Comparing expressiveness of set constructor symbols

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Abstract. In this paper we consider the relative expressive power of two very common operators applicable to sets and multisets: the *with* and the *union* operators. For such operators we prove that they are not mutually expressible by means of existentially quantified formulae. In order to prove our results, canonical forms for set-theoretic and multiset-theoretic formulae are established and a particularly natural axiomatization of multisets is given and studied.
Keywords. *Sets and Multisets Theory, Unification, Constraints.*

Introduction

In the practice of programming, as well in axiomatic study of set (and multiset) theory, two families of constructors are usually employed:

- Constructors of the form (*with*) that can build a (possibly) new set by adding one element to a given set. A typical element of this family is the *cons*-constructor of lists;
- Constructors (*union*) that can build a set made by all the elements of two given sets. A typical element of this family is the *append*-constructor of lists.

Bernays in [5] was the first to use the *with* constructor symbol in his axiomatization of Set Theory. Vaught in [22] proved, by giving an essential undecidability result, that theories involving such a constructor are extremely expressive and powerful. On the other hand, classical set theories (e.g., Zermelo-Fraenkel [16]) are more often based on union-like symbols (either unary or binary) and are sufficiently powerful to define both *with* and *union*.

In this paper we analyze the relationships between these two (kinds of) operators. We show that in both a set and a multiset setting, it is impossible to express the union-like operators with existentially quantified formulae based on with-like operators and vice versa. These results hold in any set theory sufficiently expressive to introduce the above operators.

With-like constructors can be associated with an equational theory containing two axioms (left commutativity (C_ℓ) and absorption (A_b)—cf., e.g., [13]) while union-like symbols are associated to $ACI1$ equational theories (see, e.g., [4]).

A by-product of the results of this paper is a systematic proof of the fact that the classes of formulae and problems that can be expressed with an $(A_b)(C_\ell)$ unification (constraint) problem and those concerned with $ACI(1)$ unification (constraint) problems with constants can not be (trivially) reduced to each other.

Other consequences of the results of this paper are criteria for choosing the classes of constraints that can be managed by a constraint solver (e.g., for programming with constraints, or when analyzing programs by making use of constraints), or for choosing the basic operators for dealing with sets in programming languages.

In Section 1 we formally introduce the problem faced in this paper and we fix the notation. In Section 2 we show that it is impossible to express the *with* using union-like constraints in a set framework. In Section 3 we show the vice versa. Section 4 shows how to apply the results obtained for expressiveness of unification problems and constraints. In Section 5 the results are repeated in the context of multisets. Finally, some conclusions are drawn.

1 Preliminaries

We assume standard notions and notation of first-order logic (cf. [20]). We use $\mathcal{L}$ as a meta-variable for first-order languages with equality whose variables are denoted by capital letters. We write $\varphi(X_1, \dots, X_n)$ for a formula of $\mathcal{L}$ with $X_1, \dots, X_n$ as free variables; when the context is clear, we denote a list $Z_1, \dots, Z_n$ of variables by $\boldsymbol{Z}$. The symbol false stands for a generic unsatisfiable formula, such as $X \neq X$. A formula without quantifiers is said *open* and FV is a function returning the set of free variables of a formula.

Definition of the problem. Let $\mathbb{T}$ be a first-order theory over the language $\mathcal{L}$ and Φ a class of (generally open) $\mathcal{L}$-formulae. Let f be a function symbol such that $\mathbb{T} \models \forall \boldsymbol{X} \exists Y\, (Y = f(\boldsymbol{X}))$. We say that f can be *existentially expressed* by Φ in $\mathbb{T}$ if there is $\psi \in \Phi$ such that

$$\mathbb{T} \models \forall \boldsymbol{X} Y\, (Y = f(\boldsymbol{X}) \leftrightarrow \exists \boldsymbol{Z} \psi(\boldsymbol{X}, Y, \boldsymbol{Z}))$$

Assume $\mathcal{L}$ contains equality '$=$' and membership '$\in$' as binary predicate symbols, the constant symbol $\emptyset$ for the empty set, and the binary function symbols $\{\cdot \mid \cdot\}$ for set construction *with*,[1] and $\cup$ for set union. Assume that $\mathbb{T}$ is any set theory such that these three symbols are governed by the following axioms:[2]

$$\begin{array}{lll} (N) & \forall X & (X \notin \emptyset) \\ (W) & \forall Y\, V\, X & (X \in \{\, Y \mid V \,\} \leftrightarrow (X \in V \vee X = Y)) \\ (\cup) & \forall Y\, V\, X & (X \in Y \cup V \leftrightarrow (X \in Y \vee X \in V)) \end{array}$$

[1] The interpretation of $\{x \mid y\}$ is $\{x\} \cup y$, hence $\{\cdot \mid \cdot\}$ is unnecessary whenever $\cup$ and the singleton set operator are present. Notice that $\{\cdot \mid \cdot\}$ is a list-like symbol and not the intensional set former $\{x : \varphi(x)\}$.

[2] These axioms are explicitly introduced in minimal set theories (e.g. [5, 22, 17]) while they are consequence of other axioms in stronger ones (e.g. ZF, cf. [16]).

and also assume that the extensionality principle (E) and the regularity axiom (R) hold in $\mathbb{T}$:

$$
\begin{array}{lll}
(E) & \forall X\, Y & (\forall Z\, (Z \in X \leftrightarrow Z \in Y) \rightarrow X = Y) \\
(R) & \forall X\, \exists Z\, \forall Y & (Y \in X \rightarrow (Z \in X \wedge Y \notin Z)).
\end{array}
$$

Let $\Phi_\cup$ and $\Phi_{\texttt{with}}$ be the classes of open formulae built using $\emptyset, \cup, \in, =$ and $\emptyset, \{\,\cdot \mid \cdot\,\}, \in, =$, respectively. We will prove that for any $\mathbb{T}$ satisfying the above axioms, the symbol $\{\,\cdot \mid \cdot\,\}$ can not be existentially expressed by $\Phi_\cup$ in $\mathbb{T}$ and $\cup$ can not be existentially expressed by $\Phi_{\texttt{with}}$ in $\mathbb{T}$.

Let $\mathbb{HF}$ (the class of hereditarily finite well-founded sets) be the model whose domain $\mathcal{U}$ can be inductively defined as $\mathcal{U} = \bigcup_{i \geq 0} U_i$ with $U_0 = \emptyset, U_{i+1} = \wp(U_i)$, (where $\wp$ stands for the power-set operator) and whose interpretation for $\emptyset, \{\,\cdot \mid \cdot\,\}, \cup, \in, =$ is the natural one. $\mathbb{HF}$ is a submodel of any model of a set theory satisfying the above axioms and any formula of $\Phi_\cup$ and $\Phi_{\texttt{with}}$ is satisfiable if and only if it is satisfiable over $\mathbb{HF}$ (cf. [6]). Elements of $\mathcal{U}$ can be represented by directed acyclic graphs where edges represent the membership relation.

