Learning automata

Gabriele Puppis

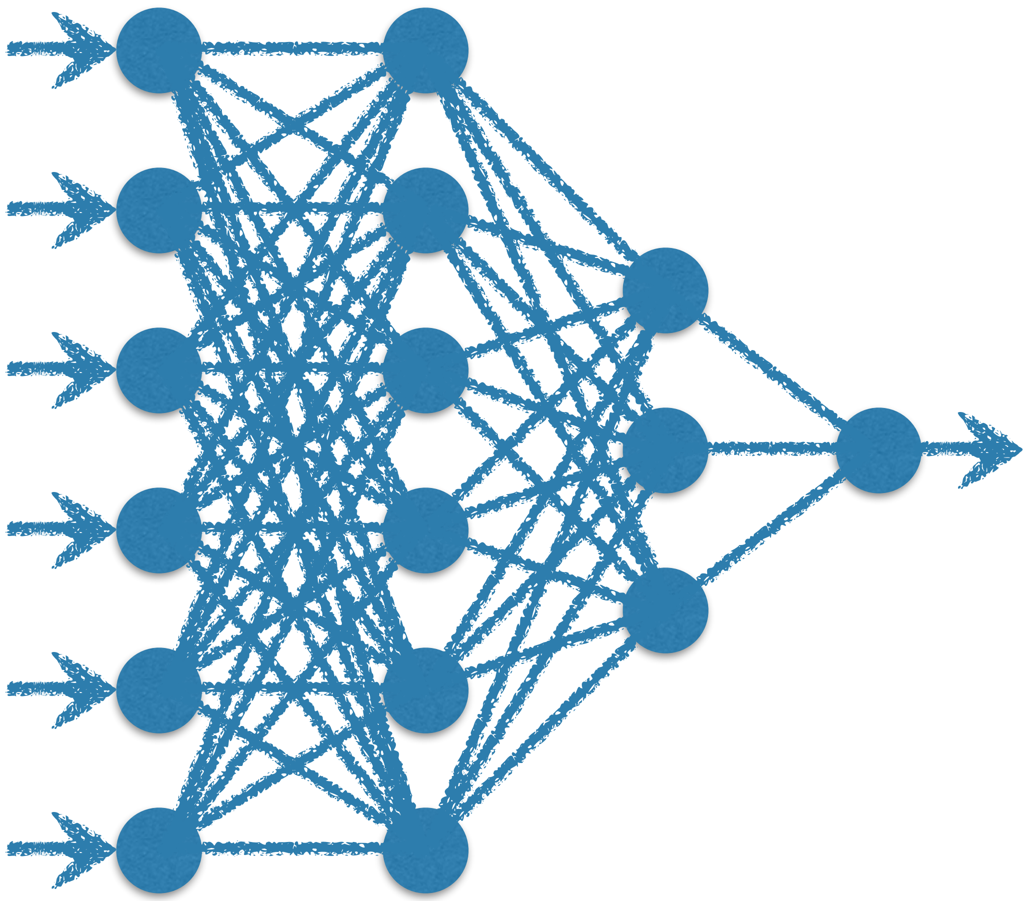
Learning automata ...and generalisations!

Gabriele Puppis

Learning scenarios

Learning scenarios

NNs



Learning scenarios

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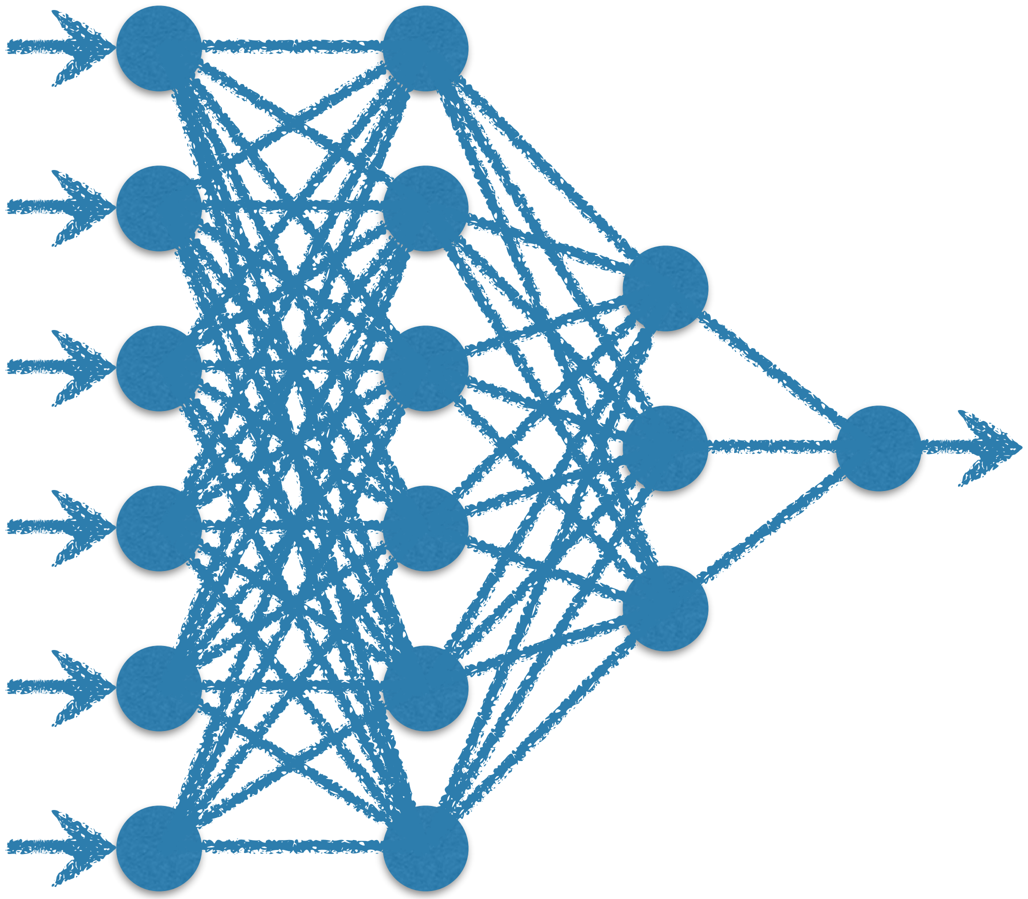
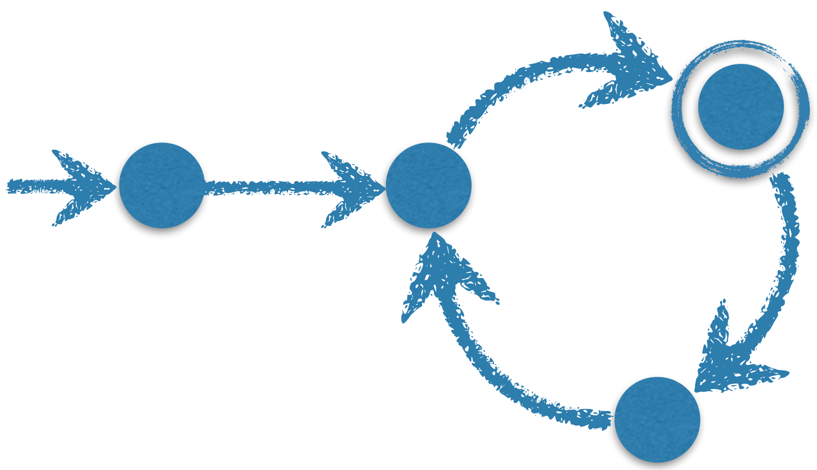


Automata



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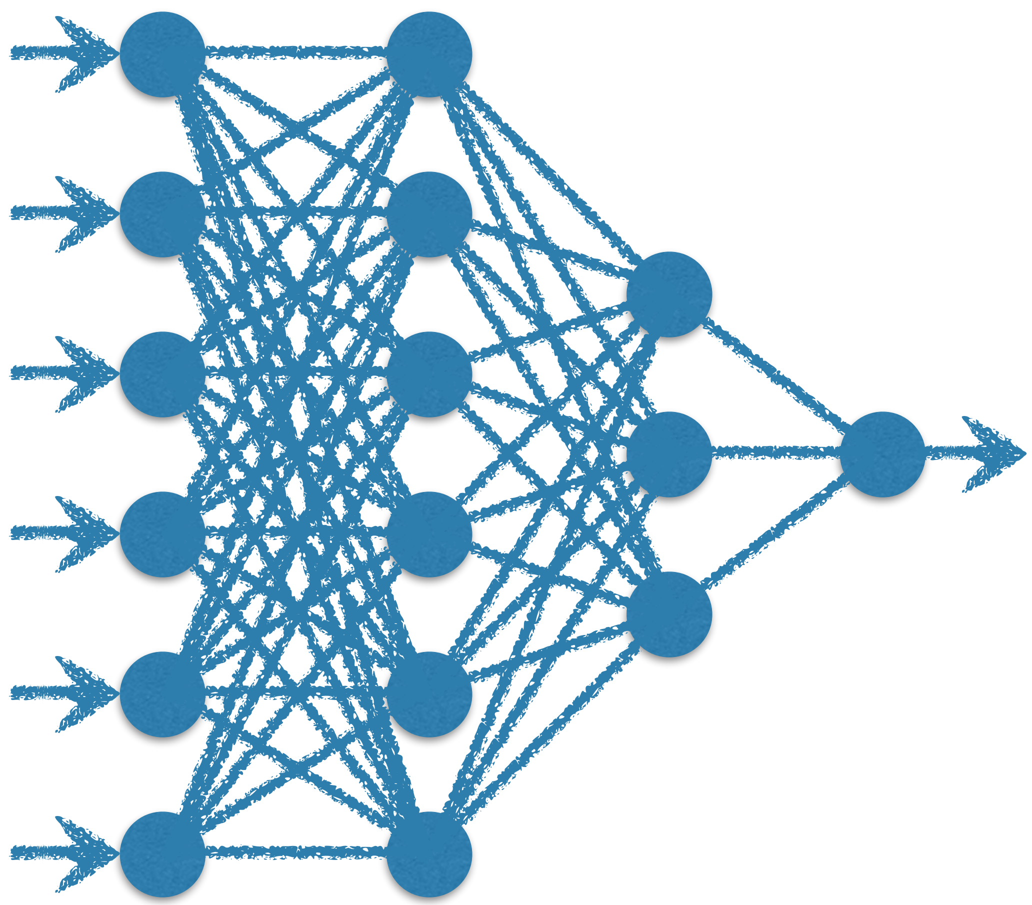

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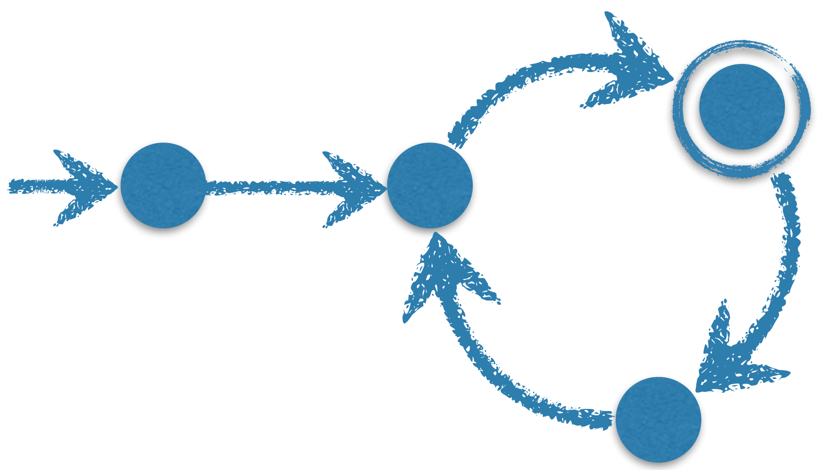
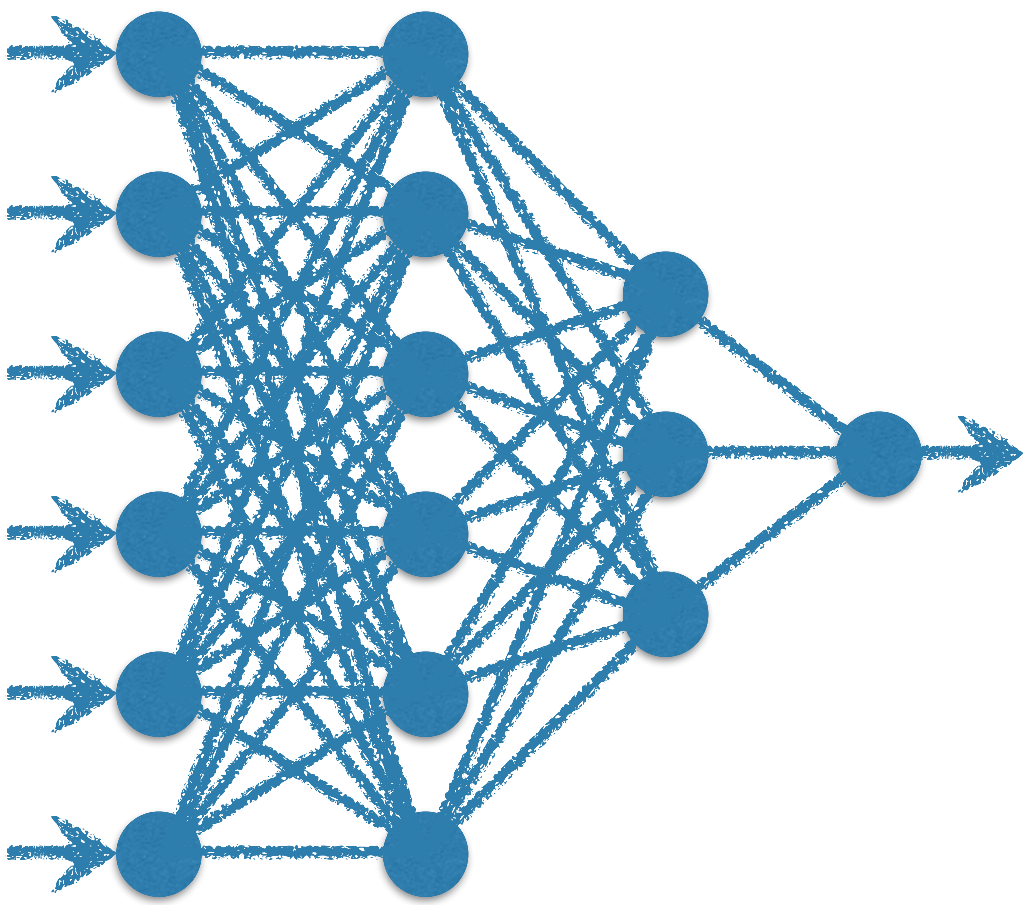


simple data, e.g. $\bar{\mathbf{x}} \in \mathbb{R}^k$

complex data, e.g. $w \in \Sigma^*$

Learning scenarios

NNs



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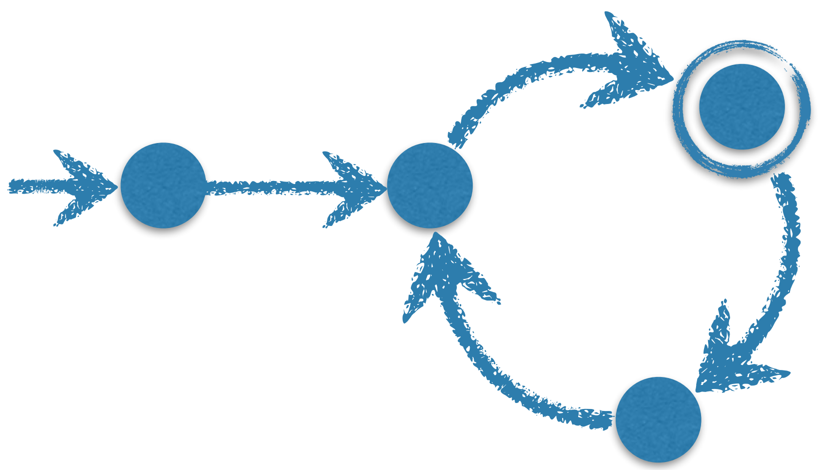

