A Guided Tour through Interval Temporal Logics
Lecture 1: a General Overview

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Outline of Lecture 1

- an introduction to interval temporal logics
- the logic of Allen's relations: Halpern and Shoham's modal logic of time intervals (HS)
- decidable fragments of HS
- undecidable fragments of HS
- outline of the (rest of the) course

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D. Della Monica, V. Goranko, A. Montanari, and G. Sciavicco, *Interval Temporal Logics: a Journey*, Bulletin of the European Association for Theoretical Computer Science, 105:73–99, 2011

Origins and application areas

- Philosophy and ontology of time, e.g., the choice between time instants and time intervals as the primary objects of a temporal ontology
- Linguistics: analysis of progressive tenses, semantics and processing of natural languages (quoting Kamp and Reyle, "truth, as it pertains to language in the way we use it, relates sentences not to instants but to temporal intervals")
- Artificial intelligence: temporal knowledge representation, systems for time planning and maintenance, theory of events (e.g., actions with duration)
- Computer science: temporal databases (e.g., temporal aggregations), specification and design of hardware components (e.g., Moszkowski's ITL), concurrent real-time processes (e.g., Hoare, Ravn, and Zhou's Duration Calculus), bioinformatics

Interval temporal logics and temporal ontologies

Interval temporal reasoning is subject to the same ontological dilemmas as the point-based temporal reasoning, viz., should the time structure be assumed:

- *linear or branching?*
- *discrete or dense?*
- *with or without beginning/end?*

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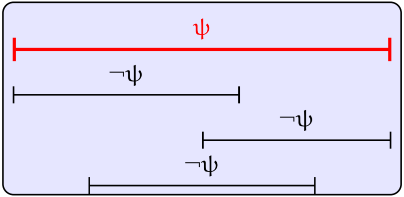
New dilemmas arise regarding the nature of the intervals:

- *How are points and intervals related? Which is the primary concept? Should an interval be identified with the set of points in it, or there is more into it?*
- *Can intervals be unbounded?*
- *Are intervals with coinciding endpoints admissible or not?*

The distinctive features of interval temporal logics

Truth of formulae is defined over intervals (not points).



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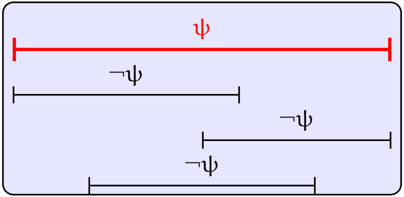

Interval temporal logics are very expressive (compared to point-based temporal logics).

In particular, formulas of interval logics express properties of pairs of time points rather than of single time points, and are evaluated as sets of such pairs, i.e., as binary relations.

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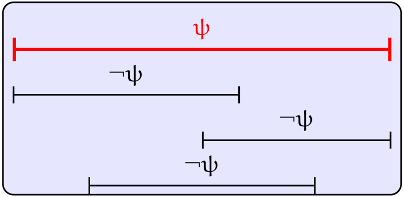

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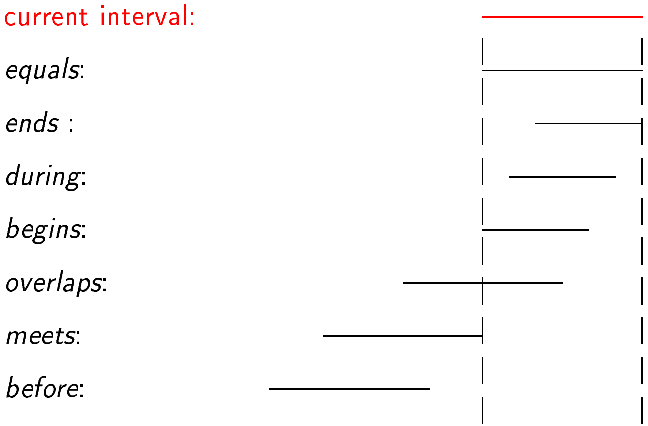
Thus, in general there is no reduction of the satisfiability/validity in interval logics to monadic second-order logic, and therefore Rabin's theorem is not applicable here.