Technical notations. We use the following syntactic convention for the set constructor symbol $\{\,\cdot \mid \cdot\,\}$: the term $\{\, s_1 \mid \{\, s_2 \mid \cdots \{\, s_n \mid t\,\} \cdots\}\}$ will be denoted by $\{s_1, \ldots, s_n \mid t\}$ or simply by $\{s_1, \ldots, s_n\}$ when t is $\emptyset$. The function *rank*, defined as: $\mathit{rank}(\emptyset) = 0, \mathit{rank}(\{t \mid s\}) = \max\{\mathit{rank}(s), 1 + \mathit{rank}(t)\}$ returns the maximum 'depth' of a ground set, while the function *find*:

$$
\mathit{find}(x,t) = \begin{cases}
\emptyset & \text{if } t = \emptyset,\ x \neq \emptyset \\
\{0\} & \text{if } t = x \\
\{1 + n : n \in \mathit{find}(x,y)\} & \text{if } t = \{y \mid \emptyset\} \\
\{1 + n : n \in \mathit{find}(x,y)\} \cup \mathit{find}(x,s) & \text{if } t = \{y \mid s\},\ s \neq \emptyset
\end{cases}
$$

returns the set of 'depths' at which a given element x occurs in the set t (there is an exception for the unique case $\mathit{find}(\emptyset, \emptyset)$). For instance, if t is $\{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$, then $\mathit{rank}(t) = 3$ (it is sufficient to compute the maximum nesting of braces), and $\mathit{find}(\emptyset, t) = \{2, 3\}, \mathit{find}(\{\emptyset\}, t) = \{1, 2\}, \mathit{find}(\{\emptyset, \{\emptyset\}\}, t) = \{1\}$. The two above-defined functions will be used in Lemma 3 to build a suitable truth valuation for (canonical) formulae over $\mathbb{HF}$. We also denote by $\{\emptyset\}^n$ the (simplest) singleton set of rank n, that is the set inductively defined as: $\{\emptyset\}^0 = \emptyset, \{\emptyset\}^{n+1} = \{\, \{\emptyset\}^n\}$.

2 Union vs *with*

In this section we show show that the function symbol $\{\,\cdot \mid \cdot\,\}$ can not be existentially expressed by the class of formulae $\Phi_\cup$. To prove this fact it is sufficient to verify that it is impossible to express the singleton operator, namely the formula $X = \{Y\}$, i.e., $X = \{Y \mid \emptyset\}$:

Lemma 1. *Let φ be a satisfiable formula of $\Phi_\cup$. Then there is a satisfying valuation σ over $\mathbb{HF}$ for φ s.t. for all $X \in FV(\varphi)$ either $\sigma(X) = \emptyset$ or $|\sigma(X)| \geq 2$.*

Proof. Without loss of generality, we can restrict our attention to flat formulae in DNF, namely disjunctions of conjunctions of literals of the form: $X \; op \; Y, X = Y \cup Z, X = \emptyset$, where op can be $=, \neq, \in, \notin$, and X, Y, Z are variables. If φ is satisfiable, then it is satisfiable over $\mathbb{HF}$; let σ be a valuation which satisfies φ over $\mathbb{HF}$ and consider a disjunct ψ satisfied by σ. Assume σ does not fulfill the requirements (otherwise we have directly the thesis) and consider the term substitution $\theta = [X/\emptyset : \sigma(X) = \emptyset]$; let ψ' be the formula obtained by

1. removing from $\psi\theta$ all the literals of the form $\emptyset = \emptyset, \emptyset = \emptyset \cup \emptyset, X \notin \emptyset$, $\emptyset \notin \emptyset$,
2. replacing the literals of the form $X = Y \cup \emptyset, X = \emptyset \cup Y$ with $X = Y$, and
3. replacing all literals of the form $\emptyset \neq X$ with $X \neq \emptyset$.

All the literals of ψ' are of the form $X = Y \cup Z, X = Y, X \neq Y, X \neq \emptyset, X \in Y, \emptyset \in Y, X \notin Y, \emptyset \notin Y$ and $\sigma(X) \neq \emptyset$ for all $X \in FV(\psi')$.[3]

Let $\bar{n} = \max_{X \in FV(\psi')} rank(\sigma(X))$ and let c be a set of rank $\bar{n}$. We obtain a satisfying valuation σ' such that $\sigma'(X)$ is not a singleton for all $X \in FV(\psi')$, by adding c to all the non-empty elements in the transitive closure of the $\sigma(X)$'s. More in detail, given the membership graphs denoting the sets associated by σ to the variables in ψ, we can build the (unique) global minimum graph $\mathcal{G}_\sigma$. Now, let $\mathcal{G}_c$ be the minimum (rooted) graph denoting the set c. Add an edge from all the nodes of $\mathcal{G}_\sigma$, save the unique leaf denoting $\emptyset$ to (the root of) $\mathcal{G}_c$, obtaining a new graph $\mathcal{G}_{\sigma'}$ (the entry points of the variables remain the same). We prove that all literals of ψ' are satisfied by $\mathcal{G}_{\sigma'}$ (i.e. by σ'):

$X \neq \emptyset, \emptyset \notin Y, \emptyset \in Y$: These literals are fulfilled by $\mathcal{G}_\sigma$ and we have added no edges to the leaves.

$X = Y$: X and Y have the same entry point in the graph $\mathcal{G}_\sigma$, and thus in $\mathcal{G}_{\sigma'}$.

$X = Y \cup Z$: We have added the same entity to all sets. Equality remains true.

$X \in Y$: This means that there is an edge from the node associated to Y to that associated to X in $\mathcal{G}_\sigma$. The edge remains in $\mathcal{G}_{\sigma'}$.

$X \neq Y$: By contradiction. Let ν be a minimal (w.r.t. the membership relation) node of the graph $\mathcal{G}_\sigma$ such that there is another node ν' in $\mathcal{G}_\sigma$ bisimilar to ν in $\mathcal{G}_{\sigma'}$. This means that:
- a node μ $\in$-successor of ν is different from a a node μ', $\in$-successor of ν' in $\mathcal{G}_\sigma$ and they are equal in $\mathcal{G}_{\sigma'}$: this is absurdum since ν is a $\in$-minimal node with this property; or
- a node μ $\in$-successor of ν or a node μ' $\in$-successor of ν' are equal in $\mathcal{G}_{\sigma'}$ to the root of $\mathcal{G}_c$: this is absurdum since either they represent $\emptyset$ or $\mathcal{G}_c$ is also an element of the set represented by them and the overall graph is acyclic.

$X \notin Y$: In $\mathcal{G}_\sigma$ there is no edge from the node associated to Y to that associated to X. Using the same considerations of the previous point, it is impossible that now the former collapse with a node reached by an outgoing edge from the node associated to Y.

[3] Observe that, for instance, a literal $X = \emptyset$ cannot be in ψ' since σ satisfies ψ and, by hypothesis, X is mapped to a non-empty set.

