# Some summation rules related to the Riordan arrays 

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#### Abstract

In this paper, the concept of Riordan array is used to propose three theorems on combinatorial sums, provided that $\sum_{k=j}^{n} F(n, k) d_{k, j}=\varphi(n, j)$. A large number of useful identities tying together the coefficients in various combinatorial function expansions are obtained by proving the equivalence of two combinatorial sums. (c) 2003 Elsevier B.V. All rights reserved.


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## 1. Introduction

In 1991, Shapiro et al. [7] examined and further generalized the concept of renewal array [6] under the name of Riordan array. Recently, Sprugnoli [8,9] used it to obtain the closed form of many combinatorial sums and generalize the well-known identities of Abel and Gould.

For sums of hypergeometric terms, an approach was widely studied by Wilf, Zeilberger and Petkovšek [5]. Based on Gosper's algorithm [2], Wilf and Zeilberger [5,10,11] developed a theory of "WZ-pairs" that allows extremely elegant certification of the truth of a certain class of combinatorial identities. Petkovšek [5] developed an algorithm for deciding whether a given linear recurrence with polynomial coefficients has a simple solution as a linear combination of hypergeometric terms. The algorithms

[^0]demonstrated in $[5,10,11]$ give a decision procedure for closed form evaluation of definite hypergeometric summation.

On the contrary, Hsu [4] proposed a method for proving combinatorial identities, in which two combinatorial sums were certified equivalent, irrespective of the fact that they have a closed form, or not. In the present paper, we try to combine the work of Shapiro, Sprugnoli and Hsu to propose some summation rules. Lots of useful identities are developed as some applications of the summation rules.

## 2. Summation rules and examples

In [8], the Riordan array was defined as follows:
Let $d(t), h(t)$ be two real functions and $d(t)=\sum_{k} d_{k} t^{k}, h(t)=\sum_{k} h_{k} t^{k}$ with $h_{0} \neq$ 0 . For a fixed $k$, if $d_{n, k}=\left[t^{n}\right] d(t)(t h(t))^{k}, 0 \leqslant k \leqslant n$, then the infinite lower triangle $\left\{d_{n, k}\right\}$ is called a Riordan array and denoted by $D=(d(t), h(t))=\left\{d_{n, k}\right\}$. In addition, $f_{k}=\left[t^{n}\right] f(t)$ denotes the coefficients of $t^{n}$ in the expansion of $f(t)$ in $t$.

Theorem 1. Let $(d(t), h(t))$ be a Riordan array, $F(n, k)$ be a bivariate function defined for integers $n, k \geqslant 0$ and $f(t)=\sum_{k \geqslant 0} f_{k} t^{k}$. If

$$
\begin{equation*}
\sum_{k=j}^{n} F(n, k) d_{k, j}=\varphi(n, j) \tag{1}
\end{equation*}
$$

then

$$
\begin{equation*}
\sum_{k=0}^{n} F(n, k)\left[t^{k}\right](g(t) f(t h(t)))=\sum_{j=0}^{n} f_{j} \varphi(n, j) . \tag{2}
\end{equation*}
$$

Proof. By Theorem 1 in [8], we have

$$
\begin{aligned}
\sum_{k=0}^{n} F(n, k)\left[t^{k}\right](g(t) f(t h(t))) & =\sum_{k=0}^{n} F(n, k) \sum_{j=0}^{k} d_{k, j} f_{j} \\
& =\sum_{j=0}^{n} f_{j} \sum_{k=j}^{n} F(n, k) d_{k, j} \\
& =\sum_{j=0}^{n} f_{j} \varphi(n, j) .
\end{aligned}
$$

Example 1. Let

$$
d(t)=h(t)=\frac{1}{1-t}, \quad f(t)=\sum_{k \geqslant 0} f_{k} t^{k} .
$$

Then

$$
(d(t), h(t))=\left(\frac{1}{1-t}, \frac{1}{1-t}\right), \quad d_{n, k}=\binom{n}{k} .
$$