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complex functions, e.g. f: $\mathbb{R}^k \to \{0,1\}$
representing pictures of dogs

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representing regular $L \subseteq \Sigma^*$

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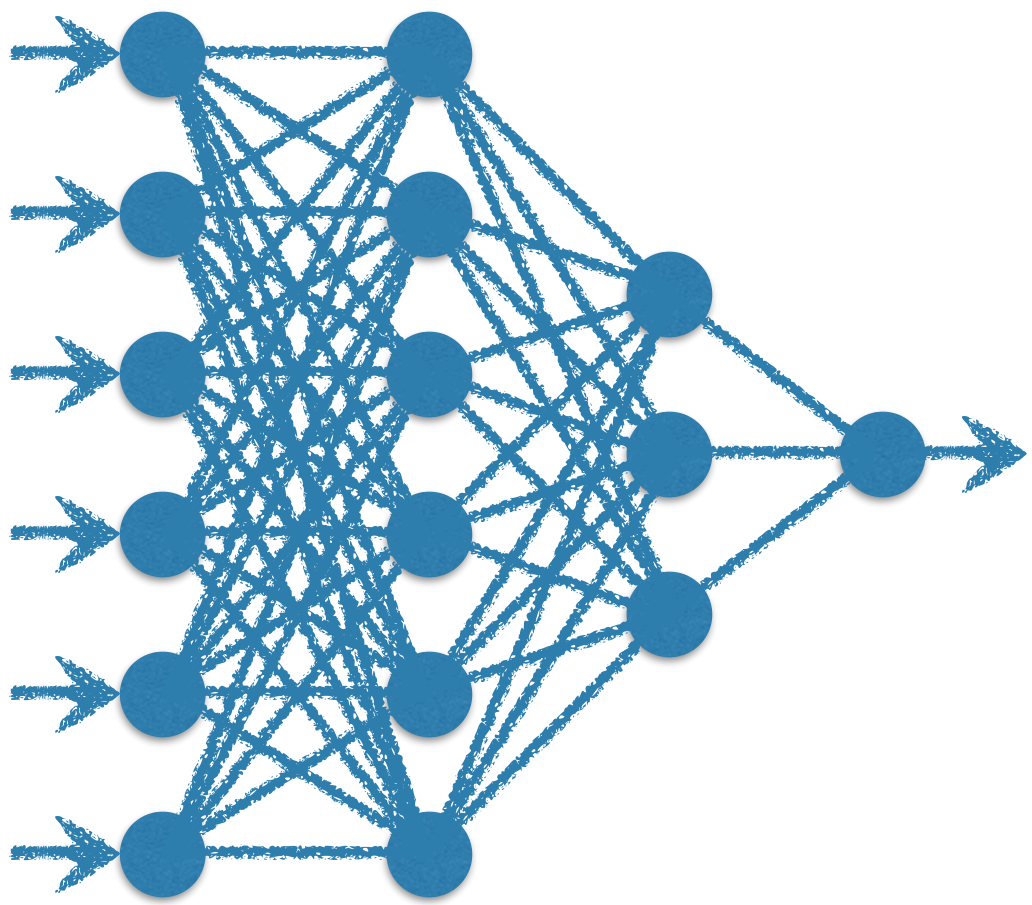

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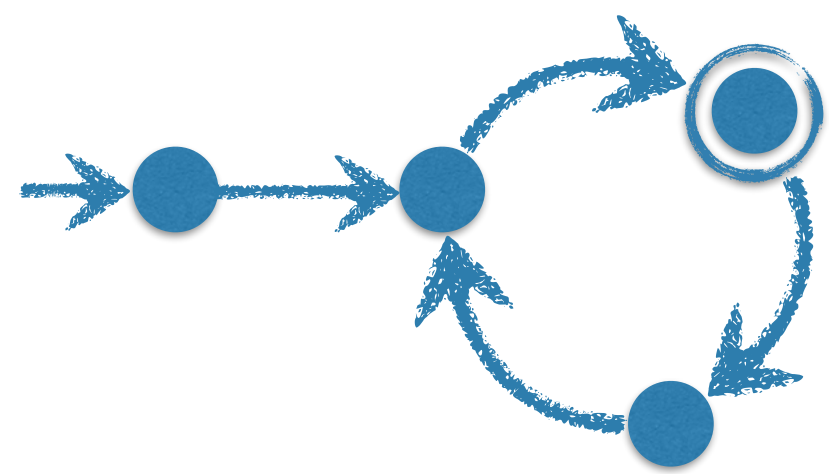

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structure (architecture) is fixed

complex data, e.g. $w \in \Sigma^*$

simple functions, e.g. f: $\Sigma^* \rightarrow \{0,1\}$
representing regular $L \subseteq \Sigma^*$

structure (states+transitions) is learned

Learning scenarios

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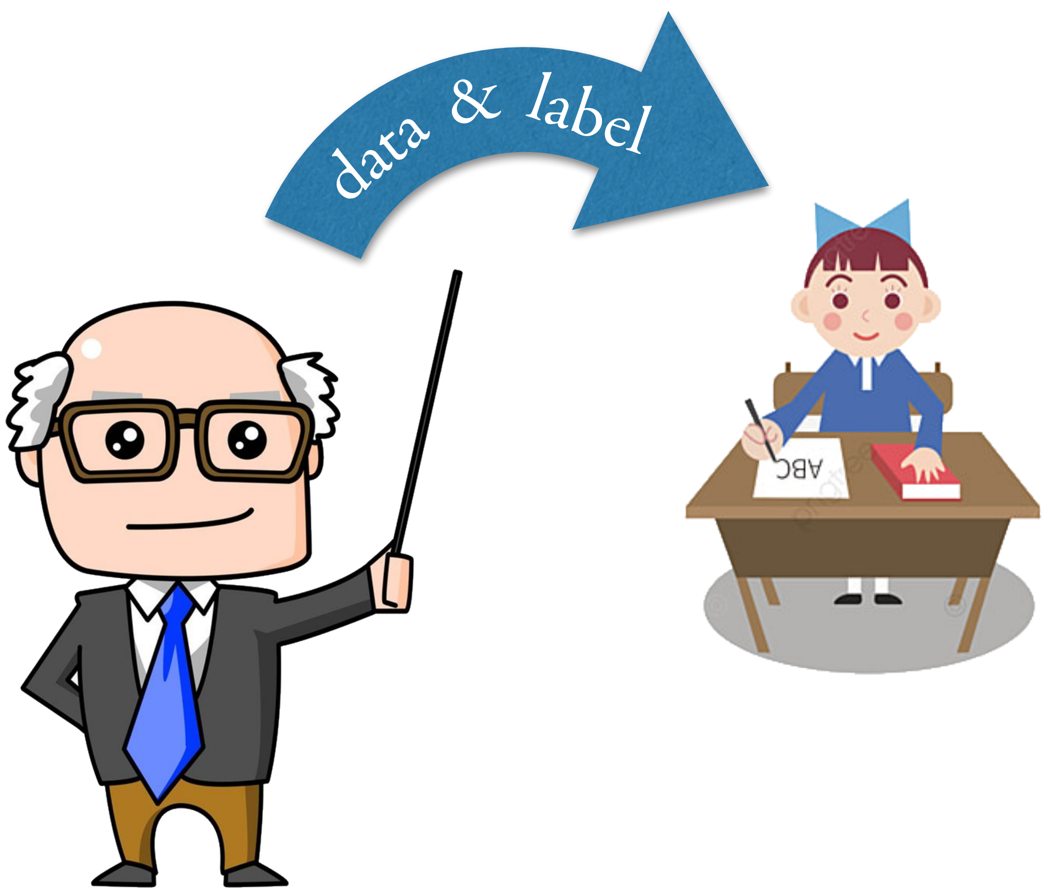
Passive



