NOTES ON LOCALITIES AND LINKING SYSTEMS

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These notes are intended as an introduction to linking systems as partial groups, and their connection with fusion systems. The development will, I hope, be not too fast and not too slow - in other words: just exactly at the right pace. Alas ! We live in a fallen world ! (How could it be otherwise ?) So, we will all have to make adjustments for the inevitable short-comings of the approach taken here.

Section 1: Partial groups

For any set X write $\mathbf{W}(X)$ for the free monoid on X. Thus, an element of $\mathbf{W}(X)$ is a finite sequence of (or *word in*) the elements of X, and the multiplication in $\mathbf{W}(X)$ consists of concatenation of sequences (denoted $u \circ v$). So, if $u = (x_1, \cdots, x_m)$ and $v = (y_1, \cdots, y_n)$ then $u \circ v = (x_1, \cdots, x_m, y_1, \cdots, y_n)$. The *length* $\ell(w)$ of the word $w = (x_1, \cdots, x_n)$ is n. The "empty word" is the word $(\emptyset)$ of length 0, and it is the identity element of the monoid $\mathbf{W}(X)$. We make no careful distinction between the set X and the set of words of length 1. That is, we regard X as a subset of $\mathbf{W}(X)$ via the identification $x \mapsto (x)$.

The use of the same symbol "$\circ$" for concatenation of sequences and (later on) for composition of functions should cause no confusion.

Definition 1.1. Let $\mathcal{L}$ be a non-empty set, let $\mathbf{W} = \mathbf{W}(\mathcal{M})$ be the free monoid on $\mathcal{L}$, and let $\mathbf{D} = \mathbf{D}(\mathcal{L})$ be asubset of $\mathbf{W}$ such that:

(1) $\mathcal{L} \subseteq \mathbf{D}$ (i.e. $\mathbf{D}$ contains all words of length 1), and

$$u \circ v \in \mathbf{D} \implies u, v \in \mathbf{D}.$$

(Notice that (1) implies that also the empty word is in $\mathbf{D}$.) A mapping $\Pi : \mathbf{D} \to \mathcal{M}$ is a *product* if:

(2) Π restricts to the identity map on $\mathcal{L}$, and

(3) $u \circ v \circ w \in \mathbf{D} \implies u \circ (\Pi(v)) \circ w \in \mathbf{D}$, and $\Pi(u \circ v \circ w) = \Pi(u \circ (\Pi(v)) \circ w)$.

An *inversion* on $\mathcal{M}$ consists of an involutory bijection $f \mapsto f^{-1}$ on $\mathcal{M}$, together with the mapping $u \mapsto u^{-1}$ on $\mathbf{W}$ (also an involutory bijection) given by

$$(x_1, \cdots, x_n) \mapsto (x_n^{-1}, \cdots x_1^{-1}).$$

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A *partial group* consists of a product $\Pi : \mathbf{D} \to \mathcal{L}$, together with an inversion $(-)^{-1}$ on $\mathcal{L}$, such that:

(4) $u \in \mathbf{D} \implies u^{-1} \circ u \in \mathbf{D}$ and $\Pi(u^{-1} \circ u) = \mathbf{1}$,

where $\mathbf{1}$ denotes the image of the empty word under Π. (Notice that (1) and (4) yield $u^{-1} \in \mathbf{D}$ if $u \in \mathbf{D}$, and since $(u^{-1})^{-1} = u$ it follows that the condition (4) is in fact symmetric.)

Example/exercise (a partial group which isn't a group). Let $\mathcal{L}$ be the 3-element set $\{\mathbf{1}, a, b\}$ and let $\mathbf{D}$ be the subset of $\mathbf{W}(\mathcal{L})$ consisting of words w such that the word obtained from w by deleting all entries equal to $\mathbf{1}$ is an alternating string of a's and b's (of odd or even length, beginning with a or beginning with b). Define $\Pi : \mathbf{D} \to \mathcal{L}$ by the formula: $\Pi(w) = \mathbf{1}$ if the number of a-entries in w is equal to the number of b's; $\Pi(w) = a$ if the number of a's exceeds the number of b's (necessarily by 1); and $\Pi(w) = b$ it the number of b's exceeds the number of a's. Define inversion on $\mathcal{L}$ by $\mathbf{1}^{-1} = \mathbf{1}$, $a^{-1} = b$, and $b^{-1} = a$. Now check that $\mathcal{L}$ with these structures is a partial group.

It will be convenient to forget for the moment, that the notion of "group" is a familiar one, and to make the definitions: A *group* is a partial group in which $\mathbf{D} = \mathbf{W}$, and a *binary group* is the more familiar thing - a set G with a binary operation $(g, h) \mapsto gh$, such that the usual "axioms" hold (associativity, existence of an identity element, and existence of inverses). The connection between the two notions is provided by the following lemma.

Lemma 1.2 (exercise).

(a) *Let G be a binary group, and let $\Pi : \mathbf{W}(G) \to G$ be the "multivariable product" given by $(g_1, \cdots, g_n) \mapsto g_1 \cdots g_n$. Then G, together with Π and the inversion in G, is a partial group, with $\mathbf{D}(G) = \mathbf{W}(G)$.*

(b) *Let $\mathcal{L}$ be a partial group for which $\mathbf{W} = \mathbf{D}$. Then $\mathcal{L}$ is a*

(1) *binary group with respect to the binary operation given by restricting Π to words of length 2, and with respect to the inversion in $\mathcal{L}$. Moreover, Π is then the multivariable product on $\mathcal{L}$ defined as in (a).*

We list some elementary consequences of definition 1.1, as follows.

Lemma 1.3. *Let $\mathcal{L}$ (with $\mathbf{D}$, Π, and inversion) be a partial group.*

(a) *Π is $\mathbf{D}$-multiplicative. That is, if $u \circ v$ is in $\mathbf{D}$ then the word $(\Pi(u), \Pi(v))$ of length 2 is in $\mathbf{D}$, and*

$$\Pi(u \circ v) = \Pi(u)\Pi(v),$$

where $\Pi(u)\Pi(v)$ is an abbreviation for $\Pi((\Pi(u), \Pi(v))$.

(b) *Π is $\mathbf{D}$-associative. That is:*

$$u \circ v \circ w \in \mathbf{D} \implies \Pi(u \circ v)\Pi(w) = \Pi(u)\Pi(v \circ w).$$

(c) *If $u \circ v \in \mathbf{D}$ then $u \circ (\mathbf{1}) \circ v \in \mathbf{D}$ and $\Pi(u \circ (\mathbf{1}) \circ v) = \Pi(u \circ v)$.*

(d) *If* $u \circ v \in \mathbf{D}$ *then both* $u^{-1} \circ u \circ v$ *and* $u \circ v \circ v^{-1}$ *are in* $\mathbf{D}$, $\Pi(u^{-1} \circ u \circ v) = \Pi(v)$, *and* $\Pi(u \circ v \circ v^{-1}) = \Pi(u)$.
(e) *The cancelation rule: If* $u \circ v$, $u \circ w \in \mathbf{D}$, *and* $\Pi(u \circ v) = \Pi(u \circ w)$, *then* $\Pi(v) = \Pi(w)$ (*and similarly for right cancellation*).
(f) *If* $u \in \mathbf{D}$ *then* $u^{-1} \in \mathbf{D}$, *and* $\Pi(u^{-1}) = \Pi(u)^{-1}$. *In particular,* $\mathbf{1}^{-1} = \mathbf{1}$.
(g) *The uncancelation rule: Let* $u, v, w \in \mathbf{W}$, *and suppose that both* $u \circ v$ *and* $u \circ w$ *are in* $\mathbf{D}$ *and that* $\Pi(v) = \Pi(w)$. *Then* $\Pi(u \circ v) = \Pi(u \circ w)$. (*Similarly for right uncancellation.*)

Proof. Let $u \circ v \in \mathbf{D}$. Then 1.1(3) applies to $(\emptyset) \circ u \circ v$ and yields $(\Pi(u)) \circ v \in \mathbf{D}$ with $\Pi(u \circ v) = \Pi((\Pi(u)) \circ v)$. Now apply 1.1(3) to $(\Pi(u)) \circ v \circ (\emptyset)$, to obtain (a).

Let $u \circ v \circ w \in \mathbf{D}$. Then $u \circ v$ and w are in $\mathbf{D}$ by 1.1(1), and $\mathbf{D}$-multiplicativity yields $\Pi(u \circ v \circ w) = \Pi(u \circ v)\Pi(w)$. Similarly, $\Pi(u \circ v \circ w) = \Pi(u)\Pi(v \circ w)$, and (b) holds.

Since $\mathbf{1} = \Pi(\emptyset)$, point (c) is immediate from 1.1(3).

Let $u \circ v \in \mathbf{D}$. Then $v^{-1} \circ u^{-1} \circ u \circ v \in \mathbf{D}$ by 1.1(4), and then $u^{-1} \circ u \circ v \in \mathbf{D}$ by 1.1(1). Multiplicativity then yields

$$\Pi(u^{-1} \circ u \circ v) = \Pi(u^{-1} \circ u)\Pi(v) = \mathbf{1}\Pi(v) = \Pi(\emptyset)\Pi(v) = \Pi(\emptyset \circ v) = \Pi(v).$$

As $(w^{-1})^{-1} = w$ for any $w \in \mathbf{W}$, one obtains $w \circ w^{-1} \in \mathbf{D}$ for any $w \in \mathbf{D}$, and $\Pi(w \circ w^{-1}) = \mathbf{1}$. From this one easily completes the proof of (d).

Now let $u \circ v$ and $u \circ w$ be in $\mathbf{D}$, with $\Pi(u \circ v) = \Pi(u \circ w)$. Then (d) (together with multiplicativity and associativity, which will not be explicitly mentioned hereafter) yield

$$\Pi(v) = \Pi(u^{-1} \circ u \circ v) = \Pi(u^{-1})\Pi(u)\Pi(v) = \Pi(u^{-1})\Pi(u)\Pi(w) = \Pi(u^{-1} \circ u \circ w) = \Pi(w),$$

and (e) holds.

Let $u \in \mathbf{D}$. Then $u \circ u^{-1} \in \mathbf{D}$, and then $\Pi(u)\Pi(u^{-1}) = \mathbf{1}$. But also $(\Pi(u), \Pi(u)^{-1}) \in \mathbf{D}$, and $\Pi(u)\Pi(u)^{-1} = \mathbf{1}$. Now (f) follows by 1.1(2) and cancellation.

Let u, v, w be as in (g). Then $u^{-1} \circ u \circ v$ and $u^{-1} \circ u \circ w$ are in $\mathbf{D}$ by (d). By two applications of (d), $\Pi(u^{-1} \circ u \circ v) = \Pi(v) = \Pi(w) = \Pi(u^{-1} \circ u \circ w)$, so $\Pi(u \circ v) = \Pi(u \circ w)$ by (e). That is, $\Pi(u)\Pi(v) = \Pi(u)\Pi(w)$, and (g) holds. □

It is often convenient to eliminate the symbol "Π" and to speak of "the product $f_1 \cdots f_n$" instead of $\Pi(f_1, \cdots, f_n)$. More generally, if $\{X_i\}_{1 \le i \le n}$ is a collection of subsets of $\mathcal{L}$ then the "product set $X_1 \cdots X_n$" is by definition the image under Π of the set of words $(f_1, \cdots, f_n) \in \mathbf{D}$ such that $f_i \in X_i$ for all i. If $X_i = \{f_i\}$ is a singleton then we may write f_i in place of X_i in such a product. Thus, for example, the product Xfg stands for the set of all $\Pi(x, f, g)$ with $(x, f, g) \in \mathbf{D}$, and with $x \in X$.

A word of urgent warning: in writing products in the above way one may be led, mistakenly, into imagining that "associativity" holds in a stronger sense than that which is given by 1.3(b). For example, one should not suppose, if $(f, g, h) \in \mathbf{W}$, and both (f, g) and (fg, h) are in $\mathbf{D}$, that (f, g, h) is in $\mathbf{D}$. That is, it may be that "the product fgh" is undefined, even though the product $(fg)h$ is defined. Of course, one is tempted to simply extend the domain $\mathbf{D}$ to include such triples (f, g, h), and to "define" the product fgh to be $(fg)h$. The trouble is that it may also be the case that gh and $f(gh)$ are defined (via $\mathbf{D}$), but that $(fg)h \neq f(gh)$.

For $\mathcal{L}$ a partial group and $f \in \mathcal{L}$, write $\mathbf{D}(f)$ for the set of all $x \in \mathcal{L}$ such that the product $f^{-1}xf$ is defined. There is then a mapping

$$c_f : \mathbf{D}(f) \to \mathcal{L}$$

given by $x \mapsto f^{-1}xf$ (and called *conjugation by* f). Our preference is for right-hand notation for mappings, so we write

$$x \mapsto (x)c_f \quad \text{or} \quad x \mapsto x^f$$

for conjugation by f.

The following result provides an illustration of the preceding notational conventions, and introduces a theme which will be developed further as we pass from partial groups to entities (objective partial groups, linking systems) which are more narrowly defined.

Lemma 1.4. *Let $\mathcal{L}$ be a partial group, let $f, g \in \mathcal{L}$ with $f \in \mathbf{D}(g)$ and with $g \in \mathbf{D}(f)$, and suppose that $f^g = f$. Then $fg = gf$ and $g^f = g$.*

Proof. We're given $(g^{-1}, f, g) \in \mathbf{D}$, $(f^{-1}, g, f) \in \mathbf{D}$, and $g^{-1}fg = f$. Then $(g, g^{-1}, f, g) \in \mathbf{D}$ by 1.(4), $(g, f) \in \mathbf{D}$ by 1.1(1), and $gg^{-1}fg = gf$ by uncancelation. Thus $fg = gf$ (1.3(c)). A further application of uncancelation yields $f^{-1}fg = f^{-1}gf$, and so $g^f = g$. $\square$

Notational Convention. In any given partial group $\mathcal{L}$, usage of the symbol "x^f" (for x and f in $\mathcal{L}$) shall be taken to imply $x\mathbf{D}(f)$. More generally, for X a subset of $\mathcal{L}$ and $f \in \mathcal{L}$, usage of "X^f" shall be taken to mean that $X \subseteq \mathbf{D}(f)$; whereupon X^f is by definition the set of all x^f with $x \in X$.

At this early point, and in the context of arbitrary partial groups, one can say very little about the maps c_f. The cancelation rule 1.2(e) implies that each c_f is injective, but beyond that, the following lemma may be the best that can be obtained.

Lemma 1.5. *Let $\mathcal{L}$ be a partial group and let $f \in \mathcal{L}$. Then the following hold.*

(a) $\mathbf{1} \in \mathbf{D}(f)$ *and* $\mathbf{1}^f = \mathbf{1}$.
(b) $\mathbf{D}(f)$ *is closed under inversion, and* $(x^{-1})^f = (x^f)^{-1}$ *for all* $x \in \mathbf{D}(f)$.
(c) c_f *is a bijection* $\mathbf{D}(f) \to \mathbf{D}(f^{-1})$, *and* $c_{f^{-1}} = (c_f)^{-1}$.
(d) $\mathcal{L} = \mathbf{D}(\mathbf{1})$, *and* $x^{\mathbf{1}} = x$ *for each* $x \in \mathcal{L}$.

Proof. By 1.1(4), $f \circ \emptyset \circ f^{-1} = f \circ f^{-1} \in \mathbf{D}$, so $\mathbf{1} \in \mathbf{D}(f)$ and then $\mathbf{1}^f = \mathbf{1}$ by 1.3(a). Thus (a) holds. Now let $x \in \mathbf{D}(f)$ and set $w = (f^{-1}, x, f)$. Then $w \in \mathbf{D}$, and $w^{-1} = (f^{-1}, x^{-1}, f)$ by definition in 1.1. Then 1.1(4) yields $w^{-1} \circ w \in \mathbf{D}$, and so $w^{-1} \in \mathbf{D}$ by 1.1(1). This shows that $\mathbf{D}(f)$ is closed under inversion. Also, 1.1(4) yields $\mathbf{1} =$

$\Pi(w^{-1} \circ w) = (x^{-1})^f x^f$, and then $(x^{-1})^f = (x^f)^{-1}$ by 1.3(f). This completes the proof of (b).

As $w \in \mathbf{D}$, 1.3(d) implies that $f \circ w$ and then $f \circ w \circ f^{-1}$ are in $\mathbf{D}$. Now 1.1(3) and two applications of 1.3(d) yield

$$fx^f f^{-1} = \Pi(f, f^{-1}, x, f, f^{-1}) = \Pi((f, f^{-1}, x) \circ f \circ f^{-1}) = \Pi(f, f^{-1}, x) = x.$$

Thus $x^f \in \mathbf{D}(f^{-1})$ with $(x^f)^{f^{-1}} = x$, and thus (c) holds.