To complete the valuation, map to $\emptyset$ all variables occurring only in the other disjuncts. $\square$

Theorem 1. *Let $\mathbb{T}$ be any set theory implying $NW \cup ER$. Then the functional symbol $\{\cdot\,|\,\cdot\}$ can not be existentially expressed by $\Phi_{\cup}$ in $\mathbb{T}$.*

Proof. By contradiction, assume that ψ in $\Phi_{\cup}$ existentially expresses $\{\cdot\,|\,\cdot\}$ in $\mathbb{T}$. Then ψ it is satisfiable over $\mathbb{HF}$. The result follows from Lemma 1 and the fact that $\mathbb{HF}$ is a submodel of any model of $NW \cup ER$. $\square$

3 *with* vs *union*

In this section we show that it is impossible to existentially express the function symbol $\cup$ using the class of formulae $\Phi_{\mathtt{with}}$. We make use of the following notion: a conjunction φ of $\Phi_{\mathtt{with}}$-literals is said to be in *canonical form* if $\varphi \equiv \mathsf{false}$ or each literal is either of the form $A = t, r \notin B, C \neq s$ where A, B, C are variables and A does not occur neither in t nor elsewhere in φ, B does not occur in r, C does not occur in s.

Lemma 2 ([8, 9]). *Let $\varphi(\boldsymbol{X})$ be a formula in $\Phi_{\mathtt{with}}$, then there is a formula $\varphi'(\boldsymbol{X}, \boldsymbol{Y}) = \bigvee_i \varphi_i(\boldsymbol{X}_i, \boldsymbol{Y}_i)$ s.t. $\{\boldsymbol{X}_i\} \subseteq \{\boldsymbol{X}\}$, $\{\boldsymbol{Y}\} = \bigcup_i \{\boldsymbol{Y}_i\}$, and each φ_i is (a conjunction of literals) in canonical form. Moreover, $\mathbb{HF} \models \forall \boldsymbol{X}\ (\varphi \leftrightarrow \exists \boldsymbol{Y} \bigvee_i \varphi_i)$.*

The above lemma, guaranteeing that any formula can be rewritten as a disjunction of canonical formulae, goes a long way towards providing a satisfiability test. In fact, consider the following

Lemma 3. *1. If $\varphi(\boldsymbol{X})$ is in canonical form and different from the formula* false*, then $\mathbb{HF} \models \exists \boldsymbol{X}\, \varphi$.*
2. If $\varphi(X, Y, \boldsymbol{Z})$ is in canonical form, different from false*, and there are neither atoms of the form $X = s$ nor of the form $Y = t$ in φ, then there is a valuation γ such that $\mathbb{HF} \models \varphi\gamma$, but $\mathbb{HF} \models (X \not\subseteq Y)\gamma$.*
3. Let φ be a formula in canonical form, different from false*, in which there is at least one of the atoms:*

$$X = \{s_1, \dots, s_m \,|\, A\}, Y = \{t_1, \dots, t_n \,|\, B\}$$

where A and B are different variables and $B \not\equiv X, A \not\equiv Y$. Then there is a valuation γ such that $\mathbb{HF} \models \varphi\gamma$ and $\mathbb{HF} \models (X \not\subseteq Y)\gamma$.

Proof. We prove (1) finding a particular valuation that satisfies the condition (2) and helps us in proving (3).

1) We split φ into $\varphi^{=}$, $\varphi^{\neq}$, and $\varphi^{\notin}$, containing $=, \neq$, and $\notin$ literals, respectively.

$\varphi^{=}$ has the form $X_1 = t_1 \wedge \cdots \wedge X_m = t_m$ and for all $i = 1, \dots, m$, X_i appears uniquely in $X_i = t_i$ and $X_i \notin FV(t_i)$. We define the mapping $\theta_1 = [X_1/t_1, \dots, X_m/t_m]$.

$\varphi^{\notin}$ has the form $r_1 \notin Y_1 \wedge \cdots \wedge r_n \notin Y_n$ (Y_i does not occur in r_i) and $\varphi^{\neq}$ has the form $Z_1 \neq s_1 \wedge \cdots \wedge Z_p \neq s_p$ (Z_i does not occur in s_i). Let $W_1, \ldots, W_h$ be the variables occurring in φ other than $X_1, \ldots, X_m, Y_1, \ldots, Y_n, Z_1, \ldots, Z_p$; we define $\theta_2 = [W_1/\{\emptyset\}^1, \ldots, W_h/\{\emptyset\}^h]$.

Let $\bar{s} = \max\{\mathit{rank}(t) : \ t \text{ occurs in } \varphi\theta_1\theta_2\} + 1$ and $R_1, \ldots, R_j$ be the variables occurring in $(\varphi^{\notin} \wedge \varphi^{\neq})\theta_2$ (actually, the variables $\boldsymbol{Y}$ and $\boldsymbol{Z}$) and $n_1, \ldots, n_j$ auxiliary variables ranging over $\mathbb{N}$. We build an integer disequation system E in the following way:

1. $E = \{n_i > \bar{s} : \forall i \in \{1, \ldots, j\}\} \cup \{n_{i_1} \neq n_{i_2} : \forall i_1, i_2 \in \{1, \ldots, j\}, i_1 \neq i_2\}$.
2. For each literal $(R_{i_1} \neq t)$ in $\varphi^{\neq}\theta_2$

$$E = E \cup \{n_{i_1} \neq n_{i_2} + c : \forall i_2 \neq i_1, \forall c \in \mathit{find}(R_{i_2}, t)\}$$

3. For each literal $(t \notin R_{i_1})$ in $\varphi^{\notin}\theta_2$

$$E = E \cup \{n_{i_1} \neq n_{i_2} + c + 1 : \forall i_2 \neq i_1, \forall c \in \mathit{find}(R_{i_2}, t)\}$$

A system of this form admits always integer solutions. Let $\{n_1 = \bar{n}_1, \ldots, n_j = \bar{n}_j\}$ be a solution, define $\theta_3 = [R_i/\{\emptyset\}^{\bar{n}_i} : \forall i \in \{1, \ldots, j\}]$.

Let $\gamma = \theta_1\theta_2\theta_3$, and observe that $\varphi\gamma$ is a conjunction of ground literals. We show that $\mathbb{HF} \models \varphi\gamma$. We analize each literal of φ.

$X = t$: $X\theta_1$ coincides syntactically with $t\theta_1 = t$. Thus, $X\gamma = X\theta_1\theta_2\theta_3 = t\theta_2\theta_3 = t\gamma$. Hence, a literal of this form is true in any model of the equality.

$r \notin Y$: Two cases are possible:

1. if $r = \emptyset$ or r is one of the variables W_i, then $r\gamma = r\theta_2 = \{\emptyset\}^{i_1}$, with $i_1 < \bar{s}$. Thus $r\gamma$ can not belong to $Y\gamma = \{\emptyset\}^{i_2}$ since $i_2 > \bar{s} \geq i_1 + 1$;
2. Otherwise, r is a term containing at least one of the variables Y_i or Z_i. From the solution to the integer system E, we obtain $\mathit{rank}(r\gamma) \neq \mathit{rank}(Y\gamma) - 1$. Since $Y\gamma$ is a term denoting a singleton set, this is sufficient to force this literal to be true in each well-founded model of membership.

$Z \neq s$: Similar to the case above.

2) Consider the formula φ as in point (2) of the statement. Since X and Y are different variables and are not among the $X_1, \ldots, X_m$, then $X\gamma$ and $Y\gamma$ are two different singleton sets. Thus, it can not be that $X\gamma \subseteq Y\gamma$.
3) Consider the formula φ as in point (3) of the statement.