Let $F_{1}(n, k)=1$. Then

$$
\sum_{k=j}^{n} F_{1}(n, k) d_{k, j}=\sum_{k=j}^{n}\binom{k}{j}=\binom{n+1}{j+1}
$$

and, by Theorem 1, we have

$$
\begin{equation*}
\sum_{k=0}^{n}\left[t^{k}\right]\left(\frac{1}{1-t} f\left(\frac{t}{1-t}\right)\right)=\sum_{j=0}^{n}\binom{n+1}{j+1} f_{j} \tag{3}
\end{equation*}
$$

For various functions $f(t)$, we may find various identities via (3). For example:
For $f(t)=1 /(1-t)$, by (3), we have

$$
\sum_{k=0}^{n} 2^{k}=\sum_{j=0}^{n}\binom{n+1}{j+1}
$$

For $f(t)=t /(1-t)^{2}=\sum_{k=0}^{\infty} k t^{k}$, then

$$
\left[t^{k}\right]\left(\frac{1}{1-t} f\left(\frac{t}{1-t}\right)\right)=\left[t^{k}\right] \frac{t}{(1-2 t)^{2}}=k 2^{k-1}
$$

and, by (3), we have

$$
\sum_{k=0}^{n} k 2^{k-1}=\sum_{j=1}^{n} j\binom{n+1}{j+1}
$$

For

$$
f(t)=\frac{t^{m}}{(1-t)(1-2 t) \cdots(1-m t)}
$$

then

$$
f_{j}=\left\{\begin{array}{l}
j \\
m
\end{array}\right\}, \quad\left[t^{k}\right]\left(\frac{1}{1-t} f\left(\frac{t}{1-t}\right)\right)=\left\{\begin{array}{l}
k+1 \\
m+1
\end{array}\right\} \quad[8, \mathrm{p} .274]
$$

and, by (3), we have

$$
\sum_{k=0}^{n}\left\{\begin{array}{l}
k+1 \\
m+1
\end{array}\right\}=\sum_{j=0}^{n}\left\{\begin{array}{l}
j \\
m
\end{array}\right\}\binom{n+1}{j+1}
$$

If

$$
F_{2}(n, k)=H_{k},
$$

where

$$
H_{k}=1+\frac{1}{2}+\cdots+\frac{1}{k} \quad(k \geqslant 1)
$$

then

$$
\sum_{k=j}^{n} F_{2}(n, k) d_{k, j}=\sum_{k=j}^{n} H_{k}\binom{k}{j}=\binom{n+1}{j+1}\left(H_{n+1}-\frac{1}{j+1}\right)
$$

and, by Theorem 1, we have

$$
\begin{equation*}
\sum_{k=0}^{n} H_{k}\left[t^{k}\right]\left(\frac{1}{1-t} f\left(\frac{t}{1-t}\right)\right)=\sum_{j=0}^{n} f_{j}\binom{n+1}{j+1}\left(H_{n+1}-\frac{1}{j+1}\right) . \tag{4}
\end{equation*}
$$

In particular, for $f(t)=1 /(1-t), t /(1-t)^{2}, t^{m} /(1-t)(1-2 t) \cdots(1-m t)$, by (4), we have the following identities:

$$
\begin{aligned}
& \sum_{k=0}^{n} 2^{k} H_{k}=\sum_{j=0}^{n}\binom{n+1}{j+1}\left(H_{n+1}-\frac{1}{j+1}\right) \\
& \sum_{k=0}^{n} k 2^{k-1} H_{k}=\sum_{j=1}^{n} j\binom{n+1}{j+1}\left(H_{n+1}-\frac{1}{j+1}\right), \\
& \sum_{k=0}^{n}\left\{\begin{array}{c}
k+1 \\
m+1
\end{array}\right\} H_{k}=\sum_{j=0}^{n}\left\{\begin{array}{c}
j \\
m
\end{array}\right\}\binom{n+1}{j+1}\left(H_{n+1}-\frac{1}{j+1}\right),
\end{aligned}
$$

where $\left\{\begin{array}{c}j \\ m\end{array}\right\}$ denotes the Stirling numbers of the second kind.
From [3], we have the following identities:

$$
\begin{align*}
& \sum_{k=j}^{[n / 2]}\binom{n}{2 k}\binom{k}{j}=2^{n-2 j-1} \frac{n}{n-j}\binom{n-j}{j} \quad[3,(3.120)],  \tag{5}\\
& \sum_{k=j}^{[n / 2]}\binom{n+1}{2 k+1}\binom{k}{j}=2^{n-2 j}\binom{n-j}{j} \quad[3,(3.121)],  \tag{6}\\
& \sum_{k=j}^{n}\binom{n-k}{s}\binom{k}{j}=\binom{n+1}{s+j+1} \quad[3,(3.3)],  \tag{7}\\
& \sum_{k=j}^{n}\binom{s+k}{s}\binom{k}{j}=\frac{n+1-j}{s+1+j}\binom{n+1}{j}\binom{n+1+s}{s}  \tag{8}\\
& \sum_{k=j}^{n}(-4)^{k}\binom{n+k}{2 k}\binom{k}{j}=(-1)^{n} 2^{2 j} \frac{2 n+1}{2 j+1}\binom{n+j}{2 j}  \tag{9}\\
& \sum_{k=j}^{n}(-4)^{k} \frac{n}{n+k}\binom{n+k}{2 k}\binom{k}{j}=(-1)^{n} 2^{2 j} \frac{n}{n+j}\binom{n+j}{2 j} \tag{10}
\end{align*}
$$

$$
\begin{align*}
& \sum_{k=j}^{[n / 2]}(-1)^{k}\binom{n-k}{k} 2^{n-2 k}\binom{k}{j}=(-1)^{j}\binom{n+1}{2 j+1} \quad[3,(3.179)],  \tag{11}\\
& \sum_{k=j}^{n}\binom{\alpha}{k}\binom{\beta}{n-k}\binom{k}{j}=\binom{\alpha}{j}\binom{\alpha+\beta-j}{n-j} \quad[3,(6.14)],  \tag{12}\\
& \sum_{k=j}^{n}(-1)^{k}\binom{n}{k}\binom{2 n-k}{n}\binom{k}{j}=(-1)^{j}\binom{n}{j} \quad[3,(6.21)],  \tag{13}\\
& \sum_{k=j}^{[n / 2]}\binom{n}{2 k}\binom{2 k}{k} 2^{n-2 k}\binom{k}{j}=\binom{2 n-2 j}{n}\binom{n}{j} \quad[3,(6.33)] . \tag{14}
\end{align*}
$$

Let $\left\{d_{n, k}\right\}=(1 /(1-t), 1 /(1-t))$, we may find various special identities via (2). For example, from Eqs. (5)-(14), just as (3) and (4), we can obtain the following identities:

$$
\begin{align*}
& \sum_{k=0}^{[n / 2]}\binom{n}{2 k}\left[t^{k}\right]\left(\frac{1}{1-t} f\left(\frac{t}{1-t}\right)\right)=\sum_{j=0}^{[n / 2]} 2^{n-2 j-1} \frac{n}{n-j}\binom{n-j}{j} f_{j},  \tag{15}\\
& \sum_{k=0}^{[n / 2]}\binom{n+1}{2 k+1}\left[t^{k}\right]\left(\frac{1}{1-t} f\left(\frac{t}{1-t}\right)\right)=\sum_{j=0}^{[n / 2]} 2^{n-2 j}\binom{n-j}{j} f_{j},  \tag{16}\\
& \sum_{k=0}^{n}\binom{n-k}{s}\left[t^{k}\right]\left(\frac{1}{1-t} f\left(\frac{t}{1-t}\right)\right)=\sum_{j=0}^{n}\binom{n+1}{s+j+1} f_{j},  \tag{17}\\
& \sum_{k=0}^{n}\binom{s+k}{s}\left[t^{k}\right]\left(\frac{1}{1-t} f\left(\frac{t}{1-t}\right)\right) \\
& \quad=\sum_{j=0}^{n} \frac{n+1-j}{s+1+j}\binom{n+1}{j}\binom{n+1+s}{s} f_{j},  \tag{18}\\
& \sum_{k=0}^{n}(-4)^{k}\binom{n+k}{2 k}\left[t^{k}\right]\left(\frac{1}{1-t} f\left(\frac{t}{1-t}\right)\right) \\
& \quad=\sum_{j=0}^{n}(-1)^{n} 2^{2 j} \frac{2 n+1}{2 j+1}\binom{n+j}{2 j} f_{j},  \tag{19}\\
& \sum_{k=0}^{n}(-4)^{k} \frac{n}{n+k}\binom{n+k}{2 k}\left[t^{k}\right]\left(\frac{1}{1-t} f\left(\frac{t}{1-t}\right)\right) \\
& =\sum_{j=0}^{n}(-1)^{n} 2^{2 j} \frac{n}{n+j}\binom{n+j}{2 j} f_{j}, \tag{20}
\end{align*}
$$