Finally, $\mathbf{1} = \mathbf{1}^{-1}$ by 1.3(f), and $\emptyset \circ x \circ \emptyset = x \in \mathbf{D}$ for any $x \in \mathcal{M}$, proving (d). $\square$

Definition 1.6. Let $\mathcal{L}$ be a partial group and let $\mathcal{H}$ be a non-empty subset of $\mathcal{L}$. Then $\mathcal{H}$ is a *partial subgroup* of $\mathcal{L}$ if $\mathcal{H}$ is closed under inversion ($f \in \mathcal{H}$ implies $f^{-1} \in \mathcal{H}$) and with respect to products. The latter condition means that $\Pi(w) \in \mathcal{H}$ whenever $w \in \mathbf{W}(\mathcal{H}) \cap \mathbf{D}$. The partial subgroup $\mathcal{H}$ is a *subgroup* of $\mathcal{L}$ if $\mathbf{W}(\mathcal{H}) \subseteq \mathbf{D}$. A partial subgroup $\mathcal{N}$ of $\mathcal{L}$ is *normal* in $\mathcal{L}$ if $x^g \in \mathcal{N}$ for all pairs $(x, g) \in \mathcal{N} \times \mathcal{L}$ for which $x \in \mathbf{D}(g)$. (Another - equivalent - way to state the condition for normality is to say that the partial subgroup $\mathcal{N}$ of $\mathcal{L}$ is normal in $\mathcal{L}$ is $g^{-1}\mathcal{N}g \subseteq \mathcal{N}$ for all $g \in \mathcal{L}$. Note that this formulation relies on one of the notational conventions introduced above, for interpreting product sets XYZ.) We shall write

$$\mathcal{H} \leq \mathcal{L}$$

to indicate that $\mathcal{H}$ is a partial subgroup of $\mathcal{L}$, and write

$$\mathcal{N} \trianglelefteq \mathcal{L}$$

to indicate that $\mathcal{N}$ is a partial normal subgroup of $\mathcal{L}$.

Lemma 1.7 (Exercise). *Let $\mathcal{H}$ and $\mathcal{K}$ be partial subgroups of a partial group $\mathcal{L}$, and let $\{\mathcal{H}_i\}_{i \in I}$ be a set of partial subgroups of $\mathcal{L}$.*

(a) *Each partial subgroup of $\mathcal{H}$ is a partial subgroup of $\mathcal{L}$.*
(b) *If $\mathcal{K} \subseteq \mathcal{H}$ then $\mathcal{K}$ is a partial subgroup of $\mathcal{H}$.*
(c) *If $\mathcal{H}$ is a subgroup of $\mathcal{L}$ then $\mathcal{H} \cap \mathcal{K}$ is a subgroup of $\mathcal{H}$ and of $\mathcal{K}$.*
(d) *Suppose $\mathcal{K} \trianglelefteq \mathcal{L}$. Then $\mathcal{H} \cap \mathcal{K} \trianglelefteq \mathcal{H}$. Moreover, $\mathcal{H} \cap \mathcal{K}$ is a normal subgroup of $\mathcal{H}$ if $\mathcal{H}$ is a subgroup of $\mathcal{L}$.*
(e) *$\cap\{\mathcal{H}_i \mid i \in I\}$ is a partial subgroup of $\mathcal{L}$, and is normal in $\mathcal{L}$ (i.e. is a partial normal subgroup of $\mathcal{L}$) if $\mathcal{H}_i \trianglelefteq \mathcal{L}$ for all i.*

$\square$

For any subset X of a partial group $\mathcal{L}$ we define the partial subgroup $\langle X \rangle$ of $\mathcal{L}$ *generated by X* to be the intersection of the set of all partial subgroups of $\mathcal{L}$ containing X. Then $\langle X \rangle$ is itself a partial subgroup of $\mathcal{L}$ by 1.7(e).

Lemma 1.8. *Let X be a subset of $\mathcal{L}$ such that X is closed under inversion ($x \in X \implies x^{-1} \in X$). Set $X_0 = X$ and recursively define X_n for $n > 0$ by*

$$X_n = \{\Pi(w) \mid w \in \mathbf{W}(X_{n-1}) \cap \mathbf{D}\}.$$

Then $\langle X\rangle = \bigcup\{X_n\}_{n\geq 0}$.

Proof. Let Y be the union of the sets X_i. Each X_i is closed under inversion by 1.3(f), and $Y \neq \emptyset$ since $\mathbf{1} = \Pi(\emptyset)$. Since Y is closed under products by construction, we get $Y \leq \langle X\rangle$, and then $Y = \langle X\rangle$ by the definition of $\langle X\rangle$. $\square$

Lemma/Exercise 1.9 (Dedekind Lemma). *Let $\mathcal{L}$ be a partial group, let $\mathcal{H}$, $\mathcal{K}$, and $\mathcal{A}$ be partial subgroups of $\mathcal{L}$, and suppose that $\mathcal{L} = \mathcal{H}\mathcal{K}$.*

(a) *If $\mathcal{K} \leq \mathcal{A}$ then $\mathcal{A} = (\mathcal{A}\cap\mathcal{H})\mathcal{K}$.*
(b) *If $\mathcal{H} \leq \mathcal{A}$ then $\mathcal{A} = \mathcal{H}(\mathcal{A}\cap\mathcal{K})$.*

$\square$

Definition 1.10. Let $\mathcal{L}$ and $\mathcal{L}'$ be partial groups, let $\beta : \mathcal{L} \to \mathcal{L}'$ be a mapping, and let $\beta^* : \mathbf{W} \to \mathbf{W}'$ be the induced mapping of free monoids. Then β is a *homomorphism* (*of partial groups*) if:

(H1) $\mathbf{D}\beta^* \subseteq \mathbf{D}'$, and
(H2) $(\Pi(w))\beta = \Pi'(w\beta^*)$ for all $w \in \mathbf{D}$.

The *kernel* of β is the set $Ker(\beta)$ of all $g \in \mathcal{L}$ such that $g\beta = \mathbf{1}'$. We say that β is an *isomorphism* if there exists a homomorphism $\beta' : \mathcal{L}' \to \mathcal{L}$ such that $\beta\circ\beta'$ and $\beta'\circ\beta$ are identity mappings.

We end this section with a few exercises (or lemmas). Some of these are perhaps not much more than observations.

Lemma/Exercise 1.11. *Let $\beta : \mathcal{L} \to \mathcal{L}'$ be a homomorphism of partial groups. Then $\mathbf{1}\beta = \mathbf{1}'$, and $(f^{-1})\beta = (f\beta)^{-1}$ for all $f \in \mathcal{L}$.* $\square$

Lemma/Exercise 1.12. *Let $\beta : \mathcal{L} \to \mathcal{L}'$ be a homomorphism of partial groups, and set $\mathcal{N} = Ker(\beta)$. Then $\mathcal{N}$ is a partial normal subgroup of $\mathcal{L}$.* $\square$

Lemma/Exercise 1.13. *Let $\beta : \mathcal{L} \to \mathcal{L}'$ be a homomorphism of partial groups, and let M be a subgroup of $\mathcal{L}$. Then $M\beta$ is a subgroup of $\mathcal{L}'$. (What might go wrong if M is merely a partial subgroup of $\mathcal{L}$? I.e. why shouldn't $M\beta$ be a partial subgroup of $\mathcal{L}'$ in that case ?)* $\square$

Lemma/Exercise 1.14. *Let $\alpha : G \to G'$ be a homomorphism of "binary groups" (i.e. a homomorphism of groups in the usual sense - see 1.2). Then α is a homomorphism of partial groups.* $\square$

Section 2: Objective partial groups and localities

Recall that if X is a subset of a partial group $\mathcal{L}$, then any statement involving the expression "X^f" should be understood as being based on the tacit hypothesis that $X \subseteq \mathbf{D}(f)$.

Throughout this section $\mathcal{L}$ is a partial group. For subgroups X and Y of $\mathcal{L}$, set

$$N_{\mathcal{L}}(X,Y) = \{f \in \mathcal{L} \mid X \subseteq \mathbf{D}(f) \text{ and } X^f \leq Y\},$$

and define the *normalizer*

$$N_{\mathcal{L}}(X) = \{f \in \mathcal{L} \mid X \subseteq \mathbf{D}(f) \text{ and } X^f = X\}.$$

The *centralizer* $C_{\mathcal{L}}(X)$ is defined to be the set of all $f \in N_{\mathcal{L}}(X)$ such that $x^f = x$ for all $x \in X$.

Definition 2.1. Let $\mathcal{L}$ be a partial group and let Δ be a collection of subgroups of $\mathcal{L}$. Define $\mathbf{D}_\Delta$ to be the set of all $w = (f_1, \cdots, f_n) \in \mathbf{W}(\mathcal{L})$ such that:

(*) there exists $(X_0, \cdots, X_n) \in \mathbf{W}(\Delta)$ with $(X_{i-1})^{f_i} = X_i$ for all i $(1 \leq i \leq n)$.

Then $(\mathcal{L}, \Delta)$ is an *objective partial group* (in which Δ is the set of *objects*), if the following three conditions hold.

(O1) $\mathbf{D} = \mathbf{D}_\Delta$.

(O2) Δ is overgroup closed. That is, whenever X, Z are objects, and Y is a subgroup of Z containing X, then Y is an object.

(O3) Δ is closed under "$\mathcal{L}$-fusion". That is whenever X and Y are objects, and $g \in N_{\mathcal{L}}(X, Y)$ is given, with the property that X^g is a subgroup of Y, then X^g is an object.

It should be emphasized that in the condition (O3) it is required that X^g be a subgroup of Y, in order to conclude that X^g be an object. Notice that the two conditions (O1) and (O2) may be summarized by saying:

(O2') Every subgroup of $\mathcal{L}$ which contains an $\mathcal{L}$-conjugate of an object, and which is contained in an object, is itself an object.

Example. Let G be a finite group, B a subgroup of G, and let Δ be a collection of subgroups of B such that $X^g \in \Delta$ whenever $X \in \Delta$ and $g \in G$ with $X^g \leq B$. Assume also that Δ is closed with respect to overgroups in B. Let $\mathcal{L}$ be the set of all $g \in G$ such that $B \cap B^g \in \Delta$, and let $\mathbf{D}$ be the subset $\mathbf{D}_\Delta$ of $\mathbf{W}(\mathcal{L})$. Then $\mathcal{L}$ is a partial group (via the multivariable product in G and the inversion in G), and $(\mathcal{L}, \Delta)$ is an objective partial group. Specifically:

(a) If $\Delta = \{B\}$ then $\mathcal{L} = N_G(B)$, and $\mathcal{L}$ is a group.

(b) (Exercise) Take $G = GL_3(2)$, $B \in Syl_2(G)$, and let M_1 and M_2 be the two maximal subgroups of G containing B. Set $\Delta = \mathcal{F}_B(G)^c$. Then $\mathcal{L} = M_1 \cup M_2$. On the other hand, if Δ is taken to be the set of all non-identity subgroups of B then $\mathcal{L} = M_1 M_2 \cup M_2 M_1$.

In an objective partial group $(\mathcal{L}, \Delta)$ we say that the word $w = (f_1, \cdots, f_n)$ is *in* $\mathbf{D}$ *via* $(X_0, \cdots, X_n)$ if the condition (*) in 2.1 applies specifically to w and $(X_0, \cdots, X_n)$. We may also say, more simply, that w *is in* $\mathbf{D}$ *via* X_0, since the sequence $(X_0, \cdots, X_n)$ is determined by w and X_0.

Examples/Exercises.

(1) Let G be a group and let Δ be a non-empty collection of subgroups of G. Assume that Δ satisfies (O3') with respect to G: Every subgroup of G which contains a

G-conjugate of a member of Δ and which is contained in a member of Δ is itself a member of Δ. Set

$$\mathcal{L} = \{g \in G \mid \exists X, Y \in \Delta \text{ with } X^g = Y\},$$

set $\mathbf{D} = \mathbf{D}_\Delta$, and let $\Pi_G : \mathbf{W}(G) \to G$ be the multivariable product in G. Then the restriction Π of Π_G to $\mathbf{D}$ maps $\mathcal{D}$ into $\mathcal{L}$, $\mathcal{L}$ is a partial group with respect to Π and the inversion in G, and $(\mathcal{L}, \Delta)$ is objective.

(2) If, in (1), there exists $X \in \Delta$ such that every G-conjugate of X is in Δ, then $\mathcal{L}$ is the group G (see 1.2 for the definition of "group"). In particular, this will be the case if the identity subgroup of G - or any other normal subgroup of G - is in Δ.

(3) Let G be the group $O_4^+(2)$. That is, let G be a semidirect product $V \rtimes S$ where V is an elementary abelian, normal subgroup of G of order 9, and where S is a dihedral subgroup of G of order 8 acting faithfully on V. (Another description of G is that G is the "wreath product" of the symmetric group of degree 3 with a cyclic group of order 2.) Let Δ be the set of all non-identity subgroups of the fixed Sylow 2-subgroup S of G, and form $\mathcal{L}$ as in (1). Check that $S \cap S^g \neq 1$ for all $g \in G$, and conclude that $G = \mathcal{L}$ as *sets*. But $\mathcal{L}$ is not a group (i.e. $\mathbf{D}(\mathcal{L}) \neq \mathbf{W}(\mathcal{L})$).

(4*) Google "John Conway M(13)" to find out about the "puzzle" M_{13}. Then figure out how to view M_{13} as an objective partial group $\mathcal{L}$ in which each $X \in \Delta$ is isomorphic to the Mathieu group M_{12} (and where the cardinality of Δ is 13).

(5) Find out about "centric linking systems" from [BLO] or [AKO]. Thus, a centric linking system $\mathcal{L}^c$ is a category whose set Δ of objects is a set of subgroups of a group S. Moreover, $\mathcal{L}^c$ comes equipped with "inclusion morphisms" $\iota_{P,\overline{P}}$, for objects P and $\overline{P}$ such that $P \leq \overline{P}$. Because of this, one can define what it means for an $\mathcal{L}^c$-isomorphism $\phi : P \to Q$ to *extend* to an $\mathcal{L}^c$-isomorphism $\overline{\phi} : \overline{P} \to \overline{Q}$. It just means that the diagram

$$\begin{array}{ccc} \overline{P} & \xrightarrow{\overline{\phi}} & \overline{Q} \\ {\scriptstyle \iota_{P,\overline{P}}}\uparrow & & \uparrow{\scriptstyle \iota_{Q,\overline{Q}}} \\ P & \xrightarrow{\phi} & Q \end{array}$$

commutes. Let $\approx$ be the weakest equivalence relation on the set $Iso(\mathcal{L}^c)$ of $\mathcal{L}^c$-isomorphisms, such that $\phi \approx \psi$ if ϕ extends to ψ. The set $Iso(\mathcal{L}^c)/\approx$ of equivalence classes is then a partial group $\mathcal{L}$ via composition in the category $\mathcal{L}^c$ (one has to show that $\approx$ respects composition in order to establish that the product is well defined) and via inversion of isomorphisms. (Again, one has to show that $\approx$ is compatible with inversion.) Moreover, $(\mathcal{L}, \Delta)$ is objective. The details of this are somewhat lengthy, and may be found in the Appendix to [Ch1]. The only point to mentioning this "example" here, is to indicate how partial groups and objective partial groups were originally conceived.

Lemma 2.2. *Let $(\mathcal{L}, \Delta)$ be an objective partial group.*

(a) *$N_{\mathcal{L}}(X)$ is a subgroup of $\mathcal{L}$ for each $X \in \Delta$.*

(b) *Let $g \in \mathcal{L}$ and let $X \in \Delta$ with $Y := X^g \in \Delta$. Then $N_{\mathcal{L}}(X) \subseteq \mathbf{D}(g)$, and*

$$c_g : N_{\mathcal{L}}(X) \to N_{\mathcal{L}}(Y)$$

is an isomorphism of groups.

(c) *Let $w = (g_1, \cdots, g_n) \in \mathbf{D}$ via $(X_0, \cdots, X_n)$. Then*

$$c_{g_1} \circ \cdots \circ c_{g_n} = c_{\Pi(w)}$$

as maps (isomorphisms) from X_0 to X_n.