If only the atom $X = \{s_1, \ldots, s_m \,|\, A\}$ is in φ, then also $\varphi' \equiv \varphi \wedge A \neq Y$ is in canonical form, since $A \not\equiv Y$. Thus, by (1), there is a valuation γ which satisfies φ' and $A\gamma = \{\emptyset\}^a$ and $Y\gamma = \{\emptyset\}^y$ with $a \neq y$. Hence, $\{\emptyset\}^{a-1} \in X\gamma$ and $\{\emptyset\}^{a-1} \notin Y\gamma$: $\mathbb{HF} \models \neg(X \subseteq Y)\gamma$.

If only the atom $Y = \{t_1, \ldots, t_n \,|\, B\}$ is in φ, then consider

$$\varphi' \equiv \varphi \wedge t_1 \notin X \wedge \cdots \wedge t_n \notin X \wedge B \neq X$$

Remove all the literals $t_i \notin X$ from φ' when $X \in FV(t_i)$ and obtain φ'', equivalent to φ' and in canonical form, since $X \not\equiv B$. Thus, by (1), there is a valuation γ

which satisfies φ' and $X\gamma = \{\emptyset\}^x$ and $B\gamma = \{\emptyset\}^b$ with $x \neq b$. Since γ is a solution, then $t_i\gamma \notin X\gamma$. Therefore $\{\emptyset\}^{x-1} \in X\gamma$ and $\{\emptyset\}^{x-1} \notin Y\gamma$: $\mathbb{HF} \models \neg(X \subseteq Y)\gamma$.

If the two atoms $X = \{s_1, \ldots, s_m \,|\, A\}, Y = \{t_1, \ldots, t_n \,|\, B\}$ are in φ and φ is in canonical form, then A and B are different variables not in those occurring as left hand side of an atom in $\varphi^=$. Let $X_1 = t_1, \ldots, X_m = t_m$ be the remaining atoms of $\varphi^=$ and consider

$$\varphi' \equiv \varphi \wedge t_1 \notin A \wedge \cdots \wedge t_n \notin A \wedge A \neq B\,.$$

Remove all the literals $t_i \notin A$ from φ' when $A \in FV(t_i)$ and obtain φ'', equivalent to φ' and in canonical form. Thus, by (1), there is a valuation γ which satisfies φ' and $A\gamma = \{\emptyset\}^a$ and $B\gamma = \{\emptyset\}^b$ with $a \neq b$. Since γ is a solution, then $t_i\gamma \notin A\gamma$. Therefore $\{\emptyset\}^{a-1} \in X\gamma$ and $\{\emptyset\}^{a-1} \notin Y\gamma$: $\mathbb{HF} \models \neg(X \subseteq Y)\gamma$. □

Lemma 4. *Let $\varphi_i(X, Y, \boldsymbol{Z}_i)$ $i \in \{1, \ldots, c\}$ be canonical formulae containing:*

- *an atom of the form $X = s_i$, where s_i is neither a variable of $X, \boldsymbol{Z}_i$, nor $\emptyset$,*
- *or an atom of the form $Y = \{t^i_1, \ldots, t^i_{m_i} \,|\, X\}$, or $Y = \{t^i_1, \ldots, t^i_{m_i}\}$, $m_i \geq 0$ (when $m_i = 0$ the equations become $Y = X$, and $Y = \emptyset$, respectively).*

Then, there are hereditarily finite sets x and y such that for all hereditarily finite sets $z_1, \ldots, z_n$, it holds that $\gamma = [X/x, Y/y, Z_1/z_1, \ldots, Z_n/z_n]$ implies:

$$\mathbb{HF} \models (X \subseteq Y)\gamma \text{ and } \mathbb{HF} \models ((\neg\varphi_1) \wedge \ldots \wedge (\neg\varphi_c))\gamma$$

Proof. Let $\hat{m} = \max\{m_1, \ldots, m_c\}$. We prove that $\gamma = [X/\emptyset, Y/\{\{\emptyset\}^0, \ldots, \{\emptyset\}^{\hat{m}}\}]$ fulfills the requirement.

Clearly $(X \subseteq Y)\gamma$ holds. Since they differ by $\hat{m}+1$ elements, every atom of the form $Y = \{t^i_1, \ldots, t^i_{m_i} \,|\, X\}$ can not be true, thus if φ_i contains one of these atoms, it is all right. Otherwise, we need to prove that $(X = s_i)\gamma$ is false. If s_i is Y then it is the same as above. Otherwise, $s_i = \{s \,|\, t\}$ for some s and t. For any valuation γ of the variables in it, $\{s \,|\, t\}\gamma$ is different from $\emptyset$. □

Theorem 2. *Let $\mathbb{T}$ be any set theory implying $NW \cup ER$. There is no formula $\varphi(X, Y, \boldsymbol{Z})$ in $\Phi_{\mathtt{with}}$ such that: $\mathbb{HF} \models \forall XY\,(X \subseteq Y \leftrightarrow \exists \boldsymbol{Z}\varphi)$.*

Proof. $X \subseteq Y$ is equivalent to $(X = \emptyset) \vee (X \neq \emptyset \wedge X \subseteq Y)$. Thus, if there is a φ equivalent to $X \neq \emptyset \wedge X \subseteq Y$, then $X = \emptyset \vee \varphi$ is equivalent to $X \subseteq Y$.

Without loss of generality (cf. Lemma 2), we can assume that φ to be a disjunction of canonical formulae $\bigvee_{i=1}^{c} \varphi_i$, each of them different from false. Moreover, we can assume that no atom of the form $X = \emptyset$ is in φ_i (since φ_i should imply $X \neq \emptyset$), and that also atoms of the form $X = Z, Z = X, Y = Z, Z = Y$ with Z in $\boldsymbol{Z}$ is not in φ_i, since Z is a new variable and therefore, such a conjunct can be eliminated after the application of the substitutions Z/X (Z/Y) to the remaining part of φ_i.

Assume first $c = 1$, i.e., φ be a canonical formula. We prove that one of the following two sentences holds:

$$(a)\ \mathcal{D} \models \exists XY\boldsymbol{Z}\,(\varphi(X, Y, \boldsymbol{Z}) \wedge X \not\subseteq Y)$$
$$(b)\ \mathcal{D} \models \exists XY\,(X \subseteq Y \wedge \forall \boldsymbol{Z}\neg\varphi(X, Y, \boldsymbol{Z}))$$

1. If X and Y occur only in negative literals or in the r.h.s. of some equality atom of φ, then by Lemma 3(2) we are in case (a).
2. If $Y = \emptyset$ or $X = Y$ or $Y = X$ or $X = \{s_1, \ldots, s_m\}$ or $Y = \{t_1, \ldots, t_n\}$ or $X = \{s_1, \ldots, s_m \,|\, Y\}$ are in φ, then it is easy to find two values for X and Y fulfilling the inclusion but invalidating φ for any possible evaluation for $\boldsymbol{Z}$. Thus, we are in case (b).
3. If $Y = \{t_1, \ldots, t_n \,|\, X\}$ is in φ, then, by Lemma 4 (with a unique disjunct, i.e. $c = 1$) we are in case (b).
4. If $X = \{s_1, \ldots, s_m \,|\, Z_i\}$ and Y occurs only in negative literals or in the r.h.s. of some equality atom of φ, then by Lemma 3(3) we are in case (a).
5. If $Y = \{t_1, \ldots, t_n \,|\, Z_j\}$ and X occurs only in negative literals or in the r.h.s. of some equality atom of φ, then by Lemma 3(3) we are in case (a).
6. $X = \{s_1, \ldots, s_m \,|\, Z_i\}$ and $Y = \{t_1, \ldots, t_n \,|\, Z_j\}$, $m, n > 0$ are in φ.
   If Z_i and Z_j are the same variable, then we are in case (b) since we can find two sets s and t, $s \subseteq t$, differing for more than $n - m$ elements.
   When Z_i and Z_j are different variables, by Lemma 3(3) we are in case (a).

