COMPLEXITY OF SETS AND BINARY RELATIONS IN CONTINUUM THEORY: A SURVEY

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Contents

In the last few years there have been many applications of descriptive set theory to the study of continua. In this paper we focus on classification results for the complexity of natural sets of continua and binary relations between continua. Recent papers in this area include [Dar00, Kru02, Kru03, Kru04, CDM05, DM04]. However the subject is much older, as witnessed by papers such as [Kur31, Maz31]. On one hand continuum theory provides natural examples for many phenomena of descriptive set theory (e.g. natural sets requiring the difference hierarchy for their classification occur quite frequently in this area — see Theorems 2.35, 2.36 and 2.41 below). On the other hand descriptive set theory sheds some light on continuum theory, e.g. by explaining why some classes of continua do not have simple topological characterizations. In this paper we will survey both classical and recent results,

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and state some open problems. The present paper can be viewed as an update of a portion (pp. 9–11) of the survey by Becker ([Bec92]), which was much broader in scope.

The relationship between descriptive set theory and continuum theory involves many other topics not covered in this survey: these include universal sets and pairs of inseparable sets. Other striking applications of descriptive set theory to the study of continua include the results by Solecki ([Sol02b], see [Sol02a, §4] for a survey) about the space of composants of an indecomposable continuum, and by Becker and others ([Bec98, BP01]) about path components.

Nadler's monograph [Nad92] and Kechris's textbook [Kec95] are our main references for continuum theory and descriptive set theory, respectively.

Let us recall the basic notions of continuum theory (as in [Nad92], we will be concerned exclusively with continua that are metrizable, i.e. with *metric continua*):

- A *continuum* is a compact and connected metric space.
- A *subcontinuum* of a continuum C is a subset of C which is also a continuum.
- A continuum is *nondegenerate* if it contains more than one point (and hence it has the cardinality of the continuum).
- A continuum is *planar* if it is homeomorphic to a subset of $\mathbb{R}^2$.

Section 1 contains the necessary background in descriptive set theory: the reader already familiar with the basics of the subject can safely skip it and refer back to it when needed. In Section 2 we consider the complexity of natural classes of continua and sketch a few proofs illustrating some basic techniques. Subsection 2.4 includes full details of some unpublished proofs of H. Becker. Section 3 deals with binary relations among continua, and in particular with quasi-orders and equivalence relations; this section includes also results about the complexity of equivalence classes of an equivalence relation and of initial segments of a quasi-order.

1. Descriptive set theory

1.1. **Spaces of continua.** We start by describing how descriptive set theory deals with continua. If X is a compact metric space with (necessarily complete) metric d, we denote by $\mathsf{K}(X)$ the hyperspace of nonempty compact subsets of X, equipped with the Vietoris topology which is generated by the Hausdorff metric, denoted by d_H. Recall that if $K, L \in \mathsf{K}(X)$ we have

$$d_H(K,L) = \max\left\{\max_{x\in K} d(x,L), \max_{x\in L} d(x,K)\right\}.$$

$\mathsf{K}(X)$ is a compact metric space ([Kec95, §4.F] or [Nad92, chapter IV]). We denote by $\mathsf{C}(X)$ the subset of $\mathsf{K}(X)$ which consists of all connected elements, i.e. of all continua included in X. $\mathsf{C}(X)$ is closed in $\mathsf{K}(X)$ and, therefore, it is a compact metric space. In particular $\mathsf{C}(X)$ is separable and completely metrizable, i.e. a *Polish space*, and thus the typical ambient space for descriptive set theory.

Let I be the closed interval $[0,1]$. Every compact metric space, and in particular every continuum, is homeomorphic to a closed subset of the Hilbert cube I^ω. Hence $\mathsf{C}(I^\omega)$ is a compact metric space containing a homeomorphic copy of every continuum. We say that $\mathsf{C}(I^\omega)$ is the *Polish space of continua*. Similarly, $\mathsf{C}(I^2)$ is a compact metric space containing a homeomorphic copy of every planar continuum, and we say that $\mathsf{C}(I^2)$ is the *Polish space of planar continua*. Now we can study subsets of $\mathsf{C}(I^\omega)$, $\mathsf{C}(I^2)$, $\mathsf{C}(I^3)$, $\mathsf{C}(I^\omega) \times \mathsf{C}(I^\omega)$, etc., with the tools and techniques of descriptive set theory.

Suppose we are given a class of continua which is topological, i.e. invariant under homeomorphisms (a typical example is the class of continua which are locally connected). In light of the previous discussion it makes sense to identify the class with the set $\mathcal{P} \subseteq \mathsf{C}(I^\omega)$ of all subcontinua of I^ω belonging to the class, so that $\mathcal{P}$ can be studied with the tools and techniques of descriptive set theory. Similarly, by considering $\mathcal{P} \cap \mathsf{C}(I^2)$ as a subset of $\mathsf{C}(I^2)$ we can study the set of planar continua belonging to the class.

In a similar fashion we can translate a relationship between continua (typical examples are homeomorphism and continuous embeddability) into a binary relation on $\mathsf{C}(I^\omega)$ or $\mathsf{C}(I^2)$.

In some situations we are interested in studying metric, rather than topological, properties of continua (see e.g. §3.6 below). Since the space $\mathsf{C}(I^\omega)$ obviously does not contain an isometric copy of every continuum, it is no longer the appropriate setting for this study. We denote by M the Urysohn space: it is the unique, up to isometry, Polish space which contains an isometric copy of every Polish space. $\mathsf{C}(M)$ is Polish and contains an isometric copy of every continuum: we say that it is the *Polish space of metric continua*. We now view classes of continua which are isometric, i.e. invariant under isometries, as subsets of $\mathsf{C}(M)$.

We are interested in classes and relations which have been studied for their own sake in continuum theory (rather than being built *ad hoc* so that the corresponding set exhibits certain descriptive set theoretic features) and in this case we often say that the class or the relation is natural (this is a sociological, rather than mathematical, notion).

1.2. **Descriptive set theoretic hierarchies.** The main goal of the research surveyed in this paper is to establish the position in the descriptive set theoretic hierarchies of sets of continua arising from natural classes. We recall the basic definitions of the hierarchies of descriptive set theory (for more details see e.g. [Kec95]).

If X is a separable metric space we denote by $\mathbf{\Sigma}^0_1(X)$ the family of open subsets of X. Then for an ordinal $\alpha > 0$, $\mathbf{\Pi}^0_\alpha(X)$ is the family of all complements of sets in $\mathbf{\Sigma}^0_\alpha(X)$, while, for $\alpha > 1$, $\mathbf{\Sigma}^0_\alpha(X)$ is the class of countable unions of elements of $\bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta(X)$. At the lowest stages we have that $\mathbf{\Pi}^0_1(X)$ is the family of closed subsets of X, while sets in $\mathbf{\Sigma}^0_2(X)$ and $\mathbf{\Pi}^0_2(X)$ are respectively the F_σ and G_δ subsets of X. Moving a bit further in the hierarchy the $\mathbf{\Pi}^0_4$ sets are the $G_{\delta\sigma\delta}$ sets (i.e. countable intersections of countable unions of G_δ sets). It is straightforward to check that $\mathbf{\Sigma}^0_\alpha \cup \mathbf{\Pi}^0_\alpha \subseteq \mathbf{\Sigma}^0_\beta \cap \mathbf{\Pi}^0_\beta$ whenever $\alpha < \beta$. If X is an uncountable Polish space and $0 < \alpha < \beta < \omega_1$ we have $\mathbf{\Sigma}^0_\alpha \cup \mathbf{\Pi}^0_\alpha \neq \mathbf{\Sigma}^0_\beta \cap \mathbf{\Pi}^0_\beta$.

The fact that $\bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha(X) = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha(X)$ is exactly the collection of all Borel subsets of X leads to the name *Borel hierarchy*.

We then denote by $\mathbf{\Sigma}^1_1(X)$ the family of subsets of X which are continuous images of a Polish space. For $n > 0$, $\mathbf{\Pi}^1_n(X)$ is the class of all complements of sets in $\mathbf{\Sigma}^1_n(X)$, and $\mathbf{\Sigma}^1_{n+1}(X)$ is the family of continuous images of a set in $\mathbf{\Pi}^1_n(Y)$ for some Polish space Y. Again $\mathbf{\Sigma}^1_n \cup \mathbf{\Pi}^1_n \subseteq \mathbf{\Sigma}^1_m \cap \mathbf{\Pi}^1_m$ whenever $n < m$, and for uncountable Polish spaces the inclusion is strict. This hierarchy is the *projective hierarchy*. $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ sets are called resp. *analytic* and *coanalytic* sets.

We will also use the *difference hierarchy*: we restrict our discussion to the very first step of this hierarchy (see [Kec95, §22.E] for a complete definition of the hierarchy). A set is in $\mathrm{D}_2(\mathbf{\Sigma}^i_\alpha)$ if it is the intersection of a $\mathbf{\Sigma}^i_\alpha$ set and a $\mathbf{\Pi}^i_\alpha$ set. A set is in $\check{\mathrm{D}}_2(\mathbf{\Sigma}^i_\alpha)$ if it is the complement of a set in $\mathrm{D}_2(\mathbf{\Sigma}^i_\alpha)$ or, equivalently, if it is the union of a $\mathbf{\Sigma}^i_\alpha$ set and a $\mathbf{\Pi}^i_\alpha$ set.