Proof. (a) Let $X \in \Delta$ and let $u \in \mathbf{W}(N_{\mathcal{L}}(X)$. Then $u \in \mathbf{D}$ via X, $\mathbf{1} \in N_{\mathcal{L}}(X)$ (1.5(d)), and $N_{\mathcal{L}}(X)^{-1} = N_{\mathcal{L}}(X)$ (1.5(c)). This shows that $N_{\mathcal{L}}(X)$ is a subgroup of $\mathcal{L}$.

(b) Let $x, y \in N_{\mathcal{L}}(X)$ and set $v = (g^{-1}, x, g, g^{-1}, y, g)$. Then $v \in \mathbf{D}$ via Y, and then $\Pi(v) = (xy)^g = x^g y^g$ (using points (a) and (b) of 1.3). Thus, the conjugation map $c_g : N_{\mathcal{L}}(X) \to N_{\mathcal{L}}(Y)$ is a homomorphism of "binary groups" (see 1.2), and hence a homomorphism of partial groups (1.14). Since $c_{g^{-1}} = c_g^{-1}$ by 1.5(c), it follows that c_g is an isomorphism of groups.

(c) Let $x \in N_{\mathcal{L}}(X_0)$, set $u_x = w^{-1} \circ (x) \circ w$, and observe that $u_x \in \mathbf{D}$ via X_n. Then $\Pi(u_x)$ can be written as $(\cdots (x)_1^g \cdots)_n^g$, and this yields (c). $\square$

The next lemma provides two basic computational tools (and indicates that objective partial groups are perhaps "closer" to being groups than are partial groups in general).

Lemma 2.3. *Let $(\mathcal{L}, \Delta)$ be an objective partial group.*

(a) *Let $(a, b, c) \in \mathbf{D}$, and set $d = abc$. Then $bc = a^{-1}d$ and $ab = dc^{-1}$ (and all of these products are defined).*

(b) *Let $(f, g) \in \mathbf{D}$ and let $X \in \Delta$. Suppose that both X^f and Y^{fg} are in Δ. Then $X^{fg} = (X^f)^g$.*

Proof. (a) As $(a, b, c) \in \mathbf{D} = \mathbf{D}_\Delta$, with $abc = d$, it follows from 2.2(c) that there is a commutative diagram of conjugation maps:

$$\begin{tikzcd} B \arrow[r, "b"] & C \\ A \arrow[u, "a"] \arrow[r, "d"'] & D \arrow[u, "c"'] \end{tikzcd}$$

in which the arrows are labelled by the conjugating elements, and where A, B, C, D are objects. Since $c_x^{-1} = (c_x)^{-1}$ for any $x \in \mathcal{L}$, one may read off from the diagram that

$(a, a^{-1}, d) \in \mathbf{D}$, and then $aa^{-1}d = d$ by 1.3. Since also $abc = d$, left cancelation yields $a^{-1}d = bc$. Similarly for $ab = dc^{-1}$.

(b) As $(f, g) \in \Delta$ we have also $(f^{-1}, f, g) \in \Delta$, and $g = \Pi(f^{-1}, f, g) = f^{-1}(fg)$. Now observe that $(f^{-1}, fg) \in \mathbf{D}$ via P^f, and apply 2.2(c) to obtain $P^{fg} = ((P^f)^{f^{-1}})^{fg} = (P^f)^g$. $\square$

The following corollary should be compared with 1.4.

Corollary/Exercise 2.4. *Let $(\mathcal{L}, \Delta)$ be an objective partial group, let $f, g \in \mathcal{L}$, and suppose that $f^g = f$. Then $g^f = g$ and $fg = gf$.*

The following result is, in some sense, the first non-trivial result in these notes. It is fundamental to everything that follows.

Proposition 2.5. *Let $(\mathcal{L}, \Delta)$ be an objective partial group, and suppose that there exists $S \in \Delta$ such that Δ is a set of subgroups of S. For each $f \in \mathcal{L}$, define S_f to be the set of all $x \in \mathbf{D}(f) \cap S$ such that $x^f \in S$. Then $S_f \in \Delta$, and $S_{f^{-1}} = (S_f)^f$.*

Proof. Fix $f \in \mathcal{L}$. Every word of length 1 is in $\mathbf{D}$ by 1.1(2). As $\mathbf{D} = \mathbf{D}_\Delta$ by (O1), there then exists $X \in \Delta$ such that $Y := X^f \in \Delta$. Let $a \in S_f$ and set $b = a^f$. Then X^a and X^b are subgroups of S (as $a, b \in S$) and so X^a and Y^b are in Δ by (O2). Then $(a^{-1}, f, b) \in \mathbf{D}$ via X^a, and $(f, b) \in \mathbf{D}$ via X. Also, $(a, f) \in \mathbf{D}$ via $X^{a^{-1}}$. Since $f^{-1}af = b$ we get $af = fb$ by cancelation, and hence

$$a^{-1}fb = a^{-1}(fb) = a^{-1}(af) = (a^{-1}a)f = f$$

by $\mathbf{D}$-associativity. Since $a^{-1}fb$ conjugates X^a to Y^b, we draw the following conclusion.

(1) $X^a \le S_f$ and $(X^a)^f \in \Delta$ for all $a \in S_f$, and for all $X \in \Delta$ for which $X^f \in \Delta$.

Now let $c, d \in S_f$. Then (1) shows that both X^c and X^{cd} are members of Δ which are conjugated to members of Δ by f. Setting $w = (f^{-1}, c, f, f^{-1}, d, f)$, we conclude (by following X^f along the chain of conjugations given by w) that $w \in \mathbf{D}$ via X^f. One then observes via 1.1(3) that

$$\Pi(w) = (cd)^f = c^f d^f. \tag{2}$$

Thus $cd \in S_f$. Since S_f is closed under inversion by 1.5(b), we conclude that S_f is a subgroup of S. As $X \le S_f \le S$, where X and S are in Δ, (02) now yields $S_f \in \Delta$. Since $c_{f^{-1}} = (c_f)^{-1}$ it follows that $S_{f^{-1}} = (S_f)^f$. $\square$

Lemma 2.6. *Assume the hypothesis of 2.5, and let $w = (g_1, \cdots, g_n) \in \mathbf{W}(\mathcal{L})$. Set*

$$\Sigma_w = \{(x_0, \cdots, x_n) \in \mathbf{W}(S) \mid (x_{i-1})^{g_i} = x_i \textit{ for all } i \textit{ with } 1 \le i \le n\},$$

and set

$$S_w = \{x_0 \in S \mid \exists (x_0, \cdots, x_n) \in \Sigma_w\}.$$

Then S_w is a subgroup of S, and $S_w \in \Delta$ if and only if $w \in \mathbf{D}$.

Proof. Let $x_0, y_0 \in S_w$ and let $\sigma = (x_0, \cdots, x_n)$ and $\tau = (y_0, \cdots, y_n)$ be the corresponding sequences in Σ_w (determined by x, y, and w). Set $S_j = S_{g_j}$ $(1 \leq j \leq n)$. Then x_{i-1} and y_{i-1} are elements of S_{i-1}, and so $x_{i-1}y_{i-1} \in S_{i-1}$ by 2.5. As c_{g_i} restricts to a homomorphism on S_{i-1} (see 2.2(b)) it follows that $x_i y_i \in S_i$. Thus S_w is closed under the binary multiplication in S. That S_w is closed under inversion is given by 1.5(b), so S_w is a subgroup of S. If $S_w \in \Delta$ then $w \in \mathbf{D}$ via S_w. Conversely, if $w \in \mathbf{D}$ then $P \leq S_w$ for some $P \in \Delta$, and then $S_w \in \Delta$ by (O2). □

We are now going to narrow the focus considerably, by restricting attention to objective partial groups whose underlying set of elements is finite, and which satisfy further conditions related to a prime p.

Definition 2.7. Let p be a prime, let $\mathcal{L}$ be a finite partial group, let S be a p-subgroup of $\mathcal{L}$, and let Δ be a set of subgroups of S. Then $(\mathcal{L}, S)$ is a *locality* if the following two conditions hold.

(L1) $(\mathcal{L}, \Delta)$ is objective, and
(L2) S is maximal in the poset (ordered by inclusion) of finite p-subgroups of $\mathcal{L}$.

For any locality $(\mathcal{L}, \Delta, S)$ there is associated fusion system $\mathcal{F}_S(\mathcal{L})$ on S. Namely, $\mathcal{F}_S(\mathcal{L})$ is defined to be the fusion system on S which is generated by the conjugation maps $c_g : S_g \to S$ $(g \in \mathcal{L})$. Recall (?) that this means that the $\mathcal{F}_S(\mathcal{L})$-homomorphisms are the mappings between subgroups of S that can be expressed as compositions of restrictions of the maps c_g for $g \in \mathcal{L}$.

Example/Lemma 2.8. *[RESTATE IN TERMS OF EARLIER EXAMPLE.] Let G be a finite group, let S be a Sylow p-subgroup of G, set $\mathcal{F} = \mathcal{F}_S(G)$, and let Γ be a non-empty $\mathcal{F}$-invariant collection of subgroups of S, such that Γ is overgroup closed in S. Define $\mathcal{L}$ to be the set of all $g \in G$ such that $S \cap S^g \in \Gamma$, and set $\mathbf{D} = \mathbf{D}_\Gamma$. Then $\mathcal{L}$ is a partial group via the restriction of the multivariable product in G to $\mathbf{D}$, as in the examples (1) through (3) following 2.1. Moreover, one may verify $(\mathcal{L}, \Gamma, S)$ is a locality; to be denoted $\mathcal{L}_\Gamma(G)$.*

Proof. ————————————-

If $g \in \mathcal{L}$ then $(S \cap S^{g^{-1}})^g = S \cap S^g \in \Gamma$, and then $(S \cap S^{g^{-1}}) \in \Gamma$ since Γ is $\mathcal{F}$-invariant. Thus $\mathcal{L} \subseteq \mathbf{D}$, and $\mathcal{L}$ is contained in the partial group $\mathcal{M} = \mathcal{M}(M, \Gamma)$ given by example 2.4(2). In that example, $\mathcal{M}$ is the set of all $g \in M$ such that there exists $P \in \Gamma$ with $P^g \in \Gamma$. Such an element g has the property that $S \cap S^g \in \Gamma$ since Γ is overgroup closed, and so $\mathcal{L} = \mathcal{M}$. Example 2.4(2) now shows that $\mathcal{L}$ is a partial group with respect to the multivariable product and the inversion in G. The condition (O1) for objectivity is given by the definition of $\mathbf{D}$, while (O2) is immediate from the assumption that Γ is overgroup closed and $\mathcal{F}$-invariant. Thus, $(\mathcal{L}, \Gamma)$ is objective. All members of Γ are subgroups of S, and S is maximal in the poset of p-subgroups of G, so $(\mathcal{L}, S)$ is a locality via Γ. □

Lemma 2.9. *Let $(\mathcal{L}, \Delta, S)$ be a locality and let $P \in \Delta$. Then there exists $g \in N_{\mathcal{L}}(P, S)$ such that $N_S(P^g) \in Syl_p(P^g)$.*

proof. Since S is maximal in the poset of p-subgroups of $\mathcal{L}$, S is a Sylow p-subgroup of the (finite) group $N_{\mathcal{L}}(S)$, and so the lemma holds for $P = S$ (and with $g = 1$).

Assume now that the lemma is false, and among all $P \in \Delta$ for which no there exists no $g \in N_{\mathcal{L}}(P, S)$ such that $N_S(P^g)$ is a Sylow subgroup of $N_{\mathcal{L}}(P^g)$, choose P so that $|P|$ is as large as possible, and then so that $|N_S(P)|$ is as large as possible. Set $R = N_S(P)$ and let R^* be a Sylow p-subgroup of $N_{\mathcal{L}}(P)$ containing R. Then $R < R^*$ (proper subgroup), and then also $R < N_{R^*}(R)$. Observe that since $P \neq S$ we have $P < R$, and the maximality of $|P|$ then yields the existence of an element $f \in N_{\mathcal{L}}(R, S)$ with $N_S(R^f) \in Syl_p(N_{\mathcal{L}}(R^f)$.

By 2.2(b) there is an isomorphism

$$N_{\mathcal{L}}(R) \xrightarrow{c_f} N_{\mathcal{L}}(R^f)$$

induced by conjugation by f. Apply Sylow's Theorem to $N_{\mathcal{L}}(R^f)$ to obtain an element $x \in N_{\mathcal{L}}(R^f)$ such that $(N_{R^*}(R)^f)^x \leq N_S(R^f)$. Here $(f, x) \in \mathbf{D}$ via R, so 2.2(c) yields $(N_{R^*}(R)^f)^x = N_{R^*}(R)^{fx}$. Thus, by replacing f with fx, we may assume that f was chosen to begin with so that $N_{R^*}(R)^f \leq N_S(R^f)$. Since R^* normalizes P, and since c_f is an isomorphism, it follows that $N_{R^*}(R)^f$ normalizes P^f, and thus $|N_S(P^f)| > |N_S(P)|$. The maximality of $|N_S(P)|$ in the choice of R then implies that P^f is not a counter-example to the lemma. Set $Q = P^f$. Thus, there exists $h \in N_{\mathcal{L}}(Q, S)$ such that $N_S(Q^h)$ is a Sylow subgroup of $N_{\mathcal{L}}(Q^h)$. Since $(f, h) \in \mathbf{D}$ via P we have $Q^h = P^g$ where $g = fh$, and thus P is not a counter-example. $\square$

At this point it may be helpful to review the peculiar definition of saturation, and of Δ-saturation, from section 0.

Proposition 2.10. *Let $(\mathcal{L}, \Delta, S)$ be a locality and set $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$. Then $\mathcal{F}$ is Δ-saturated.*

Proof. We first show that every $P \in \Delta$ has a fully normalized $\mathcal{F}$-conjugate. Namely, by 2.9 we may assume (after possibly conjugating P by a suitable element of $N_{\mathcal{L}}(P, S)$) that $N_S(P)$ is a Sylow p-subgroup of $N_{\mathcal{L}}(P)$. Now let Q be an $\mathcal{F}$-conjugate of P. Thus $Q = P\phi$ where ϕ is a composition of restrictions of $\mathcal{L}$-conjugation maps. All images of P in S under $\mathcal{L}$-conjugation are objects, by (O3), so 2.2(c) implies that in fact $Q = P^f$ for some $f \in \mathcal{L}$. Note that, as in the proof of 2.9, $c_{f^{-1}} : N_{\mathcal{L}}(Q) \to N_{\mathcal{L}}(P)$ is an isomorphism, and there exists $x \in N_{\mathcal{L}}(P)$ such that $N_S(P)^{f^{-1}x} \leq N_S(P)$. Setting $g = f^{-1}x$ (product defined via Q) we obtain $Q^g = P$ and $N_S(Q)^g \leq N_S(P)$. Since c_g is an $\mathcal{F}$-homomorphism $N_S(Q) \to S$ we conclude that P is fully normalized in $\mathcal{F}$.

In order to complete the proof of Δ-saturation it remains to show that for each $P \in \Delta$ with P fully normalized in $\mathcal{F}$, there exists a group M_P such that $P \trianglelefteq M_P$, $N_S(P) \in Syl_p(M_P)$, and such that

$$(*) \qquad \mathcal{F}_{N_S(P)}(M_P) = N_{\mathcal{F}}(P).$$

The obvious candidate for M_P is $N_{\mathcal{L}}(P)$ - and indeed it remains only to verify (*). The inclusion $\mathcal{F}_{N_S(P)}(M_P) \subseteq N_{\mathcal{F}}(P)$ of fusion systems is immediate from the definition of $\mathcal{F}$. In order to prove the opposite inlusion, let ϕ be a homomorphism in in $N_{\mathcal{F}}(P)$. Then ϕ is the restriction of an $\mathcal{F}$-homomorphism $\psi : R \to R'$ where $P \trianglelefteq R$, $P \trianglelefteq R'$, and where $P\psi = P$. By 2.2(c), ψ is given by conjugation by an element $g \in \mathcal{L}$, and then the same is true of ϕ. Since $g \in N_{\mathcal{L}}(P)$ we conclude that ϕ is an $\mathcal{F}_{N_S(P)}(M_P)$-homomorphism; and so the proof of (*), and of Δ-saturation, is complete. $\square$

Proposition 2.11. *Let $(\mathcal{L}, \Delta, S)$ be a locality and let H be a subgroup of $\mathcal{L}$.*

(a) *There exists an object $P \in \Delta$ such that $H \leq N_{\mathcal{L}}(P)$. [All subgroups are "local" subgroups.]*
(b) *If H is a p-group then there exists $g \in \mathcal{L}$ such that $H^g \leq S$. [S is a "Sylow subgroup" of $\mathcal{L}$.]*

Proof. (a) For any $w = (h_1, \cdots, h_n) \in \mathbf{W}(H)$ let w' be the word $(g_1, \cdots, g_n)$ defined by $g_i = h_1 \cdots h_i$. As $\mathcal{L}$ is finite, so is H, and one may therefore choose $w \in \mathbf{W}(H)$ so that the cardinality of the set $X = \{g_1, \cdots, g_n\}$ is as large as possible. Suppose $X \neq H$, let $g \in H - X$, and set $h = \Pi(w)^{-1}g$. Then the set of entries of $(w \circ (h))'$ is $X \cup \{g\}$, contrary to the maximality of X. Thus $X = H$.