$$
\begin{align*}
& \sum_{k=0}^{[n / 2]}(-1)^{k}\binom{n-k}{k} 2^{n-2 k}\left[t^{k}\right]\left(\frac{1}{1-t} f\left(\frac{t}{1-t}\right)\right) \\
& \quad=\sum_{j=0}^{[n / 2]}(-1)^{j}\binom{n+1}{2 j+1} f_{j},  \tag{21}\\
& \sum_{k=0}^{n}\binom{\alpha}{k}\binom{\beta}{n-k}\left[t^{k}\right]\left(\frac{1}{1-t} f\left(\frac{t}{1-t}\right)\right)=\sum_{j=0}^{n}\binom{\alpha}{j}\binom{\alpha+\beta-j}{n-j} f_{j},  \tag{22}\\
& \sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\binom{2 n-k}{n}\left[t^{k}\right]\left(\frac{1}{1-t} f\left(\frac{t}{1-t}\right)\right)=\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}^{2} f_{j},  \tag{23}\\
& \sum_{k=0}^{[n / 2]}\binom{n}{2 k}\binom{2 k}{k} 2^{n-2 k}\left[t^{k}\right]\left(\frac{1}{1-t} f\left(\frac{t}{1-t}\right)\right) \\
& \quad=\sum_{j=0}^{[n / 2]}\binom{2 n-2 j}{n}\binom{n}{j} f_{j} . \tag{24}
\end{align*}
$$

For the Stirling numbers of the both kinds $\left[\begin{array}{l}n \\ k\end{array}\right]$ and $\left\{\begin{array}{l}n \\ k\end{array}\right\}$, we have (see [8])

$$
\begin{aligned}
& {\left[t^{n}\right]\left(\ln \frac{1}{1-t}\right)^{m}=\frac{m!}{n!}\left[\begin{array}{l}
n \\
m
\end{array}\right],} \\
& {\left[t^{n}\right]\left(\mathrm{e}^{t}-1\right)^{m}=\frac{m!}{n!}\left\{\begin{array}{l}
n \\
m
\end{array}\right\}}
\end{aligned}
$$

and

$$
\sum_{k=j}^{n}\left[\begin{array}{l}
n \\
m
\end{array}\right]\left\{\begin{array}{l}
k \\
j
\end{array}\right\}=\frac{n!}{j!}\binom{n-1}{j-1} .
$$

Now consider Riordan array $\left(1,\left(\mathrm{e}^{t}-1\right) / t\right)$. Then

$$
d_{n, k}=\frac{k!}{n!}\left\{\begin{array}{l}
n \\
k
\end{array}\right\} .
$$

If $F(n, k)=\left[\begin{array}{l}n \\ k\end{array}\right] k$ !, we have

$$
\sum_{k=j}^{n} F(n, k) d_{k, j}=\sum_{k=j}^{n}\left[\begin{array}{l}
n \\
k
\end{array}\right] k!\frac{j!}{k!}\left\{\begin{array}{l}
k \\
j
\end{array}\right\}=n!\binom{n-1}{j-1} .
$$

So, by Theorem 1, we obtain

$$
\sum_{k=0}^{n}\left[\begin{array}{l}
n  \tag{25}\\
k
\end{array}\right] k!\left[t^{k}\right] f\left(\mathrm{e}^{t}-1\right)=\sum_{j=1}^{n} n!\binom{n-1}{j-1} f_{j} .
$$