By establishing the position of a natural class of continua in the Borel and projective hierarchies (i.e. the smallest family to which the set belongs), we obtain some information about the complexity of the class. This gives lower limits for the possibility of characterizing the class: e.g. if the class is $\mathbf{\Pi}^0_4$ and not $\mathbf{\Sigma}^0_4$, then any proposed characterization of the class as a countable union of countable intersections of F_σ sets is bound to be incorrect. In some cases the classification of a class of continua has immediate continuum theoretic consequences. To state an instance of this phenomena recall that a *model* for a class of continua is a continuum C_0 such that the continua in the class are exactly the continuous images of C_0 (the typical example here is the class of locally connected continua, which —by the Hahn-Mazurkiewicz theorem— has I as a model). Then a class of continua which is not $\mathbf{\Sigma}^1_1$ cannot have a model.

The main tool for establishing lower bounds on the complexity of a set is Wadge reducibility.

Definition 1.1. If X and Y are metric spaces, $A \subseteq X$, and $B \subseteq Y$, we say that A is *Wadge reducible* to B (and write $A \leq_{\mathrm{W}} B$) if there exists a continuous function $f : X \to Y$ such that for every $x \in X$, $x \in A$ if and only if $f(x) \in B$.

Notice that if e.g. B is $\mathbf{\Sigma}^0_\alpha$ and $A \leq_{\mathrm{W}} B$ then A is also $\mathbf{\Sigma}^0_\alpha$. Thus, proving that $A \leq_{\mathrm{W}} B$ for some A of known complexity yields a lower bound on the complexity of B.

Definition 1.2. If $\mathbf{\Gamma}$ is a class of sets in Polish spaces (like the classes $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$, $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$ introduced above), Y is a Polish space and $A \subseteq Y$, we say that A is $\mathbf{\Gamma}$*-hard* if $B \leq_{\mathrm{W}} A$ for every $B \subseteq X$ which belongs to $\mathbf{\Gamma}$, where X is a zero-dimensional Polish space. We say that A is $\mathbf{\Gamma}$*-complete* if, in addition, $A \in \mathbf{\Gamma}$.

If A is $\mathbf{\Gamma}$-hard and $A \leq_{\mathrm{W}} B$, then B is $\mathbf{\Gamma}$-hard: this is the typical way to prove $\mathbf{\Gamma}$-hardness.

It turns out that a set is $\mathbf{\Sigma}^0_\alpha$-complete if and only if it is $\mathbf{\Sigma}^0_\alpha$ but not $\mathbf{\Pi}^0_\alpha$, and similarly interchanging $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$. If a set is $\mathbf{\Pi}^1_n$-complete then it is not $\mathbf{\Sigma}^1_n$, and similarly interchanging $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$. If a set is $\mathrm{D}_2(\mathbf{\Sigma}^i_\alpha)$-complete then it is not $\check{\mathrm{D}}_2(\mathbf{\Sigma}^i_\alpha)$.

Most results we will survey in §2 state that a natural set of continua is $\mathbf{\Gamma}$-complete for some $\mathbf{\Gamma}$, and thus pinpoint the complexity of that particular set by showing that it belongs to $\mathbf{\Gamma}$ and not to any simpler class, in particular those of the complements of elements of $\mathbf{\Gamma}$.

We are not including in this survey sharpenings of the classification results, as those showing that a set of continua which is $\mathbf{\Gamma}$-complete for some $\mathbf{\Gamma}$, is actually homeomorphic to a well-known $\mathbf{\Gamma}$-complete set. Results of this kind are included e.g. in [DR94, CDGvM95, Sam03, KS].

Krupski (e.g. in [Kru02, Kru04]) and others have also studied the complexity of classes of continua within $\mathsf{C}(X)$, for X a compact metric space. A typical result of this kind states that, for a set of continua $\mathcal{P}$, a certain topological condition on X is sufficient for $\mathcal{P} \cap \mathsf{C}(X)$ to have in $\mathsf{C}(X)$ the same complexity that $\mathcal{P}$ has in $\mathsf{C}(I^\omega)$. These results are also not included in this survey, where we confine ourselves to subsets of $\mathsf{C}(I^n)$ for $2 \leq n \leq \omega$.

1.3. **Descriptive set theory and binary relations.** A binary relation on a Polish space X can be viewed as a subset of $X \times X$, and can be studied as such, e.g. by establishing its position in the descriptive set theoretic hierarchies described above. However this approach is much too crude, in that it neglects to take into account the particular features of a binary relation (actually what follows applies as well to n-ary relations for any $n > 1$). The following definition has been introduced and studied in depth in the context of equivalence relations, giving rise to the rich subject of "Borel reducibility for equivalence relations" ([FS89] is a pioneering paper, [BK96], [Hjo00] and [Kec02] are more recent accounts). More recently Louveau and

Rosendal ([LR]) extended its use to arbitrary binary relations, focusing in particular on quasi-orders.

Recall that a *quasi-order* is a binary relation which is reflexive and transitive. Therefore an equivalence relation is a quasi-order which is also symmetric. Moreover a quasi-order R on a set X induces naturally an equivalence relation $\sim_R$ on X defined by $x \sim_E y$ if and only if $x \, R \, y$ and $y \, R \, x$.

Definition 1.3. If R and S are binary relations on sets X and Y respectively, *a reduction of R to S* is a function $f : X \to Y$ such that

$$\forall x_0, x_1 \in X (x_0 \, R \, x_1 \iff f(x_0) \, S \, f(x_1)).$$

If X and Y are Polish spaces and f is Borel we say that R *is Borel reducible to* S, and we write $R \leq_{\mathrm{B}} S$. If $R \leq_{\mathrm{B}} S \leq_{\mathrm{B}} R$ then we say that R and S are *Borel bireducible*.

In practice we do not need the ambient spaces X and Y of Definition 1.3 to be Polish: it suffices that they are standard Borel spaces, i.e. that their Borel sets coincide with the Borel sets of a Polish topology on the same space. The basic fact we will implicitly use is that any Borel subset of a Polish space is standard Borel. This allows to study the behavior of a binary relation restricted to a Borel subset of $\mathsf{C}(I^\omega)$.

It is easy to see that if S is a quasi-order (resp. an equivalence relation) and $R \leq_{\mathrm{B}} S$, then R is a quasi-order (resp. an equivalence relation) as well. Moreover a reduction of the quasi-order R to the quasi-order S is also a reduction of the induced equivalence relation $\sim_R$ to the induced equivalence relation $\sim_S$.

If E and F are equivalence relations such that $E \leq_{\mathrm{B}} F$ we also say that the effective cardinality of the quotient space X/E is less than or equal to the effective cardinality of the quotient space Y/F: this is because there is a one-to-one function from X/E to Y/F that can be lifted to a Borel map from X to Y, i.e. the reduction of E to F. Another way of describing the fact that $E \leq_{\mathrm{B}} F$ is the following: we can assign in a Borel way F-equivalence classes as complete invariants for the equivalence relation E. Therefore the classification problem for F is at least as complicated as the classification problem for F.

As we already said, the research on Borel reducibility for equivalence relations has focused mainly on equivalence relations induced by continuous Polish group actions (or Borel bireducible to such an equivalence relation), under the headline of "descriptive dynamics". It is immediate to check that an equivalence relation of this kind on the Polish space X is $\boldsymbol{\Sigma}^1_1$ (as a subset of $X \times X$), and Miller proved (see [Kec95, Theorem 15.14]) that each equivalence class is Borel. Even in this restricted setting the structure of $\leq_{\mathrm{B}}$ is rich and complicated (e.g. see [AK00]).

Here we list only the definitions we will need to state the results surveyed in this paper.

Definition 1.4. An equivalence relation on a standard Borel space is *smooth* (or *concretely classifiable*, or *tame*) if it is Borel reducible to equality on some Polish space.

A smooth equivalence relation is considered to be very simple, since it admits "concrete" objects (i.e. elements of a Polish space) as complete invariants.

Definition 1.5. For $\mathcal{L}$ a countable relational language, let $X_{\mathcal{L}}$ be the Polish space of (codes for) $\mathcal{L}$-structures with universe $\mathbb{N}$ (see [Kec95, §16.C]). Let $\cong_{\mathcal{L}}$ denote isomorphism on $X_{\mathcal{L}}$. If E is an equivalence relation on a standard Borel space, E is *classifiable by countable structures* if $E \leq_{\mathrm{B}} \cong_{\mathcal{L}}$ for some $\mathcal{L}$; E is S_∞*-universal* if, in addition, $\cong_{\mathcal{L}} \leq_{\mathrm{B}} E$ for every $\mathcal{L}$.

The reason for the terminology "S_∞-universal" is that such an equivalence relation is as complicated as any equivalence relation induced by a continuous action of the infinite symmetric group S_∞ can be. An example of an equivalence relation which is S_∞-universal is homeomorphism on compact subsets of the Cantor space ([CG01]).

In [LR] Louveau and Rosendal started the study of Borel reducibility for equivalence relations induced by natural $\mathbf{\Sigma}^1_1$ quasi-orders and showed that several of these are $\mathbf{\Sigma}^1_1$-complete or K_σ-complete in the following sense. (Recall that a subset of a Polish space is K_σ if it is the countable union of compact sets.)

Definition 1.6. A quasi-order R on a Polish space is $\mathbf{\Sigma}^1_1$-*complete* (resp. K_σ-*complete*) if it is $\mathbf{\Sigma}^1_1$ (resp. K_σ) and $S \leq_{\mathrm{B}} R$ for any $\mathbf{\Sigma}^1_1$ (resp. K_σ) quasi-order S.

