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On the reciprocal sums of the generalized Fibonacci sequences

Han Zhang and Zhengang Wu*

*Correspondence:
sky.wzgfff@163.com
Department of Mathematics,
Northwest University, Xi'an, Shaanxi,
P.R. China

Abstract

The Fibonacci sequence has been generalized in many ways. One of them is defined by the relation $u_n = au_{n-1} + u_{n-2}$ if n is even, $u_n = bu_{n-1} + u_{n-2}$ if n is odd, with initial values $u_0 = 0$ and $u_1 = 1$, where a and b are positive integers. In this paper, we consider the reciprocal sum of u_n and then establish some identities relating to $\left\| \left(\sum_{k=n}^{\infty} \frac{1}{u_k} \right)^{-1} \right\|$, where $\|x\|$ denotes the nearest integer to x .

MSC: Primary 11B39

Keywords: generalized Fibonacci sequence; infinite sum; reciprocal sum

1 Introduction

For any integer $n \geq 0$, the well-known Fibonacci sequence F_n is defined by the second-order linear recurrence sequence $F_{n+2} = F_{n+1} + F_n$, where $F_0 = 0$ and $F_1 = 1$. The Fibonacci sequence has been generalized in many ways, for example, by changing the initial values, by changing the recurrence relation, and so on. Edson and Yayenie [1] defined a further generalized Fibonacci sequence u_n depending on two real parameters used in a non-linear recurrence relation, namely,

$$u_n = \begin{cases} au_{n-1} + u_{n-2} & \text{if } n \text{ is even and } n \geq 2, \\ bu_{n-1} + u_{n-2} & \text{if } n \text{ is odd and } n \geq 1, \end{cases} \quad (1)$$

with initial values $u_0 = 0$ and $u_1 = 1$, where a, b are positive integers. This new sequence is actually a family of sequences where each new choice of a and b produces a distinct sequence. When $a = b = 1$, we have the classical Fibonacci sequence and when $a = b = 2$, we obtain the Pell numbers. Even further, if we set $a = b = k$ for some positive integer k , we obtain the k -Fibonacci numbers.

Various properties of the Fibonacci numbers and related sequences have been studied by many authors, see [2–4]. Recently, Ohtsuka and Nakamura [5] studied the partial infinite sums of reciprocal Fibonacci numbers and proved that

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2} & \text{if } n \text{ is even and } n \geq 2, \\ F_{n-2} - 1 & \text{if } n \text{ is odd and } n \geq 1, \end{cases}$$

where $\lfloor x \rfloor$ (the floor function) denotes the greatest integer less than or equal to x .

Some related works can also be found in [6–13]. In particular, in [13], the authors studied a problem which is a little different from that of [5], namely that of determining the

nearest integer to $(\sum_{k=n}^{\infty} \frac{1}{v_k})^{-1}$. Specifically, suppose that $\|x\| = \lfloor x + \frac{1}{2} \rfloor$ (the nearest integer function), and $\{v_n\}_{n \geq 0}$ is an integer sequence satisfying the recurrence formula

$$v_n = a_1 v_{n-1} + a_2 v_{n-2} + \cdots + a_s v_{n-s} \quad (s \geq 2)$$

for any positive integer a_1, a_2, \dots, a_m , with the initial conditions $v_0 \geq 0, v_k \in \mathbf{N}, 1 \leq k \leq s-1$. Then, provided $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$, we can conclude that there exists a positive integer n_0 such that

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{v_k} \right)^{-1} \right\| = v_n - v_{n-1}$$

for all $n > n_0$.

Because the Fibonacci sequence has been generalized to a higher-order recursive sequence, any study on linear recursive sequences has little significance in this context, and we have to consider other non-linear recursive sequences. The main purpose of this paper is concerned with finding expressions for

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{u_k} \right)^{-1} \right\|.$$

In fact, this problem is difficult because each item of this sequence relies on the previous relation. In order to resolve the question, we consider the reciprocal sums in two directions: on the one hand to the subsequence u_{pk+q} and on the other to the product form $u_k u_{k+2c+1}$, where p, q, c are non-negative integers and $p \geq 2$. The results are as follows.

Theorem 1 Let $\{u_n\}$ be a second-order sequence defined by (1). For any even $p \geq 2$ and non-negative integer $q < p$, there exists a positive integer n_1 such that

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{u_{pk+q}} \right)^{-1} \right\| = u_{pn+q} - u_{pn-p+q}$$

for all $n \geq n_1$.

Theorem 2 Let $\{u_n\}$ be a second-order sequence defined by (1). For any integer $c \geq 0$, there exists a positive integer n_2 such that

$$\left\| \left(\sum_{k=n}^{\infty} \frac{a^k b^{k+2c+1}}{u_k u_{k+2c+1}} \right)^{-1} - \left(\frac{u_n u_{n+2c+1}}{a^n b^{n+2c+1}} - \frac{u_{n-1} u_{n+2c}}{a^{n-1} b^{n+2c}} \right) \right\| = 0$$

for all $n \geq n_2$.