We have $\mathbf{W}(\mathcal{H}) \subseteq \mathbf{D}$ by our definition of subgroup, so $w \in \mathbf{D}$ via some $P \in \Delta$. Then $P^{g_i} = P^{h_1 \cdots h_i} \leq S$ for all i, and so $P^h \leq S$ for all $h \in H$. Set $U = \langle P^h \mid h \in H \rangle$ (the subgroup of S generated by the union of all P^h for $h \in H$). Then $U \in \Delta$ by (O2), so it now suffices to show that $H \leq N_{\mathcal{L}}(U)$. For this it suffices to observe that, by 2.3(b), $(P^f)^g$ is defined and is equal to P^{fg} for all $f, g \in H$.

(b) By point (a) there exists $U \in \Delta$ with $H \leq N_{\mathcal{L}}(U)$, and by 2.9 there exists $V \in \Delta$ and $g \in \mathcal{L}$ such that $V = U^g$ and such that $N_S(V) \in Syl_p(N_{\mathcal{L}}(V))$. Let $c_g : N_{\mathcal{L}}(U) \to N_{\mathcal{L}}(V)$ be the isomorphism given by 2.2(b). Thus H^g is a p-subgroup of $N_{\mathcal{L}}(V)$, so there exists $x \in N_{\mathcal{L}}(V)$ with $(H^g)^x \leq N_S(V)$. Since $(g, x) \in \mathbf{D}$ via U we may apply 2.2(c), obtaining $(H^g)^x = H^{gx}$. Thus (b) holds with gx in the role of g. $\square$

Recall (?) that for any subgroup X of a partial group $\mathcal{L}$, the *normalizer* $N_{\mathcal{L}}(X)$ is the set of all elements $g \in \mathcal{L}$ such that $X^g = X$.

Lemma 2.12. *Let $(\mathcal{L}, \Delta, S)$ be a locality, let T be a subgroup of S, and set$\Delta_T = \{N_P(T) \mid T \leq P \in \Delta\}$.*

(a) *$N_{\mathcal{L}}(T)$ is a partial subgroup of $\mathcal{L}$.*
(b) *If $\Delta_T \subseteq \Delta$, then $(N_{\mathcal{L}}(T), \Delta_T)$ is an objective partial group.*
(c) *If $\Delta_T \subseteq \Delta$, and $|N_S(T)| \geq |N_S(U)|$ for every $\mathcal{L}$-conjugate U of T in S, then $(N_{\mathcal{L}}(T), N_S(T))$ is a locality via Δ_T.*

Proof. [FIX THE REFERENCES.] Let $w = (f_1, \cdots, f_n) \in \mathbf{W}(N_{\mathcal{L}}(T))$, and suppose that $w \in \mathbf{D} := \mathbf{D}(\mathcal{L})$ via a sequence $(P_0, \cdots, P_n)$ of objects. Then $\langle P_{i-1}, T \rangle \leq S_{f_i}$ for all i, by completeness, and then

$$\langle P_{i-1}, T \rangle^{f_i} = \langle P_i, T \rangle.$$

Thus, $T \leq S_w$, and we may assume for the sake of simplicity that $T \leq P_i$ for all i. Set $f = \Pi(w)$. Then 2.8(c) yields $T^f = T$, and so $N_{\mathcal{L}}(T)$ is closed under products. One observes that if $f \in N_{\mathcal{L}}(T)$ and $x \in T$, with $(f^{-1}, x, f) \in \mathbf{D}$ via $P \in \Delta$, then $(f, x^{-1}, f^{-1}) \in \mathbf{D}$ via P^{x^f}. Since an analogous statement holds when x is replaced by x^{-1}, it follows that $N_{\mathcal{L}}(T)$ is closed under inversion, and so (a) is proved.

For the remainder of the proof, we may assume that $\Delta_T \subseteq \Delta$. Set

$$\mathbf{D}_T = \mathbf{D}_\Delta \cap \mathbf{W}(N_{\mathcal{L}}(T))$$

(where $\mathbf{D}_\Delta$ is defined in 2.6). With w and $(P_0, \cdots, P_n)$ as in the proof of (a), we may then replace P_i with $N_{P_i}(T)$, and this shows that $\mathbf{D}_T$ is contained in the subset $\mathbf{D}_{\Delta_T}$ of $\mathbf{W}(N_{\mathcal{L}}(T))$. The reverse inclusion is obvious, so $(N_{\mathcal{L}}(T), N_S(T))$ satisfies the condition (O1) for objectivity. Any overgroup in $N_S(T)$ of an element of Δ_T is again in Δ_T, so the condition (O2) is satisfied, and $(N_{\mathcal{L}}(T), \Delta_T)$ is an objective partial group. Thus, (b) holds.

Now assume further that T has been chosen so that $|N_S(T)| \geq |N_S(U)|$ for each $\mathcal{L}$-conjugate U of T in S. In order to show that $(N_{\mathcal{L}}(T), N_S(T))$ is a locality via Δ_T, it suffices to show that $N_S(T)$ is maximal in the poset of p-subgroups of $N_{\mathcal{L}}(T)$. Set $R = N_S(T)$, let R_1 be a p-subgroup of $N_{\mathcal{L}}(T)$ containing R, and set $R_2 = N_{R_1}(R)$. As $R \in \Delta$, there exists $f \in \mathcal{L}$ such that $Q := R^f$ is fully normalized in $\mathcal{F}_S(\mathcal{L})$, by 2.17(a). Then $N_S(Q)$ is a Sylow p-subgroup of $N_{\mathcal{L}}(Q)$, and so there exists $g \in N_{\mathcal{L}}(Q)$ such that $(R_2)^{fg} \leq N_S(Q)$. But $(R_2)^{fg} \leq N_S(T^{fg})$, and the maximality condition on R then yields $R = R_2$ and $R = R_1$. This completes the proof of (c). □

Definition 2.13. Let $(\mathcal{L}, \Delta, S)$ be a locality, and let $\Gamma \subseteq \Delta$ be a non-empty subset such that Γ is both overgroup-closed in S and $\mathcal{F}_S(\mathcal{L})$-invariant. Set $\mathbf{D} = \mathbf{D}(\mathcal{L})$, set

$$\mathbf{D} \mid_\Gamma := \{w \in \mathbf{D} \mid S_w \in \Gamma\},$$

and let $\mathcal{L} \mid_\Gamma$ be the set of words of length 1 in $\mathbf{D} \mid_\Gamma$, regarded as a subset of $\mathcal{L}$. The *restriction of* $\mathcal{L}$ *to* Γ consists of $\mathcal{L} \mid_\Gamma$ together with the restriction to $\mathbf{D} \mid_\Gamma$ of the product in $\mathcal{L}$, and the restriction to $\mathcal{L} \mid_\Gamma$ of the inversion in $\mathcal{L}$.

Lemma 2.14. *[FIX THE REFERENCES.] Let* $(\mathcal{L}, \Delta, S)$ *be a locality, and let* Γ *be a non-empty subset of* Δ*, such that* Γ *is both overgroup-closed in* S *and* $\mathcal{F}_S(\mathcal{L})-$*invariant.*

(a) $\mathbf{D} \mid_\Gamma$ *is the set* $\mathbf{D}_\Gamma$ *of 2.6, and* $(\mathcal{L} \mid_\Gamma, \Gamma, S)$ *is a locality.*
(b) *If* $\mathcal{L}$ *is a group* M*, then* $\mathcal{L} \mid_\Gamma$ *is the locality* $\mathcal{L}_\Gamma(M)$ *given by 2.10.1.*

Proof. Set $\mathcal{M} = \mathcal{L} \mid_\Gamma$. For any $w \in \mathbf{W}$, the condition that S_w be in Γ is the defining condition for $\mathbf{D} \mid_\Gamma$, and in view of 2.13(a) it is also the defining condition for $\mathbf{D}_\Gamma$. These subsets of $\mathbf{W}$ are therefore identical, and $(\mathcal{M}, \Gamma)$ satisfies the condition (O1) for objectivity. Condition (O2) is given by the assumption that Γ is closed in $\mathcal{F}_S(\mathcal{L})$, so $(\mathcal{M}, \Gamma)$ is objective. All members of Γ are subgroups of S, and S is maximal in the poset of p-subgroups of $\mathcal{M}$ since the corresponding statement holds in $\mathcal{L}$. As $\mathcal{L}$ is finite, so is $\mathcal{M}$, so $\mathcal{M}$ is a locality, and (a) holds.

Suppose that $\mathcal{L}$ is in fact a group M, and set $\mathcal{K} = \mathcal{L}_\Gamma(M)$. By definition, an element g of M is in $\mathcal{K}$ if and only if $S \cap S^g \in \Gamma$. The latter condition means that $S_g = S \cap S^{g^{-1}}$, so $g \in \mathcal{K}$ if and only if $S_g \in \Gamma$. Similarly, $w \in \mathbf{D}(\mathcal{K})$ if and only if $S_w \in \Gamma$. This shows that $\mathbf{D}(\mathcal{K}) = \mathbf{D}_\Gamma$, and then (b) follows from (a). $\square$

We refer to the locality $(\mathcal{L}\,|_\Gamma, \Gamma, S)$ as the restriction of $\mathcal{L}$ to Γ.

Section 3: Partial normal subgroups of localities

This section contains three results (3.5 through 3.7) which enable the construction of quotient localities. Throughout, we fix a locality The following hypothesis, and notation, will be assumed throughout. $\mathcal{L} = (\mathcal{L}, \Delta, S)$ and a partial normal subgroup $\mathcal{N}$ of $\mathcal{L}$. Set $T = S \cap \mathcal{N}$, and set $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$.

Lemma 3.1.

(a) *T is strongly closed in $\mathcal{F}$, and T is maximal in the poset of all p-subgroups of $\mathcal{N}$.*
(b) *Let $x \in \mathcal{N}$ and let P be a subgroup of S_x. Then $PT = P^xT$.*
(c) *T is maximal in the poset of p-subgroups of $\mathcal{N}$.*

Proof. (a) Let $g \in \mathcal{L}$ and let $t \in S_g \cap T$. Then $t^g \in S$, and $t^g \cap \mathcal{N}$ as $\mathcal{N} \trianglelefteq \mathcal{L}$. Thus $t^g \in T$. Since $\mathcal{F}$ is generated by the conjugation maps $c_g : S_g \to S$, point (a) follows.

(b) Let $a \in P$. Then $(P^x)^a \leq S$ and $P^a = P$. Setting $w = (a^{-1}, x^{-1}, a, x)$ we then have $w \in \mathbf{D}$ via P^{xa}. Now $\Pi(w) = a^{-1}a^x \in S$, while also $\Pi(w) = (x^{-1})^a x \in \mathcal{N}$, and so $\Pi(w) \in T$. Then $a^x \in aT$, and we have thus shown that $P^x \leq PT$. Then $P^xT \leq PT$. The equality $P^xT = PT$ can then be deduced from (a) (which implies that $|P^x \cap T| = |P \cap T|$), or from symmetry with x^{-1} and P^x in place of x and P.

(c) Let R be a p-subgroup of $\mathcal{N}$ containing T. By 2.9(b) there exists $g \in \mathcal{L}$ with $R^g \leq S$, and then $R^g \leq S \cap \mathcal{N} = T$. As conjugation by g is injective (and since the groups we are working with are finite) we conclude that $T = R$. $\square$

Lemma 3.2. *Let $x, y \in \mathcal{N}$ and let $f \in N_{\mathcal{L}}(T)$.*

(a) *If $(x, f) \in \mathbf{D}$ then $(f, f^{-1}, x, f) \in \mathbf{D}$, $xf = fx^f$, and $S_{(x,f)} = S_{(f,x^f)} = S_x \cap S_f$.*
(b) *If $(f, y) \in \mathbf{D}$ then $(f, y, f^{-1}, f) \in \mathbf{D}$, $fy = y^{f^{-1}}y$, and $S_{(f,y)} = S_{(y^{f^{-1}},y)} = S_{y^{f^{-1}}} \cap S_f$.*

Proof. For point (a): Set $Q = S_{(x,f)}$ and note that $T \leq S_f$ by hypothesis. We have $Q^xT = QT$ by 3.1(b), so $Q \leq S_f$. Thus $Q \leq P := S_x \cap S_f$. But also $P^xT = PX$, so $P = Q$. Moreover, we now have $(f, f^{-1}, x, f) \in \mathbf{D}$ via Q, and then $\Pi(f, f^{-1}, x, f) = xf = fx^f$. Thus, (a) holds.

For point (b): Set $R = S_{(f,y)}$. Then $R^{fy}T = R^fT \leq S_{f^{-1}}$, so $(f, y, f^{-1}, f) \in \mathbf{D}$ via R, and $fy = y^{f^{-1}}f$. The remainder of (b) now follows as an application of (a) to $(y^{f^{-1}}, f)$. $\square$

Definition 3.3. Let $\mathcal{L} \circ \Delta$ be the set of all pairs $(f, P) \in \mathcal{L} \times \Delta$ such that $P \le S_f$. Define a relation $\uparrow$ on $\mathcal{L} \circ \Delta$ by $(f, P) \uparrow (g, Q)$ if there exist elements $x \in N_{\mathcal{N}}(P, Q)$ and $y \in N_{\mathcal{N}}(P^f, Q^g)$ such that $xg = fy$.

This relation may be indicated by means of a commutative diagram:

$$(*)\qquad \begin{tikzcd} Q \arrow[r, "g"] & Q^g \\ P \arrow[u, "x"] \arrow[r, "f"] & P^f \arrow[u, "y"'] \end{tikzcd}$$

of conjugation maps, labeled by the conjugating elements, and in which the horizontal arrows are isomorphisms and the vertical arrows are injective homomorphisms. The relation $(f, P) \uparrow (g, Q)$ may also be expressed by:

$$w := (x, g, y^{-1}, f^{-1}) \in \mathbf{D} \text{ via } P, \text{ and } \Pi(w) = \mathbf{1}.$$

It is easy to see that $\uparrow$ is a reflexive and transitive relation on $\mathcal{L} \circ \Delta$. We say that (f, P) is *maximal* in $\mathcal{L} \circ \Delta$ if $(f, P) \uparrow (g, Q)$ implies that $|P| = |Q|$. As S is finite there exist maximal elements in $\mathcal{L} \circ \Delta$. It's clear that $(f, P) \uparrow (f, S_f)$ for $(f, P) \in \mathcal{L} \circ \Delta$, by taking $x = y = 1$ in the diagram (*); so $P = S_f$ for every maximal (f, P). For this reason, we will say that an element $f \in \mathcal{L}$ is $\uparrow$*-maximal* if (f, S_f) is maximal in $\mathcal{L} \circ \Delta$.

Lemma/Exercise/Observation 3.4. *Let $f \in \mathcal{L}$.*

(a) *If $f \in N_{\mathcal{L}}(S)$ then f is $\uparrow$-maximal.*
(b) *f is $\uparrow$-maximal $\iff$ f^{-1} is $\uparrow$-maximal.*
(c) *$\uparrow$ is a transitive relation.*
(d) *For each $(f, P) \in \mathcal{L} \circ \Delta$ there exists a maximal (f', P') such that $(f, P) \uparrow (f', P')$.*

$\square$

The first main result of this section is as follows.

Proposition 3.5. *Let $f \in \mathcal{L}$ and suppose that f is $\uparrow$-maximal. Then $T \le S_f$.*

The proof requires two preliminary lemmas.

Lemma 3.5.1. *Let $(h, P) \uparrow (h', P')$, and assume that $T \le P'$. Then there exists $y \in \mathcal{N}$ such that $h = yh'$. Moreover, we then have the following.*

(a) *$y \in N_{\mathcal{N}}(P, P')$, and $P \le S_{(y,h')}$.*
(b) *If $N_T(P^h) \in Syl_p(N_{\mathcal{N}}(P^h))$ then $N_T(P^y) \in Syl_p(N_{\mathcal{N}}(P^y))$.*

Proof. Set $R = P^h$ and $R' = (P')^{h'}$. We are given a commutative diagram

$$\begin{tikzcd} P' \arrow[r, "h'"] & R' \\ P \arrow[u, "u"] \arrow[r, "h"'] & P' \arrow[u, "v"'] \end{tikzcd}$$

as in the definition of the relation $\uparrow$. (So, $u, v \in \mathcal{N}$, and $uh' = hv$). Since $T \leq P'$ we get $T = T^{h'} \leq R'$ from 3.1(a), and we get $R^v T = RT$ from 3.1(b). Thus $RT \leq R'$, and similarly $PT \leq R$.