An equivalence relation E on a Polish space is $\mathbf{\Sigma}^1_1$-*complete* (resp. K_σ-*complete*) if it is $\mathbf{\Sigma}^1_1$ (resp. K_σ) and $F \leq_{\mathrm{B}} E$ for any $\mathbf{\Sigma}^1_1$ (resp. K_σ) equivalence relation F.

Since every equivalence relation is a quasi-order, if the quasi-order R is $\mathbf{\Sigma}^1_1$-complete then the induced equivalence relation $\sim_R$ is $\mathbf{\Sigma}^1_1$-complete among equivalence relations, and similarly for K_σ-complete. A $\mathbf{\Sigma}^1_1$-complete equivalence relation is immensely more complicated than any equivalence relation induced by any Polish group action (e.g. uncountably many of its equivalence classes are not Borel). An example of a $\mathbf{\Sigma}^1_1$-complete quasi-order is isometric embeddability between Polish spaces ([LR]). The same quasi-order, restricted to Heine-Borel Polish spaces (a metric space is Heine-Borel if its closed bounded subsets are compact), is K_σ-complete ([LR]).

2. Sets of continua

2.1. **Decomposability and Baire category arguments.** The following notions provide a basic distinction between continua.

Definition 2.1. A continuum C is *decomposable* if $C = C_1 \cup C_2$ where C_1 and C_2 are proper subcontinua of C. If a continuum is not decomposable then it is *indecomposable*.

Indecomposable continua form a dense $\mathbf{\Pi}^0_2$ in $\mathsf{C}(I^n)$ for $2 \leq n \leq \omega$ ([Nad92, Exercise 1.17]). Since the set of decomposable continua is also dense, a simple Baire category argument shows:

Fact 2.2. *The set of indecomposable continua is $\mathbf{\Pi}^0_2$-complete in $\mathsf{C}(I^n)$ for $2 \leq n \leq \omega$. The set of decomposable continua is $\mathbf{\Sigma}^0_2$-complete in $\mathsf{C}(I^n)$ for $2 \leq n \leq \omega$.*

Definition 2.3. A continuum is *hereditarily decomposable*, if all its nondegenerate subcontinua are decomposable. A continuum is *hereditarily indecomposable*, if all its subcontinua are indecomposable.

There exists a (necessarily unique) continuum (the *pseudoarc*) such that its homeomorphism class is dense $\mathbf{\Pi}^0_2$ in $\mathsf{C}(I^n)$ for $2 \leq n \leq \omega$ ([Bin51], see [Nad92, Exercise 12.70]). The pseudoarc has a fascinating history, briefly sketched in [Nad92, p. 228–229]. The pseudoarc is hereditarily indecomposable, and hereditarily indecomposable continua form also a dense $\mathbf{\Pi}^0_2$ set in $\mathsf{C}(I^n)$ for $2 \leq n \leq \omega$ ([Maz30], see [Nad92, Exercise 1.23.d]). Again using Baire category we immediately obtain:

Fact 2.4. *The set of hereditarily indecomposable continua is $\mathbf{\Pi}^0_2$-complete in $\mathsf{C}(I^n)$ for $2 \leq n \leq \omega$.*

Analogous Baire category arguments also establish the following easy results which can be considered more or less folklore (e.g. [CDM05] suggests the proofs, and [Kru03] contains a detailed proof of (b)):

Fact 2.5. *The following sets of continua are* $\mathbf{\Pi}^0_2$*-complete in* $\mathsf{C}(I^n)$ *for* $2 \leq n \leq \omega$*:*

(a) the set of unicoherent continua (a continuum C *is unicoherent if* $C_1 \cap C_2$ *is a continuum whenever* C_1 *and* C_2 *are subcontinua of* C *such that* $C = C_1 \cup C_2$*);*

(b) the set of hereditarily unicoherent continua (a continuum C *is hereditarily unicoherent if every subcontinuum of* C *is unicoherent or, equivalently, if the intersection of any two subcontinua of* C *is connected);*

(c) for $n \geq 2$ *the set of irreducible continua between* n *points (a continuum is irreducible between* n *points if it contains a set of* n *points which is not contained in any proper subcontinua);*

(d) the set of hereditarily irreducible continua (a continuum is hereditarily irreducible if all its nondegenerate subcontinua are irreducible between 2 *points).*

2.2. **Hereditarily decomposable continua and generalizations of Darji's argument.** While the classification of the sets of decomposable, indecomposable and hereditarily indecomposable continua is quite old, the precise classification of the set of hereditarily decomposable continua was obtained more recently by Darji ([Dar00]). There exists "nice" characterizations for indecomposable continua ([IC68]) and for hereditarily indecomposable continua ([Pro72]), but Theorem 2.6 below implies that nothing similar is possible for hereditarily decomposable continua.

Theorem 2.6. *The set* $\mathcal{HD}$ *of hereditarily decomposable continua is* $\mathbf{\Pi}^1_1$*-complete in* $\mathsf{C}(I^n)$ *for* $2 \leq n \leq \omega$*.*

Sketch of proof. We sketch Darji's proof that $\mathcal{HD}$ is $\mathbf{\Pi}^1_1$-hard (it follows immediately from Fact 2.2 that it is $\mathbf{\Pi}^1_1$). Let $D = \{\, \alpha \in 2^\omega \mid \exists m\, \forall n > m\ \alpha(n) = 0 \,\}$. D is a countable dense subset of the Cantor space 2^ω and it is a classical result due to Hurewicz (see e.g. [Kec95, Theorem 27.5]) that $A = \{\, C \in \mathsf{K}(2^\omega) \mid C \subseteq D \,\}$ is $\mathbf{\Pi}^1_1$-complete. Darji's proof consists in showing that $A \leq_{\mathrm{W}} \mathcal{HD}$.

To this end for every $\alpha \in 2^\omega$ we construct, using the basic continuum theoretic technique of nested intersections, a continuum $X_\alpha \subseteq I^2$ with the property that if $\alpha \in D$ then X_α is an arc (i.e. homeomorphic to I), while if $\alpha \notin D$ then X_α is nondegenerate indecomposable. Moreover there exists a point p_0 such that $\alpha \neq \alpha'$ implies $X_\alpha \cap X_{\alpha'} = \{p_0\}$, and if $\alpha \in D$ then p_0 is one of the endpoints of the arc X_α. The whole construction is done in such a way that the map $\alpha \mapsto X_\alpha$ is continuous, and this implies that if $C \in \mathsf{K}(2^\omega)$ then $X_C = \bigcup_{\alpha \in C} X_\alpha$ is a continuum and that the map $\mathsf{K}(2^\omega) \to \mathsf{C}(I^2)$, $C \mapsto X_C$ is continuous.

To show that the latter map is the desired reduction, we need to show that $C \in A$ if and only if $X_C \in \mathcal{HD}$. The backward direction is obvious (if $C \notin A$ fix $\alpha \in C \setminus D$: then X_α is a nondegenerate indecomposable subcontinuum of X_C). For the forward direction it suffices to notice that if $C \subseteq D$ then X_C is a countable union of arcs pairwise intersecting in a common endpoint: such a continuum is hereditarily decomposable. □

The construction sketched above has the property that when $C \in A$ then X_C is a quite simple continuum which enjoys more properties than just being hereditarily decomposable, and when $C \notin A$ then X_C is quite complicated. This immediately shows that many sets of continua are $\mathbf{\Pi}^1_1$-hard and hence, when they are $\mathbf{\Pi}^1_1$, are indeed $\mathbf{\Pi}^1_1$-complete. Some of these fairly easy consequences of Darji's proof where noticed in [CDM05] and in [Kru03]:

Corollary 2.7. *The following sets of continua are* $\mathbf{\Pi}^1_1$*-complete in* $\mathsf{C}(I^n)$ *for* $2 \leq n \leq \omega$*:*

(a) the set of continua which have no nondegenerate hereditarily indecomposable subcontinua;

(b) *the set of uniquely arcwise connected continua (a continuum C is uniquely arcwise connected if for all distinct $x, y \in C$ there exists a unique arc contained in C with end points x and y);*
(c) *the set of dendroids (a continuum is a dendroid if it is arcwise connected and hereditarily unicoherent);*
(d) *the set of λ-dendroids (a continuum is a λ-dendroid if it is hereditarily decomposable and hereditarily unicoherent).*

The technique of Darji's proof of Theorem 2.6 has also been generalized to obtain other results. One result of this kind is due to Darji and Marcone ([DM04]):

Theorem 2.8. *The set $\mathcal{HLC}$ of hereditarily locally connected continua is $\mathbf{\Pi}^1_1$-complete in $\mathsf{C}(I^n)$ for $2 \leq n \leq \omega$ (a continuum is hereditarily locally connected if all its subcontinua are locally connected).*

Sketch of proof. We use the notation of the sketch of proof of Theorem 2.6 and sketch the proof that $\mathcal{HLC}$ is $\mathbf{\Pi}^1_1$-hard (it follows from Theorem 2.10 below that it is $\mathbf{\Pi}^1_1$). We show that $A \leq_{\mathrm{W}} \mathcal{HLC}$ by constructing, using nested intersections, for every $\alpha \in 2^\omega$ a continuum X_α and considering the map $C \mapsto X_C$ as before. Again X_α is an arc when $\alpha \in D$, and a nondegenerate indecomposable continuum otherwise. Since nondegenerate indecomposable continua are not locally connected, if $C \notin A$ then $X_C \notin \mathcal{HLC}$.