Similarly, consider Riordan array

$$
\left(1, \frac{1}{t} \ln \frac{1}{1-t}\right)=\left\{d_{n, k}\right\}=\left\{\frac{k!}{n!}\left[\begin{array}{l}
n \\
k
\end{array}\right]\right\}
$$

and the following formula (see [8])

$$
\sum_{k=j}^{n+1} \frac{(-1)^{k}}{k}\left\{\begin{array}{c}
n \\
k-1
\end{array}\right\}\left[\begin{array}{l}
k \\
j
\end{array}\right]=\frac{(-1)^{j}}{n+1}\binom{n+1}{j} B_{n-j+1}
$$

where $B_{n}$ is the $n$th Bernoulli number. We have

$$
\begin{align*}
\sum_{k=1}^{n+1} & (-1)^{k}(k-1)!\left\{\begin{array}{c}
n \\
k-1
\end{array}\right\}\left[t^{k}\right] f\left(\ln \frac{1}{1-t}\right) \\
& =\sum_{j=0}^{n+1} \frac{(-1)^{j} j!f_{j}}{n+1}\binom{n+1}{j} B_{n-j+1} . \tag{26}
\end{align*}
$$

When $f(t)=t$, formula (26) specializes to

$$
\sum_{k=1}^{n+1}(-1)^{k+1} \frac{(k-1)!}{k}\left\{\begin{array}{c}
n \\
k-1
\end{array}\right\}=B_{n}
$$

i.e.

$$
\sum_{k=0}^{n}\left\{\begin{array}{l}
n \\
k
\end{array}\right\} \frac{(-1)^{k} k!}{k+1}=B_{n}
$$

a way to define the Bernoulli number in terms of the Stirling numbers of the second kind.

For various functions $f(t)$, we may obtain many identities by formulas from (15) through (26) in a like way as Example 1.

Theorem 2. Let $(d(t), h(t))$ be a Riordan array. Let $F(n, k)$ be a bivariate function defined for integers $n, k \geqslant 0, f(t)=\sum_{n \geqslant 0} f_{n} t^{n}$ and $\hat{f}_{(h)}(t)=\left[f(y) \mid y=t h(y)^{-1}\right]=$ $\sum_{n \geqslant 0} \hat{f}_{n} t^{n}$. If

$$
\sum_{k=j}^{n} F(n, k) d_{k, j}=\varphi(n, j),
$$

then

$$
\begin{equation*}
\sum_{k=0}^{n} F(n, k)\left[t^{k}\right](d(t) f(t))=\sum_{j=0}^{n} \hat{f}_{j} \varphi(n, j) \tag{27}
\end{equation*}
$$

If $h(0)=1$, then, with the Lagrange inversion formula, we have

$$
\hat{f}_{j}=\frac{1}{j!}\left[t^{j-1}\right]\left(f^{\prime}(t)(h(t))^{-j}\right)
$$

and

$$
\begin{equation*}
\sum_{k=0}^{n} F(n, k)\left[t^{k}\right](d(t) f(t))=\sum_{j=0}^{n} \varphi(n, j) \frac{1}{j!}\left[t^{j-1}\right]\left(f^{\prime}(t)(h(t))^{-j}\right) . \tag{28}
\end{equation*}
$$

Proof. By Theorem 3.1 in [9] and the Lagrange inversion formula [1], we have

$$
\begin{aligned}
\sum_{k=0}^{n} F(n, k)\left[t^{k}\right](d(t) f(t)) & =\sum_{k=0}^{n} F(n, k) \sum_{j=0}^{k} d_{k, j} \hat{f}_{j} \\
& =\sum_{j=0}^{n} \hat{f}_{j} \sum_{k=j}^{n} F(n, k) d_{k, j} \\
& =\sum_{j=0}^{n} \hat{f}_{j} \varphi(n, j) \\
& =\sum_{j=0}^{n} \varphi(n, j) \frac{1}{j!}\left[t^{j-1}\right]\left(f^{\prime}(t)(h(t))^{-j}\right)
\end{aligned}
$$

Example 2. For Riordan array $(1 /(1-t), 1 /(1-t))$, we have

$$
h(t)=\frac{1}{1-t}
$$

and, from $y=\operatorname{th}(y)^{-1}$,

$$
y=\frac{t}{1+t} .
$$

Therefore

$$
\hat{f}_{(h)}(t)=\left[f(y) \mid y=\operatorname{th}(y)^{-1}\right]=f\left(\frac{t}{1+t}\right)=\sum_{n=0}^{\infty} \hat{f}_{n} t^{n}
$$

and

$$
\hat{f}_{j}=\left[t^{j}\right] f\left(\frac{t}{1+t}\right)
$$

Let $F(n, k)=1$, then, by (27), we have

$$
\begin{equation*}
\sum_{k=0}^{n}\left[t^{k}\right] \frac{f(t)}{1-t}=\sum_{j=0}^{n} \hat{f}_{j}\binom{n+1}{j+1}=\sum_{j=0}^{n}\binom{n+1}{j+1}\left[t^{j}\right] f\left(\frac{t}{1+t}\right) . \tag{29}
\end{equation*}
$$

