structure: \( \langle I(D), R, L \rangle \), where \( I(D) \) is a linear interval structure and \( R, L \) are the binary relations ‘right neighbor’ and ‘left neighbor’ in \( I(D) \), i.e.:

- \( iRj \) holds if and only the interval \( j \) is a right neighbor of the interval \( i \), i.e. \( i = [d_0, d_1] \) and \( j = [d_1, d_2] \) for some \( d_0, d_1, d_2 \in D \) such that \( d_0 \leq d_1 \leq d_2 \).

- \( L \) is the inverse of \( R \), i.e., \( iLj \iff jRi \).

Strict interval neighborhood structure: \( \langle I^-(D), R, L \rangle \).
Interval neighborhood structure: \( \langle \mathbb{I}(D), R, L \rangle \), where \( \mathbb{I}(D) \) is a linear interval structure and \( R, L \) are the binary relations ‘right neighbor’ and ‘left neighbor’ in \( \mathbb{I}(D) \), i.e.:

- \( \textbf{i} R \textbf{j} \) holds if and only the interval \( \textbf{j} \) is a right neighbor of the interval \( \textbf{i} \), i.e. \( \textbf{i} = [d_0, d_1] \) and \( \textbf{j} = [d_1, d_2] \) for some \( d_0, d_1, d_2 \in D \) such that \( d_0 \leq d_1 \leq d_2 \).

- \( L \) is the inverse of \( R \), i.e., \( \textbf{i} L \textbf{j} \) iff \( \textbf{j} R \textbf{i} \).

Strict interval neighborhood structure: \( \langle \mathbb{I}^{-}(D), R, L \rangle \).

Thus, interval neighborhood structures correspond to the interval relation ‘meet’ and its inverse.
Neighborhood frames
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Neighborhood frame (NF):

\( \mathbf{F} = \langle I, R, L \rangle \) where \( I \neq \emptyset \) and \( R, L \subseteq I^2 \).
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\( F = \langle I, R, L \rangle \) where \( I \neq \emptyset \) and \( R, L \subseteq I^2 \).

We denote:

- \( B_F = \{ w \in I \mid \text{there is no } v \in I \text{ such that } wLv \} \),
Neighborhood frames

Neighborhood frame (NF): \( \mathbf{F} = \langle \mathbf{I}, R, L \rangle \) where \( \mathbf{I} \neq \emptyset \) and \( R, L \subseteq \mathbf{I}^2 \).

We denote:

- \( \mathbf{B}_F = \{ w \in \mathbf{I} \mid \text{there is no } v \in \mathbf{I} \text{ such that } wLv \} \),
- \( \mathbf{B}_F^2 = \{ w \in \mathbf{I} \mid \text{there are no } u, v \in \mathbf{I}, \text{ with } u \neq v, \text{ such that } wLv \text{ and } wLu \} \),
Neighborhood frames

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- \( E_F \) and \( E^2_F \) are defined likewise, by swapping \( L \) with \( R \).
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For every sequence \( S_1, ..., S_k \in \{ R, L \} \), we denote the composition of the relations \( S_1, ..., S_k \) by \( S_1...S_k \).
Interval neighborhood frames
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Interval neighborhood frame: neighborhood frame \( F = \langle I, R, L \rangle \) satisfying:
Interval neighborhood frames

Interval neighborhood frame: neighborhood frame $F = \langle I, R, L \rangle$ satisfying:

$(NF1)$ $R$ and $L$ are mutually inverse.
Interval neighborhood frames

Interval neighborhood frame: neighborhood frame $NF$ \( F = \langle I, R, L \rangle \) satisfying:

**NF1** \( R \) and \( L \) are mutually inverse.

**NF2** \( \forall x \forall y (\exists z (xLz \land zRy) \rightarrow \forall z (xLz \rightarrow zRy)) \), and \( \forall x \forall y (\exists z (xRz \land zLy) \rightarrow \forall z (xRz \rightarrow zLy)) \).
Interval neighborhood frames

**Interval neighborhood frame**: neighborhood frame $\mathbf{F} = \langle I, R, L \rangle$ satisfying:

- **(NF1)** $R$ and $L$ are mutually inverse.

- **(NF2)** $\forall x \forall y (\exists z (xLz \land zRy) \rightarrow \forall z (xLz \rightarrow zRy))$, and $\forall x \forall y (\exists z (xRz \land zLy) \rightarrow \forall z (xRz \rightarrow zLy))$.

- **(NF3')** $RL \subseteq LRR \cup LLR \cup E$ on $I - B^2_F$, where $E$ is the equality, i.e.,
Interval neighborhood frames

Interval neighborhood frame: neighborhood frame $NF_F$ satisfying:

(NF1) $R$ and $L$ are mutually inverse.