Assume now that φ be $\varphi_1 \vee \cdots \vee \varphi_c$. We prove again that one of the two sentences holds:

$$(a)\ \mathcal{D} \models \exists XY\boldsymbol{Z}\,((\varphi_1 \vee \cdots \vee \varphi_n) \wedge X \not\subseteq Y)$$
$$(b)\ \mathcal{D} \models \exists XY\,(X \subseteq Y \wedge \forall \boldsymbol{Z}(\neg\varphi_1 \wedge \cdots \wedge \neg\varphi_c))$$

If there is one of the φ_i that fulfills case (a), then the result holds for the disjunction. Otherwise, assume that all the φ_is fulfill case (b). By the case analysis above, all these cases are those dealt by Lemma 4. Thus, we are globally in the case (b). □

Theorem 3. *Let $\mathbb{T}$ be any set theory implying $NW \cup ER$. Then $\{\cdot\,|\,\cdot\}$ can not be existentially expressed by $\Phi_{\texttt{with}}$ in $\mathbb{T}$.*

Proof. If a formula $\varphi(X, Y, Z, \boldsymbol{W})$ in $\Phi_{\texttt{with}}$ s.t. $\mathbb{HF} \models \forall XYZ\,(X = Y \cup Z \leftrightarrow \exists \boldsymbol{W}\varphi)$ exists, then also $\varphi \wedge X = Z$ is in $\Phi_{\texttt{with}}$ and it is equivalent to $Y \subseteq X$. A contradiction to Theorem 2. □

A similar result can be obtained for $\cap$ since $X = Y \cap X \leftrightarrow X \subseteq Y$.

4 Independence results for equational theories

The two set constructor symbols analyzed in this paper have been studied in the context of unification theory [4] and constraints. In this contexts the properties of the constructors are usually given by equational axioms. $\{\cdot\,|\,\cdot\}$, $\cup$, and $\emptyset$ are governed by the axioms of Fig. 1.

As far as the theory $(A_b)(C_\ell)$ is concerned, in [10] it is presented the first unification algorithm for the *general* case (other free function symbols are admitted [4]). The algorithms in [2, 21] reduces redundancies of unifiers, while those

(A_b)	$\{X \mid X \mid Y\}\} = \{X \mid Y\}$
(C_ℓ)	$\{X \mid \{Y \mid Z\}\} = \{Y \mid \{X \mid Z\}\}$
(A)	$(X \cup Y) \cup Z = X \cup (Y \cup Z)$
(C)	$X \cup Y = Y \cup X$
(I)	$X \cup X = X$
(1)	$\emptyset \cup X = X$

Fig. 1. Equational axioms for $\emptyset, \{\cdot \mid \cdot\}$, and $\cup$

in [9, 1, 7] are ensured to remain in the class NP. In [8, 9, 11] positive and negative constraints of this theory has been studied and solved.

In [3] a unification algorithm for the $ACI1$ unification *with constants* (no other free function symbols, save constants, are admitted [4]) is presented.

The expressiveness results of the previous sections allows us to prove the independence of the class of formulae that can be expressed using these two families of problems:

1. From Theorem 3 we know that $X = Y \cup Z$ can not be expressed by an open *with*-based formula. Thus, in particular for simple formulae made only by disjunctions of conjunctions of equations.
2. From Theorem 1 we know that it is impossible to express $X = \{Y \mid \emptyset\}$ using *union*-like formulae. Again, this implies the result.

A similar result holds for constraints problem involving positive and negative equality literals.

Notice that in the framework of *general* $ACI1$ unification, we can encode all $(A_b)(C_\ell)$ unification problems, since, if $f(\cdot)$ is a unary (free) function symbol, then $\{X \mid Y\}$ is equivalent to $f(X) \cup Y$ (cf. also Footnote 1 and [19]). In [12] $ACI1$ constraints (including unification) for the general case are studied and solved.

5 Multisets

Expressiveness results similar to those of the previous sections can be obtained in a multiset framework. However, while the meaning of set operators is a common and unambiguous knowledge, there is no uniform and universally accepted view of multisets. Thus, we begin by recalling an existing axiomatizazion; then we discuss the introduction of new axioms that help us to formalize multisets in a reasonable way.

Let $\mathcal{L}^m$ be a first-order language having $=$ and $\in$ as binary predicate symbols, $\emptyset$ as constant, and $\{\!\{\cdot \mid \cdot\}\!\}$ as binary function symbols, whose behavior is regulated by axioms $(N), (R)$, and (cf. [13]):

$$
\begin{array}{ll}
(W^m) & \forall Y\, V\, X\ (X \in \{\!\{\, Y \mid V\,\}\!\} \leftrightarrow X \in V \vee X = Y) \\
(E^m_p) & \forall XYZ\ \ \{\!\{X, Y \mid Z\}\!\} = \{\!\{Y, X \mid Z\}\!\}.
\end{array}
$$

The extensionality axiom (E) for sets has been replaced by (E_p^m), which does not imply the absorption property, i.e.: $NW^mE_p^mR \not\models \{\!\{ X, X \,|\, Y \}\!\} = \{\!\{ X \,|\, Y \}\!\}$. The elements of the models of $NW^mE_p^mR$ are called *multisets*.

The usual definition for inclusion between sets $X \subseteq Y \leftrightarrow \forall Z(Z \in X \to Z \in Y)$ does not capture the intended meaning of inclusion between multisets, since, for instance, $\{\!\{ \emptyset, \emptyset \}\!\} \subseteq \{\!\{ \emptyset \}\!\}$ oppositely to intuition. A similar consideration can be done for the union symbol if interpreted in the same way as for sets. A more tuned definition for multiset-inclusion can be recursively described by:

$$
\begin{aligned}
X \sqsubseteq Y \leftrightarrow X = \emptyset \vee & \\
& \exists VWZ\,(X = \{\!\{ V \,|\, W \}\!\} \wedge Y = \{\!\{ V \,|\, Z \}\!\} \wedge W \sqsubseteq Z)
\end{aligned}
$$

Multisets are often called bags in literature. A bag is the intuitive way to imagine a multiset: consider two bags, one containing 1 apple and 2 oranges and another containing 2 apples and 3 oranges. The former is a sub-bag $\sqsubseteq$ of the latter. This second definition of inclusion can be given in a non-recursive way using a language $\mathcal{L}^{mn}$, more natural to deal with multisets, in which $\in$ is replaced by an infinite set of predicate symbols: $\in^0, \in^1, \in^2, \ldots$ Intuitively, the meaning of $X \in^i Y$, with $i \geq 0$, is that there are at least i occurrences of X in the multiset Y. Thus, $X \in^0 Y$ is always satisfied.