Binary ordering relations over intervals

The thirteen binary ordering relations between two intervals on a linear order (those below and their inverses) form the set of *Allen's interval relations*:



HS: the modal logic of Allen's interval relations

Allen's interval relations give rise to corresponding unary modalities over frames where intervals are primitive entities:
Halpern and Shoham's modal logic of time intervals HS, interpreted over interval structures (not to be confused with Allen's Interval Algebra)

J.Y. Halpern and Y. Shoham, *A Propositional Modal Logic of Time Intervals*, Journal of the ACM, 38:279–292, 1991

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The satisfiability/validity problem for HS is highly undecidable over all standard classes of linear orders. What about its fragments?

HS fragments

More than four thousands fragments of HS (over the class of all linear orders) can be identified by choosing a different subset of the set of basic modal operators. However, only 1347 genuinely different ones exist

D. Della Monica, V. Goranko, A. Montanari, and G. Sciavicco, *Expressiveness of the Interval Logics of Allen's Relations on the Class of all Linear Orders: Complete Classification*, IJCAI 2011

To show non-definability of a given modality in a specific fragment, one can use a standard technique in modal logic, based on the notion of *bisimulation* and the invariance of modal formulae with respect to bisimulations

(Un)decidability of HS fragments: main parameters

Research agenda:

- search for maximal decidable HS fragments;
- search for minimal undecidable HS fragments.

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(Un)decidability of HS fragments depends on two factors:

- the set of interval modalities;
- the class of interval structures (linear orders) over which the logic is interpreted.

A real character: the logic D

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I. Shapirovsky, *On PSPACE-decidability in Transitive Modal Logic*, Advances in Modal Logic 2005

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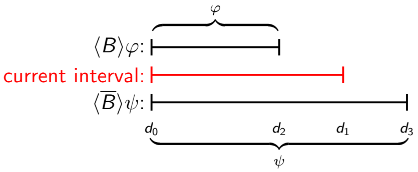
Aside: it is unknown, when D is interpreted over the class of all linear orders.

An easy case: the logic $\mathrm{B}\overline{\mathrm{B}}$

Consider the fragment $\mathrm{B}\overline{\mathrm{B}}$.



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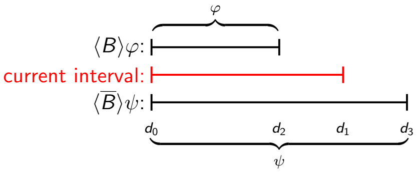

The decidability of $\mathrm{B}\overline{\mathrm{B}}$ can be shown by embedding it into the propositional temporal logic of linear time LTL$[\mathrm{F}, \mathrm{P}]$: formulas of $\mathrm{B}\overline{\mathrm{B}}$ can be translated into formulas of LTL$[\mathrm{F}, \mathrm{P}]$ by replacing $\langle \mathrm{B}\rangle$ with P (sometimes in the past) and $\langle \overline{\mathrm{B}}\rangle$ with F (sometimes in the future):

LTL$[\mathrm{F}, \mathrm{P}]$ has the small (pseudo)model property and is decidable

The case of $\mathrm{E}\overline{\mathrm{E}}$ is similar.

A well-behaved fragment: the logic $A\overline{A}$

Formulas of the logic $A\overline{A}$ of Allen's relations *meets* and *met by* are recursively defined by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle A \rangle \varphi \mid \langle \overline{A} \rangle \varphi \ ([A] = \neg\langle A \rangle\neg; \text{ same for } [\overline{A}])$$

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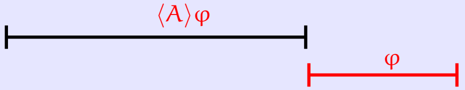