(NF2) $\forall x \forall y (\exists z(xLz \land zRy) \rightarrow \forall z(xLz \rightarrow zRy))$, and $\forall x \forall y (\exists z(xRz \land zLy) \rightarrow \forall z(xRz \rightarrow zLy))$.

(NF3') $RL \subseteq LRR \cup LLR \cup E$ on $I - B_F^2$, where $E$ is the equality, i.e., $\forall x \forall y (\exists z \exists u(xLz \land zLu) \land \exists z(xRz \land zLy) \rightarrow x = y \lor \exists w \exists z((xLw \land wRz \land zRy) \lor (xLw \land wLz \land zRy)))$. 
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(NF3') $RL \subseteq LRR \cup LLR \cup E$ on $I - F^2$, where $E$ is the equality, i.e.,

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(NF3'') Likewise, $LR \subseteq RLL \cup RRL \cup E$ on $I - E^2_F$. 

Interval neighborhood frames

**Interval neighborhood frame**: neighborhood frame $NF F = \langle I, R, L \rangle$ satisfying:

1. **(NF1)** $R$ and $L$ are mutually inverse.

2. **(NF2)** $\forall x \forall y (\exists z (x L z \land z R y) \rightarrow \forall z (x L z \rightarrow z R y))$, and $\forall x \forall y (\exists z (x R z \land z L y) \rightarrow \forall z (x R z \rightarrow z L y))$.

3. **(NF3')** $R L \subseteq L R R \cup L L R \cup E$ on $I - B^2_F$, where $E$ is the equality, i.e., $\forall x \forall y (\exists z \exists u (x L z \land z L u) \land \exists z (x R z \land z L y) \rightarrow x = y \lor \exists w \exists z ((x L w \land w R z \land z L y) \lor (x L w \land w L z \land z R y)))$.

4. **(NF3'')** Likewise, $L R \subseteq R L L \cup R R L \cup E$ on $I - E^2_F$.

5. **(NF4)** $R R R \subseteq R R$, i.e., $\forall w \forall x \forall y \forall z (w R x \land x R y \land y R z \rightarrow \exists u (w R u \land u R z))$. 
Some properties of interval neighborhood frames
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An interval neighborhood frame $\mathbf{F} = \langle I, R, L \rangle$ is said to be:
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An interval neighborhood frame $F = \langle I, R, L \rangle$ is said to be:

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- **rich**, if $F \models \forall x (\exists y (xRy \land yRy) \land \exists y (xLy \land yLy))$;
- **normal**, if $F \models \forall x \forall y (\forall z (zRx \leftrightarrow zRy) \land \forall z (zLx \leftrightarrow zLy) \rightarrow x = y)$;
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- **tight**, if $\mathbf{F} \models \forall x \forall y ((xRRy \land yRRx) \rightarrow x = y)$;
- **weakly left-connected** (resp., **weakly right-connected**) if the relation $LR \cup LRR \cup LLR$ (resp., $RL \cup RRL \cup RLL$) is an equivalence relation on $I - B_F$ (resp., $I - E_F$);
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An interval neighborhood frame $F = \langle I, R, L \rangle$ is said to be:

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An interval neighborhood frame $F = \langle I, R, L \rangle$ is said to be:

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- **left-connected** (resp., **right-connected**) if that relation is the universal relation on $I - B_F$ (resp., $I - E_F$);
- **weakly connected** if each of the relations $LR \cup LRR \cup LLR$ and $RL \cup RRL \cup RLL$ is an equivalence relation on $I$; **connected**, if each of these relations is the universal relation on $I$. 
Representation theorems for interval neighborhood frames
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3. Every connected, open, strict and normal interval neighborhood frame is isomorphic to a strict unbounded interval neighborhood structure.

Other representation theorems for classes of interval frames
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- for point-based structures with a quaternary relation encoding meeting of two intervals
  

There are still various unexplored representation problems
Summary

• Every partial order has an associated interval structure.
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• Representation theorems characterize up to isomorphism (or isomorphic embedding) the class of concrete relational interval structures of a given type.
Summary

• Every partial order has an associated interval structure.

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• Respectively, a large variety of relational interval structures and frames.