However now we must make sure that $X_C \in \mathcal{HLC}$ when $C \in A$. The construction of the proof of Theorem 2.6 does not work for this, since in that case we have $X_C \notin \mathcal{HLC}$ whenever C is infinite. To solve this problem we must make sure that $X_\alpha \cap X_{\alpha'}$ is quite large, yet X_α and $X_{\alpha'}$ are still distinct. This goal is achieved by putting quite detailed requirements on the nested intersections used to define the X_α's. □

Another generalization of Darji's construction has been introduced by Krupski in [Kru03, §3]. Krupski proves a general lemma stating that any set of continua enjoying some properties with respect to a construction made with inverse limits of polyhedra is $\mathbf{\Pi}^1_1$-hard, and then considers some specific examples of his construction. Here is one of his results:

Theorem 2.9. *The set of strongly countable-dimensional continua is $\mathbf{\Pi}^1_1$-complete in $\mathsf{C}(I^\omega)$ (a continuum is strongly countable-dimensional if it is the countable union of compact finite-dimensional spaces).*

2.3. **Continua with strong forms of connectedness.** Sets of continua enjoying some strong form of connectedness (as the one classified by Corollary 2.7.(b) and Theorem 2.8) are obviously quite important.

The classification of locally connected continua (also called Peano continua, because of the well-known Hahn-Mazurkiewicz theorem stating that a continuum C is locally connected if and only if there exists $f : I \to C$ continuous and onto) is a classical result due independently to Kuratowski ([Kur31]) and Mazurkiewicz ([Maz31]). A modern proof is included in [CDM05].

Theorem 2.10. *The set of locally connected continua is $\mathbf{\Pi}^0_3$-complete in $\mathsf{C}(I^n)$ for $2 \leq n \leq \omega$.*

The following theorem is due independently to Ajtai and Becker ([Bec92], see [Kec95, Theorem 37.11] for a proof: there the results are stated for compacta, but the proofs actually deal with continua).

Theorem 2.11. *The set of arcwise connected continua is $\mathbf{\Pi}^1_2$-complete in $\mathsf{C}(I^n)$ for $3 \leq n \leq \omega$.*

Theorem 2.11 provides a quite rare example of a natural set whose classification involves the second level of the projective hierarchy, but fails to classify the set of planar arcwise connected continua. The following problem is of obvious importance:

Problem 2.12. Classify the complexity of the set of arcwise connected continua in $\mathsf{C}(I^2)$.

This set is clearly $\mathbf{\Pi}^1_2$ and Becker's proof of Theorem 2.11 shows that it is $\mathbf{\Sigma}^1_1$-hard and hence not $\mathbf{\Pi}^1_1$; Darji's proof of Theorem 2.6 shows that the set is $\mathbf{\Pi}^1_1$-hard and hence not $\mathbf{\Sigma}^1_1$ (the latter result was obtained independently by Just, unpublished). Putting together these proofs (e.g. with the technique used in the proof of Theorem 2.14 below) it can be shown that the set of arcwise connected continua is $\mathrm{D}_2(\mathbf{\Sigma}^1_1)$-hard.

2.4. **Simply connected continua.** In the 1980's Becker also studied the set $\mathcal{SC}$ of simply connected continua (a continuum C is simply connected if it is arcwise connected and has no holes, i.e. every continuous function from the unit circle into C can be extended to a continuous function from the closed unit disk into C). It is immediate that $\mathcal{SC}$ is $\mathbf{\Pi}^1_2$ in $\mathsf{C}(I^n)$ for every n. It turns out that the precise classification of $\mathcal{SC}$ depends on the dimension of the ambient space.

Theorem 2.13. *The set of simply connected continua $\mathcal{SC}$ is $\mathbf{\Pi}^1_1$-complete in $\mathsf{C}(I^2)$.*

A proof that $\mathcal{SC}$ is $\mathbf{\Pi}^1_1$-hard can be found in [Kec95, Theorem 33.17]. On the other hand Becker's original proof that the set of planar simply connected continua is $\mathbf{\Pi}^1_1$ involves a quite delicate argument based on the techniques of so-called *effective* descriptive set theory ([Bec86], see [Bec92] for a sketch of the proof). It has however been suggested (e.g. by the anonymous referee of this paper) that the ideas and results of [BP01] could be used to obtain a completely *classical*, and probably simpler, proof that $\mathcal{SC}$ is $\mathbf{\Pi}^1_1$ in $\mathsf{C}(I^2)$. The details of this new proof are still to be worked out.

The exact classification of $\mathcal{SC}$ in $\mathsf{C}(I^3)$ is unknown. The best known upper bound is the obvious one, namely $\mathbf{\Pi}^1_2$. The following theorem of Becker ([Bec87]) establishes a lower bound and implies that $\mathcal{SC}$ is more complex in $\mathsf{C}(I^3)$ than in $\mathsf{C}(I^2)$. We include here Becker's proof, which has never been published in print.

Theorem 2.14. *The set $\mathcal{SC}$ of simply connected continua is $\mathrm{D}_2(\mathbf{\Sigma}^1_1)$-hard in $\mathsf{C}(I^3)$.*

We first establish the following weaker result:

Lemma 2.15. *$\mathcal{SC}$ is $\mathbf{\Sigma}^1_1$-hard in $\mathsf{C}(I^3)$.*

Proof. The proof is based on the construction (also due to Becker, see [Bec84]) which shows that the set of arcwise connected continua is $\mathbf{\Sigma}^1_1$-hard in $\mathsf{C}(I^2)$. This construction is essentially contained in the proof that the set of arcwise connected continua is $\mathbf{\Pi}^1_2$-hard in $\mathsf{C}(I^3)$ published in [Kec95, Theorem 37.11]. Actually we do not need all details of the construction, and we summarize in the next paragraph the ones we will use.

We have a continuous function $T \mapsto L_T$ from the set of descriptive set-theoretic trees to $\mathsf{C}(I^2)$. We can assume that $L_T \subset I \times [0, 1/2]$, that the line segments $I \times \{0\}$ and $\{1\} \times [0, 1/2]$ (corresponding to l and $l_\emptyset$ in the notation of [Kec95]) are included in L_T, and that the point $(0, 1/2)$ (corresponding to r in Kechris's notation) belongs to L_T for any T. If T is well-founded L_T has exactly two arc components: the points $(0, 1/2)$ and $(0, 0)$ belong to these different arc components. If T is not well-founded then L_T is arcwise connected.

Let $M_T = L_T \cup (\{0\} \times [1/2, 1]) \subset I^2$: as far as arcwise connectedness is concerned, M_T has the same properties of L_T, so that in particular if T is well-founded $(0, 1)$

and $(0,0)$ belong to different arc components of M_T, and there are no other arc components of M_T.

Now we work in three-dimensional space and let $Z_0 = I^2 \times \{0\}$, $Z_1 = I^2 \times \{1\}$, $Y_0 = I \times \{0\} \times I$, $Y_1 = I \times \{1\} \times I$, $X_0 = \{0\} \times I^2$, and $X_1 = \{1\} \times I^2$. The set $A = Z_0 \cup Z_1 \cup Y_0 \cup Y_1$ is a cube with the interior of the opposite faces X_0 and X_1 removed. Let $B_T = M_T \times I$ and notice that half of each X_0 and X_1 is included in B_T for each T. Let $C_T = B_T \cup A$: the function $T \mapsto C_T$ from the set of descriptive set-theoretic trees to $\mathsf{C}(I^3)$ is clearly continuous and we claim that T is not well-founded if and only if $C_T \in \mathcal{SC}$. Since the set of well-founded descriptive set-theoretic trees is $\mathbf{\Pi}^1_1$-complete the claim completes the proof.

First of all notice that C_T is arcwise connected for every T, and hence we need to show that T is not well-founded if and only if C_T has no holes. It is clear that A has holes because e.g. a homeomorphism of the unit circle onto $A \cap X_0$ cannot be extended to a continuous function from the unit disk to A. Notice also that any continuous function from the unit circle to B_T can be extended to a continuous function from the unit disk to C_T with range intersecting Z_1 (the idea is that we can "lift" the circle in B_T to Z_1, where we can contract it to a single point without problems). Therefore the only possible reason for C_T being not simply connected is the hole in A, i.e. if a continuous function from the unit circle to C_T has no extension to a continuous function from the unit disk to C_T, it "goes around the hole in A".

If T is not well-founded then M_T is arcwise connected and let $X \subseteq M_T$ be an arc with endpoints $(0,1)$ and $(0,0)$. Now $X \times I \subseteq B_T$ "fills the hole" in A: e.g. the homeomorphism of the unit circle onto $A \cap X_0$ can be extended to the unit disk by a continuous function with range included in $X \times I$. Any continuous function from the unit circle to C_T which "goes around the hole in A" can be extended to a continuous function from the unit disk to C_T using $X \times I$, and therefore C_T has no holes.

If T is well-founded there is no arc in M_T with endpoints $(0,1)$ and $(0,0)$ and hence there is no arc in B_T with endpoints $(0,1,0)$ and $(0,0,0)$. This implies that each arc in C_T with endpoints $(0,1,0)$ and $(0,0,0)$ intersects either Z_0 or Z_1. It follows that the homeomorphism of the unit circle onto $A \cap X_0$ cannot be extended to a continuous function from the unit disk to C_T. Therefore C_T has holes. □

Proof of Theorem 2.14. By Lemma 2.15 and the part of Theorem 2.13 proved in [Kec95, Theorem 33.17] $\mathcal{SC}$ is both $\mathbf{\Sigma}^1_1$-hard and $\mathbf{\Pi}^1_1$-hard in $\mathsf{C}(I^3)$. This means that there exist continuous functions $T \mapsto C_T$ and $T \mapsto C'_T$ from the set of descriptive set-theoretic trees to $\mathsf{C}(I^3)$ such that T is well-founded if and only if $C_T \notin \mathcal{SC}$, if and only if $C'_T \in \mathcal{SC}$. Moreover we can assume $C_T \subseteq [0,1/3] \times I^2$, $C'_T \subseteq [2/3,1] \times I^2$, $(1/3,0,0) \in C_T$, and $(2/3,0,0) \in C'_T$ for every T.