The axioms to model multisets in this language are ($i \in \mathbb{N}$):

$$
\begin{array}{ll}
(0) & \forall XY\ (X \in^0 Y) \\
(I) & \forall XY\ (X \in^{i+1} Y \to X \in^i Y) \\
(N') & \forall X\ (X \notin^1 \emptyset) \\
(W') & \forall XYZ \left(X \in^{i+1} \{\!\{ Y \,|\, Z \}\!\} \leftrightarrow \begin{array}{l} (X = Y \wedge X \in^i Z) \vee \\ (X \neq Y \wedge X \in^{i+1} Z) \end{array} \right) \\
(E') & \forall XY\ (X = Y \leftrightarrow (\forall i > 0) \forall Z (Z \in^i X \leftrightarrow Z \in^i Y)) \\
(R') & \forall X\, \exists Z\, \forall Y\ (Y \in^1 X \to (Z \in^1 X \wedge Y \notin^1 Z))
\end{array}
$$

With these axioms, and making use of arithmetic, we can define usual predicates and operators, such as subset, union, difference. Before doing that, some choices must be made. In [14], for instance, the authors propose two kinds of union symbols: $\cup$ and $\uplus$.[4] Both of them are useful in different contexts. In particular, the $\uplus$ is the more natural for performing bag-union. Consider the bags of the example above: a bag that is union of the two bags contains 3 apples and 5 oranges. $\cup$ of [14], instead, takes the maximum number of occurrences of items of the two bags, and $\cap$ the minimum. The axiom for $\uplus$ is the following:

$$
\begin{aligned}
(\uplus)\ X = Y \uplus Z \leftrightarrow (\forall i > 0) \forall W (W \in^i X \leftrightarrow & \\
\exists m_1 m_2 (W \in^{m_1} Y \wedge W \in^{m_2} Z \wedge i = m_1 + m_2))
\end{aligned}
$$

In Section 5.2 we also discuss the relationships between bags and infiniteness. We define $\Phi^{mn}_{\mathtt{with}}$ as the set of open first-order formulae of $\mathcal{L}^{mn}$ involving $\emptyset, \{\!\{ \cdot \,|\, \cdot \}\!\}, \in^i$, $=$ and $\Phi^{mn}_{\uplus}$ as the set of open first-order formulae of $\mathcal{L}^{mn}$ involving $\emptyset, \uplus, \in^i, =$.

[4] In [14] a bag constructor of any finite arity is used to build bags. However, the binary functor symbol $\{\!\{ \cdot \,|\, \cdot \}\!\}$ adopted in this paper is sufficient to perform that task.

5.1 Multiset *with* vs union

From [9] one can deduce a result for canonical formulae in a multiset context similar to that described in Lemma 2. The only difference is that $\notin^1$ must be used in place of $\notin$. That result (outside the scope of the paper) can be obtained using multiset unification [13, 7] and rewriting rules for constraints [9].

Lemma 5 ([9]). *Let $\varphi(\boldsymbol{X})$ be a formula of $\Phi^{mn}_{\mathtt{with}}$, then there is a formula $\varphi'(\boldsymbol{X}, \boldsymbol{Y}) = \bigvee_i \varphi_i(\boldsymbol{X}_i, \boldsymbol{Y}_i)$ such that $\{\boldsymbol{X}_i\} \subseteq \{\boldsymbol{X}\}$, $\{\boldsymbol{Y}\} = \bigcup_i \{\boldsymbol{Y}_i\}$, and each φ_i is a conjunction of literals in canonical form. Moreover, $0IN'W'E'R \models \forall \boldsymbol{X} \; (\varphi \leftrightarrow \exists \boldsymbol{Y} \bigvee_i \varphi_i)$.*

Proving two lemmata analogous to Lemma 3 and Lemma 4 (the valuations defined for these lemmata holds in the multiset case, as well), we can prove:

Theorem 4. *Let $\mathbb{T}$ be any theory implying $0IN'W'E'R$. There is no formula $\varphi(X, Y, \boldsymbol{Z})$ in $\Phi^{mn}_{\mathtt{with}}$ such that: $\mathbb{T} \models \forall XY \, (X \sqsubseteq Y \leftrightarrow \exists \boldsymbol{Z} \varphi)$.*

Proof. Same proof as Theorem 2, with references to the multiset versions of Lemmata 3 and 4. □

Theorem 5. *Let $\mathbb{T}$ be any theory implying $0IN'W'E'R$. Then the function symbol $\uplus$ can not be existentially expressed by $\Phi^{mn}_{\mathtt{with}}$ in $\mathbb{T}$.*

Proof. From Theorem 4 and the property that $X = Y \uplus Z$ implies $Y \sqsubseteq X$. □

5.2 $\uplus$ vs multiset `with`

In the previous paragraph we have shown that the operator $\uplus$ is a natural *union* operator for multisets (bags). In this section we show that $\Phi^{mn}_{\uplus}$ can not existentially express the *with* operator $\{\!\!\{ \cdot \mid \cdot \}\!\!\}$. The proof technique used for sets in Lemma 1 can not be applied to the case of multisets. There are two main problems. The first is that a constraint $X = Y \uplus Z$ is no longer satisfied when we add one element to any multiset occurring in it. The second is that it is too restrictive to analyze hereditarily finite solutions only, since simple constraints (e.g., $X = Y \uplus X \wedge Y \neq \emptyset$ or $X = \{\!\!\{ Y \mid X \}\!\!\}$) have only infinite solutions. This is a very interesting feature of multisets: infiniteness can be expressed by a single equation, while nesting of quantifiers is needed in set theory (cf. [18]). Thus, we introduce axiom (S^ω) ensuring that, given an object c, there is the infinite multiset $\{\!\!\{ c, c, c, \ldots \}\!\!\}$:

$$(S^\omega) \qquad \forall Y \exists X (X = \{\!\!\{ Y \mid X \}\!\!\} \wedge (\forall Z \in X)(Z = Y))$$

Observe that the universal quantification is not needed to ensure the existence of an infinite multiset (the first equation is sufficient). It is introduced to ensure the existence of the particular multiset containing only one element repeated an infinite number of times. However, $|\omega|$ is the only infinite cardinality we are interested in in this work. Thus, all models have domains in which bags contain at most $|\omega|$ elements. Since we are interested in infinite bags, in all axioms the numbers i must be intended to range over $0, 1, 2, \ldots, \omega$ and the symbol $\in^\omega$ to belong to the language. Let $\mathbb{T}_m = (0)(I)(N')(W')(E')(R)(\uplus)(S^\omega)$.