Set $w = (u, h', v^{-1}, (h')^{-1})$. Then $w \in \mathbf{D}$ via the sequence

$$(P, P^u, P^{uh'} = P^{hv}, P^h = R, R^{(h')^{-1}}).$$

Set $y = \Pi(w)$. Thus $y = u(v^{-1})^{(h')^{-1}}$, and so $y \in \mathcal{N}$, and indeed $y \in N_{\mathcal{N}}(P, P')$. One checks that

$$yh' = uh'v^{-1} = hvv' = h$$

(by checking that the products are defined), so we have produced the required element y, and we've established point (a).

Notice that $N_{\mathcal{N}}(R)$ is a normal subgroup of (the group) $N_{\mathcal{L}}(R)$, by 1.7(d). By (a) and 2.2(b) we have an isomorphism of groups

$$N_{\mathcal{L}}(P^y) \xrightarrow{c_{h'}} N_{\mathcal{L}}(R),$$

which then restricts to an isomorphism

$$N_{\mathcal{N}}(P^y) \longrightarrow N_{\mathcal{N}}(R),$$

and which (since $T \leq P'$) restricts further to an isomorphism $N_T(P^y) \to N_T(R)$. Assuming now that $N_T(R)$ is a Sylow subgroup of $N_{\mathcal{N}}(R)$, we conclude that $N_Y(P^y)$ is a Sylow subgroup of $N_{\mathcal{N}}(P^y)$. That is, (b) holds. $\square$

Lemma 3.5.2. *Suppose that f is $\uparrow$-maximal, and let $y \in N_{\mathcal{N}}(S_f, S)$. Then $|T \cap S_f| = |T \cap (S_f)^y|$, and $(f, S_f) \uparrow (y^{-1}f, (S_f)^y)$. In particular, $y^{-1}f$ is $\uparrow$-maximal.*

Proof. Set $P = S_f$. Then $P^y T = PT$, by 3.1(b). Then

$$|P^y : P^y \cap T| = |P^y T : T| = |PT : T| = |P : P \cap T|,$$

and so $|T \cap P| = |T \cap P^y|$. The following diagram

$$\begin{array}{ccc} P^y & \xrightarrow{y^{-1}f} & P^f \\ {\scriptstyle y}\uparrow & & \uparrow{\scriptstyle \mathbf{1}} \\ P & \xrightarrow[f]{} & P^f \end{array}$$

shows that $(f, P) \uparrow (y^{-1}f, P^y)$. Transitivity of $\uparrow$ now implies that $(y^{-1}f, P^y)$ is maximal in $\mathcal{L} \circ \Delta$. That is, $y^{-1}f$ is $\uparrow$-maximal. $\square$

Proof of Proposition 3.5. Let f be $\uparrow$-maximal, set $P = S_f$, and set $Q = P^f$.

STEP 1: Suppose first that $N_T(P) \in Syl_p(N_{\mathcal{N}}(P))$, and consider the isomorphism

$$N_{\mathcal{L}}(P) \xrightarrow{c_f} N_{\mathcal{L}}(Q)$$

and its restriction (as in the proof of 3.5.1) to an isomorphism

$$N_{\mathcal{N}}(P) \longrightarrow N_{\mathcal{N}}(Q).$$

Then $N_T(P)^f \in Syl_p(N_{\mathcal{N}}(Q)$, and Sylow's Theorem yields an element $x \in N_{\mathcal{N}}(Q)$ such that $N_T(Q) \leq (N_T(P)^f)^x$. Here $(f, x) \in \mathbf{D}$ via P, so $(N_T(P)^f)^x = N_T(P)^{fx}$.

Set $\widetilde{P} = N_T(Q)^{(fx)^{-1}}P$, and ovserve now that $(f, P) \uparrow (fx, \widetilde{P})$ via the following diagram.

$$\begin{tikzcd}
\widetilde{P} \arrow[r, "fx"] & N_T(Q)Q \\
P \arrow[u, "1"] \arrow[r, "f"'] & Q \arrow[u, "x"']
\end{tikzcd}$$

Maximality of (f, P) yields $|P| = |\widetilde{P}|$, so $P = \widetilde{P}$ and $Q = N_T(Q)Q$. Thus $N_T(Q) \leq Q$, whence $T \leq Q$, and $T \leq P$. We have thus shown:

(*) For any $g \in \mathcal{L}$ such that g is $\uparrow$-maximal, and such that $N_T(P) \in Syl_p(N_{\mathcal{N}}(P)$, we have $T \leq S_f$.

STEP 2: Among all counter-examples to the proposition, assume that f has been chosen so that first $|P \cap T|$ is as large as possible, and then so that $|P|$ is as large as possible. As above, write $P = S_f$, $Q = P^f$. By 2.9 we may choose $g \in N_{\mathcal{L}}(Q, S)$ such that $N_S(Q^g) \in Syl_p(N_{\mathcal{L}}(Q^g))$. Set $R = Q^g$. Since $N_{\mathcal{N}}(R)$ is a normal subgroup of $N_{\mathcal{L}}(R)$ we obtain

(**) $N_T(R) \in Syl_p(N_{\mathcal{N}}(R))$.

(The intersection of a normal subgroup N of a finite group G with any given Sylow subgroup of G is a Sylow subgroup of N.)

Set $h = fg$ (defined via P) and let (h', P') be maximal in $\mathcal{L} \circ \Delta$, with $(h, P) \uparrow (h', P')$ (3.4(d)). In the usual way, we express this relation by a diagram.

$$\begin{tikzcd}
P' \arrow[r, "h'"] & R' \\
P \arrow[u, "u"] \arrow[r, "h"'] & R \arrow[u, "v"']
\end{tikzcd}$$

There are now two possibilities.

Case (A): Assume $T \not\leq P'$. Then h' is a counter-example to the proposition. Since $|P \cap T| \leq |P' \cap T|$ it follows from the maximality of $(|P \cap T|, |P|)$ in our choice of f that

$|P| = |P'|$. Then h is $\uparrow$-maximal, and h^{-1} is $\uparrow$-maximal (3.4(b)). Then (**) and (*) yield $T \leq R$, and so $T \leq P$ (3.1(a)). That is, f is not a counter-example to the proposition.

Case (B): Assume $T \leq P'$. Then 3.5.1 applies, and yields an element $y \in N_{\mathcal{N}}(P, P')$ such that $h = yh'$. Then 3.5.2 applies and shows that $y^{-1}f$ is $\uparrow$-maximal and $(f, P) \uparrow (f^{-1}y, P^y)$. As (f, P) is $\uparrow$-maximal, and since $(f^{-1}y, P^y) \uparrow (f^{-1}y, S_{f^{-1}y})$, the transitivity of $\uparrow$ yields $P^y = S_{y^{-1}f}$.

Since $N_S(P^h) \in Syl_p(N_{\mathcal{L}}(P^h))$ by (**), we have $N_S(P^y) \in Syl_p(N_{\mathcal{L}}(P^y)$ by 3.5.1(b), and then $N_T(P^y) \in Syl_p(N_{\mathcal{N}}(P^y)$. Then (*) applies to $y^{-1}f$ in the role of f, and yields $T \leq P^y$. Then $T \leq P$ and, once again, f is not a counter-example. This completes the proof. $\square$

Corollary 3.6 (Frattini Lemma). *Let $\mathcal{L} = (\mathcal{L}, \Delta, S)$ be a locality, let $\mathcal{N}$ be a partial normal subgroup of $\mathcal{L}$, and set $T = S \cap \mathcal{N}$. Then $\mathcal{L} = \mathcal{N}N_{\mathcal{L}}(T) = N_{\mathcal{L}}(T)\mathcal{N}$.*

Proof. Let $f \in \mathcal{L}$, set $P = S_f$, and choose $(g, Q) \in \mathcal{L} \circ \Delta$ so that $(f, P) \uparrow (g, Q)$ and so that g is $\uparrow$-maximal. By transitivity of $\uparrow$, we may take $Q = S_g$. Then $T \leq Q$ by 4.5, and then 4.5.1 yields an element $y \in N_{\mathcal{N}}(P, Q)$ with $f = yg$. Here $g \in N_{\mathcal{L}}(T)$ by 3.1(a). Thus $f \in \mathcal{N}N_{\mathcal{L}}(T)$. By 3.2 we have also $f = fy^f \in N_{\mathcal{L}}(T)\mathcal{N}$. $\square$

Lemma 3.7 (Splitting Lemma). *Let $(x, f) \in \mathbf{D}$ with $x \in \mathcal{N}$ and with f $\uparrow$-maximal. Then $S_{(x,f)} = S_{xf} = S_{(f,x^f)}$.*

Proof. Appealing to 3.2, set $y = x^f$ and $g = xf$ (so that also $g = fy$), and set $Q = S_{(x,f))}$ (so that also $Q = S_{(f,y)})$. Thus $Q \leq S_f \cap S_g$. Set

$$P_0 = N_{S_f}(Q), \quad P_1 = N_{S_g}(Q), \quad P = \langle P_0, P_1 \rangle,$$

and set $R = P_0 \cap P_1$. Then $Q \leq R$. In fact, 2.3(b) shows that $y = f^{-1}g$ and that $(R^f)^y = R^g$, so $R \leq Q$, and thus $P_0 \cap P_1 = Q$. Assume now that (x, f) is a counter-example to the lemma. That is, assume $Q < S_g$ (proper inclusion). Then $Q < P_1$ and so $P_1 \not\leq P_0$. Thus:

(*) $P_1 \not\leq S_f$.

Among all counter-examples, take (x, f) so that $|Q|$ is as large as possible.

CASE 1: The case $x \in N_{\mathcal{L}}(T)$.

As $f \in N_{\mathcal{L}}(T)$ (3.5) we get $T \leq Q$, and then $x \in N_{\mathcal{L}}(Q)$ by 3.1(b). Thus $Q^g = Q^{xf} = Q^f$. Set $Q' = Q^g$. Then 2.2(b) yields an isomorphism $c_f : N_{\mathcal{L}}(Q) \to N_{\mathcal{L}}(Q')$. Since $f = x^{-1}g$, 2.2(c) yields $c_f = c_{x^{-1}} \circ c_g$. As $x \in N_{\mathcal{N}}(Q) \trianglelefteq N_{\mathcal{L}}(Q)$, we get $(P_1)^{x^{-1}} \leq N_{\mathcal{N}}(Q)P_1$, and then

$$(P_1)^f = ((P_1)^{x^{-1}})^g \leq (N_{\mathcal{N}}(Q)P_1)^g = N_{\mathcal{N}}(Q')N_S(Q').$$

Also $(P_0)^f \leq N_S(Q')$, so

(**) $P^f \leq N_{\mathcal{N}}(Q')N_S(Q')$.

Since $T \leq Q'$, 3.1(c) implies that T is a Sylow p-subgroup of $N_{\mathcal{N}}(Q')$, and hence $N_S(Q')$ is a Sylow p-subgroup of $N_{\mathcal{N}}(Q')N_S(Q')$. Then (**) and Sylow's Theorem together yield an element $v \in N_{\mathcal{N}}(Q')$ such that $P^{fv} \leq N_S(Q')$. In particular, we have $P \leq S_{fv}$ (and where fv is defined via Q).

Set $u = v^{f^{-1}}$. Then 3.2 yields $(u, f) \in \mathbf{D}$ and $S_{(u,f)} = S_{(f,v)}$. If $S_{(f,v)} = S_{uf}$ then $(f, v) \in \mathbf{D}$ via P, so that $P \leq S_f$, contrary to (*). Thus $S_{(f,v)} =\neq S_{uf}$, and so (u, f) is a counter-example to the lemma. As $Q \leq P$, the maximality of $|Q|$ in our choice of (x, f) implies that $Q = P$, so $Q = P_1$, and we again contradict (*). Thus, we are reduced to:

CASE 2: The case $x \in N_{\mathcal{L}}(T)$.

Let h be $\uparrow$-maximal, with $(g, S_g) \uparrow (h, S_h)$. Then $T \leq S_h$ by 3.5, and then 3.5.1 yields an element $r \in \mathcal{N}$ with $g = rh$ and (3.5.1(a)) with $S_g \leq S_{(r,h)}$. Set $w = (f^{-1}, x^{-1}, r, h)$ and observe that $w \in \mathbf{D}$ via Q^g and that $\Pi(w) = (f^{-1}x^{-1})(rh) = g^{-1}g = \mathbf{1}$. Then 2.3 yields $h = r^{-1}xf$. Since both f and h are in $N_{\mathcal{L}}(T)$, 2.2(c) yields $r^{-1}x \in N_{\mathcal{L}}(T)$, and so $r^{-1}x \in N_{\mathcal{N}}(T)$. Then Case 1 applies to $(r^{-1}x, f)$, and thus $S_h = S_{(r^{-1}x,f)} \leq S_f$ (using 3.2). By definition of $\uparrow$ there exist $a, b \in \mathcal{N}$ such that one has the usual sort of "commutative diagram":

$$\begin{tikzcd} S_h \arrow[r, "h"] & S_{h^{-1}} \\ S_g \arrow[u, "a"] \arrow[r, "g"'] & S_{g^{-1}} \arrow[u, "b"'] \end{tikzcd} .$$

As $T \leq S_h$, 3.1(b) yields

$$S_g \leq S_g T = (S_g)^a T \leq S_h,$$

and so $S_g \leq S_f$. This again contradicts (*), and completes the proof. $\square$

The Splitting Lemma yields a useful criterion for partial normality, as follows.

Corollary 3.8. *Let $\mathcal{L}$ be a locality, let $\mathcal{N} \trianglelefteq \mathcal{L}$, and let $\mathcal{K} \trianglelefteq \mathcal{N}$ be a partial normal subgroup of $\mathcal{N}$. Suppose that $\mathcal{K}$ is $N_{\mathcal{L}}(T)$-invariant. I.e. suppose that $x^h \in \mathcal{K}$ for all $(h^{-1}, x, h) \in \mathbf{D}$ such that $x \in \mathcal{K}$ and $h \in N_{\mathcal{L}}(T)$.) Then $\mathcal{K} \trianglelefteq \mathcal{L}$.*

Proof. Let $x \in \mathcal{K}$ and let $f \in \mathcal{L}$ such that x^f is defined. By the Frattini Lemma we may write $f = xh$ with $x \in \mathcal{N}$ and with h $\uparrow$-maximal, and then the Splitting Lemma yields $S_f = S_{(x,h)}$. Set $u = (f^{-1}, x, f)$ and $v = (h^{-1}, g^{-1}, x, g, h)$. Then $S_u = S_v \in \Delta$, and $x^f = \Pi(u) = \Pi(v) = (x^g)^h$. Thus $x^f \in \mathcal{K}$, and $\mathcal{K} \trianglelefteq \mathcal{L}$. $\square$

Lemma 3.9. *Let h be $\uparrow$-maximal. Then $\mathcal{N}h = h\mathcal{N} = \{g \in \mathcal{L} \mid (g, S_g) \uparrow (h, S_h)\}$.*

Proof. Given $(g, S_g) \uparrow (h, S_h)$ with the usual diagram:

$$\begin{tikzcd} S_h \arrow[r, "h"] & S_{h^{-1}} \\ S_g \arrow[u, "u"] \arrow[r, "g"'] & S_{g^{-1}} \arrow[u, "v"'] \end{tikzcd} ,$$

one sees (as in 2.3) that the word $w = (u, h, v^{-1}, h^{-1}, h, g^{-1}) \in \mathbf{D}$ via P, that $\Pi(w) = \mathbf{1}$, and then that $g = u(v^{-1})^{h^{-1}} h \in \mathcal{N}h$. Then also $g \in h\mathcal{N}$ by 3.2.