We define a continuous function $(T,S) \mapsto D_{T,S}$ from the product of the set of descriptive set-theoretic trees with itself to $\mathsf{C}(I^3)$ by setting $D_{T,S} = C_T \cup ([1/3,2/3] \times \{0\} \times \{0\}) \cup C'_S$ (i.e. we are joining with a segment C_T and C'_S). $D_{T,S}$ is a continuum and it is simply connected if and only if T is not well-founded and S is well-founded. Since the set $\{\, (T,S) \mid T \text{ is not well-founded and } S \text{ is well-founded} \,\}$ is easily seen to be $\mathrm{D}_2(\mathbf{\Sigma}^1_1)$-complete, this completes the proof. □

The following problem is therefore still open.

Problem 2.16. Classify the complexity of $\mathcal{SC}$ in $\mathsf{C}(I^3)$.

Becker ([Bec87]) showed that the obvious upper bound for $\mathcal{SC}$ is sharp in $\mathsf{C}(I^n)$ for $n > 3$. Again we include the proof, which has not appeared in print elsewhere.

Theorem 2.17. *The set $\mathcal{SC}$ of simply connected continua is $\mathbf{\Pi}^1_2$-complete in $\mathsf{C}(I^n)$ for $4 \leq n \leq \omega$.*

Proof. We already noticed that $\mathcal{SC}$ is $\mathbf{\Pi}^1_2$ in $\mathsf{C}(I^n)$ for any n. The idea of the proof that $\mathcal{SC}$ is $\mathbf{\Pi}^1_2$-hard in $\mathsf{C}(I^4)$ is to lift up one dimension the proof of Lemma 2.15. (This is analogous to the technique used to prove that the set of arcwise connected continua is $\mathbf{\Pi}^1_2$-hard in $\mathsf{C}(I^3)$ starting from the proof that the set of arcwise connected continua is $\mathbf{\Sigma}^1_1$-hard in $\mathsf{C}(I^2)$.)

It suffices to show that $A \leq_{\mathrm{W}} \mathcal{SC} \cap \mathsf{C}(I^4)$ for any $\mathbf{\Pi}^1_2$ subset A of the Baire space $\mathbb{N}^\omega$. Any such A is of the form

$$\{\, x \in \mathbb{N}^\omega \mid \forall y \in 2^\omega \; \exists z \in \mathbb{N}^\omega \; \forall n \; (x[n], y[n], z[n]) \in S \,\}$$

for some descriptive set-theoretic tree S on $\mathbb{N} \times 2 \times \mathbb{N}$ (here $x[n]$ is the initial segment of x with length n). If $x \in \mathbb{N}^\omega$ and $y \in 2^\omega$ let

$$S(x,y) = \left\{\, s \in \mathbb{N}^{<\omega} \mid (x[\operatorname{lh} s], y[\operatorname{lh} s], s) \in S \,\right\}$$

(here $\operatorname{lh} s$ is the length of the finite sequence s). The function $(x,y) \mapsto S(x,y)$ from $\mathbb{N}^\omega \times 2^\omega$ to the set of descriptive set-theoretic trees is continuous, and we have $x \in A$ if and only if for every $y \in 2^\omega$ the tree $S(x,y)$ is not well-founded.

By Lemma 2.15 there exists a continuous function $T \mapsto C_T$ from the set of descriptive set-theoretic trees to $\mathsf{C}(I^3)$ such that T is not well-founded if and only if $C_T \in \mathcal{SC}$. We may assume $(0,0,0) \in C_T$ for every T. Moreover we can identify the Cantor space 2^ω with Cantor's middle third set, which is a subset of I. For each $x \in \mathbb{N}^\omega$ and $y \in 2^\omega$ let $U_{x,y} = C_{S(x,y)} \times \{y\}$. The function $\mathbb{N}^\omega \times 2^\omega \to \mathsf{C}(I^4)$, $(x,y) \mapsto U_{x,y}$ is clearly continuous, and $U_{x,y} \in \mathcal{SC}$ if and only if $S(x,y)$ is not well-founded.

For every $x \in \mathbb{N}^\omega$ let $E_x = \bigcup_{y \in 2^\omega} U_{x,y} \cup (\{0\}^3 \times I)$. To check that E_x is a continuum, notice that compactness follows from the continuity of $y \mapsto U_{x,y}$, and E_x is connected because the various $U_{x,y}$'s are joined by a segment. The function $x \mapsto E_x$ is easily seen to be continuous from $\mathbb{N}^\omega$ to $\mathsf{C}(I^4)$. Since $E_x \in \mathcal{SC}$ if and only if each $U_{x,y} \in \mathcal{SC}$, it is straightforward to check that $x \in A$ if and only if $E_x \in \mathcal{SC}$. □

Theorem 2.17 provides another example of a natural set which is $\mathbf{\Pi}^1_2$-complete.

2.5. **Continua which do not contain subcontinua of a certain kind.** The technique used to prove Theorem 2.9 was used by Krupski in [Kru03, §3] to prove results dealing with sets of continua which do not contain a copy of a fixed continuum:

Theorem 2.18. *The set of continua which do not contain the pseudo-arc is $\mathbf{\Pi}^1_1$-complete in $\mathsf{C}(I^n)$ for $2 \leq n \leq \omega$.*

The pseudo-arc can be characterized as the unique hereditarily indecomposable continuum which is the inverse limit of a sequence of arcs. A hereditarily indecomposable continuum which is the inverse limit of a sequence of circles but not the inverse limit of a sequence of arcs is called a *pseudo-solenoid*. There exists a unique (up to homeomorphism) planar pseudo-solenoid, which is called the *pseudo-circle*. Krupski used the technique discussed above to prove:

Theorem 2.19. *The set of continua which do not contain any pseudo-solenoid is $\mathbf{\Pi}^1_1$-complete in $\mathsf{C}(I^n)$ for $2 \leq n \leq \omega$. The set of continua which do not contain the pseudo-circle is $\mathbf{\Pi}^1_1$-complete in $\mathsf{C}(I^n)$ for $2 \leq n \leq \omega$.*

A similar result is the following, which is stated explicitly in [Kru03], but whose proof is based on Becker's construction for the hardness part of the proof of Theorem 2.13 (see [Kec95, p. 256–257]).

Theorem 2.20. *The set of continua which do not contain any circle is* $\mathbf{\Pi}^1_1$*-complete in* $\mathsf{C}(I^n)$ *for* $2 \leq n \leq \omega$.

In [Kru03, §4] Krupski proves, using techniques not directly related to Darji's construction, another result of this kind:

Theorem 2.21. *The set of continua which do not contain any arc is* $\mathbf{\Pi}^1_1$*-complete in* $\mathsf{C}(I^n)$ *for* $2 \leq n \leq \omega$.

This theorem is sharpened by the following two results, again proved in [Kru03, §4]:

Theorem 2.22. *The set of* λ*-dendroids which do not contain any arc is* $\mathbf{\Pi}^1_1$*-complete in* $\mathsf{C}(I^n)$ *for* $2 \leq n \leq \omega$.

Theorem 2.23. *The set of hereditarily decomposable continua which do not contain any arc is* $\mathbf{\Pi}^1_1$*-complete in* $\mathsf{C}(I^n)$ *for* $2 \leq n \leq \omega$.

In the same vein the following problem is interesting:

Problem 2.24. Classify the complexity of the set of continua which do not contain any hereditarily decomposable subcontinuum in $\mathsf{C}(I^n)$ for $2 \leq n \leq \omega$.

A straightforward computation shows that this set is $\mathbf{\Pi}^1_2$, and, as noticed by Krupski in [Kru03], the proof of Theorem 2.21 shows that it is $\mathbf{\Pi}^1_1$-hard.

A classification result concerning continua which do not contain point exhibiting a certain behavior is the following theorem of Krupski ([Kru02]):

Theorem 2.25. *The set of locally connected continua which do not contain any local cut point is* $\mathbf{\Pi}^0_3$*-complete in* $\mathsf{C}(I^n)$ *for* $2 \leq n \leq \omega$ *(*$x \in C$ *is a local cut point of the continuum* C *if there exists an open neighborhood* U *of* x *such that* $U \setminus \{x\}$ *is not connected).*

2.6. **More results by Krupski.** In [Kru04] Krupski studies other natural classes of continua. Here are some of his results:

Theorem 2.26. *The set of continua with the property of Kelley is* $\mathbf{\Pi}^0_3$*-complete in* $\mathsf{C}(I^n)$ *for* $2 \leq n \leq \omega$ *(a continuum* C *with compatible metric* d *has the property of Kelley if for every* $\epsilon > 0$ *there exists* $\delta > 0$ *such that whenever* $x, y \in C$ *are such that* $d(x,y) < \delta$ *and* D *is a subcontinuum of* C *with* $x \in D$ *there exists a subcontinuum* E *of* C *with* $y \in E$ *and* $d_H(D,E) < \epsilon$*).*

Theorem 2.27. *The set of arc continua is* $\mathbf{\Pi}^1_1$*-complete in* $\mathsf{C}(I^n)$ *for* $3 \leq n \leq \omega$ *(an arc continuum is a continuum such that all its proper nondegenerate subcontinua are arcs).*

Since planar arc continua do exist, the following problem is natural:

Problem 2.28. Classify the complexity of the set of arc continua in $\mathsf{C}(I^2)$.