Learning scenarios

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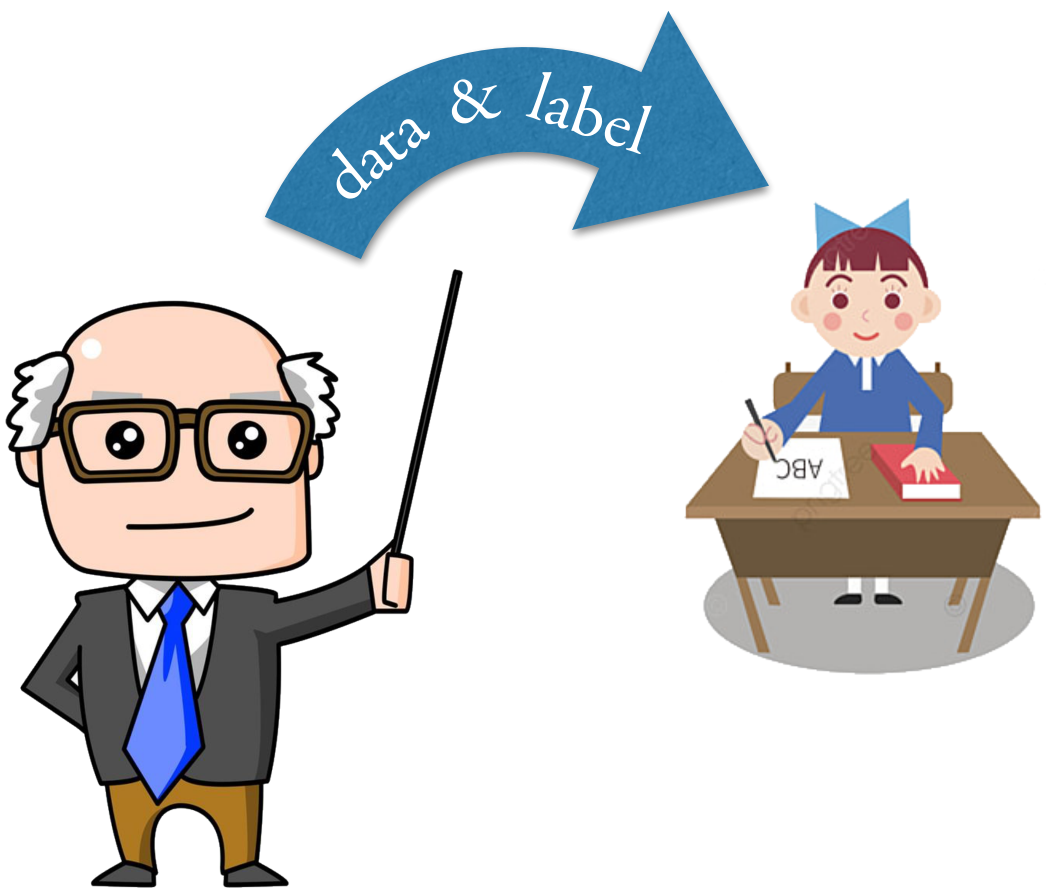
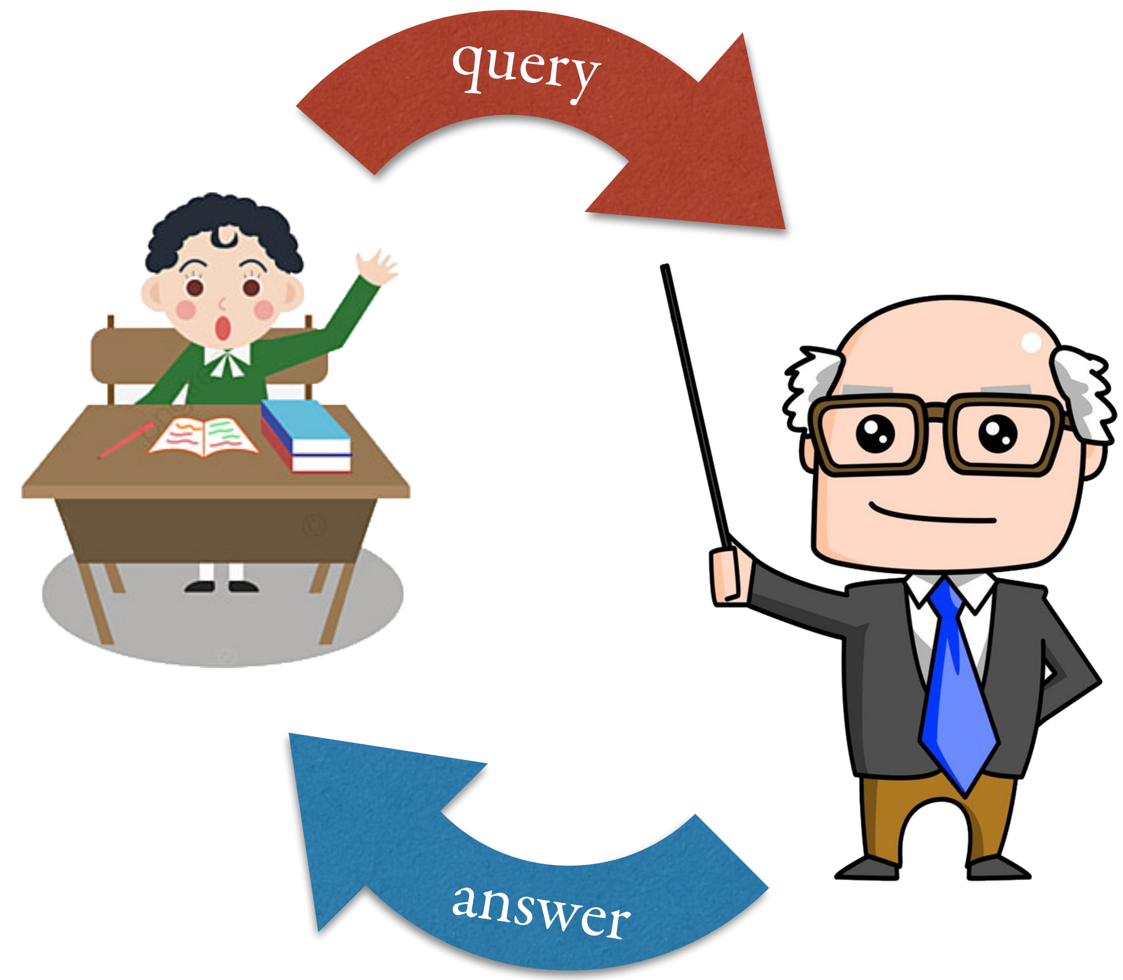
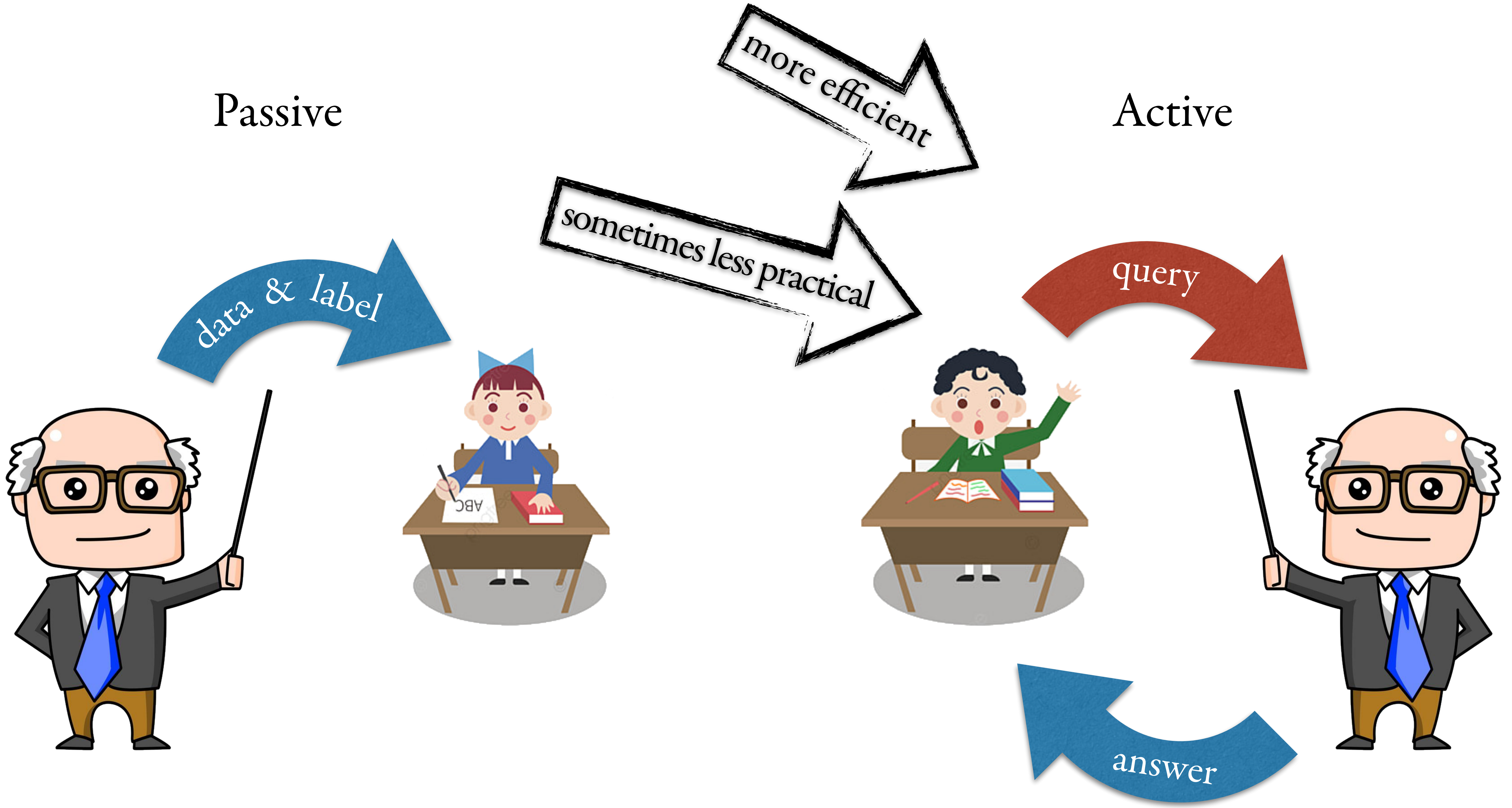


Active



Learning scenarios



A bit of history on automata learning

‘56	Moore	*Gedanken-experiments on sequential machines*
‘67	Gold	*passive learning in the limit*
‘87	Angluin	*active learning with queries*
’93…	Pitt et al.	*PAC-learning, cryptographic hardness*
’95…	Maler et al.	*learning regular ω-languages*
’96…	Vilar et al.	*learning word transformations*
’00…	Beimel et al.	*learning weighted and multiplicity automata*
’10…	Lemay et al.	*learning tree transformations*
’12…	Howar et al.	*learning languages over infinite alphabets*
’14…	Maier et al.	*learning timed languages*
’15…	Balle et al.	*spectral techniques for learning*

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Edward F. Moore

INTRODUCTION

This paper is concerned with finite automata[1] from the experimental point of view. This does not mean that it reports the results of any experimentation on actual physical models, but rather it is concerned with what kinds of conclusions about the internal conditions of a finite machine it is possible to draw from external experiments. To emphasize the conceptual nature of these experiments, the word "gedanken-experiments" has been borrowed from the physicists for the title.

The sequential machines considered have a finite number of states, a finite number of possible input symbols, and a finite number of possible output symbols. The behavior of these machines is strictly deterministic (i.e., no random elements are permitted in the machines) in that the present state of a machine depends only on its previous input and previous state, and the present output depends only on the present state.

The point of view of this paper might also be extended to probabilistic machines (such as the noisy discrete channel of communication theory[2]), but this will not be attempted here.

EXPERIMENTS

There will be two kinds of experiments considered in this paper. The first of these, called a simple experiment, is depicted in Figure 1.

Learning as a game

1) Teacher has a secret regular language L_0, e.g. represented by DFA A_0
Learner initially only knows the underlying alphabet Σ

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   otherwise gives a shortest counter-example $w \in (L_0 - L(A)) \cup (L(A) - L_0)$
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- membership queries alone are not sufficient for Learner to win

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Observations

- membership queries alone are not sufficient for Learner to win
- instead, equivalence queries alone are sufficient to win (why? how quick?)

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in the form of an algorithm (L^* algorithm)

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MAT (Minimally Adequate Teacher)

Sometimes answering an equivalence query is not practical: if Teacher knew A_0, he could pass this information directly to Learner, so why bothering querying?

Equivalence queries can however be *approximated* by a series of membership queries: as long as membership in A matches membership in A_0, Learner assumes he made the correct guess.

This latter setting is often called
Black-box learning

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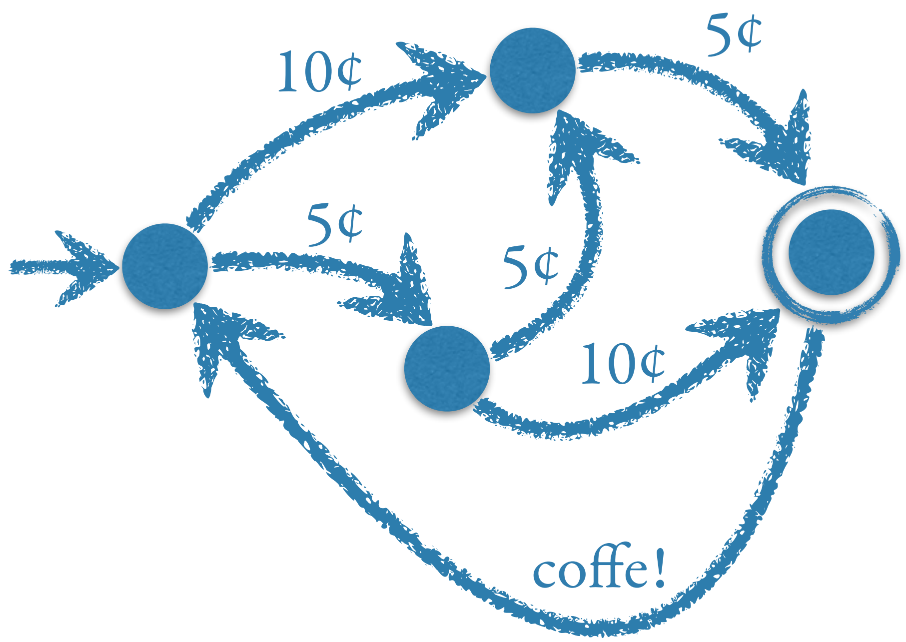

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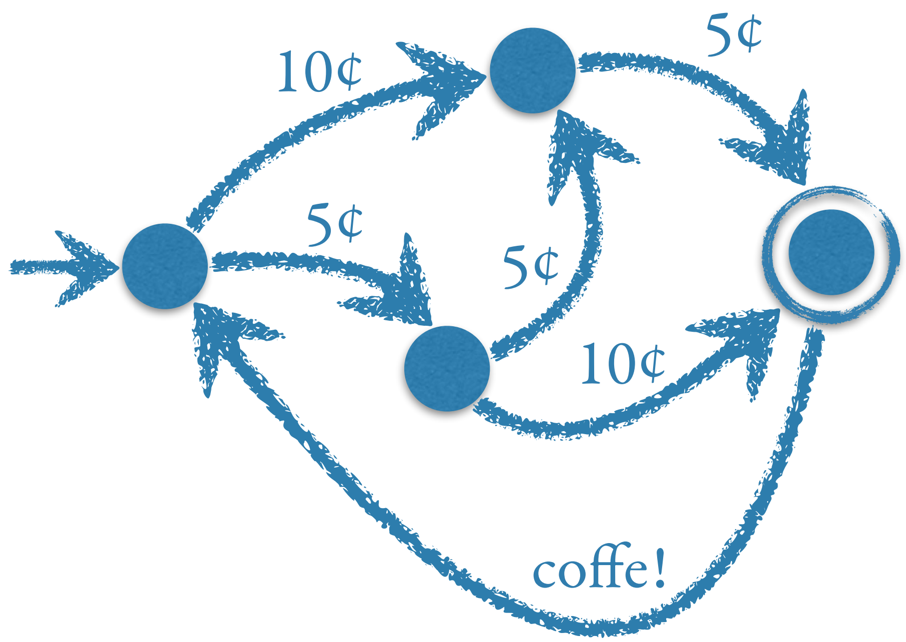

Verification: model-learning
(in contrast to model-checking)

Control theory:
system identification,
diagram inference

Language theory:
grammar inference,
regular extrapolation

Security:
protocol state fuzzing

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Learning as a killer app of Myhill-Nerode theorem

Myhill-Nerode equivalence: $u \sim_{L_0} v$ if ....?

Learning as a killer app of Myhill-Nerode theorem

Myhill-Nerode equivalence: $u \sim_{L_0} v$ if $\forall t \in \Sigma^* \;\; u\,t \in L_0 \leftrightarrow v\,t \in L_0$

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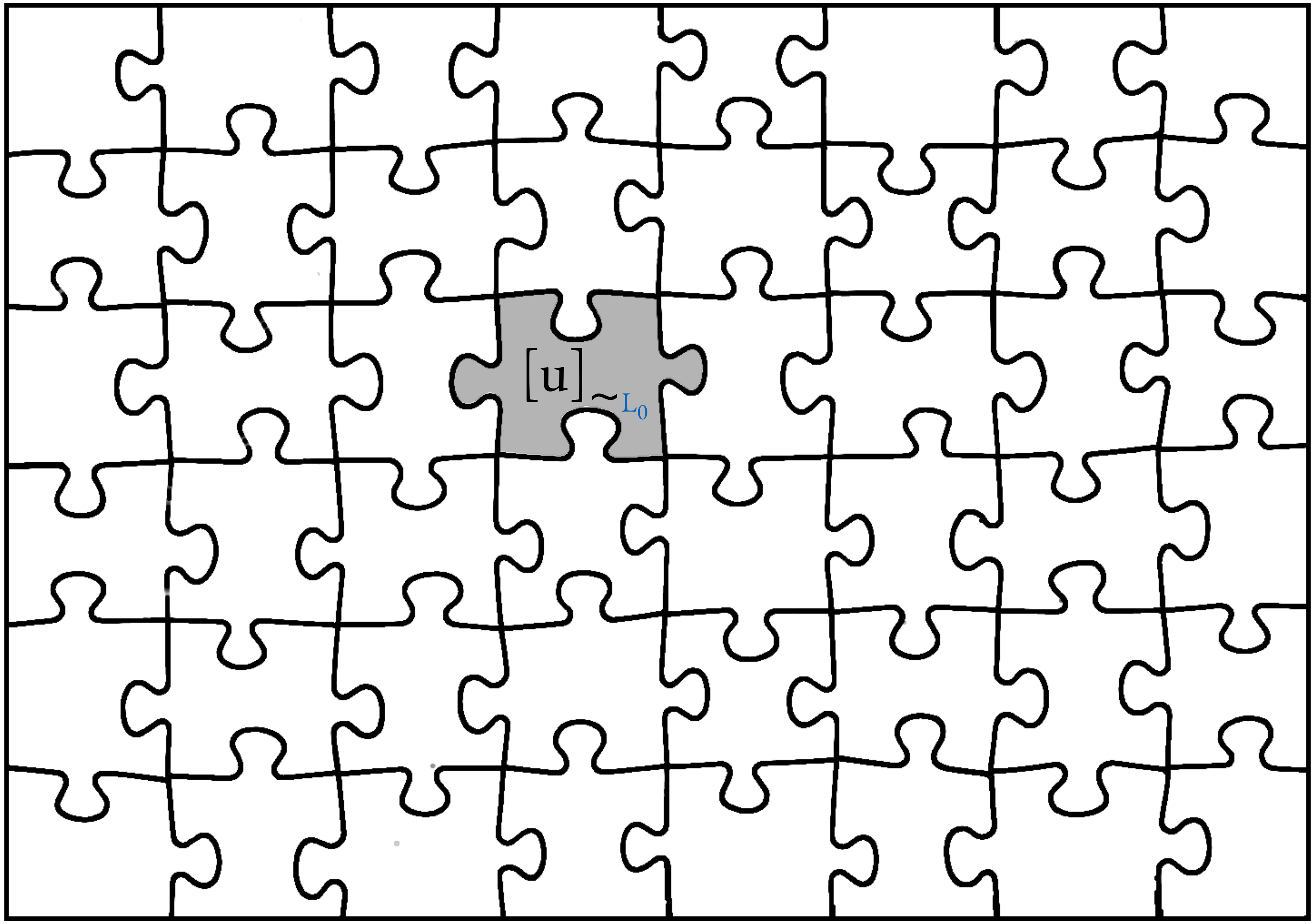