Conversely, let g be an element of $\mathcal{N}h$. So, $g = xh$ with $x \in \mathcal{N}$, and where $S_g = S_{(x,h)}$ by th Splitting Lemma. Then $S_g \le S_h$ by 3.2, and we get

$$(S_g)^x \le (S_g)^x T = S_g T \le S_h,$$

using 3.1(b) and 3.5. Similarly, since $g^{-1} = h^{-1}x^{-1} = (x^{-1})^h h^{-1}$ one finds that $S_{g^{-1}}T \le S_{h^{-1}}$. Then

$$\begin{tikzcd} S_h \arrow[r, "h"] & S_{h^{-1}} \\ S_g \arrow[u, "x"] \arrow[r, "g"'] & S_{g^{-1}} \arrow[u, "\mathbf{1}"'] \end{tikzcd},$$

is a diagram which expresses $(g, S_g) \uparrow (h, S_h)$. $\square$

Lemma 3.10. *The following hold.*

(a) *$f\mathcal{N} = \mathcal{N}f$ for all $f \in N_{\mathcal{L}}(T)$.*
(b) *If $(g, S_g) \uparrow (h, S_h)$ and h is $\uparrow$-maximal then $\mathcal{N}_g \subseteq \mathcal{N}h$.*
(c) *$\mathcal{L}$ is partitioned by the set of all subsets $\mathcal{N}f$ such that f is $\uparrow$-maximal.*
(d) *Let $u = (g_1, \cdots, g_n) \in \mathbf{D}$, and let $v = (h_1, \cdots, h_n)$ be a sequence of up-maximal elements of $\mathcal{L}$ such that $g_i \in \mathcal{N}h_i$ for all i. Then $v \in \mathbf{D}$, and $TS_u \le S_v$.*

Proof. Point (a) is contained in 3.2, and is included here for emphasis. Now let h be $\uparrow$-maximal and let $g \in \mathcal{N}h$. Thus $g = xh$ for some $x \in \mathcal{N}$, and then the Splitting Lemma (3.7) yields $S_g = S_{(x,h)}$, and $S_g \le S_h$ by 3.2. Let $y \in \mathcal{N}$ with $(y, g) \in \mathbf{D}$. Then $(y, x, h) \in \mathbf{D}$ via $S_{(y,g)}$, and $yg = (yx)h \in \mathcal{N}h$. Thus $\mathcal{N}g \subseteq \mathcal{N}h$, and we have (b).

Next, let f and g be $\uparrow$-maximal in $\mathcal{N}$ with $\mathcal{N}f \cap \mathcal{N}g \ne \emptyset$, and let $h \in \mathcal{N}f \cap \mathcal{N}g$. Then $\mathcal{N}h \subseteq \mathcal{N}f \cap \mathcal{N}g$ by point (b). Write $h = xf$ with $x \in \mathcal{N}$. The Splitting Lemma yields $(x^{-1}, h) \in \mathbf{D}$ via $(S_h)^x$, so $f = x^{-1}h \in \mathcal{N}h$, and so $f \in \mathcal{N}g$ and $\mathcal{N}f \subseteq \mathcal{N}g$ by (b). Similarly $\mathcal{N}g \le \mathcal{N}f$, and (c) holds.

Finally, let u and v be given as in (d), and write $g_i = x_i h_i$ with $x_i \in \mathcal{N}$. Set $w = (x_1, h_1, \cdots, x_n, h_n)$. Then $S_u = S_w$ by splitting (and by induction on n). Since $T \le S_v$ by 3.5, it suffices to show that $S_u \le S_v$ in order to prove (d).

Set $P = S_u$ and proceed by induction on n. Suppose $geq2$ and set $u_0 = (g_1, \cdots, g_{n-1})$ and $v_0 = (h_1, \cdots, h_{n-1})$. Then $P \le S_{u_0}$, and so $P \le S_{v_0}$ by the inductive hypothesis. Set $g = \Pi(u_0)$ and set $Q = P^g$. Then $Q \le S_{g_n} = S_{(x_n,h_n)}$, and in this way we are reduced to the case $n = 1$. Here $P^{x_1} \le PT$ by 3.1(b), and so $P^{x_1} \le S_{h_1}$ by 3.2. Thus $P \le S_v$, as required. $\square$

We refer to the sets $\mathcal{N}f$ with f $\uparrow$-maximal as the *maximal cosets of* $\mathcal{N}$. Notice that $\mathcal{N}f$ is a maximal coset if and only if $\mathcal{N}f$ is maximal in the poset of all subsets $\mathcal{N}g$ (partially ordered by set-theoretic inclusion), by points (b) and (c) of 3.10. We write

$\mathcal{L}/\mathcal{N}$, or simply $\overline{\mathcal{L}}$, for the set of maximal cosets of $\mathcal{N}$. Let $\rho : \mathcal{L} \to \overline{\mathcal{L}}$ be the quotient map, sending $g \in \mathcal{L}$ to the unique maximal coset $[g]$ of $\mathcal{N}$ containing g.

Set $\mathbf{W} := \mathbf{W}(\mathcal{L})$ and $\overline{\mathbf{W}} = \mathbf{W}(\overline{\mathcal{L}})$, and let $\rho^* : \mathbf{W} \to \overline{\mathbf{W}}$ be the induced mapping of free monoids. For any subset or element X of $\mathbf{W}$, write $\overline{X}$ for the image of X under ρ^*, and similarly if Y is a subset or element of $\mathcal{L}$ write $\overline{Y}$ for the image of Y under ρ. In particular, $\overline{\mathbf{D}}$ is the image of $\mathbf{D}$ under ρ^*. Set $\overline{\Delta} = \{\overline{P} \mid P \in \Delta\}$.

For $w \in \mathbf{W}$, we shall say that w is $\uparrow$-maximal if every entry of w is $\uparrow$-maximal.

Lemma 3.11. *There is a unique mapping $\overline{\Pi} : \overline{\mathbf{D}} \to \overline{\mathcal{L}}$, a unique involutory bijection $\overline{f} \mapsto \overline{f}^{-1}$ on $\overline{\mathcal{L}}$, and a unique element $\overline{\mathbf{1}}$ of $\overline{\mathcal{L}}$ such that $\overline{\mathcal{L}}$, with these structures, is a partial group, and such that ρ is a homomorphism of partial groups.*

Proof. Let $u = (g_1, \cdots, g_n)$ and $v = (h_1, \cdots, h_n)$ be members of $\mathbf{D}$ such that $\overline{u} = \overline{v}$. By 3.10(d) there exists for each i an $\uparrow$-maximal $f_i \in \mathcal{L}$ with $g_i, h_i \in \mathcal{N}f_i$. Set $w = (f_1, \cdots, f_n)$. Then $w \in \mathbf{D}$ by 3.13(e), and then 3.3(a) shows that $\Pi(u)$ and $\Pi(v)$ are elements of $\mathcal{N}\Pi(w)$. Thus $\overline{\Pi(u)} = \overline{\Pi(w)} = \overline{\Pi(v)}$, and there is a well-defined mapping $\overline{\Pi} : \overline{\mathbf{D}} \to \overline{\mathcal{L}}$ given by

$$(*) \qquad \overline{\Pi}(w) = \overline{\Pi(w)}.$$

For any subset X of $\mathcal{L}$ write X^{-1} for the set of inverses of elements of X. For any $f \in \mathcal{L}$ we then have $(\mathcal{N}f)^{-1} = f^{-1}\mathcal{N}^{-1}$ by 1.1(4). Here $\mathcal{N}^{-1} = \mathcal{N}$ as $\mathcal{N}$ is a partial group, and then $(\mathcal{N}f)^{-1} = \mathcal{N}f^{-1}$ by 3.13(a). The inversion map $\mathcal{N}f \mapsto \mathcal{N}f^{-1}$ is then well-defined, and is an involutory bijection on $\overline{\mathcal{L}}$. Set $\overline{\mathbf{1}} = \mathcal{N}$.

We now check that the axioms in 2.1, for a partial group, are satisfied by the above structures. Since $\overline{\mathbf{D}}$ is the image of $\mathbf{D}$ under ρ^*, we get $\overline{\mathcal{L}} \subseteq \overline{\mathbf{D}}$. Now let $\overline{w} = \overline{u} \circ \overline{v} \in \overline{\mathbf{D}}$, let u, v be $\uparrow$-maximal pre-images in $\mathbf{W}$ of $\overline{u}$, and $\overline{v}$, and set $w = u \circ v$. Then w is $\uparrow$-maximal, and so $w \in \mathbf{D}$ by 3.12(e). Then u and v are in $\mathbf{D}$, and so $\overline{u}$ and $\overline{v}$ are in $\overline{\mathbf{D}}$. Thus $\overline{\mathbf{D}}$ satisfies 1.1(1). Clearly, (*) implies that $\overline{\Pi}$ restricts to the identity on $\overline{\mathcal{L}}$, so $\overline{\Pi}$ satisfies 1.1(2).

Next, let $\overline{u} \circ \overline{v} \circ \overline{w} \in \overline{\mathbf{D}}$, and choose corresponding $\uparrow$-maximal pre-images u, v, w. Set $g = \Pi(v)$. Then $\overline{g} = \overline{\Pi}(\overline{v})$ by (*). By 1.1(3) we have both $u \circ v \circ w$ and $u \circ (g) \circ w$ in $\mathbf{D}$, and these two words have the same image under Π. Applying ρ^* we obtain words in $\overline{\mathbf{D}}$ having the same image under $\overline{\Pi}$, and thus $\overline{\Pi}$ satisfies 1.1(3). By definition, $\overline{\Pi}(\emptyset) = \overline{\mathbf{1}}$, and then the condition 1.1(4) is readily verified. Thus, $\overline{\mathcal{L}}$ is a partial group.

By definition, $\overline{\mathbf{D}}$ is the image of $\mathbf{D}$ under ρ^*. So, in order to check that ρ is a homomorphism of partial groups it suffices to show that if $w \in \mathbf{D}$ then $\overline{\Pi}(w\rho^*) = \Pi(w)\rho$. But this is simply the statement (*). Moreover, it is this observation which establishes that the given partial group structure on $\overline{\mathcal{L}}$ is the unique one for which ρ is a homomorphism of partial groups. We have $f \in Ker(\gamma)$ if and only if $f\gamma = \overline{\mathbf{1}} = \mathcal{N}$. Since $\mathcal{N}f = \mathcal{N}$ implies $f = \mathbf{1}f \in \mathcal{N}$, and since $\mathcal{N}$ is the maximal coset of $\mathcal{L}$ containing $\mathbf{1}$, we obtain $Ker(\gamma) = \mathcal{N}$. $\square$

We end this section with an important example.

Proposition 3.12. *Let $(\mathcal{L}, \Delta, S)$ be a locality having the property that for each $P \in \Delta$, $C_{\mathcal{L}}(P)$ is the direct product of a p-group with a (necessarily unique) p'-group $\Theta(P)$, and set*

$$\Theta = \bigcup\{\Theta(P) \mid P \in \Delta\}.$$

Then $\Theta \trianglelefteq \mathcal{L}$.

Section 4: Quotients and products

We begin this section by showing that the image of a locality under a "strongly surjective" homomorphism β in induces the structure of a locality on the image of β.

[We need the following result on finite p-groups, to be placed in Appendix G.]

Lemma G.n. *Let S be a finite p-group, let $P \leq S$ be a subgroup, and let $\mathcal{C}$ be a set of S-conjugates of P. Set $X = \bigcup \mathcal{C}$, and suppose that $P^x \in \mathcal{C}$ for all $x \in X$. Then either $X = P$ or $N_S(P) \cap X$ properly contains P.*

Proof. Set $S = S^{(0)}$, and define $S^{(n)}$ for $n > 0$ by $S^{(n)} = [S^{(n-1)}, S]$. Then $P^g \leq PS^{(1)}$ for all $g \in S$, and so $X \subseteq PS^{(1)}$. Set $X_0 = X$, and define X_n for $n > 0$ by $X_n = \bigcup\{P^x \mid x \in X_{n-1}\}$. A straightforward argument by induction shows that $X_n \subseteq PS^{(n+1)}$, and hence there is a least index k such that $X_k = P$. Assuming that $X \neq P$ we get $k > 0$, and then $P \neq X_{k-1} \leq N_S(P)$. □

Proposition 4.1. *Let $(\mathcal{L}, \Delta, S)$ be a locality, let $\mathcal{L}'$ be a partial group, and let $\beta : \mathcal{L} \to \mathcal{L}'$ be a homomorphism of partial groups. Set $\mathbf{D} = \mathbf{D}(\mathcal{L})$ and $\mathbf{D}' = \mathbf{D}(\mathcal{L}')$, $S' = S\beta$, and set $\Delta' = \{P\beta \mid P \in \Delta\}$. Assume that $\mathbf{D}\beta^* = \mathbf{D}'$. Then $(\mathcal{L}', \Delta', S')$ is a locality, and the following hold.*

- (a) *For any subgroup M of $\mathcal{L}$, the restriction of β to M is a surjective homomorphism $\beta_M : M \to M\beta$ of groups.*
- (b) *If $w \in \mathbf{D}$ via $P \in \Delta$, then $w\beta^* \in \mathbf{D}'$ via $P\beta$.*
- (c) *Let $w' \in \mathbf{D}'$ via $P' \in \Delta'$, and let P be the pre-image of P' via the restriction β_S of β to S. Then there exists $w \in \mathbf{D}$ such that each entry of w is $\uparrow$-maximal with respect to $Ker(\beta)$ and such that $w\beta^* = w'$. Moreover, any such w is in $\mathbf{D}$ via P.*
- (d) *If $Ker(\beta) = \mathbf{1}$ then β is an isomorphism.*

Proof. We note first of all that the hypothesis $\mathbf{D}\beta^* = \mathbf{D}'$ implies that β^* maps the set of words of length 1 in $\mathcal{L}$ onto the set of words of length 1 in $\mathcal{L}'$. Thus β is surjective and, in particular, $\mathcal{L}'$ is finite.

Let M be a subgroup of $\mathcal{L}$. Then $\mathbf{W}(\mathcal{M}) \subseteq \mathbf{D}$ by 1.2. Note that $\mathbf{W}(X\gamma) = \mathbf{W}(X)\gamma^*$ for any set X and any mapping γ on X. As $\mathbf{D}\beta^* \subseteq \mathbf{D}'$ by (1), one concludes that $\mathbf{W}(M\beta) \subseteq \mathbf{D}'$, and hence $M\beta$ is a subgroup of $\mathcal{L}'$. Then β induces a homomorphism $M \to M\beta$ of groups, by 1.13. Thus (a) holds. In particular, S' is a p-group, and Δ' is a set of subgroups of S'.

Let $P \in \Delta$ and let $w \in \mathbf{D}$ via P. If $w = (\emptyset)$ is the empty word then so is $w\beta^*$, and then (see definition 2.1) $w\beta^* \in \mathbf{D}'$ via any member of Δ'. Suppose that $w = (f)$ has length 1, and set $Q = P^f$. Let $x \in P$. Then $(f^{-1}, x, f) \in \mathbf{D}$, and $x^f \in Q$. As β is a homomorphism we get $(f^{-1}, x, f)\beta^* \in \mathbf{D}'$ and $(x\beta)^{f\beta} = (x^f)\beta \in Q\beta$. Then 1.5(c) shows that conjugation by $f\beta$ defines a bijection from $P\beta$ to $Q\beta$, where $P\beta$ and $Q\beta$ are subgroups of S' by (a).

Now suppose that $w = (f_1, \cdots, f_n) \in \mathbf{D}$ via $(P_0, \cdots, P_n)$, write $w\beta^* = w' = (f_1', \cdots, f_n')$, and set $P_i' = P_i\beta$. Here $w' \in \mathbf{D}'$ since β is a homomorphism, and the argument of the preceding paragraph shows that $(P_{i-1}')^{f_i'} = P_i$ for all i with $1 \leq i \leq n$. This is all that we mean by saying that $w' \in \mathbf{D}'$ via $(P_0', \cdots, P_n')$, or via P_0' for short. Thus $(\mathcal{L}', \Delta')$ satisfies the condition (O1) for objectivity in definition 2.1.

In the remaining arguments we shall need to consider the partial normal subgroup $\mathcal{N} := Ker(\beta)$ of $\mathcal{L}$ (see 1.12), and the group $T := S \cap \mathcal{N}$. By 3.14 $\mathcal{L}$ is partitioned by the maximal cosets of $\mathcal{N}$, while 3.13(b) shows that each maximal coset is of the form $\mathcal{N}h$ where h is $\uparrow$-maximal with respect to $\mathcal{N}$. Plainly, β is constant on each coset of $\mathcal{N}$, hence every fiber of β contains an $\uparrow$-maximal representative.