Krupski studied also the following problem, which is still open:

Problem 2.29. Classify the complexity of the set of solenoids in $\mathsf{C}(I^n)$ for $3 \leq n \leq \omega$ (a solenoid is a continuum which is homeomorphic to the inverse limit of unit circles with maps $z \mapsto z^n$ for $n > 1$).

Solenoids are non-planar continua, so the problem above is not interesting in $\mathsf{C}(I^2)$. In [Kru04] it is shown that the set of solenoids is Borel and $\mathbf{\Pi}^0_3$-hard in $\mathsf{C}(I^n)$ for $3 \leq n \leq \omega$.

2.7. **Curves.** An important part of continuum theory is the theory of curves: a curve is a 1-dimensional continuum. Since a theorem of Mazurkiewicz (see e.g. [Nad92, Theorem 13.57]) asserts that every compact metric space of dimension at least 2 contains a nondegenerate indecomposable continuum, every hereditarily decomposable continuum is a curve. We already dealt with some sets of continua which are actually sets of curves: beside hereditarily decomposable continua, these include hereditarily locally connected continua and solenoids. We will now list some classification results (and an open problem) for sets of curves which are included in the set of hereditarily decomposable continua. Each statement (from Theorem 2.30 to Theorem 2.35 included) deals with a set of curves properly included in the ones considered in the statements preceding it.

Theorem 2.30. *The set of Suslinian continua is $\mathbf{\Pi}^1_1$-complete in $\mathsf{C}(I^n)$ for $2 \leq n \leq \omega$ (a continuum is Suslinian if each collection of its pairwise disjoint nondegenerate subcontinua is countable).*

This result is proved in [DM04], and it follows fairly easily from the proof of Theorem 2.6 and from the fact that a non Suslinian continuum contains a Cantor set of pairwise disjoint nondegenerate subcontinua ([CL78]).

Problem 2.31. Classify the complexity of the set of rational continua (a continuum C is *rational* if every point of C has a neighborhood basis consisting of sets with countable boundary).

The set of rational continua is easily seen to be $\mathbf{\Sigma}^1_2$ and $\mathbf{\Pi}^1_1$-hard. If it turns out to be $\mathbf{\Sigma}^1_2$-complete would be the first example of a natural set of continua with this classification. Notice that every hereditarily locally connected continuum is rational, but not viceversa.

Theorem 2.32. *The set of finitely Suslinian continua is $\mathbf{\Pi}^1_1$-complete in $\mathsf{C}(I^n)$ for $2 \leq n \leq \omega$ (a continuum C is finitely Suslinian if for every $\varepsilon > 0$ each collection of pairwise disjoint subcontinua of C with diameter $\geq \varepsilon$ is finite).*

This result is proved by Darji and Marcone in [DM04]: the hardness part follows from Theorem 2.8 since a planar continuum is hereditarily locally connected if and only if it is finitely Suslinian ([Lel71]). In general each finitely Suslinian continuum is hereditarily locally connected, but not viceversa.

The following theorem was also proved in [DM04]. We sketch the proof of the lower bound because it shares some common features with many other proofs in the area. To prove that a set of continua $\mathcal{P}$ is $\mathbf{\Gamma}$-hard we start from a continuum $C \notin \mathbf{\Gamma}$. Usually C was originally defined as a counterexample showing that $\mathcal{P}$ does not enjoy some property and/or does not coincide with some other set. We look for modifications to the construction of C which lead to a continuum belonging to $\mathcal{P}$. The modifications should depend continuously on some parameter ranging in a Polish space, and yield a member of $\mathcal{P}$ if and only if the parameter belongs to a $\mathbf{\Gamma}$-complete set. When this strategy succeeds, we have shown that $\mathcal{P}$ is $\mathbf{\Gamma}$-hard.

Theorem 2.33. *The set $\mathcal{R}$ of regular continua is $\mathbf{\Pi}^0_4$-complete in $\mathsf{C}(I^n)$ for $2 \leq n \leq \omega$ (a continuum C is regular if every point of C has a neighborhood basis consisting of sets with finite boundary).*

Sketch of proof. Recall the following characterization of regular continua due to Lelek ([Lel71]): a continuum C is regular if and only if for every $\varepsilon > 0$ there exists n such that every collection of n subcontinua of C of diameter $\geq \varepsilon$ is not pairwise disjoint. Using this characterization it can be shown fairly easily that $\mathcal{R}$ is $\mathbf{\Pi}^0_4$.

To prove that $\mathcal{R}$ is $\mathbf{\Pi}^0_4$-hard we apply the strategy discussed before the statement of the theorem. Let C be the continuum described by Nadler in [Nad92, Example

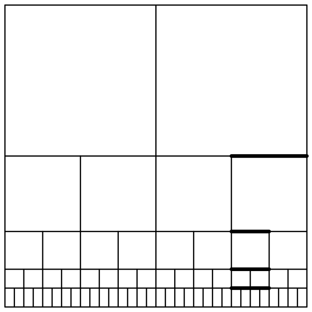

FIGURE 1. the continuum C of the proof of Theorem 2.33

10.38] and partially drawn in Figure 1 (the construction should be continued by adding infinitely many horizontal segments above the lowest segment, and 2^{n-1} new vertical segments from the n-th horizontal segment to the lowest segment). It is easy to check using Lelek's characterization that C is not regular (the horizontal segments are an infinite pairwise disjoint collection of subcontinua of the same diameter). C was originally introduced to show that a continuum which is the union of two regular continua (or even of two dendrites) need not to be regular, or even hereditarily locally connected.

If we modify C by deleting all but finitely many of the horizontal segments (and leaving all vertical segments and the bottom horizontal segment untouched), the resulting continuum will be regular. If we view the n-th horizontal segment (counting from the top down) as consisting of 2^n subsegments of equal length, to achieve regularity it suffices that for n sufficiently large we delete a portion of each of these subsegments (to make sure that the resulting set is a continuum we should delete a nonempty open connected subset of each subsegment).

We can further assign each of the subsegments to a column: a subsegment is in the first column if at least its leftmost part is in the left half of the figure, in the second column if at least its leftmost part is in the third quarter (counting from the left) of the figure, and in general belongs to the k-th column if at least its leftmost part is in the $2^k - 1$-th 2^{-k} piece (counting from the left) of the figure. In Figure 1 the subsegments belonging to the third column are thicker (notice that for $n \geq 3$, 2^{n-3} subsegments of the n-th horizontal segment belong to the third column). With this notation and using the characterization mentioned above, it can be shown that to obtain regularity it suffices that in each column only finitely many subsegments are not affected by our deletion process.

This leads to a way of reducing the $\mathbf{\Pi}^0_4$-complete set

$$A = \left\{ \alpha \in I^{\mathbb{N}\times\mathbb{N}} \mid \forall k \; \exists m \; \forall n > m \; \alpha(k,n) < 1 \right\}$$

to $\mathcal{R}$. For any $\alpha \in I^{\mathbb{N}\times\mathbb{N}}$ we define a continuum C_α obtained by deleting a (possibly empty) open connected subset of every horizontal subsegment belonging to column k and horizontal segment n. The size of the open subset we delete is dictated by $\alpha(k,n)$, so that in particular if $\alpha(k,n) = 1$ the subsegment is left untouched. This process can be set up so that the map $I^{\mathbb{N}\times\mathbb{N}} \to \mathsf{C}(I^2)$, $\alpha \mapsto C_\alpha$ is continuous.

If $\alpha \in A$ then in defining C_α we are deleting a nonempty subset of all but finitely many subsegments of each column, so that $C_\alpha \in \mathcal{R}$. If $\alpha \notin A$ there is at least a column where infinitely many subsegments are not affected by our deletion process. It follows that a subcontinuum of C_α is homeomorphic to C, and therefore $C_\alpha \notin \mathcal{R}$. Therefore $A \leq_{\mathrm{W}} \mathcal{R}$ and the proof is complete. □

This result is particularly interesting from a descriptive set theoretic viewpoint because natural sets appearing at the fourth level of the Borel hierarchy are quite rare (and the only claim to an example appearing at later levels is in [Sof02]).

Theorem 2.34. *The set of dendrites is $\mathbf{\Pi}^0_3$-complete in $\mathsf{C}(I^n)$ for $2 \leq n \leq \omega$ (a dendrite is a locally connected continuum which does not contain any circle).*

This result is proved by Camerlo, Darji and Marcone in [CDM05], essentially in the same way used in that paper for proving Theorem 2.10.

Dendrites are quite important planar continua (e.g. Nadler devotes a whole chapter of [Nad92] to their study, and the cover of the same book depicts a universal dendrite) and will play a role also in Section 3, where often the complexity of a binary relation will be studied on the set of dendrites.

Theorem 2.35. *The set of trees is $\mathrm{D}_2(\mathbf{\Sigma}^0_3)$-complete in $\mathsf{C}(I^n)$ for $2 \leq n \leq \omega$ (a tree is a dendrite which can be written as the union of finitely many arcs which pairwise intersect only in their end points).*

This theorem is also due to Camerlo, Darji and Marcone ([CDM05]) and is also particularly interesting from a descriptive set theoretic viewpoint: natural sets whose classification involves the difference hierarchy are uncommon. Recalling the definition of the difference hierarchy, Theorem 2.35 asserts that the set of trees is the intersection of a $\mathbf{\Sigma}^0_3$ and a $\mathbf{\Pi}^0_3$ set, but cannot be written as the union of a $\mathbf{\Sigma}^0_3$ and a $\mathbf{\Pi}^0_3$ set.