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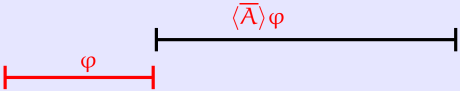

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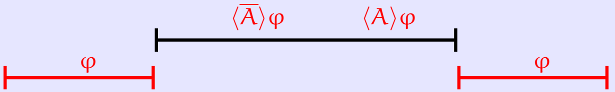
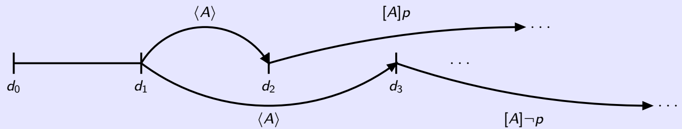


We cannot abstract way from any of the endpoints of intervals:

- contradictory formulas may hold over intervals with the same right endpoint and a different left endpoint

$\langle A\rangle[A]p \wedge \langle A\rangle[A]\neg p$ is satisfiable:



For any $d > d_3$, p holds over $[d_2, d]$ and $\neg p$ holds over $[d_3, d]$.

The importance of the past (in $A\overline{A}$)

Unlike what happens with point-based linear temporal logic, $A\overline{A}$ is strictly more expressive than its future fragment A (proof technique: invariance of modal formulas with respect to bisimulation)

There is a log-space reduction from the satisfiability problem for $A\overline{A}$ over $\mathbb{Z}$ to its satisfiability problem over $\mathbb{N}$, that turns out to be much more involved than the corresponding reduction for point-based linear temporal logic

$A\overline{A}$ is able to separate $\mathbb{Q}$ and $\mathbb{R}$, while A is not

D. Della Monica, A. Montanari, and P. Sala, *The importance of the past in interval temporal logics: the case of Propositional Neighborhood Logic*, in A. Artikis et al. (Eds.), Logic Programs, Norms and Action (Sergot Festschrift), LNAI 7360, Springer, 2012, pp. 79–102.

Expressive completeness of $A\overline{A}$ with respect to $\mathrm{FO}^2[<]$

Expressive completeness of $A\overline{A}$ with respect to the two-variable fragment of first-order logic for binary relational structures over various linearly-ordered domains $\mathrm{FO}^2[<]$

M. Otto, *Two Variable First-order Logic Over Ordered Domains*, Journal of Symbolic Logic, 66(2):685–702, 2001

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Remark. The two-variable property is a sufficient condition for decidability, but it is not a necessary one (for instance, D is decidable over dense linear orders, but it does not satisfy the two-variable property - three variables are needed)

Decidability of $A\overline{A}$

As a by-product, decidability (in fact, NEXPTIME-completeness) of $A\overline{A}$ over all linear orders, well-orders, finite linear orders, and the linear order on the natural numbers

D. Bresolin, V. Goranko, A. Montanari, and G. Sciavicco, *Propositional Interval Neighborhood Logics: Expressiveness, Decidability, and Undecidable Extensions*, Annals of Pure and Applied Logic, 161(3):289–304, 2009

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- It is far from being trivial to extract a decision procedure from Otto's proof
- Some meaningful cases are missing (dense linear orders, weakly discrete linear orders)

Tableau-based decision procedures for $A\overline{A}$ - 1

An optimal tableau-based decision procedure for the future fragment of $A\overline{A}$ (the future modality $\langle A \rangle$ only) over the natural numbers

D. Bresolin, A. Montanari, and G. Sciavicco, *An Optimal Decision Procedure for Right Propositional Neighborhood Logic*, Journal of Automated Reasoning, 38(1-3):173–199, 2007

Later extended to full $A\overline{A}$ over the integers (it can be tailored to natural numbers and finite linear orders)

D. Bresolin, A. Montanari, and P. Sala, *An Optimal Tableau-based Decision Algorithm for Propositional Neighborhood Logic*, STACS 2007

Tableau-based decision procedures for $A\overline{A}$ - 2

Recently, optimal tableau-based decision procedures for $A\overline{A}$ over all, dense, and weakly-discrete linear orders have been developed