Open problem In the light of our investigation, for any positive integer $s \geq 2$ and l , whether there exist identities for

$$\left(\sum_{k=n}^{\infty} \frac{1}{u_k^s} \right)^{-1} \quad \text{and} \quad \left(\sum_{k=n}^{\infty} \frac{1}{u_k u_{k+l}} \right)^{-1}$$

represent two interesting, albeit challenging, open problems.

2 Proofs of the theorems

We need the following lemma.

Lemma (Generalized Binet's formula) *The terms of the generalized Fibonacci sequence u_n are given by*

$$u_n = \frac{a^{2\lfloor \frac{n}{2} \rfloor - n + 1}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \cdot \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where $\alpha = \frac{ab + \sqrt{a^2 b^2 + 4ab}}{2}$, $\beta = \frac{ab - \sqrt{a^2 b^2 + 4ab}}{2}$.

Proof See Theorem 2 of [4]. □

Proof of Theorem 1 From the geometric series as $\epsilon \rightarrow 0$, we have

$$\frac{1}{1 \pm \epsilon} = 1 \mp \epsilon + O(\epsilon^2) = 1 + O(\epsilon).$$

From Lemma and the identity $\alpha\beta = -ab$, we have

$$u_{pk+q} = \begin{cases} \frac{\alpha^{pk+q} - \beta^{pk+q}}{a^{\frac{pk+q-2}{2}} b^{\frac{pk+q}{2}} (\alpha - \beta)} & \text{if } q \text{ is even (so that } pk + q \text{ is even),} \\ \frac{\alpha^{pk+q} - \beta^{pk+q}}{ab^{\frac{pk+q-1}{2}} (\alpha - \beta)} & \text{if } q \text{ is odd (so that } pk + q \text{ is odd).} \end{cases}$$

Let

$$A = \begin{cases} \frac{1}{a^{\frac{pk+q-2}{2}} b^{\frac{pk+q}{2}} (\alpha - \beta)} & \text{if } q \text{ is even,} \\ \frac{1}{ab^{\frac{pk+q-1}{2}} (\alpha - \beta)} & \text{if } q \text{ is odd.} \end{cases}$$

Thus,

$$u_{pk+q} = A\alpha^{pk+q} + O\left(\frac{|\beta|^{\frac{pk}{2}}}{\alpha^{\frac{pk}{2}}}\right).$$

Hence,

$$\begin{aligned} \frac{1}{u_{pk+q}} &= \frac{1}{A\alpha^{pk+q}(1 + O(\frac{|\beta|^{\frac{pk}{2}}}{\alpha^{\frac{3pk}{2}}}))} = \frac{1}{A\alpha^{pk+q}} \left(1 + O\left(\frac{|\beta|^{\frac{pk}{2}}}{\alpha^{\frac{3pk}{2}}}\right)\right) \\ &= \frac{1}{A\alpha^{pk+q}} + O\left(\frac{|\beta|^{\frac{pk}{2}}}{\alpha^{\frac{5pk}{2}}}\right). \end{aligned}$$

Thus,

$$\sum_{k=n}^{\infty} \frac{1}{u_{pk+q}} = \frac{\frac{1}{\alpha^{pn+q}}}{A(1 - \frac{1}{\alpha^p})} + O\left(\frac{|\beta|^{\frac{pn}{2}}}{\alpha^{\frac{5pn}{2}}}\right) = \frac{\alpha^p}{A\alpha^{pn+q}(\alpha^p - 1)} \left(1 + O\left(\frac{|\beta|^{\frac{pn}{2}}}{\alpha^{\frac{3pn}{2}}}\right)\right).$$

Taking the reciprocal of this expression yields

$$\begin{aligned} \left(\sum_{k=n}^{\infty} \frac{1}{u_{pk+q}} \right)^{-1} &= \frac{A\alpha^{pn+q}(\alpha^p - 1)}{\alpha^p} + O\left(\frac{|\beta|^{\frac{pn}{2}}}{\alpha^{\frac{pn}{2}}}\right) \\ &= A\alpha^{pn+q} - A\alpha^{pn-p+q} + O\left(\frac{|\beta|^{\frac{pn}{2}}}{\alpha^{\frac{pn}{2}}}\right) \\ &= u_{pn+q} - u_{pn-p+q} + O\left(\frac{|\beta|^{\frac{pn}{2}}}{\alpha^{\frac{pn}{2}}}\right). \end{aligned} \quad (2)$$

Therefore, for any even $p \geq 2$ and integer $0 < q < p$, there exists $n \geq n_1$ sufficiently large such that the modulus of the last error term of identity (2) becomes less than 1/2. This completes the proof of Theorem 1. \square