Lemma 6. *For each satisfiable formula φ in $\Phi_{\uplus}^{mn}$ it holds that $\mathbb{T}_m \not\models \forall(\varphi \rightarrow X = \{\!\![\, Y \,]\!\!\})$.*

Proof. As done in Lemma 1, without loss of generality we can consider a formula φ in flat form in DNF, namely a disjunction of conjunctions of literals of the form:

$$V = W \uplus Z, V = \emptyset, V = W, V \neq W, V \in^i W, V \notin^i W$$

Let $\mathcal{M} = \langle M, (\cdot)^{\mathcal{M}} \rangle$ be a model of T_m and assume there is a valuation σ of the variables of φ on M such that $\mathcal{M} \models \varphi\sigma$; we build a valuation σ' on a domain $\mathcal{M}'$ (possibly, on $\mathcal{M}$ itself) such that $\mathcal{M}' \models \varphi\sigma'$ and $\mathcal{M}' \not\models (X = \{\!\![\, Y \,]\!\!\})\sigma'$. Let ψ be a disjunct of φ that is satisfied by σ and involves variables X and Y (if there are no disjuncts of this form the result holds trivially).

If $\sigma(X)$ is $(\emptyset)^{\mathcal{M}}$ or a bag containing at least two elements, then choose $\sigma' = \sigma$.

Otherwise, consider a formula ψ' defined as follows: replace each variable V such that $\sigma(V) = (\emptyset)^{\mathcal{M}}$ with $\emptyset$ and retain only one representative for each set $V_1, \dots, V_n$ of variables such that $\sigma(V_i) = \sigma(V_j)$. Let X be the representative of its class. After performing simple rewrites on ψ modified as above it is easy to obtain a formula ψ' as conjunction of literals of the form:

$$V = W \uplus Z, V \neq W, V \neq \emptyset, V \in^i W, \emptyset \in^i W, V \in^i W, \emptyset \in^i W\,.$$

Moreover, for $V \in FV(\psi')$ it holds that $\sigma(V) \neq (\emptyset)^{\mathcal{M}}$ and if V, W are distinct variables, $\sigma(V) \neq \sigma(W)$. Observe that X belongs to $FV(\psi') = \{V_1, \dots, V_k\}$.

Let $c = \sigma(V_1) \uplus \cdots \uplus \sigma(V_k) \in M$ and c_ω be the solution on $\mathcal{M}$ of the formula $K = \{\!\![\, c \mid K \,]\!\!\} \wedge (\forall Z \in K)(Z = c)$. The existence of c_ω in M is ensured by axiom (S^ω) (informally, c_ω is the infinite bag $\{\!\![\, c, c, c, \dots \,]\!\!\}$). Observe that since $\mathcal{M}$ is a model of (R), it can not be the case that $c \in \sigma(V)$ for $V \in FV(\psi')$.

We define a Mostowski collapsing function [16] $\beta : M \longrightarrow M'$, as follows: $\beta(m) = m$ if $m \neq \sigma(V)$ for all $V \in FV(\psi')$; otherwise $\beta(m)$ is

$$\biguplus_{m' \in^h m \wedge m' \notin^{h+1} m} \{\!\![\, \underbrace{\beta(m'), \dots, \beta(m')}_{h} \,]\!\!\} \uplus \biguplus_{m' \in^\omega m} \{\!\![\, \underbrace{\beta(m'), \beta(m'), \dots}_{\omega} \,]\!\!\} \uplus c_\omega$$

The domain M' is such that $M' = M$ as long as there is in M $\beta(m)$ for all $m = \sigma(V)$. Otherwise, obtain $\mathcal{M}' = \langle M', (\cdot)^{\mathcal{M}'} \rangle$ expanding the model $\mathcal{M}$ so that to guarantee the presence of those elements. We prove that

$$m_1 \neq m_2 \rightarrow \beta(m_1) \neq \beta(m_2) \tag{1}$$

for all m_i such that $c \notin m_i$. Since M is a well-founded model of membership, we prove this fact by contradiction in this way. Let m_1 be a $\in$-minimal element such that there is m_2 such that $m_1 \neq m_2$ and $\beta(m_1) = \beta(m_2)$. The following cases must be analyzed:

$\beta(m_1) = m_1, \beta(m_2) = m_2$: then $\beta(m_1) \neq \beta(m_2)$ by hypothesis: absurdum.

$\beta(m_1) = m_1, \beta(m_2) \neq m_2$: This means that $c \in \beta(m_2)$. By hypothesis $c \notin m_1$: absurdum.

$\beta(m_1) \neq m_1, \beta(m_2) = m_2$**:** similar to the previous case.

$\beta(m_1) \neq m_1, \beta(m_2) \neq m_2$**:** by hypothesis $c \notin m_1$ and $c \notin m_2$; this means that $c \in^\omega \beta(m_1)$ and $c \in^\omega \beta(m_2)$ and c is not the element that made the two bags equal. Thus, for some elements $m'_1 \in^1 m_1$ and $m'_2 \in^1 m_2$ $m'_1 \neq m'_2$ and $\beta(m'_1) = \beta(m'_2)$. This is absurdum since m_1 is a $\in$-minimal element with this property.

We are ready to define the valuation σ' on $\mathcal{M}'$: $\sigma'(V) = \beta(\sigma(V))$ for each $V \in FV(\psi')$. It is clear that for all $V \in FV(\psi')$ $\sigma'(V)$ is an infinite bag. Thus, clearly, $\mathcal{M} \not\models (X = \{\!\{ Y \}\!\})\sigma'$.

To complete the proof we must prove that $\mathcal{M} \models (s\ op\ t)\sigma'$ for each literal $s\ op\ t$ of ψ'. By case analysis:

$V \neq \emptyset, \emptyset \in^i W, \emptyset \notin^i W$**:** These literals remain true trivially.

$V = W \uplus Z$**:** We know that $\sigma(V) = \sigma(W) \uplus \sigma(Z)$. The function β extends the common elements in the same way. We only need to notice that c is inserted ω times on the l.h.s. and $\omega + \omega = \omega$ times on the r.h.s.

$V \in^i W$**:** Again, if $\sigma(V) \in^i \sigma(W)$ then $\beta(\sigma(V)) \in^i \beta(\sigma(W))$ by construction.

$V \neq W$**:** It derives from property (1) above.

$V \notin^i W$**:** If $\sigma(V) \notin^i \sigma(W)$ and $\beta(\sigma(V)) \in^i \beta(\sigma(W))$ this means that $\sigma(V)$ is a multiset without c among its elements that collapses using β with another multiset with the same property. This is absurdum by property (1) above.

Assign $(\emptyset)^{\mathcal{M}}$ to all variables of φ occurring only in the other disjuncts. $\square$

Theorem 6. $\{\!\{ \cdot \mid \cdot \}\!\}$ *can not be existentially expressed by* $\Phi^{mn}_{\uplus}$ *in* $\mathbb{T}_m$.

Proof. Immediate from Lemma 6 since $X = \{\!\{ Y \}\!\}$ is equivalent to $X = \{\!\{ X \mid \emptyset \}\!\}$. $\square$

Remark 1. The proof of Lemma 6 can be repeated on finite models. However, there is an interesting technical point. Suppose to have $X = Y \uplus Z$ and a multiset solution σ. If we add one element, say c, to all variables, this will be no longer a solution, since X should contain two occurrences of c instead of 1. Intuitively, we have to fulfill an integer linear system of equations, obtained from the formula ψ. But we can use the fact that the system is already fulfilled by σ. Thus, modifying the definition of β, second case, with: $\{\!\{ \underbrace{c, \ldots, c}_{|\sigma(V)|} \}\!\}$ we can prove the result for finite models of bags.