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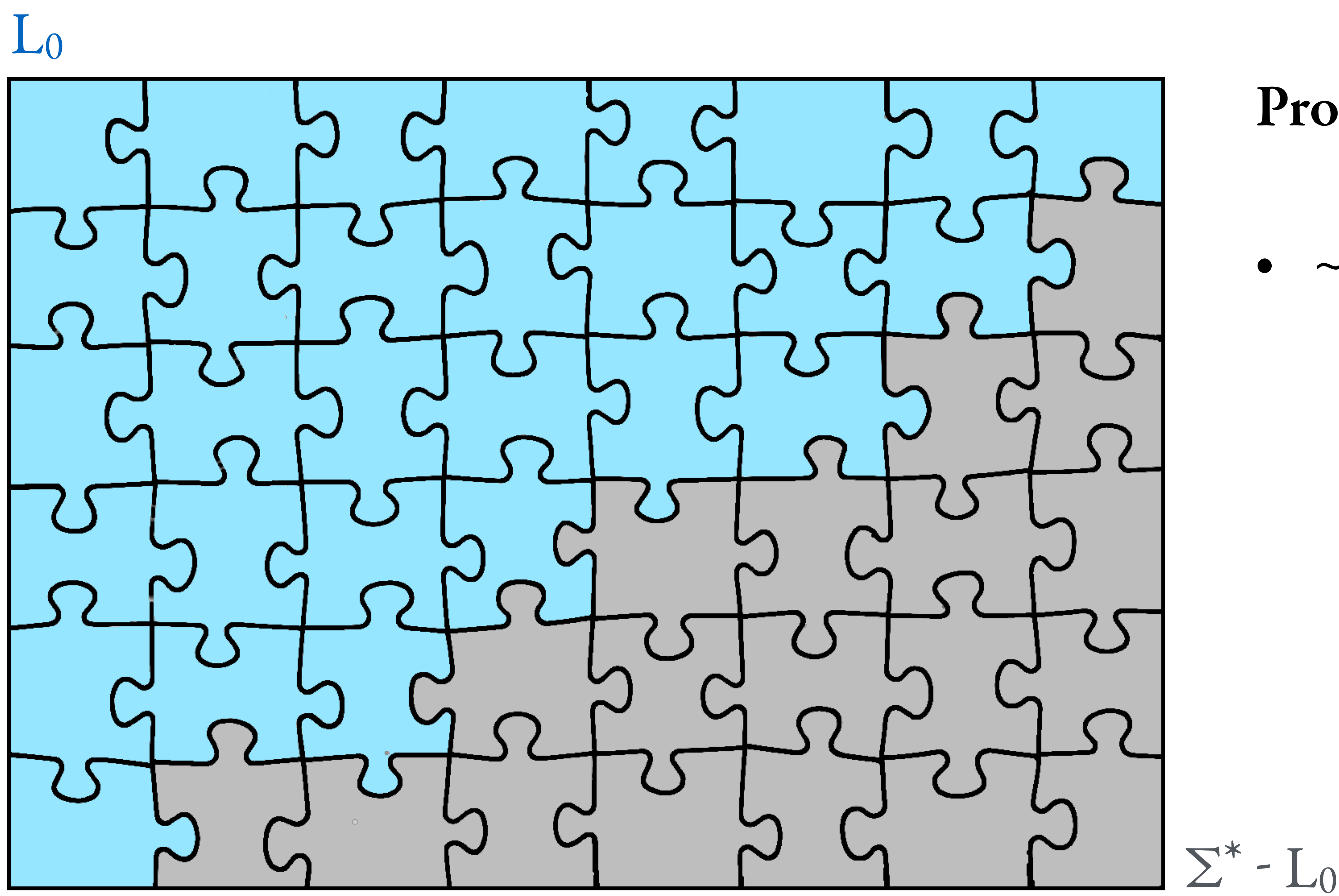

Properties:

- $\sim_{L_0}$ refines $=_{L_0}$

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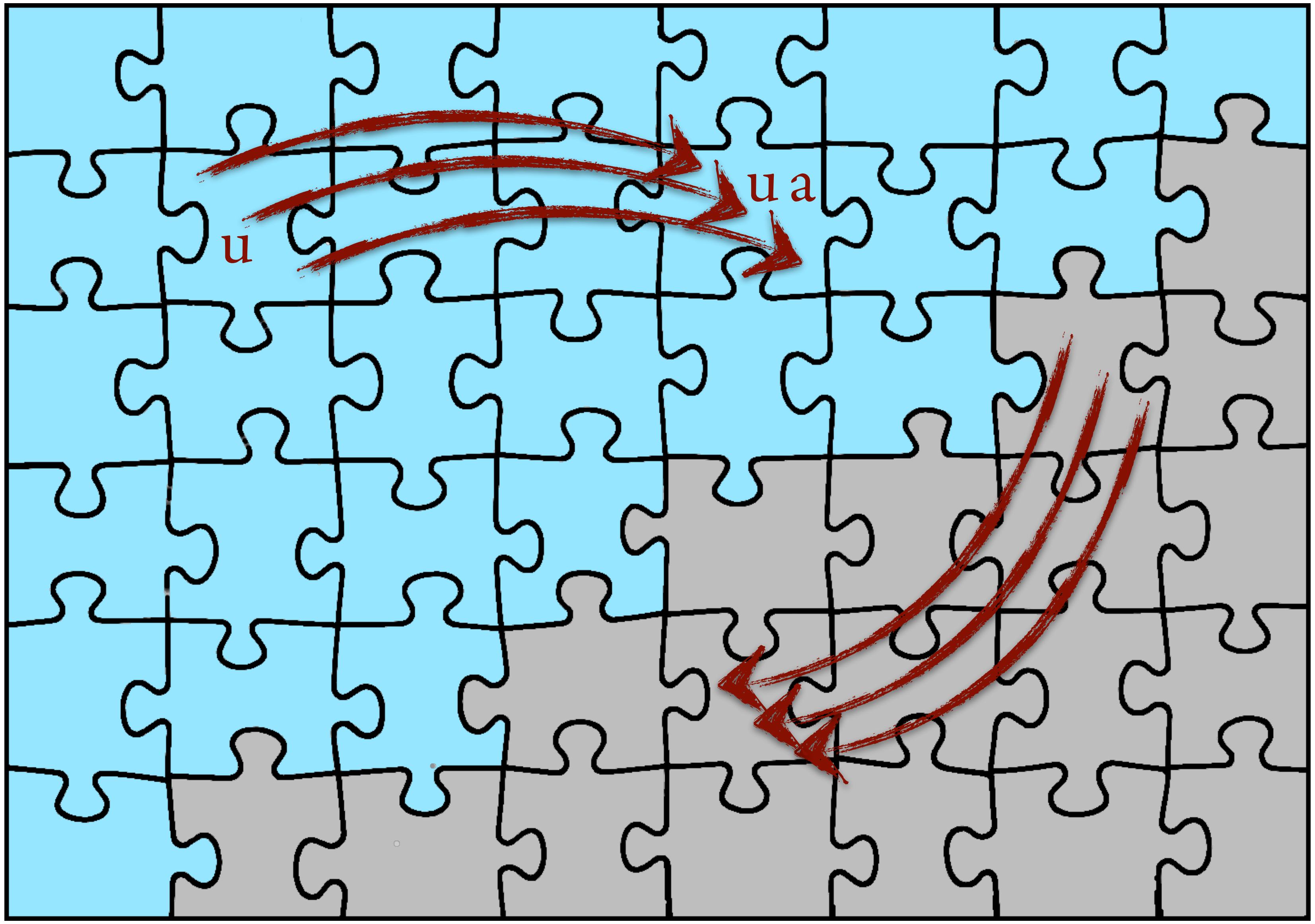

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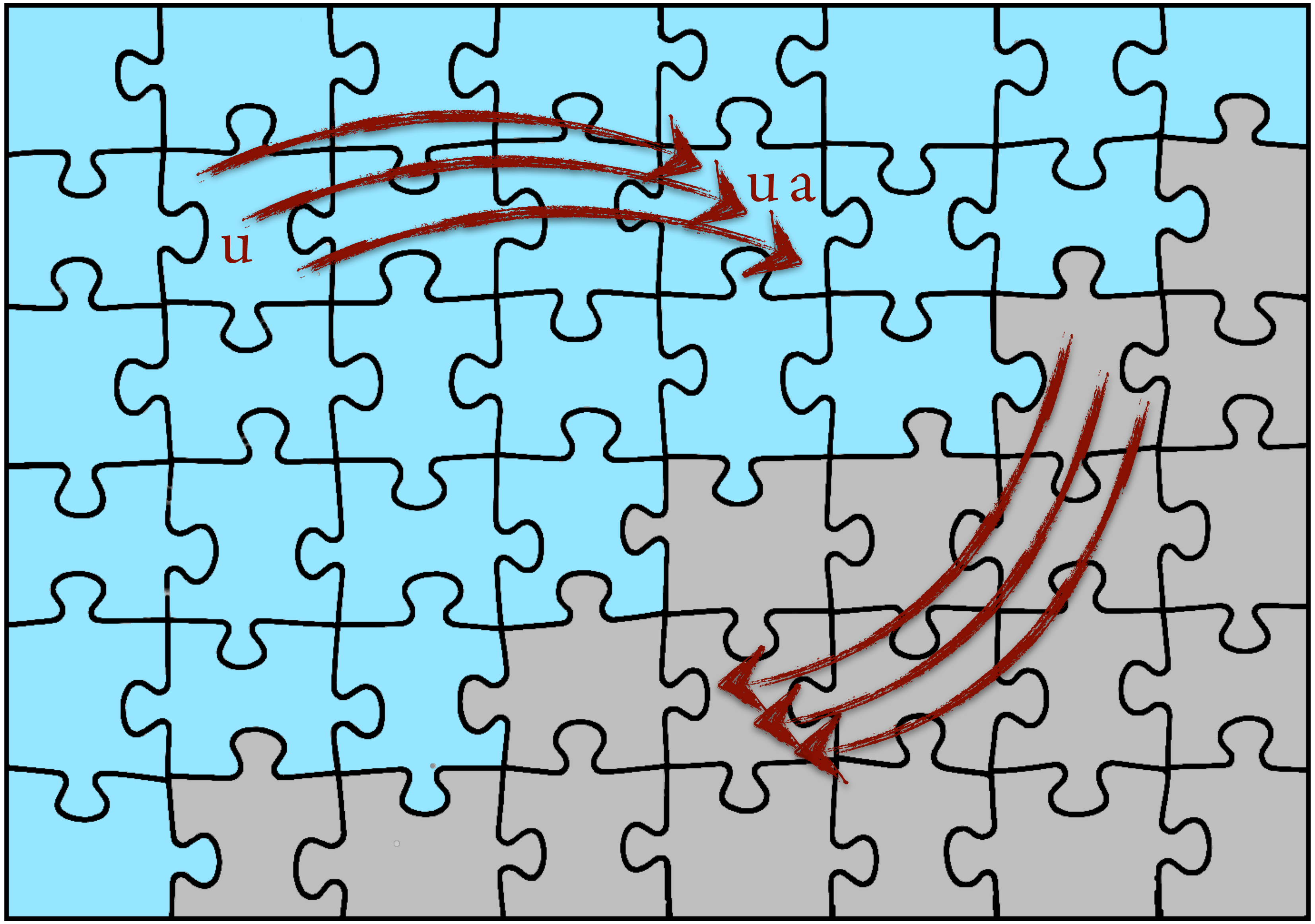