It is easy to find that (2) becomes (27) if $\left[t^{k}\right](g(t) f(t h(t)))$ is replaced by $\left[t^{k}\right](d(t) f(t))$ and $f_{j}$ by $\hat{f}_{j}$ in (2). For example, just as (29), each of formulas (4) and from (15) through (24) may be replaced, respectively, by

$$
\begin{align*}
& \sum_{k=0}^{n} H_{k}\left[t^{k}\right] \frac{f(t)}{1-t}=\sum_{j=0}^{n}\left(H_{n+1}-\frac{1}{j+1}\right)\binom{n+1}{j+1}\left[t^{j}\right] f\left(\frac{t}{1+t}\right),  \tag{30}\\
& \sum_{k=0}^{[n / 2]}\binom{n}{2 k}\left[t^{k}\right] \frac{f(t)}{1-t}=\sum_{j=0}^{[n / 2]} 2^{n-2 j-1} \frac{n}{n-j}\binom{n-j}{j}\left[t^{j}\right] f\left(\frac{t}{1+t}\right),  \tag{31}\\
& \sum_{k=0}^{[n / 2]}\binom{n+1}{2 k+1}\left[t^{k}\right] \frac{f(t)}{1-t}=\sum_{j=0}^{[n / 2]} 2^{n-2 j}\binom{n-j}{j}\left[t^{j}\right] f\left(\frac{t}{1+t}\right),  \tag{32}\\
& \sum_{k=0}^{n}\binom{n-k}{s}\left[t^{k}\right] \frac{f(t)}{1-t}=\sum_{j=0}^{n}\binom{n+1}{s+j+1}\left[t^{j}\right] f\left(\frac{t}{1+t}\right),  \tag{33}\\
& \sum_{k=0}^{n}\binom{s+k}{s}\left[t^{k}\right] \frac{f(t)}{1-t} \\
& =\sum_{j=0}^{n} \frac{n+1-j}{s+1+j}\binom{n+1}{j}\binom{n+1+s}{s}\left[t^{j}\right] f\left(\frac{t}{1+t}\right),  \tag{34}\\
& \sum_{k=0}^{n}(-4)^{k}\binom{n+k}{2 k}\left[t^{k}\right] \frac{f(t)}{1-t} \\
& =\sum_{j=0}^{n}(-1)^{n} 2^{2 j} \frac{2 n+1}{2 j+1}\binom{n+j}{2 j}\left[t^{j}\right] f\left(\frac{t}{1+t}\right),  \tag{35}\\
& \sum_{k=0}^{n}(-4)^{k} \frac{n}{n+k}\binom{n+k}{2 k}\left[t^{k}\right] \frac{f(t)}{1-t} \\
& =\sum_{j=0}^{n}(-1)^{n} 2^{2 j} \frac{n}{n+j}\binom{n+j}{2 j}\left[t^{j}\right] f\left(\frac{t}{1+t}\right),  \tag{36}\\
& \sum_{k=0}^{[n / 2]}(-1)^{k}\binom{n-k}{k} 2^{n-2 k}\left[t^{k}\right] \frac{f(t)}{1-t}=\sum_{j=0}^{[n / 2]}(-1)^{j}\binom{n+1}{2 j+1}\left[t^{j}\right] f\left(\frac{t}{1+t}\right),  \tag{37}\\
& \sum_{k=0}^{n}\binom{\alpha}{k}\binom{\beta}{-k}\left[t^{k}\right] \frac{f(t)}{1-t}=\sum_{j=0}^{n}\binom{\alpha}{j}\binom{\alpha+\beta-j}{n-j}\left[t^{j}\right] f\left(\frac{t}{1+t}\right), \tag{38}
\end{align*}
$$

$$
\begin{align*}
& \sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\binom{2 n-k}{n}\left[t^{k}\right] \frac{f(t)}{1-t}=\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}^{2}\left[t^{j}\right] f\left(\frac{t}{1+t}\right),  \tag{39}\\
& \sum_{k=0}^{[n / 2]}\binom{n}{2 k}\binom{2 k}{k} 2^{n-2 k}\left[t^{k}\right] \frac{f(t)}{1-t}=\sum_{j=0}^{[n / 2]}\binom{2 n-2 j}{n}\binom{n}{j}\left[t^{j}\right] f\left(\frac{t}{1+t}\right) . \tag{40}
\end{align*}
$$