D. Bresolin, A. Montanari, P. Sala, and G. Sciavicco, *Optimal Tableau Systems for Propositional Neighborhood Logic over All, Dense, and Discrete Linear Orders*, TABLEAUX 2011

The very last achievement in this direction is an optimal tableau-based decision procedure for $A\overline{A}$ over the reals

A. Montanari and P. Sala, *An Optimal Tableau System for the Logic of Temporal Neighborhood Over the Reals*, TIME 2012

Maximal decidable fragments

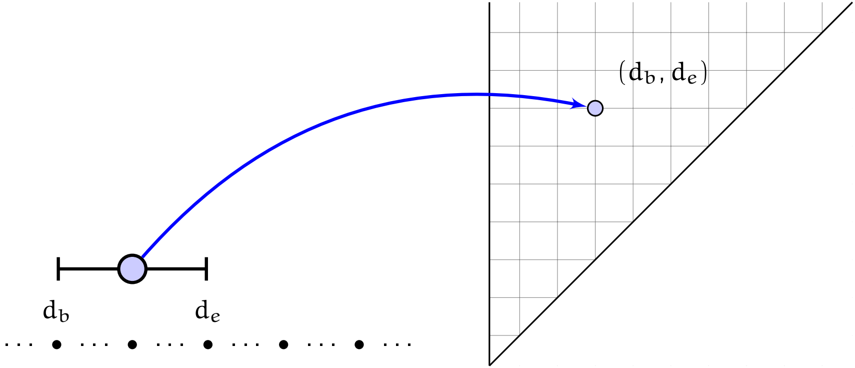
Issue: can we add other modalities from the HS repository to the logic of temporal neighborhood $A\overline{A}$ or to the logic of the subinterval relation D preserving decidability?

The search for maximal decidable fragments of HS benefitted from a natural geometrical interpretation of interval logics proposed by Venema.

Y. Venema, *Expressiveness and Completeness of an Interval Tense Logic*, Notre Dame Journal of Formal Logic, 31(4):529–547, 1990

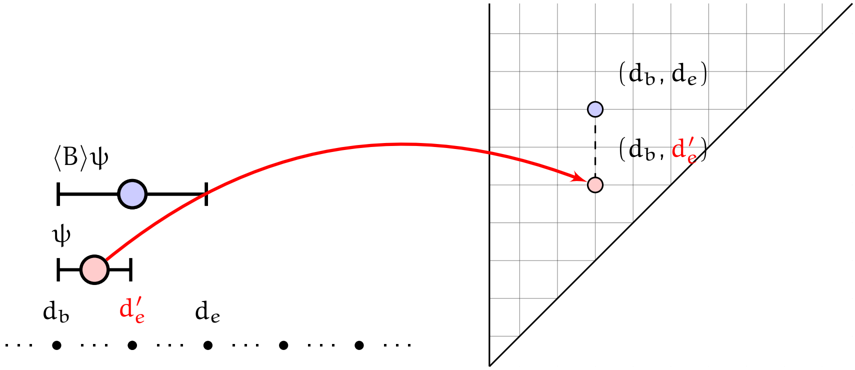
In the following, we illustrate the basic ingredients of such a geometrical interpretation, and we summarize the main results.

A geometrical account of interval logic: intervals



Every interval can be represented by a point in the second octant (in general, in the half plane $y \geqslant x$).