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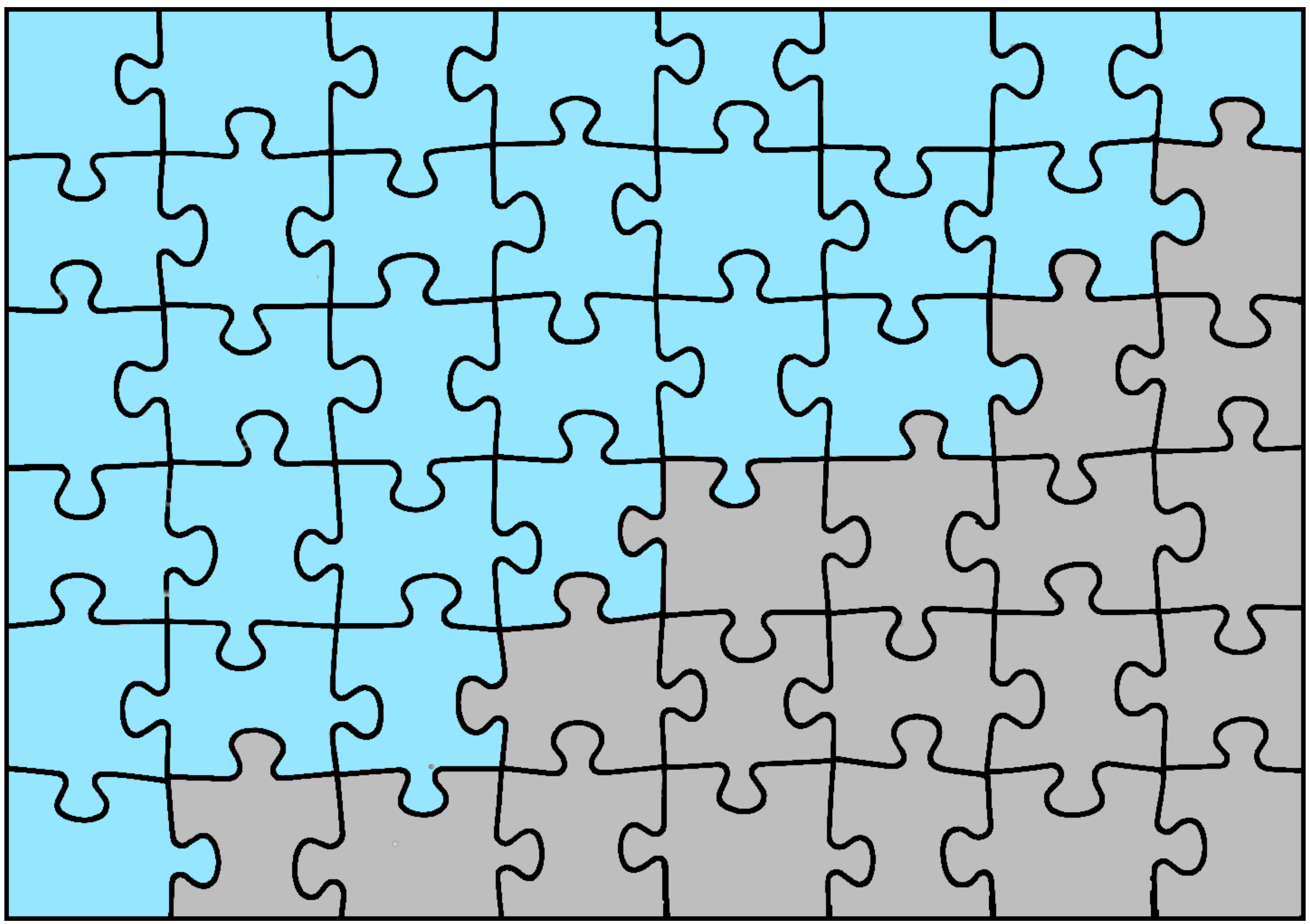

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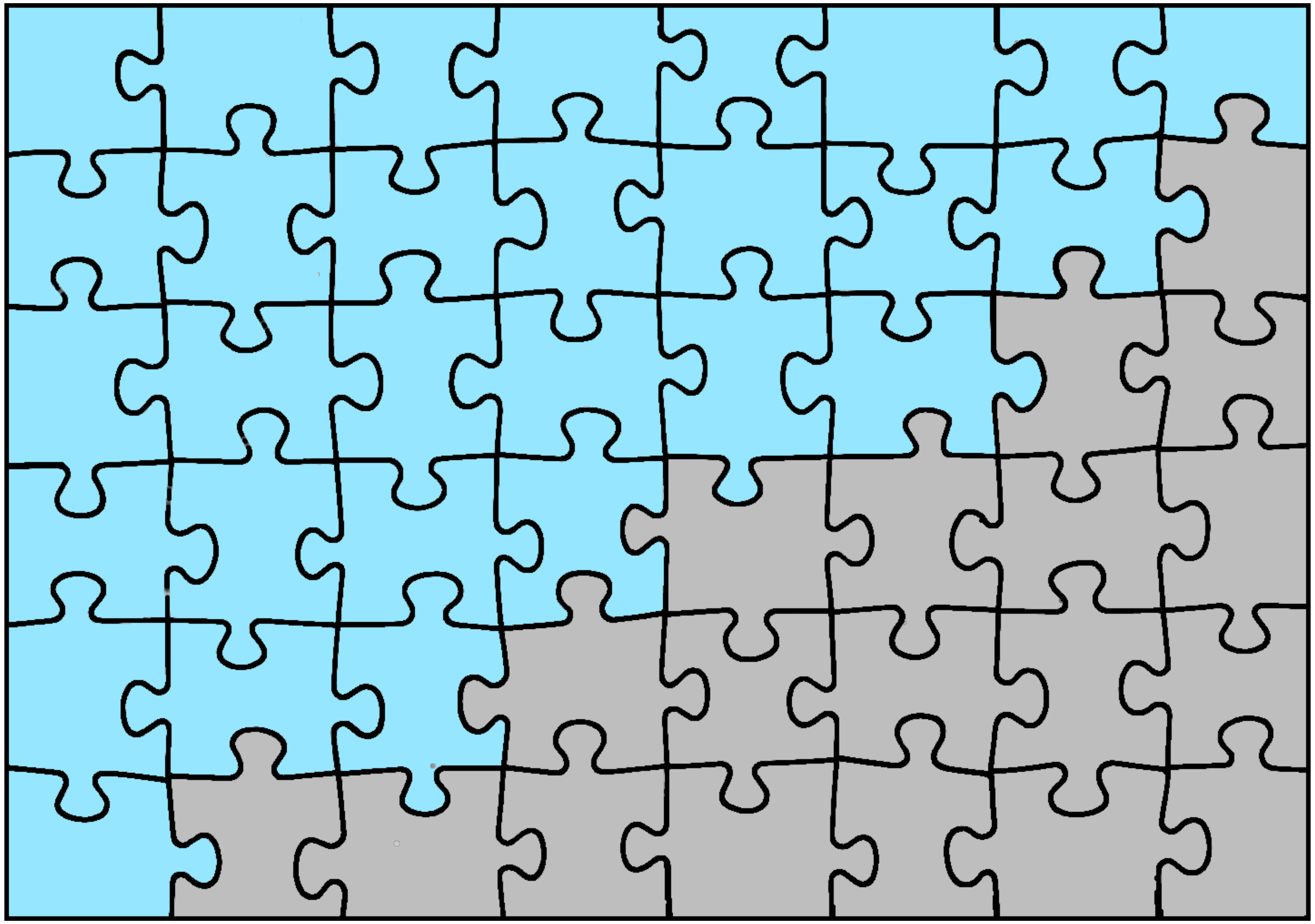

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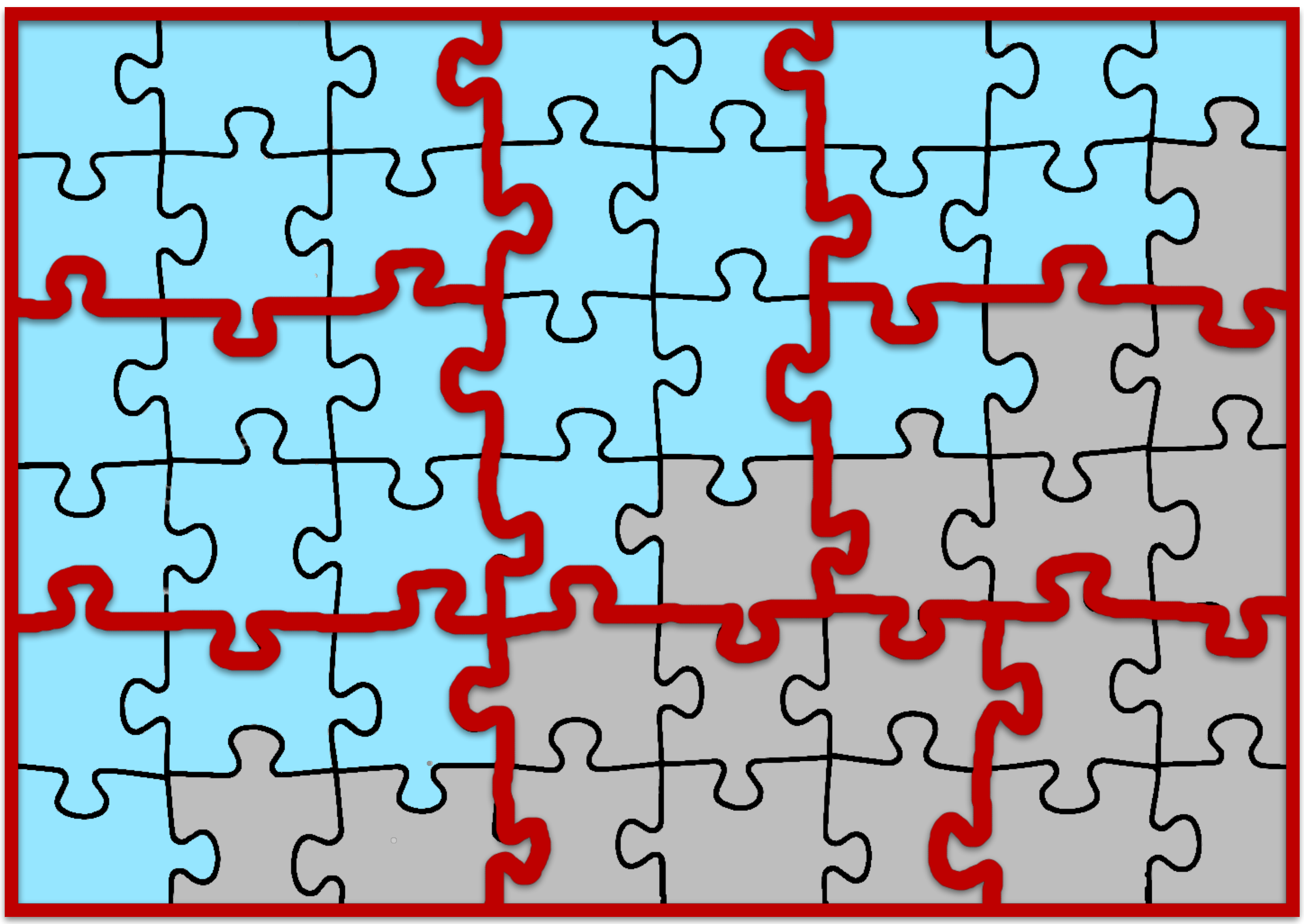

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- $\approx_{L_0,T}$ is coarser than $\sim_{L_0}$

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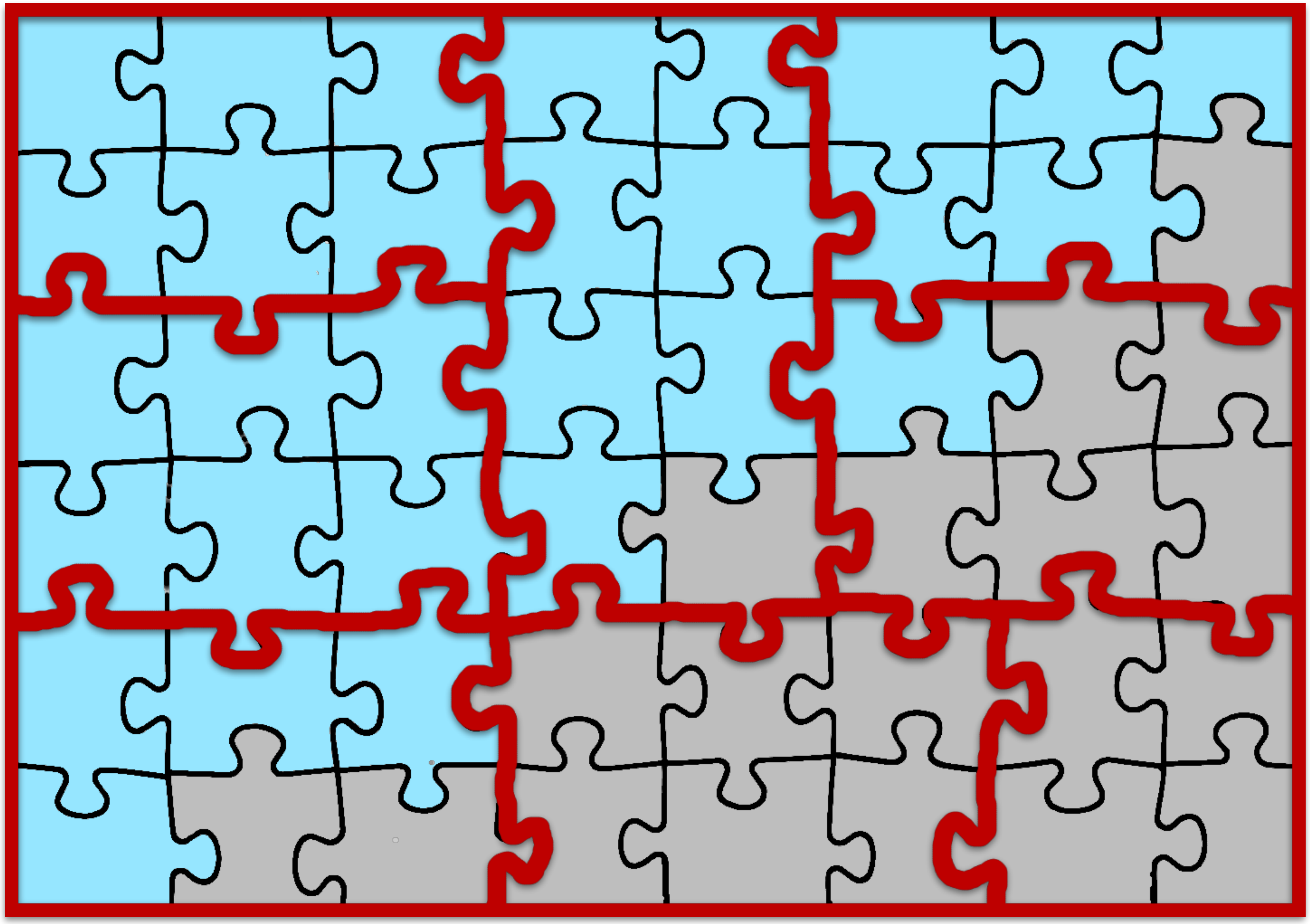