Let $f' \in \mathcal{L}'$ and let f be $\uparrow$-maximal in $\mathcal{L}$ with $f\beta = f'$. Set $P = S_f$ and $Q = P^f$, and set $P' = P\beta$ and $Q' = Q\beta$. Then $P', Q' \in \Delta'$ and $(P')^{f'} = Q'$. Define $X := S'_{f'}$ to be the set of all $x \in S'$ such that $x^{f'}$ is defined and such that $x^{f'} \in S'$. We now claim that $(P')^x \subseteq X$ for all $x \in X$. Indeed, let $x \in X$ and set $y = x^{f'}$. Then $(x^{-1}, f', y) \in \mathbf{D}'$ via $(P')^x$. Then $(f', y) \in \mathbf{D}'$, while $(x, f') \in \mathbf{D}'$ via $(P')^{x^{-1}}$. From $(f')^{-1}xf' = y$ we get $xf' = f'y$ by 1.3(e), and hence $x^{-1}f'y = x^{-1}(f'y) = x^{-1}(xf')$. As $(x^{-1}, x, f') \in \mathbf{D}'$ by 1.3(d), we conclude that $x^{-1}f'y = f'$, and thus $(P')^x \subseteq X$, as claimed.

Suppose that $P' \neq X$. Then [the above LEMMA p-GROUPS] shows that $X \cap N_{S'}(P')$ properly contains P'. Let $x \in N_{S'}(P')$ with $x \notin P'$, let a be a pre-image x under β_S, and set $A = P\langle a \rangle$. We note that $T \leq P$ by 3.8, and we observe that $T = Ker(\beta_S)$. As β_S is a homomorphism $S \to S'$ of groups, we get $A \leq N_{\mathcal{L}}(P)$. Conjugation by f induces an isomorphism $N_{\mathcal{L}}(P) \to N_{\mathcal{L}}(Q)$ by 2.3(b), so $B := A^f$ is a p-subgroup of $N_{\mathcal{L}}(Q)$. Set $K = N_{\mathcal{N}}(Q)$. Then K is a normal subgroup of the group $M := N_{\mathcal{L}}(Q)$, by 1.7(d). Moreover, $T \leq Q$ by 2.10(a), and then $T \in Syl_p(K)$ by 2.10(c). As $B\beta_M = (A\beta)^{f'}$, we find $B \leq N_S(Q)K$, and then Sylow's theorem yields $B^c \leq N_S(Q)$ for some $c \in K$. Then $(P, f) \uparrow (A, fc)$, contrary to the $\uparrow$-maximality of f. Thus:

(1) (*) $(S_f)\beta = S_{f\beta}$ for every $f \in \mathcal{L}$ such that f is $\uparrow$-maximal with respect to $\mathcal{N}$.

It is immediate from (*) that the condition (O2) of definition 2.1 holds in $\mathcal{L}'$. Since we have already verified (O1), we now conclude that $(\mathcal{L}', \Delta')$ is an objective partial group. If S' is properly contained in a p-subgroup R' of $\mathcal{L}'$ then $\beta_{N_{\mathcal{L}}(S)}$ yields a contradiction to the maximality of S among the p-subgroups of $\mathcal{L}$. Since it has already been established that $\mathcal{L}'$ is finite, we conclude that $(\mathcal{L}', \Delta', S')$ is a locality.

Let $w' \in \mathbf{D}'$ via $P' \in \Delta'$, let w be a lifting of w' via β, such that every entry of w is $\uparrow$-maximal, and let P be the pre-image of P' via β_S. Then (*) implies that $w \in \mathbf{D}$ via P, and hence (c) is proved.

Finally, assume $\mathcal{N} = \mathbf{1}$. Then every element of $\mathcal{L}$ is $\uparrow$-maximal with respect to $\mathcal{N}$, and β_S is an isomorphism. Let $g_1, g_2 \in \mathcal{L}$ with $g' := g_1\beta = g_2\beta$, and set $A_i = S_{g_i}$. Then $A_i\beta = S'_{g'}$ by (*), and hence $A_1 = A_2$. Then $(g_1^{-1}, g_2) \in \mathbf{D}$ via $(A_1)^{g_1}$, and $(g_1^{-1}g_i)\beta = \mathbf{1}$.

Thus $g_1 = g = g_2$, and β is injective.

Set $\gamma = \beta^{-1}$, and let $w' \in \mathbf{D}'$ via $P' \in \Delta'$. Then (b) yields $w'\gamma \in \mathbf{D}$, and we have

$$(\Pi(w'\gamma^*))\beta = \Pi'(w'(\gamma^* \circ \beta^*)) = \Pi'(w').$$

Now apply γ to obtain $\Pi(w'\gamma^*) = \Pi'(w')\gamma$. Thus β^{-1} is a homomorphism, completing the proof of (d). $\square$

Definition 4.2. Let $(\mathcal{L}, \Delta, S)$ and $(\mathcal{L}', \Delta', S')$ be localities, and let $\beta : \mathcal{L} \to \mathcal{L}'$ be a homomorphism of partial groups. Then β is a *projection* if the following two conditions hold.

(1) $\mathbf{D}\beta^* = \mathbf{D}'$.
(2) $\Delta' = \{P\beta \mid P \in \Delta\}$.

Theorem 4.3 (First Isomorphism Theorem). *Let $(\mathcal{L}, \Delta, S)$ be a locality, let $\mathcal{N} \trianglelefteq \mathcal{L}$ be a partial normal subgroup, and let $\rho : \mathcal{L} \to \overline{\mathcal{L}}$ be the homomorphism of partial groups given by 3.15. Set $\overline{\Delta} = \{P\rho \mid P \in \Delta\}$.*

(a) *$(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ is a locality, and $\rho : \mathcal{L} \to \overline{\mathcal{L}}$ is a projection.*
(b) *Let $(\mathcal{L}', \mathbf{D}', S')$ be a locality, let $\beta : \mathcal{L} \to \mathcal{L}'$ be a projection, and suppose that $\mathcal{N} = Ker(\beta)$. Then there is a unique isomorphism $\overline{\beta} : \mathcal{L}/\mathcal{N} \to \mathcal{L}'$ such that $\beta = \rho \circ \overline{\beta}$.*

Proof. By definition of the partial group $\overline{\mathcal{L}}$ in 3.15, the domain $\mathbf{D}(\overline{\mathcal{L}})$ of the product $\overline{\Pi}$ is the image of $\mathbf{D}(\mathcal{L})$ under ρ^*. Point (a) is therefore immediate from 4.1, and from the definition of projection (4.2).

Let $\beta : \mathcal{L} \to \mathcal{L}'$ be a projection as in (b), and set $\mathcal{N} = Ker(\beta)$. Then $\mathcal{N} \trianglelefteq \mathcal{L}$ by 1.12. Clearly, β is constant on cosets $\mathcal{N}f$ of $\mathcal{N}$. Since $\mathcal{L}$ is partitioned by the maximal cosets of $\mathcal{N}$, by 3.13(d), we may define a mapping $\overline{\beta} : \overline{\mathcal{L}} \to \mathcal{L}'$ by applying β to each maximal coset. Let $\overline{\beta}^*$ be the induced mapping $\mathbf{W}(\overline{\mathcal{L}}) \to \mathbf{W}(\mathcal{L}')$ of free monoids.

Let $\overline{w} \in \mathbf{D}(\overline{\mathcal{L}})$ and let w be an $\uparrow$-maximal word (i.e. in which each entry is $\uparrow$-maximal) in $\mathbf{W}(\mathcal{L})$ such that $w\rho^* = \overline{w}$. Then $w \in \mathbf{D}(\mathcal{L})$, and $\overline{w}\overline{\beta}^* = w\beta^*$ by definition of $\overline{\beta}$. Here $w\beta^* \in \mathbf{D}(\mathcal{L}')$ is a homomorphism, and

$$(*) \qquad \Pi'(\overline{w}\overline{\beta}^*) = \Pi'(w\beta^*) = \Pi(w)\beta.$$

Let $[\Pi(w)]$ be the unique maximal coset of $\mathcal{N}$ containing $\Pi(w)$. Then $[\Pi(w)] = \overline{\Pi}(\overline{w})$ by definition of $\overline{\Pi}$, and then (*) yields

$$\Pi'(\overline{w}\overline{\beta}^*) = \Pi(w)\beta = [\Pi(w)]\overline{\beta} = (\overline{\Pi}(\overline{w}))\overline{\beta}.$$

Thus, $\overline{\beta}$ is a homomorphism of partial groups.

We note that

$$\mathbf{D}(\overline{\mathcal{L}})\overline{\beta}^* = \mathbf{D}(\mathcal{L})\beta^* = \mathbf{D}(\mathcal{L}')$$

as β is a projection. Similarly, we find that $\overline{\Delta\beta} = \Delta'$, and hence $\overline{\beta}$ is a projection. Further, $Ker(\overline{\beta}) = \overline{\mathbf{1}}$ since $Ker(\beta) = \mathcal{N}$, so $\overline{\beta}$ is an isomorphism by 4.1(d).

Finally, let $\gamma : \overline{\mathcal{L}} \to \mathcal{L}'$ be an isomorphism such that $\rho \circ \gamma = \beta$. Let $\overline{f} \in \overline{\mathcal{L}}$ and let f be an $\uparrow$-maximal pre-image of $\overline{f}$ via ρ. Then, in the obvious way:

$$\overline{f}\gamma = f(\rho \circ \gamma) = f\beta = f(\rho \circ \overline{\beta}) = \overline{f\beta},$$

and thus $\gamma = \overline{\beta}$. This completes the proof of (b). $\square$

Proposition 4.4 (Partial Subgroup Correspondence). *Let $(\mathcal{L}, \Delta, S)$ be a locality, let $\mathcal{N} \trianglelefteq \mathcal{L}$ be a partial normal subgroup of $\mathcal{L}$, and set $T = S \cap \mathcal{N}$. Let $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ be the quotient locality $\mathcal{L}/\mathcal{N}$, and let $\rho : \mathcal{L} \to \overline{\mathcal{L}}$ be the canonical projection. Then ρ induces a bijection σ from the set $\mathfrak{H}$ of partial subgroups $\mathcal{H}$ of $\mathcal{L}$ containing $\mathcal{N}$ to the set $\overline{\mathfrak{H}}$ of partial subgroups $\overline{\mathcal{H}}$ of $\overline{\mathcal{L}}$. Moreover, for any $\mathcal{H} \in \mathfrak{H}$, we have $\mathcal{H}\sigma \trianglelefteq \mathcal{L}/\mathcal{N}$ if and only if $\mathcal{H} \trianglelefteq \mathcal{L}$.*

Proof. Any partial subgroup of $\mathcal{L}$ containing $\mathcal{N}$ is a union of maximal cosets of $\mathcal{N}$, by 3.14. Then 4.3 enables the same argument as in ordinary groups, for the proof that ρ induces a bijection $\mathfrak{H} \to \overline{\mathfrak{H}}$.

For any $g \in \mathcal{L}$ denote by $[g]$ the maximal coset of $\mathcal{N}$ containing g. Now suppose that $\mathcal{H} \in \mathfrak{H}$ with $\mathcal{H} \trianglelefteq \mathcal{L}$. Let $g \in \mathcal{L}$ and $h \in \mathcal{H}$ with $\overline{u} := ([g^{-1}], [h], [g]) \in \mathbf{D}(\mathcal{L}/\mathcal{N})$. As above (and in view of 3.7(b)) we may take h and g to be $\uparrow$-maximal, with $(g^{-1}, h, g) \in \mathbf{D}(\mathcal{L})$. Then $\Pi(g^{-1}, h, g) \in \mathcal{H}$ and $\overline{\Pi}(\overline{u}) \in \overline{\mathcal{H}}$. Thus $\overline{\mathcal{H}} \trianglelefteq \mathcal{L}/\mathcal{N}$ in this case.

Conversely, let $\overline{\mathcal{K}} \trianglelefteq \mathcal{L}/\mathcal{N}$, and let $f \in \mathcal{K}$ and $x \in \mathcal{L}$ with $(x^{-1}, f, x) \in \mathbf{D}(\mathcal{L})$. Let y be $\uparrow$-maximal in $[x]$ and let h be $\uparrow$-maximal in $[f]$. Then $([x^{-1}], [f], [x]) = ([y^{-1}], [g], [y]) \in \mathbf{D}(\mathcal{L}/\mathcal{N})$ and $[g^y] \in \overline{\mathcal{K}}$. As $[g^y] = [f^x]$, we conclude that $f^x \in \mathcal{K}$, and thus $\mathcal{K} \trianglelefteq \mathcal{L}$. $\square$

It may be worth-while to record the following essentialy trivial result.

Lemma 4.5. *Let $\mathcal{N} \trianglelefteq \mathcal{L}$ and let $\rho : \mathcal{L} \to \mathcal{L}/\mathcal{N}$ be the canonical projection. Further, let $\mathcal{H}$ be a partial subgroup of $\mathcal{L}$ containing $\mathcal{N}$ and let X be an arbitrary subset of $\mathcal{L}$. Then $(X \cap \mathcal{H})\rho = X\rho \cap \mathcal{H}\rho$.*

Proof. By 3.14, $\mathcal{H}$ is a union of maximal cosets of $\mathcal{N}$, and then $\mathcal{H}\rho$ is the set of those maximal cosets. On the other hand $X\rho$ is the set of all maximal cosets $\mathcal{N}g$ of $\mathcal{N}$ such that $U \cap \mathcal{N}g \neq \emptyset$. Thus $X\rho \cap \mathcal{H}\rho \subseteq (X \cap \mathcal{H})\rho$. The reverse inclusion is obvious. $\square$

Corollary 4.6. *Let $(\mathcal{L}, \Delta, S)$ be a locality, let $\mathcal{N} \trianglelefteq \mathcal{L}$, and let $\mathcal{M}$ be a partial normal subgroup of $\mathcal{L}$ containing $\mathcal{N}$. Let $\rho : \mathcal{L} \to \mathcal{L}/\mathcal{N}$ be the canonical projection. Then $(S \cap \mathcal{M})\rho$ is a maximal p-subgroup of $\mathcal{M}\rho$.*

Proof. Write $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ for the quotient locality given by 4.3, and set $\overline{\mathcal{M}} = \mathcal{M}\rho$. Applying 4.5 with S in the role of X, we obtain $(S \cap \mathcal{M})\rho = \overline{S} \cap \overline{\mathcal{M}}$. Since $\overline{\mathcal{M}} \trianglelefteq \overline{\mathcal{L}}$, it follows from 2.10(c) that $\overline{S} \cap \overline{\mathcal{M}}$ is maximal in the poset of p-subgroups of $\overline{\mathcal{M}}$, completing the proof. $\square$

Proposition 4.7. *Let $(\mathcal{L}, \Delta, S)$ be a locality, and let $\mathcal{M} \trianglelefteq \mathcal{L}$ and $\mathcal{N} \trianglelefteq \mathcal{L}$ be partial normal subgroups. Set $U = S \cap \mathcal{M}$ and $V = S \cap \mathcal{N}$, and assume that the following condition holds.*

(*) *$\mathcal{M}$ normalizes V, and $\mathcal{N}$ normalizes U.*

Then $\mathcal{M}\mathcal{N} = \mathcal{N}\mathcal{M} \trianglelefteq \mathcal{L}$. Moreover, if $\mathcal{L}$ is a Δ-linking system and $C_S(UV)UV \in \Delta$ then $S \cap \mathcal{M}\mathcal{N} = UV$.

The proof will require the following result on "splitting".

Lemma 4.8. *Assume the hypothesis of 4.7, and let $g \in \mathcal{M}\mathcal{N}$. Then there exists $(x, y) \in$ $\mathbf{D}$ with $x \in \mathcal{M}$, $y \in \mathcal{N}$, and with $S_g = S_{(x,y)}$.*

Proof. Among all $(x, y) \in \mathbf{D} \cap (\mathcal{M} \times \mathcal{N})$ with $xy = g$, choose (x, y) so that

$$(|U \cap S_{(x,y)}|, |V \cap S_{(x,y)}|, |S_{(x,y)}|)$$

is as large as possible in the lexicographic ordering. Set $Q = S_{(x,y)}$ and set $P = N_{S_g}(Q)$. It suffices to show that $P = Q$ in order to obtain the lemma.

By 3.2 we have $(y, y^{-1}, x, y) \in \mathbf{D}$ and $g = yx^y$. Suppose that $P \leq S_y$. Then $P^y \leq S$, and since $P^g \leq S$ we conclude that $P \leq S_{(y,x^y)}$, and hence $P = Q$, as desired. Thus we may assume:

(1) $P \nleq S_y$.