For (25) and (26), we have

$$
\begin{align*}
& \sum_{k=0}^{n}\left[\begin{array}{l}
n \\
k
\end{array}\right] k!\left[t^{k}\right] f(t)=\sum_{j=1}^{n} n!\binom{n-1}{j-1}\left[t^{j}\right] f(\ln (1+t)),  \tag{41}\\
& \sum_{k=1}^{n+1}(-1)^{k}(k-1)!\left\{\begin{array}{c}
n \\
k-1
\end{array}\right\}\left[t^{k}\right] f(t) \\
& \quad=\sum_{j=0}^{n+1} \frac{(-1)^{j} j!}{n+1}\binom{n+1}{j} B_{n-j+1}\left[t^{j}\right] f\left(1-\mathrm{e}^{-t}\right) . \tag{42}
\end{align*}
$$

The partial Bell polynomials are the polynomials $B_{n, k}=B_{n, k}\left(x_{1}, x_{2}, \ldots, x_{n-k+1}\right)$ in an infinite number of variables $x_{1}, x_{2}, \ldots$, defined by

$$
\frac{1}{k!}\left(\sum_{m \geqslant 1} x_{m} \frac{t^{m}}{m!}\right)^{k}=\sum_{n \geqslant k} B_{n, k} \frac{t^{n}}{n!}, \quad k=0,1,2, \ldots .
$$

By the definition of Riordan arrays, we have

$$
\left(1, \sum_{m \geqslant 1} x_{m} \frac{t^{m}}{m!}\right)=\left\{B_{n, k} k!\right\} .
$$

For the partial Bell polynomials $B_{n, k}$, we have the following well-known formulas (Faa di Bruno formula [1, p. 137]):

$$
\tilde{h}_{0}=\tilde{f}_{0}, \quad \tilde{h}_{n}=\sum_{k=1}^{n} \tilde{f}_{k} B_{n, k}\left(\tilde{g}_{1}, \tilde{g}_{2}, \ldots, \tilde{g}_{n-k+1}\right),
$$

where

$$
\begin{aligned}
& f(t)=\sum_{k=0}^{\infty} \tilde{f}_{k} \frac{t^{k}}{k!}, \\
& g(t)=\sum_{k=1}^{\infty} \tilde{g}_{k} \frac{t^{k}}{k!}
\end{aligned}
$$

and

$$
f(g(t))=\sum_{k=0}^{\infty} \tilde{h}_{k} \frac{t^{k}}{k!}
$$

Let $B_{n, 0}=\delta_{n, 0}$, then the above formulas may be replaced by

$$
\tilde{h}_{n}=\sum_{k=0}^{n} \tilde{f}_{k} B_{n, k}\left(\tilde{g}_{1}, \tilde{g}_{2}, \ldots, \tilde{g}_{n-k+1}\right) .
$$

Theorem 3. Let $F(n, k)$ be a bivariate function defined for integers $n, k \geqslant 0$. If

$$
\begin{equation*}
\sum_{k=j}^{n} F(n, k) B_{k, j}=M(n, j) \tag{43}
\end{equation*}
$$

then

$$
\begin{equation*}
\sum_{k=0}^{n} F(n, k) \tilde{h}_{k}=F(n, 0) \tilde{f}_{0}+\sum_{j=1}^{n} \tilde{f}_{j} M(n, j) \tag{44}
\end{equation*}
$$

Proof. By (43), we have

$$
\begin{aligned}
\sum_{k=0}^{n} F(n, k) \tilde{h}_{k} & =\sum_{k=0}^{n} F(n, k) \sum_{j=0}^{k} \tilde{f}_{j} B_{k, j} \\
& =F(n, 0) \tilde{f}_{0}+\sum_{k=1}^{n} F(n, k) \sum_{j=1}^{n} \tilde{f}_{j} B_{k, j} \\
& =F(n, 0) \tilde{f}_{0}+\sum_{j=1}^{n} \tilde{f}_{j} \sum_{k=j}^{n} F(n, k) B_{k, j} \\
& =F(n, 0) \tilde{f}_{0}+\sum_{j=1}^{n} \tilde{f}_{j} M(n, j) .
\end{aligned}
$$