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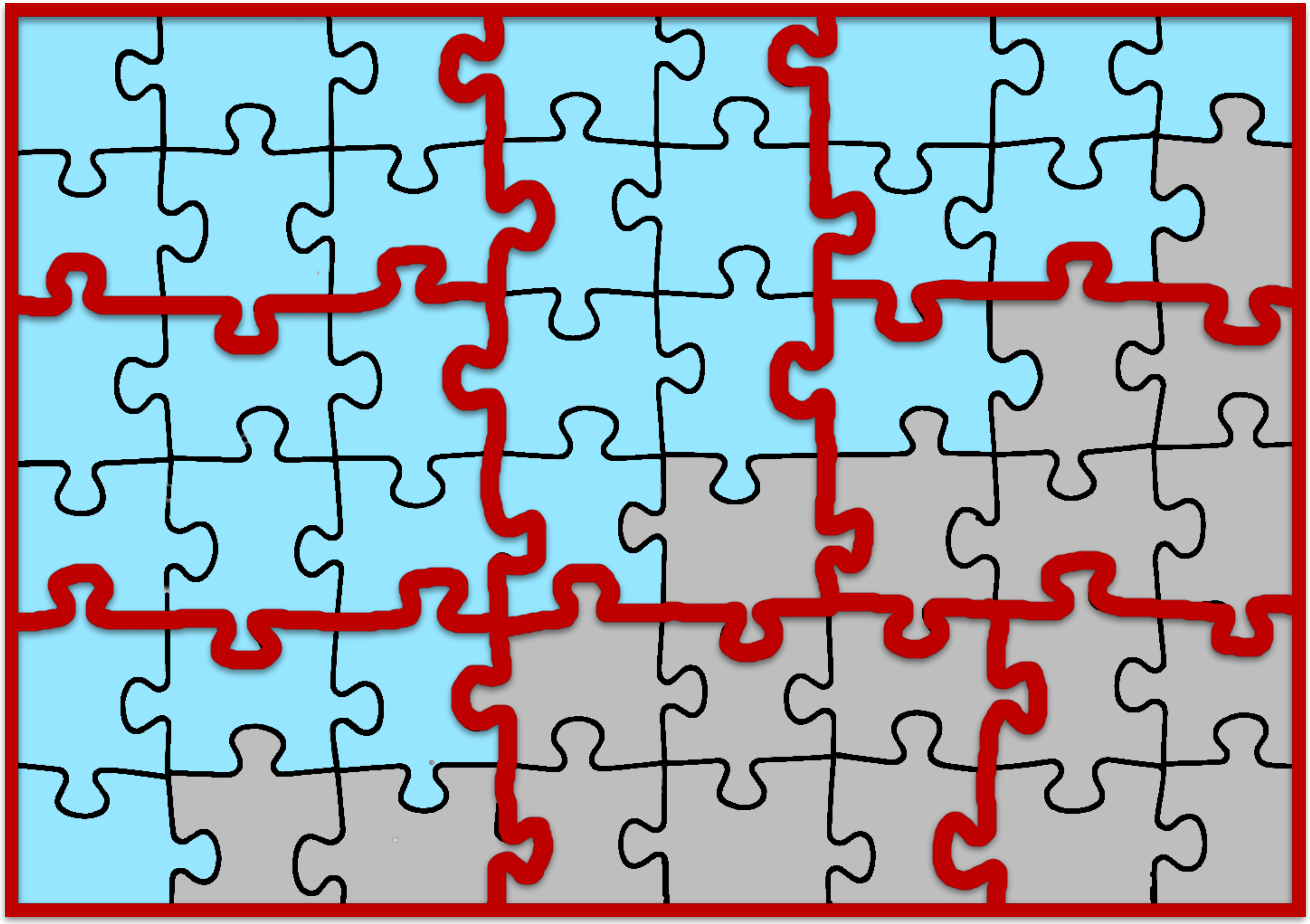

Learner constructs automata from pairs (S,T) where $S \subseteq \Sigma^*$ is T-minimal & T-complete

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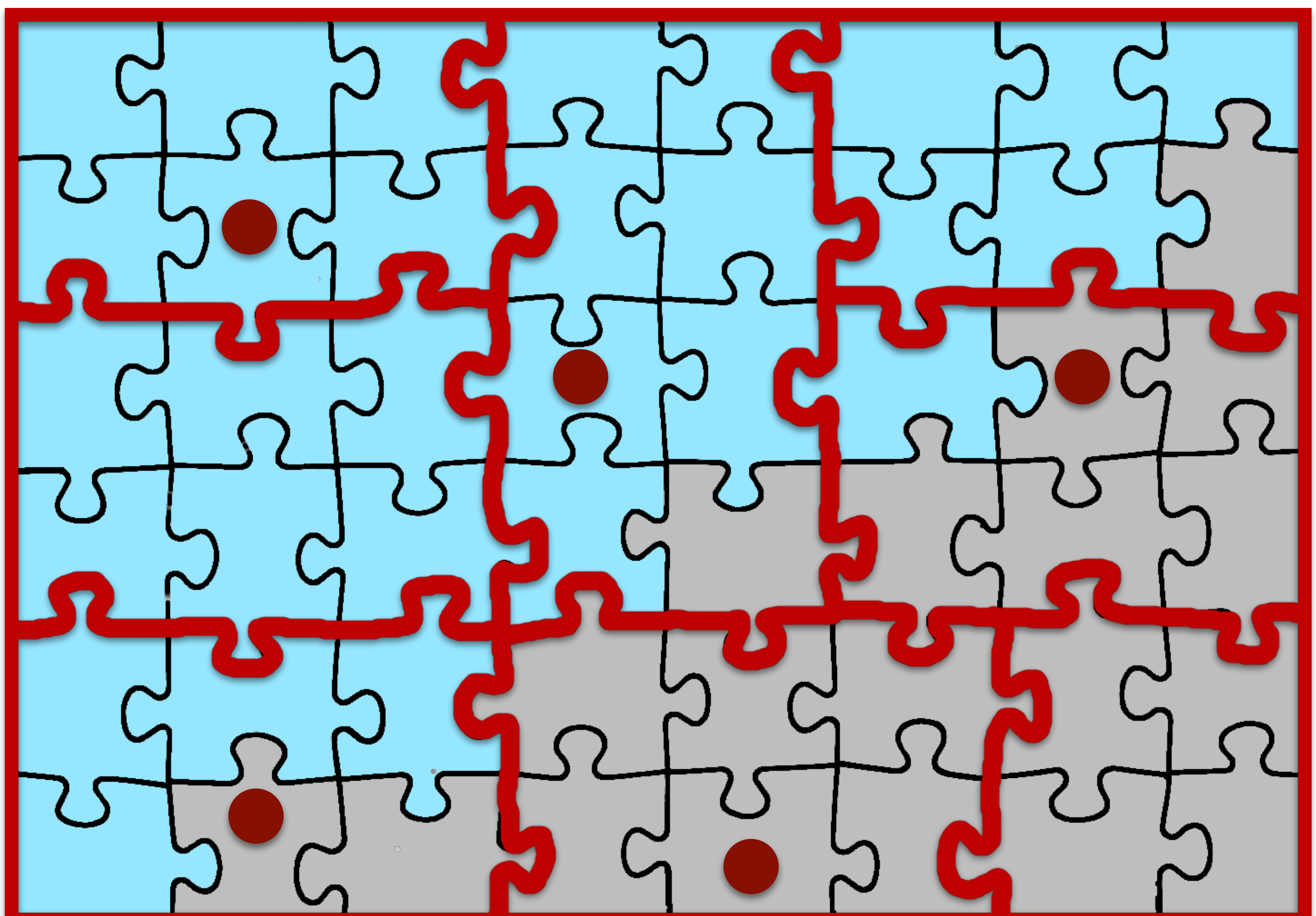

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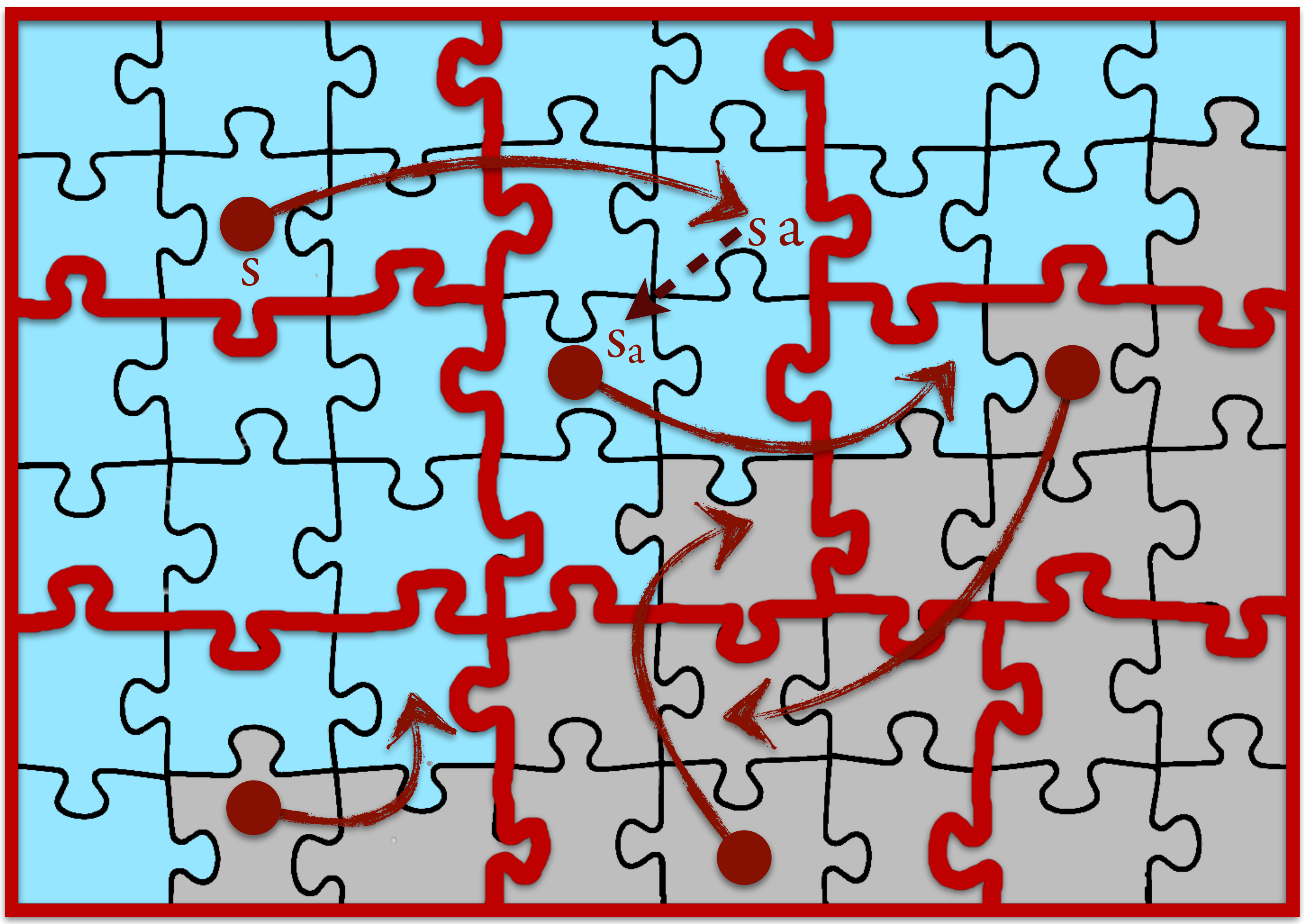

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Learner-strategy

$S = T = \{\varepsilon\}$ // S is T-minimal, possibly not T-complete

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Learner-strategy

```
S = T = {ε}   // S is T-minimal, possibly not T-complete
loop
  while S NOT T-complete
    let s ∈ S and a ∈ Σ such that
        ∀s'∈S ∃t∈T Membership(s a t) ≠ Membership(s' t)
    S = S ∪ {s a}
```

Learner constructs automata from pairs (S,T) where $S \subseteq \Sigma^*$ is T-minimal & T-complete

$S \subseteq \Sigma^*$ is T-minimal if $\forall s \neq s' \in S \;\; s \not\approx_{L_0,T} s'$

$S \subseteq \Sigma^*$ is T-complete if $\forall s \in S \;\; \forall a \in \Sigma$
$\exists\, s_a \in S \;\; s_a \approx_{L_0,T} s\,a$

Learning as a killer app of Myhill-Nerode theorem

Myhill-Nerode equivalence: $u \sim_{L_0} v$ if $\forall t \in \Sigma^* \ \ u\,t \in L_0 \leftrightarrow v\,t \in L_0$

Relativization to a test set T: $u \approx_{L_0,T} v$ if $\forall t \in T \ \ u\,t \in L_0 \leftrightarrow v\,t \in L_0$

Learner-strategy

S = T = {ε} // S is T-minimal, possibly not T-complete

loop

while S NOT T-complete

let s ∈ S and a ∈ Σ such that

∀s'∈S ∃t∈T Membership(s a t) ≠ Membership(s' t)

S = S ∪ {s a}

A = DFA with state set S, transitions $\delta(s,a) = s_a$

initial state ε, final states s' s.t. Membership(s')

Learner constructs automata from pairs (S,T) where $S \subseteq \Sigma^*$ is T-minimal & T-complete

$S \subseteq \Sigma^*$ is T-minimal if $\forall s \neq s' \in S \ \ s \not\approx_{L_0,T} s'$

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S = S ∪ {s a}

A = DFA with state set S, transitions $\delta(s,a) = s_a$

initial state ε, final states s' s.t. Membership(s')

s_a is the unique word in S such that $s_a \approx_{L_0,T} s\,a$

(S T-complete → s_a exists

S T-minimal → s_a unique)

Learner constructs automata from pairs (S,T) where $S \subseteq \Sigma^*$ is T-minimal & T-complete

$S \subseteq \Sigma^*$ is T-minimal

if $\forall s \neq s' \in S$ $s \not\approx_{L_0,T} s'$

$S \subseteq \Sigma^*$ is T-complete

if $\forall s \in S$ $\forall a \in \Sigma$

$\exists s_a \in S$ $s_a \approx_{L_0,T} s\,a$

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<u>Learner-strategy</u>

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S = S ∪ {s a}

A = DFA with state set S, transitions $\delta(s,a) = s_a$

initial state ε, final states s' s.t. Membership(s')

if Equivalence(A) // this surely happens

return A // when |S| = index($\sim_{L_0}$)

else

let w be the counter-example of equivalence

T = T ∪ {suffixes of w}

s_a is the unique word in S such that $s_a \approx_{L_0,T} s\,a$

(S T-complete → s_a exists

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$S \subseteq \Sigma^*$ is <u>T-minimal</u>

if $\forall s \neq s' \in S \;\; s \not\approx_{L_0,T} s'$

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if $\forall s \in S \;\; \forall a \in \Sigma$