Let h be $\uparrow$-maximal in the maximal coset of $\mathcal{M}$ containing g. Then 3.9 yields an element $r \in \mathcal{M}$ such that $g = rh$, and 3.11 yields $S_g = S_{(r,h)}$. Then $Q \leq S_{(r,h)}$, so $(y^{-1}, x^{-1}, r, h) \in \mathbf{D}$ via Q^g and $\Pi(y^{-1}, x^{-1}, r, h) = \Pi(g^{-1}, g) = \mathbf{1}$. Thus:

$$(*) \qquad y = x^{-1}rh \quad \text{and} \quad h = r^{-1}xy.$$

Since $y, h \in N_{\mathcal{L}}(U)$, it follows that $r^{-1}x \in N_{\mathcal{M}}(U)$, and then that $h = (r^{-1}x)y \in \mathcal{M}\mathcal{N}$.

Suppose that h does not provide a counter-example to the lemma. That is, suppose that there exists $x' \in \mathcal{M}$ and $y' \in \mathcal{N}$ such that $(x', y') \in \mathbf{D}$, $x'y' = h$, and $S_{(x',y')} = S_h$. As $r^{-1}xy = h = x'y'$ we get $xy = rx'y'$, and $(rx', y') \in \mathbf{D}$ with $rx'y' = rh = g$. The idea now is to replace (x, y) with (rx', y') and to contradict the assumption that $S_g \neq Q$. In order to achieve this, note first of all that $S_g \leq S_r$ since $S_{(r,h)} = S_{rh} = S_g$. Then note that $(S_g)^r \leq S_h$, and that $S_h = S_{(x',y')} \leq S_{x'}$. Thus $(S_g)^r \leq S_{x'}$, and so $S_g \leq S_{rx'}$. As $rx'y' = g$ we conclude that $S_g \leq S_{(rx',y')}$, which yields the desired contradiction. We conclude that h is itself a counter-example to the lemma.

Since $h = r^{-1}xy$ by (*), and since h and y are in $N_{\mathcal{L}}(U)$, we have $r^{-1}x \in N_{\mathcal{M}}(U)$, and then $U \leq S_{(r^{-1}x,y)}$ since $h \in N_{\mathcal{L}}(U)$. The maximality condition placed on Q in our choice of g then yields $U \leq Q$, and a symmetric argument yields $V \leq Q$. Setting $H = N_{\mathcal{L}}(Q)$, it now follows from 3.1(b) that $x, y \in H$.

Set $X = H \cap \mathcal{M}$ and $Y = H \cap \mathcal{N}$. Then X, Y, and UV are normal subgroups of H, and XY/UV is a p'-group. Here $P \leq H$, and in the quotient group $\overline{H} := H/(X \cap Y)UV$ we then have $[\overline{P}, \overline{g}] = 1$. As $\overline{X} \cap \overline{Y} = 1$ we have $C_{\overline{XY}}(\overline{P}) = C_{\overline{X}}(\overline{P}) \times C_{\overline{Y}}(\overline{P})$. As $\overline{g} = \overline{xy}$

it follows that $\overline{x}$ and $\overline{y}$ centralize $\overline{P}$. Thus $P^x \leq (X \cap Y)P$ and $P \in Syl_p((X \cap Y)P)$. Sylow's Theorem then yields $z \in X \cap Y$ with $P^x = P^z$. Replacing (x, y) with (xz^{-1}, zy) we get $g = (xz^{-1})(zy)$ and $P \leq S_{(xz^{-1}, zy)}$. This contradicts the maximality of Q and yields a final contradiction, proving the lemma. $\square$

Proof of 4.7. Let $w = (g_1, \cdots, g_n) \in \mathbf{W}(\mathcal{M}\mathcal{N}) \cap \mathbf{D}$ via $Q \in \Delta$. By 4.8 we may write $g_i = x_i y_i$ with $x_i \in \mathcal{M}$, $y_i \in \mathcal{N}$, and with $S_{g_i} = S_{(x_i, y_i)}$. Set $w' = (x_1, y_1, \cdots, x_n, y_n)$. Then $w' \in \mathbf{D}$ via Q and $\Pi(w) = \Pi(w')$. Since each y_i normalizes U, it follows from 3.4 that $\Pi(w') = \Pi(w'')$ for some w'' such that $w'' = u \circ v \in \mathbf{D}$, where $u \in \mathbf{W}(\mathcal{M})$, and where $v \in \mathbf{W}(\mathcal{N})$. Thus $\mathcal{M}\mathcal{N}$ is closed under Π. In order to show that $\mathcal{M}\mathcal{N} = (\mathcal{M}\mathcal{N})^{-1}$ we note that if $(x, y) \in \mathbf{D} \cap (\mathcal{M} \times \mathcal{N})$ then $(y^{-1}, x^{-1}) \in \mathbf{D}$ and that $y^{-1}x^{-1} \in \mathcal{M}\mathcal{N}$ by 3.2. Thus $\mathcal{M}\mathcal{N}$ is a partial subgroup of $\mathcal{L}$. Moreover, we have shown that $\mathcal{M}\mathcal{N} = \mathcal{N}\mathcal{M}$.

Let $g \in \mathcal{M}\mathcal{N}$ and let $f \in \mathcal{L}$ with $(f^{-1}, g, f) \in \mathbf{D}$. As usual we may write $f = hr$ with $r \in \mathcal{N}$, $h \in N_{\mathcal{L}}(V)$, and $S_f = S_{(r,h)}$. Write $g = xy$ as in 4.8. By assumption we have $(f^{-1}, g, f) \in \mathbf{D}$ via some $P \in \Delta$. Setting $v = (r^{-1}, h^{-1}, x, y, h, r)$ it follows that $u \in \mathbf{D}$ via P and that $g^f = \Pi(u)$. Here 3.2 yields $(h^{-1}, h, y, h) \in \mathbf{D}$ via $S_{(y,h)}$, so $v' := (r^{-1}, h^{-1}, x, h^{-1}, h, y, h, r) \in \mathbf{D}$ via P and

$$g^f = \Pi(v) = \Pi(v') = (x^h y^h)^r \in (\mathcal{M}\mathcal{N})^r.$$

Since $r \in \mathcal{N}$, and $\mathcal{M}\mathcal{N}$ is a partial group, we conclude that $g^f \in \mathcal{M}\mathcal{N}$. Thus $\mathcal{M}\mathcal{N} \trianglelefteq \mathcal{L}$.

Now suppose that $R := C_S(UV)UV \in \Delta$ and that $\mathcal{L}$ is a Δ-linking system. Then $N_{\mathcal{M}\mathcal{N}}(UV)$ is a subgroup of the group $D := N_{\mathcal{L}}(R)$ by 3.4, and a normal subgroup by 1.7(d). Further, $N_{\mathcal{M}}(U)$ and $N_{\mathcal{N}}(V)$ are contained in D as normal subgroups, and 2.10(c) shows that $N_{\mathcal{M}}(U)N_{\mathcal{N}}(V)/UV$ is a product of two normal p'-subgroups of D/UV, and is therefore itself a p'-group. Let $s \in N_S(UV) \cap \mathcal{M}\mathcal{N}$. Then $s \in D$, and thus the image of s in D/UV is trivial. Thus $S \cap \mathcal{M}\mathcal{N} = UV$, and the proof is complete. $\square$

Corollary 4.9. *Let $\mathcal{M}, \mathcal{N} \trianglelefteq \mathcal{L}$ and suppose that $\mathcal{M} \cap \mathcal{N} \leq S$. Then $\mathcal{M}\mathcal{N} \trianglelefteq \mathcal{L}$.*

Proof. Immediate from 4.7 and 2.17. $\square$

Proposition 4.10. *Let $(\mathcal{L}, \Delta, S)$ be a Δ-linking system, let $\mathcal{N} \trianglelefteq \mathcal{L}$ be a partial normal subgroup, and set $T = S \cap \mathcal{N}$. Let $(N_{\mathcal{L}}(T), \Delta, S)$ be the locality given by 2.14(a) and let $\mathcal{K} = \langle C_{\mathcal{L}}(T)\mathcal{N} \rangle$ be the partial subgroup of $\mathcal{L}$ generated by the set $C_{\mathcal{L}}(T)\mathcal{N}$. Assume that $C_S(T)T \in \Delta$. Then $\mathcal{K} \trianglelefteq \mathcal{L}$ and $S \cap \mathcal{K} = C_S(T)T$.*

Proof. Set $\mathcal{L}_T = N_{\mathcal{L}}(T)$, $\mathcal{C}_T = C_{\mathcal{L}}(T)$, $\mathcal{N}_T = \mathcal{N} \cap \mathcal{L}_T$, and $T^* = C_S(T)T$. Then $\mathcal{N}_T = N_{\mathcal{N}}(T^*)$ by 3.4. Since $\mathcal{C}_T$ normalizes T, and $\mathcal{N}_T$ normalizes T^*, 4.7 implies that $\mathcal{C}_T\mathcal{N}_T$ is a partial normal subgroup of $\mathcal{L}_T$ and that $S \cap \mathcal{C}_T\mathcal{N}_T = T^*$.

Let $\rho : \mathcal{L} \to \mathcal{L}/\mathcal{N}$ be the canonical projection given by 4.3, and let β be the restriction of ρ to $\mathcal{L}_T$. The definition of the product in $\mathcal{L}/\mathcal{N}$ shows that for $\overline{w} \in \mathbf{D}(\mathcal{L}/\mathcal{N})$ there exists $w \in \mathbf{D}(\mathcal{L})$ with $\widetilde{\rho}^* = \overline{w}$ and such that each entry of w is $\uparrow$-maximal with respect to $\mathcal{N}$. Moreover, any such w is in $\mathbf{D}(\mathcal{L}_T)$ by 3.8, and thus β is a projection, with kernel $\mathcal{N}_T$. Let $\rho_T : \mathcal{L}_T \to \mathcal{L}_T/\mathcal{N}_T$ be the canonical projection. Then 4.3(b) yields a unique isomorphism $\overline{\beta} : \mathcal{L}_T/\mathcal{N}_T \to \mathcal{L}/\mathcal{N}$ such that $\rho_T \circ \overline{\beta} = \beta$. This shows that the image of

$\mathcal{C}_T$ under β is a partial normal subgroup of $\mathcal{L}/\mathcal{N}$. Its pre-image in $\mathcal{L}$ is then a partial normal subgroup of $\mathcal{L}$ containing $\mathcal{N}$ by 4.4. Since $\mathcal{K}$ is the smallest partial subgroup of $\mathcal{L}$ containing both $\mathcal{N}$ and $\mathcal{C}_T$, we conclude that $\mathcal{K} \trianglelefteq \mathcal{L}$.

We have seen that $T^* = S \cap \mathcal{C}_T\mathcal{N}_T$. Then $T^*\rho_T$ is a maximal p-subgroup of $\mathcal{C}_T\mathcal{N}_T/\mathcal{N}_T$ by 4.6, and so $T^*\rho$ is a maximal p-subgroup of $\mathcal{K}/\mathcal{N}$. By 4.1(a), ρ restricts to a homomorphism $S \cap \mathcal{K} \to (S \cap \mathcal{K})\rho$ of groups, with kernel $S \cap \mathcal{N} = T$. The maximality of $T^*\rho$ then yields $T^* = S \cap \mathcal{K}$. $\square$

[THE FOLLOWING APPLICATION OF 4.3 WON'T BE USED LATER ON IN THIS PAPER. THE INTERNAL REFERENCES IN THE PROOF WILL BE FIXED UP, LATER.]

Proposition 4.11. *Let $\mathcal{L} = (\mathcal{L}, \Delta, S)$ be a locality and set $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$. For each $P \in \Delta$ set $\Theta(P) = O^p(C_{\mathcal{L}}(P))$, and set*

$$\Theta = \Theta_{\mathcal{L}} = \bigcup_{P \in \Delta} \Theta(P).$$

(a) *Assume that each $\Theta(P)$ is a p'-group. Then Θ is a partial normal subgroup of $\mathcal{L}$, $\Theta \cap S = 1$, and $C_{\mathcal{L}/\Theta}(P) \cong C_{\mathcal{L}}(P)/\Theta(P)$ is a p-group for all P in Δ.*
(b) *Assume that $\mathcal{F}$ is saturated and that $\mathcal{F}^{cr} \subseteq \Delta \subseteq \mathcal{F}^q$. Then each $\Theta(P)$ is a p'-group, and $\mathcal{L}/\Theta$ is a Δ-linking system.*

Proof. (a): Set $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$, and let $P \in \Delta$ with P fully normalized in $\mathcal{F}$. For any $g \in C_{\mathcal{L}}(P)$ the conjugation map $c_g : S_g \to S$ is an element of $C_{\mathcal{F}}(P)$. As P is $\mathcal{F}$-centric, $C_{\mathcal{F}}(P)$ is the fusion system $\mathcal{F}_U(U)$ where $U = C_S(P)$. In particular, the subsystem $\mathcal{F}_U(C_{\mathcal{L}}(P))$ of $C_{\mathcal{F}}(P)$ is the fusion system of a p-group, and so $O^p(C_{\mathcal{L}}(P)$ is a p'-group. The same is then true of any $\mathcal{F}$-conjugate of P, so $\Theta(P)$ is a p'-group for any $P \in \Delta$, and $\Theta \cap S = 1$.

Assume that each $\Theta(P)$ is a p'-group for each $P \in \Delta$. Let $f \in \Theta$, set $R = S_f$, and let Q be maximal with respect to inclusion in $\{P \in \Delta \mid f \in \Theta(P)\}$. Then $[N_R(Q), f] \leq S$ since $x^{-1}x^f \in S$ for all $x \in R$. But also $[N_R(Q), f] \leq \Theta(Q)$ since $\Theta(Q) \trianglelefteq N_{\mathcal{L}}(Q)$. But $S \cap \Theta(P) = 1$ as $\Theta(P)$ is a p'-group, so $[N_R(Q), f] = 1$, and f is an element of $C_{\mathcal{M}}(N_R(P))$ of order prime to p. That is, we have $f \in \Theta(N_R(Q))$, and the maximality of Q then yields $Q = R$. Thus:

(*) $N_\Theta(P, S) = \Theta(P)$ for all $P \in \Delta$.

Let $w = (f_1, \cdots, f_n) \in \mathbf{D}$ with $f_i \in \Theta$ for all i, and set $P = S_w$. Then (*) yields $f_i \in \Theta(P)$ for all i, and so $\Pi(w) \in \Theta(P)$. As Θ is closed under inversion we conclude that Θ is a partial subgroup of $\mathcal{M}$. Now let $g \in \mathcal{L}$, and suppose that $f \in \mathbf{D}(f)$. That is, suppose that $(g^{-1}, f, g) \in \mathbf{D}$ via some $P \in \Delta$. Since c_g is an isomorphism $N_{\mathcal{L}}(P) \to N_{\mathcal{L}}(P^f)$, by 2.7(b), c_g maps $\Theta(P)$ to $\Theta(P^f)$. Thus, $f^g \in \Theta$, and so $\Theta \trianglelefteq \mathcal{L}$.

Since $\Theta(P) \cap S = 1$ for all P, we have $\Theta \cap S = 1$, so Δ is equal to its image in the quotent locality $\mathcal{L}/\Theta$. Since $N_{\mathcal{L}}(P)/\Theta(P) \cong N_{\mathcal{L}/\Theta}(P)$ by 4.13(b), (a) is proved.

(b): Suppose now that Δ is contained in the set $\mathcal{F}^c$ of $\mathcal{F}$-centric subgroups of S, and let $P \in \Delta$. As $\mathcal{F}$ is Δ-saturated by 2.14(a), P has an $\mathcal{L}$-conjugate Q such that Q is fully normalized in $\mathcal{F}$. Then $C_S(Q) = Z(Q)$ as Q is centric, and $N_S(Q) \in Syl_p(N_{\mathcal{L}}(Q)$. Thus $Z(Q)$ is Sylow in $C_{\mathcal{L}}(Q)$, and so $C_{\mathcal{L}}(Q) = Z(Q) \times \Theta(Q)$ where $\Theta(Q)$ is a p'-group. By conjugation (and 2.7(b)), the analogous factorization holds also for $C_{\mathcal{L}}(P)$. Now (a) applies and yields $C_{\mathcal{L}/\Theta}(P) = Z(P)$. Thus $\mathcal{L}/\Theta$ is a Δ-linking system, and is a centric linking system if $\Delta = \mathcal{F}^c$. $\square$

Example 4.12. Let G be a finite group, let S be a Sylow p-subgroup of G, let $\mathcal{F}$ be the fusion system $\mathcal{F}_S(G)$, and let Δ be an $\mathcal{F}$-invariant, overgroup closed subset of $\mathcal{F}^q$ containing $\mathcal{F}^{cr}$. Let $\mathcal{T} := \mathcal{T}_\Delta(G)$ be the Δ-transporter system given by 2.5. The preceding theorem then yields a D-linking system $\mathcal{L}_\Delta(G) = (\mathbf{L}/\Theta, \Delta, S)$.