By removing from the definition of tree the requirement that they are dendrites, we obtain the important notion of a graph (a whole chapter of [Nad92] deals with graphs).

Theorem 2.36. *The set of graphs is $\mathrm{D}_2(\mathbf{\Sigma}^0_3)$-complete in $\mathsf{C}(I^n)$ for $2 \leq n \leq \omega$.*

This theorem is again proved in [CDM05] in the same way of Theorem 2.35, and again provides a rare example of a natural set whose classification needs the difference hierarchy.

The proof that the sets of trees and graphs are $\mathrm{D}_2(\mathbf{\Sigma}^0_3)$ is sketched after Theorem 3.11 below.

2.8. **Retracts.** Recall the following definition:

Definition 2.37. A separable metric space C is an *absolute retract* if whenever C is embedded as a closed subset in a separable metric space Y, there exists a continuous function $f : Y \to C$ which is the identity on C.

A compact absolute retract is a continuum, so any classification result about the set of absolute retracts in $\mathsf{K}(X)$ holds also in $\mathsf{C}(X)$.

Cauty, Dobrowolski, Gladdines, and van Mill ([CDGvM95]) studied the set of compacta which are absolute retracts in $\mathsf{K}(I^2)$.

Theorem 2.38. *The set of absolute retracts is $\mathbf{\Pi}^0_3$-complete in $\mathsf{C}(I^2)$.*

Dobrowolski and Rubin ([DR94]) studied the same set in $\mathsf{K}(I^n)$ for $3 \leq n \leq \omega$, and found the situation to be quite different.

Theorem 2.39. *The set of absolute retracts is $\mathbf{\Pi}^0_4$-complete in $\mathsf{C}(I^n)$ for $3 \leq n \leq \omega$.*

Let us also recall the definition of absolute neighborhood retract:

Definition 2.40. A separable metric space C is an *absolute neighborhood retract* if whenever C is embedded as a closed subset in a separable metric space Y, there exist a neighborhood U of C in Y and a continuous function $f : U \to C$ which is the identity on C.

Cauty, Dobrowolski, Gladdines, and van Mill in [CDGvM95] studied also the set of compacta which are absolute neighborhood retracts in $\mathsf{K}(I^2)$. Since there exist compact absolute neighborhood retracts which are not connected, one cannot immediately translate to our case results obtained in $\mathsf{K}(X)$. However in Remarque 3.10 of [CDGvM95] it is suggested how to adapt the proof of the result on compacta to obtain:

Theorem 2.41. *The set of continua which are absolute neighborhood retracts is* $\mathrm{D}_2(\mathbf{\Sigma}^0_3)$*-complete in* $\mathsf{C}(I^2)$*.*

Hence we have another example of a natural set of continua whose classification needs the difference hierarchy.

Dobrowolski and Rubin in [DR94] explicitly studied the set of absolute neighborhood retracts which are continua in dimension > 2 and found again the situation to be different than in the planar case:

Theorem 2.42. *The set of continua which are absolute neighborhood retracts is* $\mathbf{\Pi}^0_4$*-complete in* $\mathsf{C}(I^n)$ *for* $3 \leq n \leq \omega$*.*

2.9. **σ-ideals of continua.** In [Cam05] Camerlo studied σ-ideals of continua. Here is the definition:

Definition 2.43. Let X be a Polish space and $\emptyset \neq \mathcal{I} \subseteq \mathsf{C}(X)$. Then $\mathcal{I}$ is a *σ-ideal of continua* if:

(a) any subcontinuum of an element of $\mathcal{I}$ belongs to $\mathcal{I}$;
(b) if $C_n \in \mathcal{I}$ for all $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} C_n$ is a continuum, then $\bigcup_{n\in\mathbb{N}} C_n \in \mathcal{I}$.

Some of the sets of continua we mentioned before (e.g. the set of Suslinian continua and the set of strongly countable-dimensional continua) are actually σ-ideals.

The basic classification result obtained by Camerlo mirrors the dichotomy for $\mathbf{\Pi}^1_1$ σ-ideals of compact sets proved by Kechris, Louveau and Woodin ([KLW87]), stating that a $\mathbf{\Pi}^1_1$ σ-ideal of compact sets in a Polish space is either $\mathbf{\Pi}^0_2$ or $\mathbf{\Pi}^1_1$-complete. Here is Camerlo's theorem:

Theorem 2.44. *Let* $\mathcal{I} \subseteq \mathsf{C}(I^n)$ *for* $2 \leq n \leq \omega$ *be a* $\mathbf{\Pi}^1_1$ *σ-ideal of continua which is closed under homeomorphisms. Then* $\mathcal{I}$ *is either* $\mathbf{\Pi}^0_2$ *or* $\mathbf{\Pi}^1_1$*-complete.*

Using Theorem 2.44 the proofs of Theorems 2.9 and 2.30 can be simplified: to prove that the σ-ideal under consideration is $\mathbf{\Pi}^1_1$-complete it suffices to show that it is $\mathbf{\Pi}^1_1$ and $\mathbf{\Sigma}^0_2$-hard.

Kechris, Louveau and Woodin in [KLW87] also proved that a $\mathbf{\Sigma}^1_1$ σ-ideal of compact sets is $\mathbf{\Pi}^0_2$, and Camerlo mirrored also this result:

Theorem 2.45. *Let* $\mathcal{I} \subseteq \mathsf{C}(I^n)$ *for* $2 \leq n \leq \omega$ *be a* $\mathbf{\Sigma}^1_1$ *σ-ideal of continua which is closed under homeomorphisms. Then* $\mathcal{I}$ *is* $\mathbf{\Pi}^0_2$*.*

Camerlo remarks that the hypothesis in Theorems 2.44 and 2.45 that the σ-ideal is closed under homeomorphisms can be relaxed but cannot be totally deleted. [Cam05] contains several other results on σ-ideals of continua.

3. Binary relations between continua

3.1. **Homeomorphism.** The most natural equivalence relation between continua is that of homeomorphism. Kechris and Solecki proved that homeomorphism between compact metric spaces is induced by a Polish group action ([Hjo00, §4.4]). On the other hand, Hjorth ([Hjo00, §4.3]) proved the following theorem:

Theorem 3.1. *The equivalence relation of homeomorphism between locally connected continua, and a fortiori between arbitrary continua, in* $\mathsf{C}(I^n)$ *for* $3 \leq n \leq \omega$ *is strictly more complicated (in the sense of* $\leq_{\mathrm{B}}$*) than any equivalence relation classifiable by countable structures.*

(Hjorth actually talks about compacta, but his construction produces locally connected continua.) The most difficult part of the proof of Theorem 3.1 is showing that homeomorphism is not classifiable by countable structures: this is an application of Hjorth's theory of turbulence.

In contrast with Theorem 3.1, Camerlo, Darji and Marcone ([CDM05]) proved the following result:

Theorem 3.2. *The equivalence relation of homeomorphism between dendrites in* $\mathsf{C}(I^n)$ *for* $2 \leq n \leq \omega$ *is classifiable by countable structures.*

Theorem 3.2 shows that the homeomorphism type of a dendrite is easier to recognize than the homeomorphism type of an arbitrary locally connected continuum. Camerlo, Darji and Marcone showed also that the classification given by Theorem 3.2 is optimal, by showing that homeomorphism between dendrites is S_∞-universal and hence can still be considered very complicated. Their result is actually sharper and to state it we need the following definition.

Definition 3.3. If C is a continuum and $x \in C$ the *order of* x *in* C, denoted by $\mathrm{ord}(x, C)$, is the smallest cardinal number κ such that there exists a neighborhood-base for x in C consisting of open sets each with boundary of cardinality less than or equal to κ. A point $x \in C$ is a *branching point of* C if $\mathrm{ord}(x, C) > 2$.

Theorem 3.4. *The equivalence relation of homeomorphism between dendrites with all branching points belonging to an arc is* S_∞*-universal in* $\mathsf{C}(I^n)$ *for* $2 \leq n \leq \omega$.

The dendrites mentioned in this theorem are quite simple (see the remark after Fact 3.12 below). Theorem 3.4 (combined with Theorem 3.2) states that already at this level homeomorphism is as complicated as it can get when dendrites are concerned.

3.2. **Continuous embeddability.** A natural $\mathbf{\Sigma}^1_1$ quasi-order between continua is that of continuous embeddability. The sharpest result about it is due to Camerlo ([Cam04]), and improves previous results by Louveau and Rosendal ([LR]) and Marcone and Rosendal ([MR04]):

Theorem 3.5. *The quasi-order of continuous embeddability restricted to dendrites whose points have order at most* 3 *is* $\mathbf{\Sigma}^1_1$*-complete in* $\mathsf{C}(I^n)$ *for* $2 \leq n \leq \omega$.

As explained in §1.3, this implies that the equivalence relation of mutual continuous embeddability between dendrites (and, a fortiori, between arbitrary continua) is $\mathbf{\Sigma}^1_1$-complete (as an equivalence relation) and hence immensely more complicated than the equivalence relations considered in §3.1.

Theorem 3.5 implies also that for continuous embeddability the situation is different from the one illustrated in §3.1: restricting the quasi-order of continuous embeddability between continua to dendrites does not lead to a simpler quasi-order.