$\exists\, s_a \in S \;\; s_a \approx_{L_0,T} s\,a$

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Learner-strategy

```
S = T = {ε}   // S is T-minimal, possibly not T-complete
loop          // this will loop at most index(~L0) times
  while S NOT T-complete
    let s ∈ S and a ∈ Σ such that
        ∀s'∈S ∃t∈T Membership(s a t) ≠ Membership(s' t)
    S = S ∪ {s a}
  A = DFA with state set S, transitions δ(s,a) = sa
      initial state ε, final states s' s.t. Membership(s')
  if Equivalence(A)    // this surely happens
    return A           // when |S| = index(~L0)
  else
    let w be the counter-example of equivalence
    T = T ∪ {suffixes of w}  // S becomes T-incomplete
                             // and will grow at next iteration...
```

Learner constructs automata from pairs (S,T) where $S \subseteq \Sigma^*$ is T-minimal & T-complete

$S \subseteq \Sigma^*$ is T-minimal
if $\forall s \neq s' \in S$ $s \not\approx_{L_0,T} s'$

$S \subseteq \Sigma^*$ is T-complete
if $\forall s \in S$ $\forall a \in \Sigma$
$\exists s_a \in S$ $s_a \approx_{L_0,T} s\,a$

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```
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        ∀s'∈S ∃t∈T Membership(s a t) ≠ Membership(s' t)
    S = S ∪ {s a}
  A = DFA with state set S, transitions δ(s,a) = sa
      initial state ε, final states s' s.t. Membership(s')
  if Equivalence(A)   // this surely happens
    return A          // when |S| = index(~L0)
  else
    let w be the counter-example of equivalence
    T = T ∪ {suffixes of w}  // S becomes T-incomplete
                             // and will grow at next iteration...
```

Proof by contradiction:

Assume:

- $w = a_1 \dots a_n$ counter-example
- $t_i = a_{i+1} \dots a_n$ suffixes of w
- s_i state reached by A after reading prefix $a_1 \dots a_i$ of w
- S is $(T \cup \{t_0,\dots,t_n\})$-complete

Verify by induction on i that

$$w \in L_0 \text{ iff } s_i t_i \in L_0$$

Conclude $w \in L_0$ iff $w \in L(A)$

An alternative view - Hankel matrices

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Hankel matrix of L_0:

$H \in \{0,1\}^{\Sigma^* \times \Sigma^*}$

$H(s,t) = \text{Membership}(s\,t)$

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Relativization to a test set T: $u \approx_{L_0,T} v$ if $\forall t \in T$ $ut \in L_0 \leftrightarrow vt \in L_0$

	ε	a	b	aa	ab	...	aba	...
ε	1	0	0	1	0	...	0	...
a	0	1	0	0	0	...	0	...
b	0	1	0	0	0	...	0	...
aa	1	0	0	1	0	...	0	...
ab	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...
aba	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...

Hankel matrix of L_0:

$H \in \{0,1\}^{\Sigma^* \times \Sigma^*}$

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(infinite, highly redundant, but nicely structured)

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Relativization to a test set T: $u \approx_{L_0,T} v$ if $\forall t \in T \quad u\,t \in L_0 \leftrightarrow v\,t \in L_0$

	ε	a	b	aa	ab	...	aba	...
ε	1	0	0	1	0	...	0	...
a	**0**	**1**	**0**	**0**	**0**	**...**	**0**	**...**
b	0	1	0	0	0	...	0	...
aa	1	0	0	1	0	...	0	...
ab	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...
aba	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...

Hankel matrix of L_0:

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a row = a $\sim_{L_0}$-class

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Relativization to a test set T: $u \approx_{L_0,T} v$ if $\forall t \in T \;\; u\,t \in L_0 \leftrightarrow v\,t \in L_0$

	ε	a	b	aa	ab	...	aba	...
ε	1	0	0	**1**	0	...	0	...
a	0	1	0	**0**	0	...	0	...
b	0	1	0	**0**	0	...	0	...
aa	1	0	0	**1**	0	...	0	...
ab	0	0	0	**0**	0	...	0	...
...	...	...	...	**...**	...	...	...	...
aba	0	0	0	**0**	0	...	0	...
...	...	...	...	...	...	...	...	...

Hankel matrix of L_0:

$H \in \{0,1\}^{\Sigma^* \times \Sigma^*}$

$H(s,t) = \text{Membership}(s\,t)$

(infinite, highly redundant, but nicely structured)

a row = a $\sim_{L_0}$-class

a column = a test word t

An alternative view - Hankel matrices

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	ε	a	b	aa	ab	...	aba	...
ε	**1**	**0**	0	**1**	0	...	0	...
a	**0**	**1**	0	**0**	0	...	0	...
b	0	1	0	0	0	...	0	...
aa	**1**	**0**	0	**1**	0	...	0	...
ab	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...
aba	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...

Hankel matrix of L_0:

$H \in \{0,1\}^{\Sigma^* \times \Sigma^*}$

$H(s,t) = \text{Membership}(s\,t)$

(infinite, highly redundant, but nicely structured)

a row = a $\sim_{L_0}$-class

a column = a test word t

a submatrix = a pair (S,T)

An alternative view - Hankel matrices

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Relativization to a test set T: $u \approx_{L_0,T} v$ if $\forall t \in T \;\; ut \in L_0 \leftrightarrow vt \in L_0$

	ε	a	b	aa	ab	...	aba	...
ε	1	0	0	1	0	...	0	...
a	0	1	0	0	0	...	0	...
b	0	1	0	0	0	...	0	...
aa	1	0	0	1	0	...	0	...
ab	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...
aba	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...

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S = {ε} T = {ε}

	ε	a	b	aa	ab	...	aba	...
ε	1	0	0	1	0	...	0	...
a	0	1	0	0	0	...	0	...
b	0	1	0	0	0	...	0	...
aa	1	0	0	1	0	...	0	...
ab	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...
aba	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...

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	ε	a	b	aa	ab	...	aba	...
ε	1	0	0	1	0	...	0	...
a	0	1	0	0	0	...	0	...
b	0	1	0	0	0	...	0	...
aa	1	0	0	1	0	...	0	...
ab	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...
aba	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...

S = {ε} T = {ε}

expand S to make it T-complete...

An alternative view - Hankel matrices

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	ε	a	b	aa	ab	...	aba	...
ε	**1**	0	0	1	0	...	0	...
a	**0**	1	0	0	0	...	0	...
b	0	1	0	0	0	...	0	...
aa	1	0	0	1	0	...	0	...
ab	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...
aba	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...

S = {ε} T = {ε}

expand S to make it T-complete...

S = {ε, a} T = {ε}

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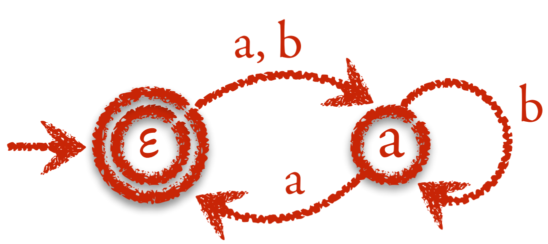
	ε	a	b	aa	ab	...	aba	...
ε	1	0	0	1	0	...	0	...
a	0	1	0	0	0	...	0	...
b	0	1	0	0	0	...	0	...
aa	1	0	0	1	0	...	0	...
ab	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...
aba	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...

S = {ε} T = {ε}

expand S to make it T-complete...

S = {ε, a} T = {ε}

build candidate automaton...



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	ε	a	b	aa	ab	...	aba	...
ε	**1**	0	0	1	0	...	0	...
a	**0**	1	0	0	0	...	0	...
b	0	1	0	0	0	...	0	...
aa	1	0	0	1	0	...	0	...
ab	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...
aba	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...

S = {ε} T = {ε}

expand S to make it T-complete...

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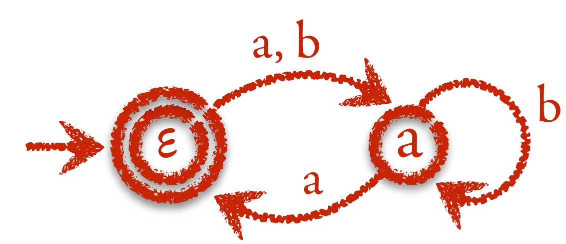
build candidate automaton...



expand T by counter-example w = aba

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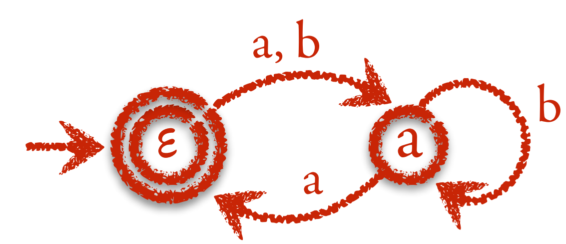
	ε	a	b	aa	ab	...	aba	...
ε	**1**	**0**	0	1	0	...	**0**	...
a	**0**	**1**	0	0	0	...	**0**	...
b	0	1	0	0	0	...	0	...
aa	1	0	0	1	0	...	0	...
ab	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...
aba	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...

S = {ε} T = {ε}

expand S to make it T-complete...

S = {ε, a} T = {ε}

build candidate automaton...



expand T by counter-example w = aba

S = {ε, a} T = {ε, a, ba, aba}

An alternative view - Hankel matrices

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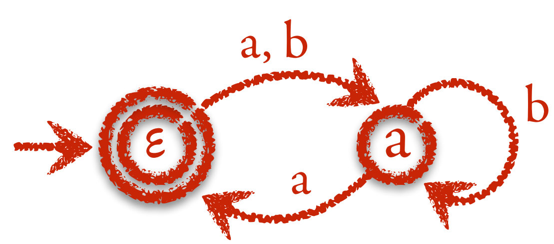
	ε	a	b	aa	ab	...	aba	...
ε	**1**	**0**	0	1	0	...	**0**	...
a	**0**	**1**	0	0	0	...	**0**	...
b	0	1	0	0	0	...	0	...
aa	1	0	0	1	0	...	0	...
ab	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...
aba	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...

S = {ε} T = {ε}

expand S to make it T-complete...

S = {ε, a} T = {ε}