Example 3. Let $g(t)=\sum_{k \geqslant 1} t^{k}=t /(1-t)$ and $F(n, k)=1 / k!$. Then

$$
B_{n, k}\left(\tilde{g}_{1}, \tilde{g}_{2}, \ldots\right)=B_{n, k}(1!, 2!, \ldots)=\binom{n-1}{k-1} \frac{n!}{k!}
$$

and

$$
\sum_{k=j}^{n} F(n, k) B_{k, j}=\sum_{k=j}^{n} \frac{1}{k!}\binom{k-1}{j-1} \frac{k!}{j!}=\frac{1}{j!} \sum_{k=j}^{n}\binom{k-1}{j-1}=\frac{1}{j!}\binom{n}{j} .
$$

By (44), we have

$$
\sum_{k=0}^{n}\left[t^{k}\right] f\left(\frac{t}{1-t}\right)=\tilde{f}_{0}+\sum_{j=1}^{n}\binom{n}{j}\left[t^{j}\right] f(t),
$$

i.e.

$$
\left[t^{n}\right] \frac{1}{1-t} f\left(\frac{t}{1-t}\right)=\tilde{f}_{0}+\sum_{j=1}^{n}\binom{n}{j}\left[t^{j}\right] f(t),
$$

because, in general,

$$
\sum_{k=0}^{n}\left[t^{k}\right] f(t)=\left[t^{n}\right] \frac{f(t)}{1-t}
$$

Furthermore

$$
\tilde{f}_{0}+\sum_{j=1}^{n}\binom{n}{j}\left[t^{j}\right] f(t)=\sum_{j=0}^{n}\binom{n}{j}\left[t^{j}\right] f(t)
$$

and, in conclusion

$$
\sum_{j=0}^{n}\binom{n}{j}\left[t^{j}\right] f(t)=\left[t^{n}\right] \frac{1}{1-t} f\left(\frac{t}{1-t}\right)
$$

This, however, is the well-known Euler's transformation, a simple result in the theory of Riordan arrays.

Example 4. Let

$$
g(t)=\sum_{m=1}^{\infty} x_{m} \frac{t^{m}}{m!}, \quad F(n, k)=\binom{n}{k} x_{n-k} .
$$

Then by [3k] in [1, p. 136], we have

$$
\sum_{k=j}^{n} F(n, k) B_{k, j}=\sum_{k=j}^{n}\binom{n}{k} x_{n-k} B_{k, j}=(j+1) B_{n, j+1}
$$

where $B_{n, k}=B_{n, k}\left(x_{1}, x_{2}, \ldots, x_{n-k+1}\right)$.
By (44), we obtain

$$
\begin{equation*}
\sum_{k=0}^{n}\binom{n}{k} x_{n-k} k!\left[t^{k}\right] f\left(\sum_{m=1}^{\infty} x_{m} \frac{t^{m}}{m!}\right)=x_{n} \tilde{f}_{0}+\sum_{j=1}^{n}(j+1)!B_{n, j+1}\left[t^{j}\right] f(t) \tag{45}
\end{equation*}
$$

For various functions $f(t)$ and $g(t)$, we can obtain various identities by (45). For example, let $f(t)=g(t)=\mathrm{e}^{t}-1$ and $B_{n, j+1}(1,1, \ldots, 1)=\left\{\begin{array}{c}n \\ j+1\end{array}\right\}$, then we have the following identity:

$$
\sum_{k=1}^{n-1} \mathscr{B}_{k}\binom{n}{k}=\sum_{j=1}^{n}(j+1)\left\{\begin{array}{c}
n \\
j+1
\end{array}\right\}
$$

where $\mathscr{B}_{k}$ is the Bell number defined by

$$
\mathrm{e}^{\mathrm{e}^{t}-1}-1=\sum_{k=1}^{\infty} \mathscr{B}_{k} \frac{t^{k}}{k!}
$$

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