5.3 Independence results for bag equational theories

As for sets, the expressiveness results proved in the two previous subsections have a consequence on the class of formulae that can be expressed using multiset unification and constraints. The symbols $\emptyset, \{\!\{ \cdot \mid \cdot \}\!\}$, and $\uplus$ fulfill the equational properties $(C_\ell), (A), (C), (1)$ as in Fig. 1, while properties (A_b) and (I) are no longer true.

In [13, 7] a general unification algorithm is presented for the theory (C_ℓ). In [9] constraint solvers for the theory are presented. For links to $AC1$ unification algorithms, see [4]. Similarly to what holds for sets (cf. Sect. 4), it is an immediate consequence of Theorems 5 and 6 that $AC1$ unification with constants can not express all general (C_ℓ) unification problems and general (C_ℓ) unification can not express all $AC1$ unification with constants problems. General $AC1$, instead, can deal with any general (C_ℓ) unification problem using the usual encoding of `with` (cf. Sect. 4).

6 Conclusions

We have analyzed the relationships between the expressive power of two very common set and multiset constructors. In particular, we have proved that *union*-like and *with*-like symbols are not mutually expressible without using universal quantification. This has many consequences, such as the fact that testing satisfiability of with-based formulae (constraints) is easier than for formulae of the other case. This is a criterion for choosing the admissible constraints ([15]) of a CLP language. The conjecture of this expressiveness result has been used in [11] to enlarge the class of admissible constraints of $CLP(\mathcal{SET})$ [8]. However, the main consequence is perhaps an independence result of two very common equational theories for handling sets ($(A_b)(C_\ell)$ and $ACI1$). The same results can be obtained for multisets theories. In particular, for proving the results we have pointed out that

- $\in$ is sufficient, for sets, for giving (clean) axioms for equality.
- For multisets, the symbols $\in^i$, $i \in \mathbb{N}$ with the meaning 'to belong at least i times' and integer arithmetic is needed to perform the same task.
- The result could be extended for lists as well, but a complex axiomatic based on the notion of 'to belong in the list at the position ith' is needed.

Observe that the canonical form of this paper is very "explicit" and it might require a lot of time to be computed. To find more implicit but efficient normal forms is crucial when developing a CLP language but not for the aim of this work in which we have used the *existence* of a solved form for a theoretical proof of expressiveness.

Acknowledgements

We thank Domenico Cantone, Enrico Pontelli, and Gianfranco Rossi for their active contribution during our discussions and the referees that greatly helped to clarify the presentation. This research is partially supported by MURST project: *Certificazione automatica di programmi mediante interpretazione astratta.*

References

1. D. Aliffi, A. Dovier, and G. Rossi. From set to hyperset unification. *Journal of Functional and Logic Programming*, 1999(10):1–48, September 1999.

2. P. Arenas-Sánchez and A. Dovier. A minimality study for set unification. *Journal of Functional and Logic Programming*, 1997(7):1–49, December 1997.
3. F. Baader and W. Büttner. Unification in commutative and idempotent monoids. *Theoretical Computer Science*, 56:345–352, 1988.
4. F. Baader and K. U. Schulz. Unification theory. In Wolfgang Bibel and Peter H. Schmidt, editors, *Automated Deduction: A Basis for Applications. Volume I, Foundations: Calculi and Methods*. Kluwer Academic Publishers, Dordrecht, 1998.
5. P. Bernays. A system of axiomatic set theory. Part I. *The Journal of symbolic logic*, 2:65–77, 1937.
6. D. Cantone, A. Ferro, and E. G. Omodeo. *Computable Set Theory, Vol. 1*. Int'l Series of Monographs on Computer Science. Clarendon Press, Oxford, 1989.
7. E. Dantsin and A. Voronkov. A nondeterministic polynomial-time unification algorithm for bags, sets and trees. In W. Thomas, editor, *FoSSaCS'99*, volume 1578 of *Lecture Notes in Computer Science*, pages 180–196. Springer-Verlag, Berlin, 1999.
8. A. Dovier. A Language with Finite Sets Embedded in the CLP Scheme. In R. Dyckhoff, editor, *Int'l Workshop on Extension of Logic Programming*, volume 798 of *Lecture Notes in Artificial Intelligence*, pages 77–93. Springer-Verlag, Berlin, 1994.
9. A. Dovier. *Computable Set Theory and Logic Programming*. PhD thesis, Università degli Studi di Pisa, March 1996. TD–1/96.
10. A. Dovier, E. G. Omodeo, E. Pontelli, and G. Rossi. {log}: A Language for Programming in Logic with Finite Sets. *J. of Logic Programming*, 28(1):1–44, 1996.
11. A. Dovier, C. Piazza, E. Pontelli, and G. Rossi. On the Representation and Management of Finite Sets in CLP-languages. In J. Jaffar, editor, *Proc. of 1998 Joint Int'l Conf. and Symp. on Logic Programming*, pages 40–54. The MIT Press, Cambridge, Mass., 1998.
12. A. Dovier, C. Piazza, E. Pontelli, and G. Rossi. ACI1 constraints. In D. De Schreye, editor, *ICLP'99, 16th International Conference on Logic Programming*, pages 573–587. The MIT Press, Cambridge, Mass., 1999.
13. A. Dovier, A. Policriti, and G. Rossi. A uniform axiomatic view of lists, multisets and the relevant unification algorithms. *Fundamenta Informaticae*, 36(2/3):201–234, 1998.
14. S. Grumbach and T. Milo. Towards tractable algebras for bags. *Journal of Computer and System Sciences*, 52(3):570–588, 1996.
15. J. Jaffar, M. Maher, K. Marriott, and P. Stuckey. The semantics of constraint logic programs. *Journal of Logic Programming*, 1–3(37):1–46, 1998.
16. K. Kunen. *Set Theory. An Introduction to Independence Proofs*. Studies in Logic. North Holland, Amsterdam, 1980.
17. E. G. Omodeo and A. Policriti. Solvable set/hyperset contexts: I. some decision procedures for the pure, finite case. *Communication on Pure and Applied Mathematics*, 9–10:1123–1155, 1995.
18. F. Parlamento and A. Policriti. The logically simplest form of the infinity axiom. *American Mathematical Society*, 103:274–276, 1988.
19. O. Shmueli, S. Tsur, and C. Zaniolo. Compilation of set terms in the logic data language (LDL). *Journal of Logic Programming*, 12(1-2):89–119, 1992.
20. J. R. Shoenfield. *Mathematical Logic*. Addison-Wesley, Reading, 1967.
21. F. Stolzenburg. An algorithm for general set unification and its complexity. *Journal of Automated Reasoning*, 22, 1999.
22. R. L. Vaught. On a Theorem of Cobham Concerning Undecidable Theories. In E. Nagel, P. Suppes, and A. Tarski, editors, *Proceedings of the 1960 International Congress*, pages 14–25. Stanford University Press, Stanford, 1962.