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• Several representation theorems have been obtained, but many interesting cases are still unexplored.
Halpern-Shoham’s modal logic of interval relations
Allen’s interval relations give rise to respective unary modal operators over relational interval structures, thus defining the multimodal logic HS introduced by Halpern and Shoham in 1991.
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In the case of non-strict semantics, it suffices to choose as primitive the modalities $\langle B \rangle, \langle E \rangle, \langle \overline{B} \rangle, \langle \overline{E} \rangle$ corresponding to the relations begins, ends, and their inverses; the other modalities then become definable.
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Thus, the formulas of HS are:

$$\phi ::= p \mid \neg \phi \mid \phi \land \psi \mid \langle B \rangle \phi \mid \langle E \rangle \phi \mid \langle \overline{B} \rangle \phi \mid \langle \overline{E} \rangle \phi.$$
Models for propositional interval logics

\( \mathcal{AP} \): a set of atomic propositions (over intervals).
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Non-strict interval model:

\[ M^+ = \langle \mathbb{I}(\mathbb{D})^+, V \rangle, \]

where $V : \mathcal{AP} \mapsto 2^{\mathbb{I}(\mathbb{D})^+}$. 
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Thus, \( V(p) \) can be viewed as a binary relation on \( D \).
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Thus, $V(p)$ can be viewed as a binary relation on $D$.

$I(\mathbb{D})$ will mean either $\mathbb{I}(\mathbb{D})^+$ or $\mathbb{I}(\mathbb{D})^-$, and $M$ will denote a strict or a non-strict interval model.
Semantics of HS

The formal semantics of these modal operators:
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$\langle B \rangle$: $M, [d_0, d_1] \models \langle B \rangle \phi$ if there exists $d_2$ such that $d_0 \leq d_2 < d_1$ and $M, [d_0, d_2] \not\models \phi$. 
Semantics of HS

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$$\langle E \rangle: \models M, [d_0, d_1] \models \langle E \rangle \phi \text{ if there exists } d_2 \text{ such that } d_0 < d_2 \leq d_1 \text{ and } M, [d_2, d_1] \models \phi.$$
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$\langle \overline{B} \rangle$: $M, [d_0, d_1] \models \langle \overline{B} \rangle \phi$ if there exists $d_2$ such that $d_1 < d_2$ and $M, [d_0, d_2] \not\models \phi$. 
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Thus, every HS-formula is interpreted in an interval model by a set
of ordered pairs of points, i.e., a binary relation.
Semantics of HS

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Thus, every HS-formula is interpreted in an interval model by a set of ordered pairs of points, i.e., a binary relation.

A useful new symbol is the modal constant \( \pi \) for point-intervals interpreted as follows:

\[ M, [d_0, d_1] \models \pi \text{ if } d_0 = d_1. \]
Semantics of HS

The formal semantics of these modal operators:

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Thus, every HS-formula is interpreted in an interval model by a set of ordered pairs of points, i.e., a binary relation.

A useful new symbol is the modal constant \(\pi\) for point-intervals interpreted as follows:

\[\mathcal{M}, [d_0, d_1] \not\models \pi \text{ if } d_0 = d_1.\]

It is definable as either \([B] \perp\) or \([E] \perp\), so it is only needed in weaker fragments of HS.
Defining the other interval modalities in HS
Defining the other interval modalities in HS

In the non-strict semantics:
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- **Right neighbor:** $\langle R \rangle \varphi := \langle E \rangle (\pi \land \langle \overline{B} \rangle \varphi)$. Also denoted $\Diamond_r$. 
Defining the other interval modalities in HS

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Defining the other interval modalities in HS

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- **After**: $\langle L \rangle \varphi := \langle R \rangle (\neg \pi \land \langle R \rangle \varphi)$.
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- **Before**: \( \langle \overline{L} \rangle \varphi := \langle L \rangle (\overline{\pi} \land \langle L \rangle \varphi) \).
- **Overlaps on the right**: \( \langle O \rangle \varphi := \langle E \rangle \langle B \rangle \varphi \);
Defining the other interval modalities in HS

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Defining the other interval modalities in HS

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What happens in the strict semantics?
Defining the other interval modalities in HS

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What happens in the strict semantics? The modalities over the neighborhood relations must be added
Some important fragments of HS
Some important fragments of HS

• Sub-interval logics
Some important fragments of HS

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- Neighborhood logics
Some important fragments of HS

• Sub-interval logics
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Each of these, considered over various classes of interval structures: all, dense, (weakly) discrete, finite, etc., with strict or non-strict semantics.
Fragments of HS: logics of sub-intervals

The generic logic of sub-intervals $\mathbf{D}$:  $\phi ::= p \mid \neg \phi \mid \phi \land \psi \mid \langle D \rangle \phi$. 
Fragments of HS: logics of sub-intervals

The generic logic of sub-intervals \( \textbf{D} \): \( \phi ::= p \mid \neg \phi \mid \phi \land \psi \mid \langle \textbf{D} \rangle \phi \).