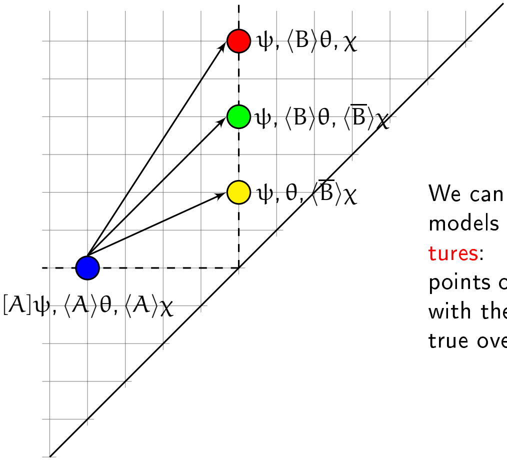
A geometrical account of interval logic: interval relations



$$d_b < d'_e < d_e$$

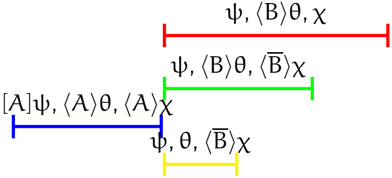
Every interval relation has a spatial counterpart.

A geometrical account of interval logic: models

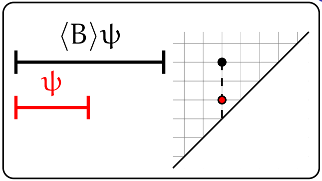


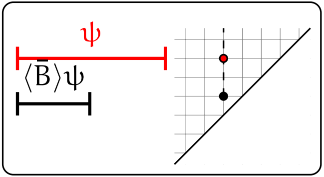
We can give a spatial interpretation to models of a formula φ as compass structures:

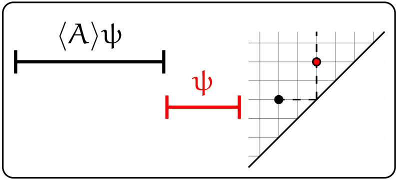
points of a compass structure are colored with the set of subformulas of φ that are true over the corresponding intervals

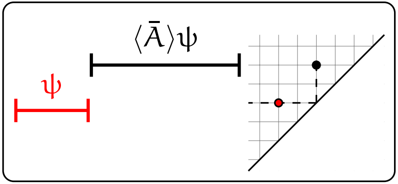


The maximal decidable fragment $AB\overline{BA}$





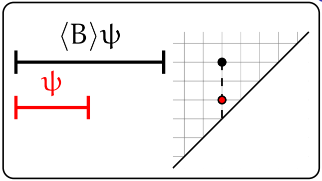


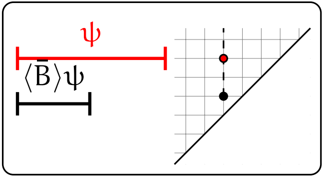


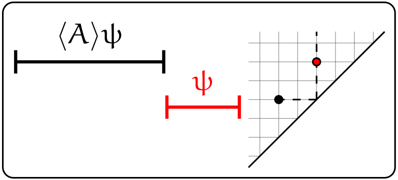
$AB\overline{BA}$ is NONPRIMITIVE RECURSIVE-hard over finite linear orders; undecidable elsewhere

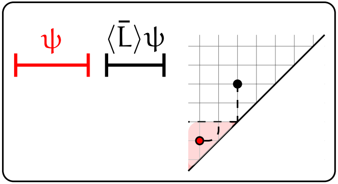
A. Montanari, G. Puppis, and P. Sala, *Maximal decidable fragments of Halpern and Shoham's modal logic of intervals*, ICALP 2010

The maximal decidable fragment $\mathsf{AB\overline{BL}}$









We replace $\langle\overline{A}\rangle$ by $\langle\overline{L}\rangle$: $\mathsf{AB\overline{BL}}$ is EXSPACE-complete over the classes of all, dense, and (weakly) discrete linear orders

D. Bresolin, A. Montanari, P. Sala, and G. Sciavicco, *What's decidable about Halpern and Shoham's interval logic? The maximal fragment* $\mathsf{AB\overline{BL}}$, LICS 2011

Paths to undecidability - 1

Undecidability results for HS fragments have been obtained by means of reductions from several undecidable problems:

- reduction from the non-halting problem for Turing machines (e.g., HS over all meaningful classes of linear orders, and BE over dense linear orders – that of BE over all linear orders immediately follows);