3.3. **Continuous surjections.** In §3.2 we considered embeddings, i.e. injective maps between continua. However Darji pointed out that in continuum theory continuous surjections (epimorphisms) are at least as important as continuous injections (see e.g. [Nad92, Theorem 3.21]), and raised the question of the complexity of the corresponding quasi-order. Let us define $C \preceq_e D$ if and only if there exists $f : D \to C$ continuous and onto. Camerlo ([Cam]) answered Darji's question by proving:

Theorem 3.6. *The quasi-order of epimorphism between continua is $\mathbf{\Sigma}^1_1$-complete in $\mathsf{C}(I^n)$ for $2 \leq n \leq \omega$.*

Actually Camerlo's proof shows that the result holds also if we restrict the kind of surjective maps we are considering (the definition of $\leq_{\mathrm{B}}$ is such that neither of these corollaries of the proof of Theorem 3.6 implies its statement or follows from it):

Corollary 3.7. *The quasi-order of monotone epimorphism between continua is $\mathbf{\Sigma}^1_1$-complete in $\mathsf{C}(I^n)$ for $2 \leq n \leq \omega$ (a continuous function between continua is monotone if the preimage of every point is a continuum).*

Corollary 3.8. *The quasi-order of weakly confluent epimorphism between continua is $\mathbf{\Sigma}^1_1$-complete in $\mathsf{C}(I^n)$ for $2 \leq n \leq \omega$ (a continuous function between continua is weakly confluent if every subcontinua of the image is image of a subcontinuum of the domain).*

As far as $\preceq_e$ is concerned it does not make sense to study dendrites (as we did in Theorems 3.2, 3.4, and 3.5). In fact the Hahn-Mazurkiewicz theorem and the fact that every continuum is homeomorphic to a subset of I^ω imply that all nondegenerate locally connected continua (and in particular all dendrites) are equivalent with respect to the equivalence relation induced by $\preceq_e$. However if we require that the surjective function is not only continuous, but also open, dendrites become interesting once more, and Camerlo ([Cam]) proved:

Theorem 3.9. *The quasi-order of open epimorphism between dendrites (and, a fortiori, between continua) is $\mathbf{\Sigma}^1_1$-complete in $\mathsf{C}(I^n)$ for $2 \leq n \leq \omega$.*

3.4. **Likeness and quasi-homeomorphism.** An important quasi-order on continua is the relation of likeness (e.g. a whole chapter of [Nad92] deals with arc-like continua).

Definition 3.10. If $\mathcal{C}$ is a class of continua and D is a continuum we say that D is *$\mathcal{C}$-like* if for every $\varepsilon > 0$ there exist $C \in \mathcal{C}$ and a continuous map $f : D \to C$ such that f is onto and $\{ x \in D \mid f(x) = y \}$ has diameter less than ε for each $y \in C$.

When $\mathcal{C} = \{C\}$ we say that D is *C-like* and write $D \preceq C$. The equivalence relation induced by likeness is called *quasi-homeomorphism*.

Camerlo, Darji and Marcone ([CDM05]) studied extensively likeness, mainly when $\mathcal{C}$ is a set of dendrites or graphs. They obtained complexity results for many initial segments of $\preceq$, including the following:

Theorem 3.11. *(a) If C is a nondegenerate dendrite or a graph, then the set of C-like continua is $\mathbf{\Pi}^0_2$-complete in $\mathsf{C}(I^n)$ for $2 \leq n \leq \omega$;*
(b) the set of graph-like continua is $\mathbf{\Pi}^0_2$-complete in $\mathsf{C}(I^n)$ for $2 \leq n \leq \omega$;
(c) the set of tree-like continua is $\mathbf{\Pi}^0_2$-complete in $\mathsf{C}(I^n)$ for $2 \leq n \leq \omega$.

In particular Theorem 3.11.(a) implies that the sets of arc-like continua and of circle-like continua are both $\mathbf{\Pi}^0_2$-complete in $\mathsf{C}(I^n)$ for $2 \leq n \leq \omega$.

Theorem 3.11 is used to obtain the upper bounds for Theorems 2.35 and 2.36. In fact, by a result of Kato and Ye ([KY00]), if C is a locally connected continuum, D a graph (resp. tree) and $C \preceq D$ then C is also a graph (resp. tree). Hence if $\{ D_n \mid n \in \mathbb{N} \}$ is a sequence of graphs (resp. trees) which contains at least a member from each quasi-homeomorphism class of graphs (resp. trees), then a continuum C is a graph (resp. tree) if and only if C is locally connected and $C \preceq D_n$ fo some n. By Theorems 2.10 and 3.11.a this shows that the set of graphs (resp. trees) is $\mathrm{D}_2(\mathbf{\Sigma}^0_3)$.

3.5. **Some homeomorphism and quasi-homeomorphism classes.** The fact that each homeomorphism class of a continuum is a Borel subset of $\mathsf{C}(I^\omega)$ is a classical result due to Ryll-Nardzewski ([RN65]). The following fact is a consequence of Theorem 3.4:

Fact 3.12. *For every $\alpha < \omega_1$ there exists a dendrite C with all branching points belonging to an arc such that the homeomorphism class of C in $\mathsf{C}(I^n)$ for $2 \leq n \leq \omega$, is not $\mathbf{\Pi}^0_\alpha$.*

This means that the set of dendrites with all branching points belonging to an arc (and a fortiori the sets of dendrites and of continua) is partitioned into homeomorphism classes of unbounded Borel complexity. (The intuitive fact that the dendrites with all branching points belonging to an arc —already mentioned in Theorem 3.4— are "simple" is made precise by the following result of [CDM05]: there exist exactly two $\preceq$-minimal quasi-homeomorphism classes of dendrites with infinitely many branching points, and one of them consists precisely of all dendrites with infinitely many branching points which all belong to an arc.)

If we fix a specific continuum we can study the complexity of the homeomorphism and quasi-homeomorphism class of that continuum. To this end the following theorem, which combines results of Segal ([Seg68], rediscovered by Kato and Ye in [KY00]) and Camerlo, Darji and Marcone ([CDM05]) is useful:

Theorem 3.13. *Let C be either a graph or a dendrite with finitely many branching points. A continuum is homeomorphic to C if and only if it is quasi-homeomorphic to C.*

In general quasi-homeomorphism is coarser than homeomorphism, but Theorem 3.13 states that for fairly simple continua the two equivalence relation coincide.

Camerlo, Darji and Marcone ([CDM05]) proved:

Theorem 3.14. *Let C be either a graph or a nondegenerate dendrite. The quasi-homeomorphism class of C is $\mathbf{\Pi}^0_3$-complete in $\mathsf{C}(I^n)$ for $3 \leq n \leq \omega$. If C is planar (e.g. if it is a dendrite) then this holds also in $\mathsf{C}(I^2)$.*

The proof of Theorem 3.14 uses the results of Theorem 3.11.

Combining the last two Theorems we obtain the following result ([CDM05]):

Theorem 3.15. *Let C be either a graph or a dendrite with finitely many branching points. The homeomorphism class of C is $\mathbf{\Pi}^0_3$-complete in $\mathsf{C}(I^n)$ for $3 \leq n \leq \omega$. If C is planar then this holds also in $\mathsf{C}(I^2)$.*

Some instances of Theorem 3.15 are much older: e.g. $\mathbf{\Pi}^0_3$-completeness of the set of all arcs follows from the results in [Kur31] and [Maz31].

The following fact follows by a Baire category argument from the remarks before Fact 2.4:

Fact 3.16. *The homeomorphism class of the pseudo-arc is $\mathbf{\Pi}^0_2$-complete in $\mathsf{C}(I^n)$ for $2 \leq n \leq \omega$.*

In [Kru02] Krupski studied the homeomorphism class of two important continua: Sierpiński universal curve M^2_1 (see [Nad92, Example 1.11]) and Menger universal curve M^3_1 (see e.g. [Eng78, p.122]). The former contains a homeomorphic copy of every 1-dimensional planar compacta, while the latter contains a homeomorphic copy of every 1-dimensional compacta. Krupski proved:

Theorem 3.17. *The homeomorphism class of the Sierpiński universal curve is $\mathbf{\Pi}^0_3$-complete in $\mathsf{C}(I^n)$ for $2 \leq n \leq \omega$.*

Theorem 3.18. *The homeomorphism class of the Menger universal curve is $\mathbf{\Pi}^0_3$-complete in $\mathsf{C}(I^n)$ for $3 \leq n \leq \omega$.*

3.6. **Isometry and Lipschitz isomorphism.** The equivalence relation of isometry between Polish metric spaces is very complicated and has been studied in depth by Gao and Kechris ([GK03]). However if we restrict ourselves to compact Polish metric spaces, Gromov ([Gro99], see [Hjo00, §4.4] for a sketch) has shown that isometry is smooth. This immediately yields (recall from page 3 that M denotes the Urysohn space):

Theorem 3.19. *The equivalence relation of isometry between continua in $\mathsf{C}(M)$ is smooth.*

Rosendal ([Ros]) studied the metric equivalence relation of Lipschitz isomorphism: two metric spaces C and C' with metrics d and d' are *Lipschitz isomorphic* if there exist a bijection $f : C \to C'$ and $c \geq 1$ such that $\frac{1}{c}d(x,y) \leq d'(f(x), f(y)) \leq c\,d(x,y)$ for every $x, y \in C$ (obviously such an f is a homeomorphism). Rosendal proved:

Theorem 3.20. *The equivalence relation of Lipschitz isomorphism between continua in $\mathsf{C}(M)$ is K_σ-complete.*

Rosendal's proof uses continua which are not locally connected, but Rosendal himself pointed out that a straightforward modification yields also:

Theorem 3.21. *The equivalence relation of Lipschitz isomorphism between dendrites in $\mathsf{C}(M)$ is K_σ-complete.*

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