Semantics: \( \mathcal{M}, [d_0, d_1] \models \langle \textbf{D} \rangle \phi \) iff there exists a sub-interval \([d_2, d_3]\) of \([d_0, d_1]\) such that \( \mathcal{M}, [d_2, d_3] \models \phi \).
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The generic logic of sub-intervals $D$: $\phi ::= p \mid \neg\phi \mid \phi \land \psi \mid \langle D \rangle \phi$.

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Variations: reflexive, proper, or strict subinterval relation.
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Variations: reflexive, proper, or strict subinterval relation.

$\mathbf{D}$ is quite expressive, e.g.: for non-trivial combinatorial relationships between width and depth of an interval, of the type:

$$\bigwedge_{i=1}^{d(n)} \langle D \rangle \left( p_i \land \bigwedge_{j \neq i} \langle D \rangle \neg p_j \right) \rightarrow \langle D \rangle^n \top$$

for a large enough $d(n)$. 
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Also, for special properties of the models, e.g.: the formula

$$\langle D \rangle \langle D \rangle \top \land [D](\langle D \rangle \top \rightarrow \langle D \rangle \langle D \rangle \top \land \langle D \rangle[D] \bot)$$

for proper subinterval relation has no discrete or dense models in the strict semantics, but is satisfiable in the Cantor space over $\mathbb{R}$. 
Fragments of HS: propositional neighborhood logics
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Neighborhood logics: interval logics based on the relation *meets* (right neighbor) and its inverse *met-by* (left neighbor).
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These modalities are denoted in HS respectively by $\langle A \rangle$ and $\langle \overline{A} \rangle$ and are based on the strict semantics.
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The language of propositional neighborhood logics for non-strict semantics $\text{PNL}^+$:

$$\phi ::= p \mid \neg \phi \mid \phi \land \psi \mid \Diamond_r \phi \mid \Diamond_l \phi.$$
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$$(\Diamond_r) \; M^+, [d_0, d_1] \models \Diamond_r \phi \text{ if there exists } d_2 \text{ such that } d_1 \leq d_2 \text{ and } M^+, [d_1, d_2] \models \phi;$$
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$$(\Diamond_l) \quad \mathbf{M}^+, [d_0, d_1] \models \Diamond_l \phi \text{ iff there exists } d_2 \text{ such that } d_2 \leq d_0 \text{ and } \mathbf{M}^+, [d_2, d_0] \models \phi,$$
Venema’s logic CDT
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Recall the ternary relation *chop*:

```
    k
   __________
  i       j
```
Venema’s logic CDT

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\begin{array}{c}
i \\
\hline
k \\
\hline
j \\
\end{array}
\]

The logic CDT contains binary modalities associated with the relation \textit{chop} and its residuals.
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The logic CDT contains binary modalities associated with the relation \textit{chop} and its residuals.

Syntax of CDT:

\[
\phi ::= \pi \mid p \mid \neg \phi \mid \phi \land \psi \mid \phi \mathsf{C} \psi \mid \phi \mathsf{D} \psi \mid \phi \mathsf{T} \psi.
\]
Semantics of CDT
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Semantics over partial orderings with the linear intervals property:
Semantics of CDT

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\[ C: \ M, [d_0, d_1] \models \phi \quad C \psi \text{ iff there exists } d_2 \in \mathbb{D} \text{ such that:} \]
\[ d_0 \leq d_2 \leq d_1, \ M, [d_0, d_2] \models \phi, \text{ and } M, [d_2, d_1] \models \psi. \]
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\[ D: \text{ } M, [d_0, d_1] \models \phi D \psi \text{ iff there exists } d_2 \in \mathbb{D} \text{ such that: } d_2 \leq d_0, \text{ } M, [d_2, d_0] \models \phi, \text{ and } M, [d_2, d_1] \models \psi. \]
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\[ \text{D: } \mathcal{M}, [d_0, d_1] \models \phi \land \psi \text{ iff there exists } d_2 \in \mathfrak{D} \text{ such that: } d_2 \leq d_0, \mathcal{M}, [d_2, d_0] \models \phi, \text{ and } \mathcal{M}, [d_2, d_1] \models \psi. \]

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\( \mathfrak{D} \) can be read as **Done**, \( \mathfrak{T} \) as **To do**.
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\[ \langle B \rangle \phi ::= \phi C(\neg \pi), \]
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In the strict semantics: replace \( \neg \pi \) by \( \top \).

Thus, CDT is at least as expressive as HS.
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On the other hand, none of $C$, $D$, $T$ is expressible in HS (CDT is strictly more expressive than HS).
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