J.Y. Halpern and Y. Shoham, *A Propositional Modal Logic of Time Intervals*, Journal of the ACM, 38:279–292, 1991

K. Lodaya, *Sharpening the Undecidability of Interval Temporal Logic*, ASIAN 2000

Paths to undecidability - 2

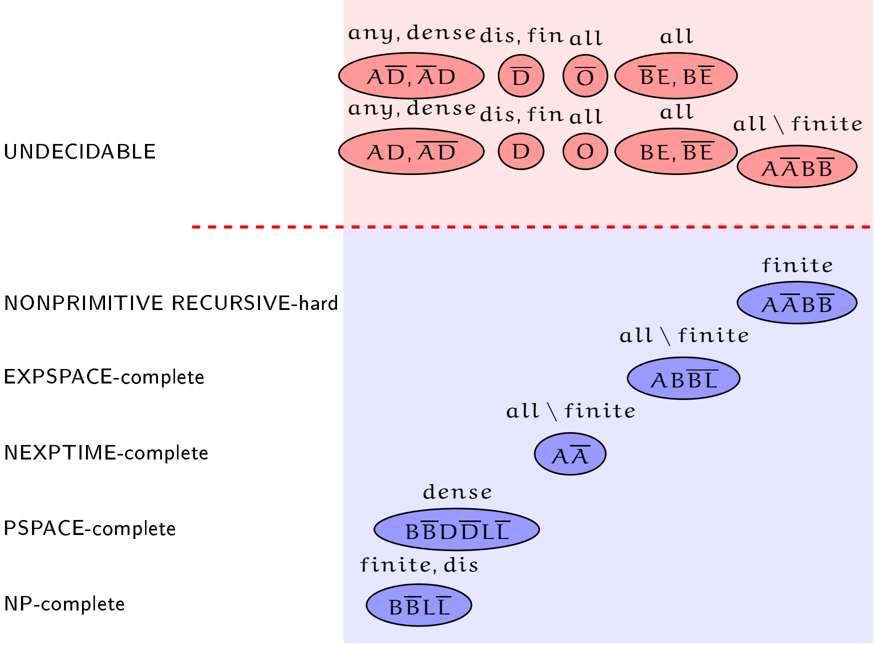
- reductions from several variants of the tiling problem, like the octant tiling problem and the finite tiling problem (O, $\overline{\mathrm{O}}$, AD, $\overline{\mathrm{A}}\mathrm{D}$, $\mathrm{A}\overline{\mathrm{D}}$, $\overline{\mathrm{AD}}$, BE, $\overline{\mathrm{B}}\mathrm{E}$, $\mathrm{B}\overline{\mathrm{E}}$, and $\overline{\mathrm{BE}}$ over any class of linear orders that contains, for each $n > 0$, at least one linear order with length greater than n)

D. Bresolin, D. Della Monica, V. Goranko, A. Montanari, and G. Sciavicco, *The dark side of Interval Temporal Logics: sharpening the undecidability border*, TIME 2011

- reduction from the halting problem for two-counter automata (e.g., D over finite and discrete linear orders).

J. Marcinkowski and J. Michaliszyn, *The Ultimate Undecidability Result for the Halpern-Shoham Logic*, LICS 2011

The (almost) complete picture



Outline of the (rest of the) course

- Lecture 2. Interval structures, relations, and logics. Interval structures and relations. Representation theorems. Interval temporal logics. The logic HS.

- Lecture 3. Interval temporal logics: languages and expressiveness. Meaningful fragments of HS. Standard translation to first-order logic (FOL). Expressiveness of interval logics. Comparing expressiveness of interval temporal logics and FOL. Expressive completeness results.

- Lecture 4. Interval temporal logics: undecidability. Undecidability of HS and of some meaningful fragments of it. Undecidability via tiling.

- Lectures 5 and 6. Interval temporal logics: decidability. Model-theoretic decidability proofs. Tableau methods for (interval) temporal logics.

- Mid- and long-term research agenda.