build candidate automaton...



expand T by counter-example w = aba

S = {ε, a} T = {ε, a, ba, aba}

expand S to make it T-complete...

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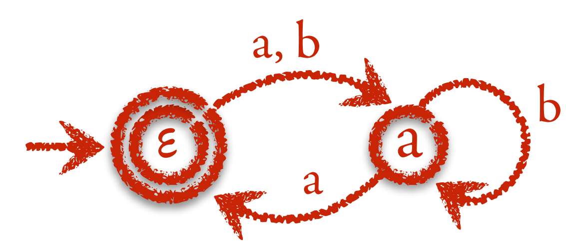
	ε	a	b	aa	ab	...	aba	...
ε	**1**	**0**	0	1	0	...	**0**	...
a	**0**	**1**	0	0	0	...	**0**	...
b	0	1	0	0	0	...	0	...
aa	1	0	0	1	0	...	0	...
ab	**0**	**0**	0	0	0	...	**0**	...
...	...	...	...	...	...	...	...	...
aba	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...

S = {ε} T = {ε}

expand S to make it T-complete...

S = {ε, a} T = {ε}

build candidate automaton...



expand T by counter-example w = aba

S = {ε, a} T = {ε, a, ba, aba}

expand S to make it T-complete...

S = {ε, a, ab} T = {ε, a, ba, aba}

An alternative view - Hankel matrices

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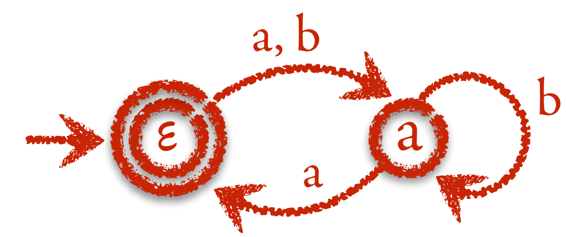
	ε	a	b	aa	ab	…	aba	…
ε	**1**	**0**	0	1	0	…	**0**	…
a	**0**	**1**	0	0	0	…	**0**	…
b	0	1	0	0	0	…	0	…
aa	1	0	0	1	0	…	0	…
ab	**0**	**0**	0	0	0	…	**0**	…
…	…	…	…	…	…	…	…	…
aba	0	0	0	0	0	…	0	…
…	…	…	…	…	…	…	…	…

S = {ε}   T = {ε}

expand S to make it T-complete...

S = {ε, a}   T = {ε}

build candidate automaton...



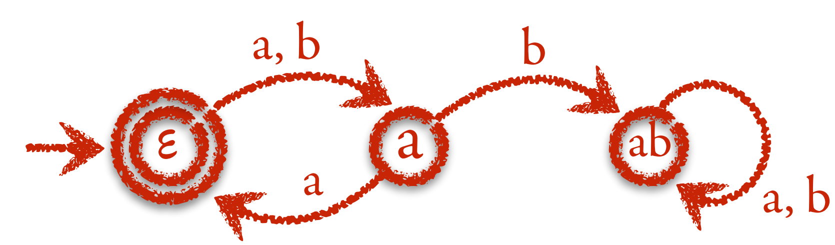
expand T by counter-example w = aba

S = {ε, a}   T = {ε, a, ba, aba}

expand S to make it T-complete...

S = {ε, a, ab}   T = {ε, a, ba, aba}

build candidate automaton...



An alternative view - Hankel matrices

Myhill-Nerode equivalence: $u \sim_{L_0} v$ if $\forall t \in \Sigma^* \; ut \in L_0 \leftrightarrow vt \in L_0$

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	ε	a	b	aa	ab	...	aba	...
ε	**1**	**0**	0	1	0	...	**0**	...
a	**0**	**1**	0	0	0	...	**0**	...
b	0	1	0	0	0	...	0	...
aa	1	0	0	1	0	...	0	...
ab	**0**	**0**	0	0	0	...	**0**	...
...	...	...	...	...	...	...	...	...
aba	0	0	0	0	0	...	0	...
...	...	...	...	...	...	...	...	...

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expand S to make it T-complete...

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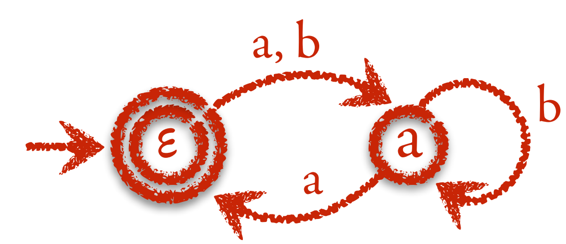

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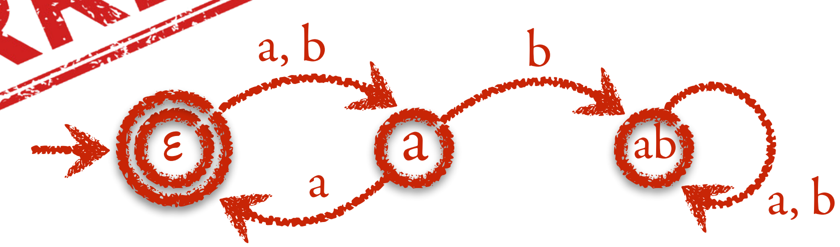
CORRECT



From word languages to word functions

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Automata can be used to represent not only languages, but also *functions on words*

$$f: \Sigma^* \to \Gamma^*$$

e.g. f(abaab) = abcaacb

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Many variants of automata with outputs. Simplest one is <u>sequential transducer</u>
i.e. $A = (\Sigma, \Gamma, Q, q_0, \delta)$ with $\delta: Q \times \Sigma \to Q \times \Gamma^*$ (e.g. $\delta(q, a)=(q', ac)$)

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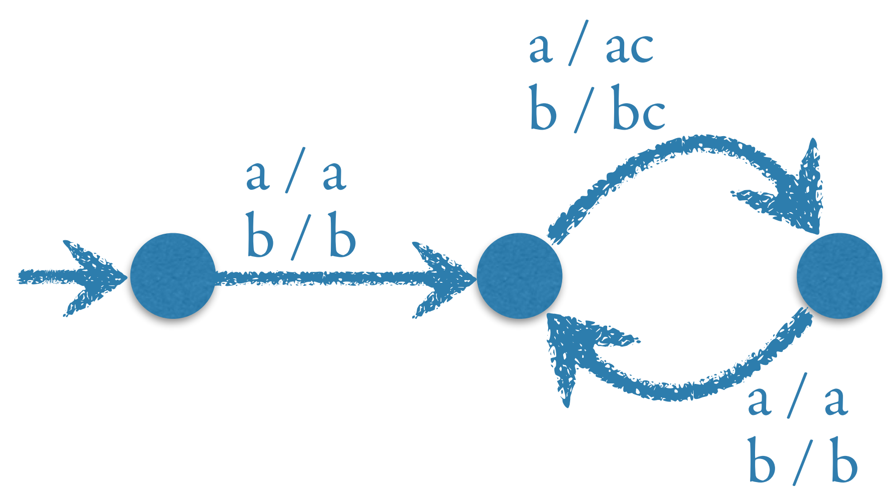

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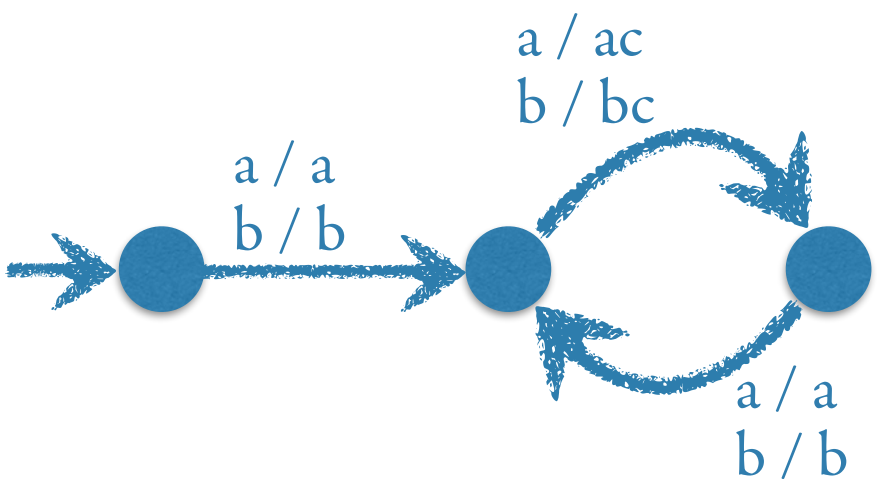

Note: sequential transducers can only compute *total, monotone* functions
(f is <u>monotone</u> if whenever w is prefix of w' then f(w) is prefix of f(w'))

Learning monotone word functions

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1) Teacher has a secret function f_0: $\Sigma^* \to \Gamma^*$, computed by a seq. transducer A_0
Learner initially only knows the input alphabet Σ

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3) Teacher answers accordingly:
   a) gives value of $f_0(w)$
   b) yes if f_0 is computed by A (requires an algorithm for testing equivalence),
   otherwise gives a shortest <u>counter-example</u> w such that $f_0(w) \neq A(w)$

Learning monotone word functions

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Myhill-Nerode equivalence: $u \sim_{f_0} v$ if $\forall t \in \Sigma^* \quad f_0(u\,t) - f_0(u) = f_0(v\,t) - f_0(v)$

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Example

f_0 inserts c between every two positions

$f_0(\varepsilon) = \varepsilon$, $f_0(a) = a$,
$f_0(aa) = aac$, $f_0(aaa) = aaca$, ...

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$[\varepsilon]_{\sim_{f_0}}$ $[a]_{\sim_{f_0}}$

E.g. $a \sim_{f_0} bba$ because $f(a\,???...) - f(a) = a\,?c??c... - a$
$f(bba\,???...) - f(bba) = bbca\,?c??c... - bbca$

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Properties:

- $\sim_{f_0}$ is right-invariant (i.e. $u \sim_{f_0} v \rightarrow u\,a \sim_{f_0} v\,a$)

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- $\sim_{f_0}$ is right-invariant (i.e. $u \sim_{f_0} v \to u\,a \sim_{f_0} v\,a$)
- $\sim_{f_0}$ has finite index iff f is computed by a seq. transducer...

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Hankel matrix of f_0:

$H \in (\Gamma^*)^{\Sigma^* \times \Sigma^*}$

$H(s,t) = f_0(s\,t) - f_0(s)$

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ε	ε	a	b	aac	abc	...	aaca	...
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aa	ε	a	b	aac	abc	…	aaca	…
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S T-minimal if $\forall s \neq s' \in S$
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…	…	…	…	…	…	…	…	…
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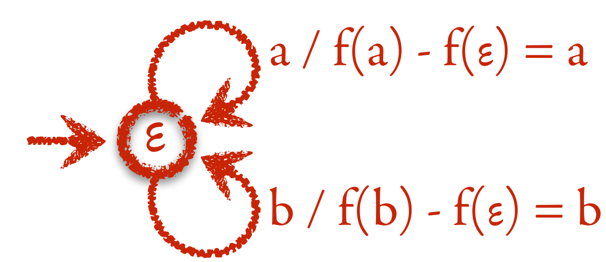

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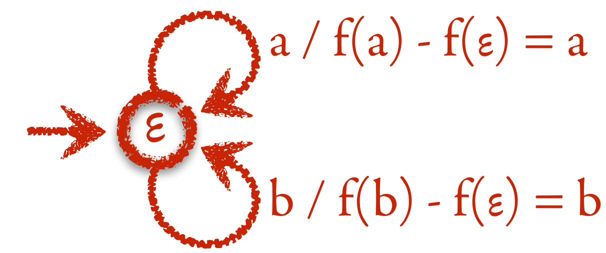

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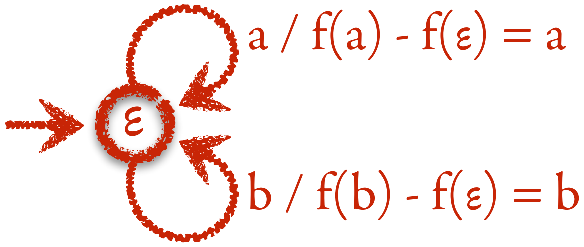

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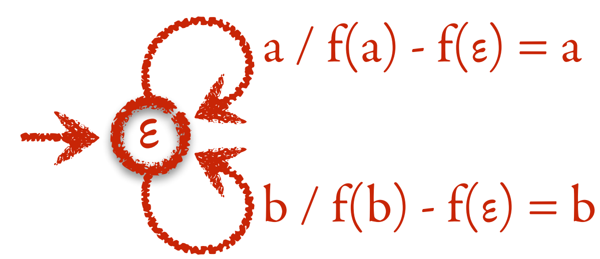

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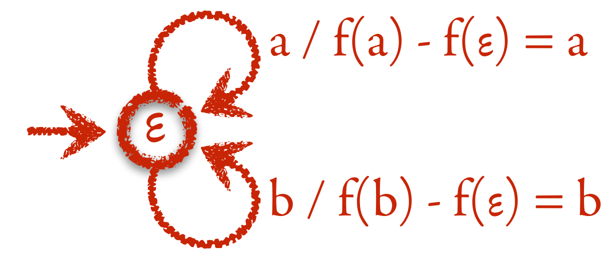

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a	**ε**	ac	**bc**	aca	**acb**	…	acaac	…
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aa	ε	a	b	aac	abc	…	aaca	…
ab	ε	a	b	aac	abc	…	aaca	…
…	…	…	…	…	…	…	…	…
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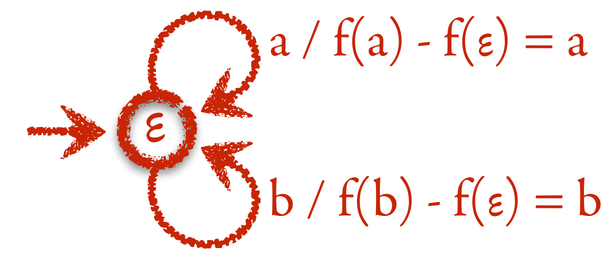

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Learning monotone word functions

Myhill-Nerode <u>equivalence</u>: $u \sim_{f_0} v$ if $\forall t \in \Sigma^*$ $f_0(u\,t) - f_0(u) = f_0(v\,t) - f_0(v)$

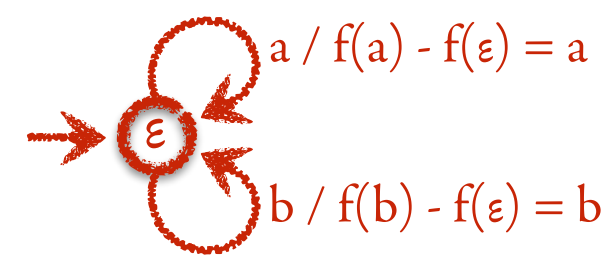
Relativization to test set T: $u \approx_{f_0, T} v$ if $\forall t \in T$ $f_0(u\,t) - f_0(u) = f_0(v\,t) - f_0(v)$

	ε	a	b	aa	ab	…	aaa	…
ε	**ε**	a	**b**	aac	**abc**	…	aaca	…
a	**ε**	ac	**bc**	aca	**acb**	…	acaac	…
b	ε	ac	bc	aca	acb	…	acaac	…
aa	ε	a	b	aac	abc	…	aaca	…
ab	ε	a	b	aac	abc	…	aaca	…
…	…	…	…	…	…	…	…	…
aaa	ε	ac	bc	aca	acb	…	acaac	…
…	…	…	…	…	…	…	…	…

start with

S = {ε} T = {ε}

build candidate transducer…



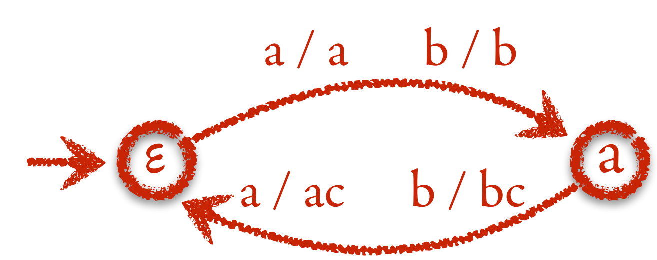
expand T by counter-example w = ab

S = {ε} T = {ε, b, ab}

expand S to make it T-complete…

S = {ε, a} T = {ε, b, ab}

build candidate transducer…



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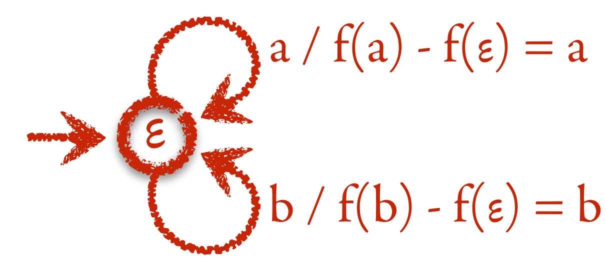
b	ε	ac	bc	aca	acb	...	acaac	...
aa	ε	a	b	aac	abc	...	aaca	...
ab	ε	a	b	aac	abc	...	aaca	...
...	...	...	...	...	...	...	...	...
aaa	ε	ac	bc	aca	acb	...	acaac	...
...	...	...	...	...	...	...	...	...

start with

S = {ε} T = {ε}

build candidate transducer...



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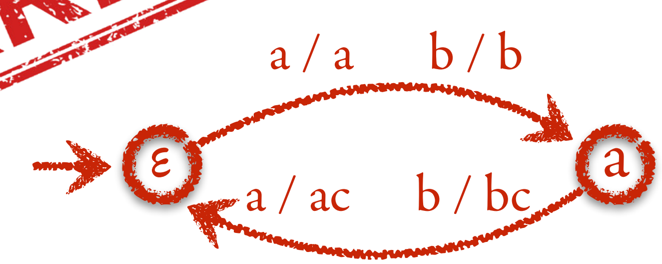
CORRECT

build candidate transducer...



From word languages to word functions… and beyond

This learning technique with Hankel matrices has been successfully applied to:

- Weighted automata (i.e. outputs given by products of weights along transitions) and, partially, to probabilistic automata
- Büchi automata
- Tree transducers (e.g. for learning XSLT transformations of XML documents)
- Timed & register automata (e.g. for processing strings with timestamps/data)

A simple yet useful real application (surprisingly, not yet done :/)

Implementing a learning algorithm for automata/transducers

would allow to automatically derive *RegEx* expressions like

```
/^(0?[1-9]|[12][0-9]|3[01])([ \/\-])(0?[1-9]|1[012])
\2([0-9][0-9][0-9][0-9])(([ -])([0-1]?[0-9]|2[0-3]):
[0-5]?[0-9]:[0-5]?[0-9])?$/
```

from positive and negative examples examples like

01/01/2000	ab/01/2000
18/10/1985	1/01/2000
…	18/10/200x
	…