Proof of Theorem 2 In the first place, suppose that $k \geq 2$ is even. From Lemma we have

$$u_k = \frac{1}{a^{\frac{k}{2}-1} b^{\frac{k}{2}}} \cdot \frac{\alpha^k - \beta^k}{\alpha - \beta},$$

and

$$u_{k+2c+1} = \frac{1}{a^{\frac{k}{2}+c} b^{\frac{k}{2}+c}} \cdot \frac{\alpha^{k+2c+1} - \beta^{k+2c+1}}{\alpha - \beta}.$$

The identities $(\alpha - \beta)^2 = a^2 b^2 + 4ab$ and $\alpha\beta = -ab$ now yield

$$\begin{aligned} u_k u_{k+2c+1} &= \frac{\alpha^{2k+2c+1} + \beta^{2k+2c+1} - (\alpha\beta)^k(\alpha^{2c+1} + \beta^{2c+1})}{a^{k+c-1} b^{k+c} (\alpha - \beta)^2} \\ &= \frac{\alpha^{2k+2c+1}}{(ab^2 + 4b)(ab)^{k+c}} + O\left(\left(\frac{\alpha\beta}{ab}\right)^k\right) \\ &= \frac{\alpha^{2k+2c+1}}{(ab^2 + 4b)(ab)^{k+c}} + O(1). \end{aligned}$$

Further, if $k \geq 1$ is odd, the same identity is similarly obtained. Thus, in both cases we have

$$\begin{aligned} \frac{1}{u_k u_{k+2c+1}} &= \frac{1}{\frac{\alpha^{2k+2c+1}}{(ab^2 + 4b)(ab)^{k+c}} + O(1)} = \frac{1}{\frac{\alpha^{2k+2c+1}}{(ab^2 + 4b)(ab)^{k+c}} \left(1 + O\left(\frac{(ab)^k}{\alpha^{2k}}\right)\right)} \\ &= \frac{(ab^2 + 4b)(ab)^{k+c}}{\alpha^{2k+2c+1}} + O\left(\frac{(ab)^{2k}}{\alpha^{4k}}\right). \end{aligned}$$

Hence,

$$\begin{aligned} \frac{a^k b^{k+2c+1}}{u_k u_{k+2c+1}} &= \frac{(ab^3 + 4b^2)a^{2k+c}b^{2k+3c}}{\alpha^{2k+2c+1}} + O\left(\frac{(ab)^{3k}}{\alpha^{4k}}\right) \\ &= \frac{a^{c+1}b^{3c+3} + 4a^c b^{3c+2}}{\alpha^{2c+1}} \cdot \frac{(ab)^{2k}}{\alpha^{2k}} + O\left(\frac{(ab)^{3k}}{\alpha^{4k}}\right). \end{aligned}$$

Let $B = \frac{a^{c+1}b^{3c+3} + 4a^cb^{3c+2}}{\alpha^{2c+1}}$, then

$$\frac{a^k b^{k+2c+1}}{u_k u_{k+2c+1}} = \frac{B(ab)^{2k}}{\alpha^{2k}} + O\left(\frac{(ab)^{3k}}{\alpha^{4k}}\right).$$

Consequently,

$$\begin{aligned} \sum_{k=n}^{\infty} \frac{a^k b^{k+2c+1}}{u_k u_{k+2c+1}} &= B \sum_{k=n}^{\infty} \frac{(ab)^{2k}}{\alpha^{2k}} + O\left(\sum_{k=n}^{\infty} \frac{(ab)^{3k}}{\alpha^{4k}}\right) \\ &= B \cdot \frac{\left(\frac{ab}{\alpha}\right)^{2n}}{1 - \left(\frac{ab}{\alpha}\right)^2} + O\left(\frac{(ab)^{3n}}{\alpha^{4n}}\right). \end{aligned}$$

Taking the reciprocal of this expression yields

$$\begin{aligned} \left(\sum_{k=n}^{\infty} \frac{a^k b^{k+2c+1}}{u_k u_{k+2c+1}} \right)^{-1} &= \frac{1}{B \cdot \frac{\left(\frac{ab}{\alpha}\right)^{2n}}{1 - \left(\frac{ab}{\alpha}\right)^2} (1 + O(\frac{|\beta|^n}{\alpha^n}))} \\ &= \frac{1 - \left(\frac{ab}{\alpha}\right)^2}{B \cdot \left(\frac{ab}{\alpha}\right)^{2n}} \left(1 + O\left(\frac{|\beta|^n}{\alpha^n}\right)\right) \\ &= \frac{1 - \left(\frac{ab}{\alpha}\right)^2}{B \cdot \left(\frac{ab}{\alpha}\right)^{2n}} + O\left(\frac{1}{\alpha^n |\beta|^n}\right) \\ &= \frac{1}{B} \cdot \left(\frac{\alpha}{ab}\right)^{2n} - \frac{1}{B} \cdot \left(\frac{\alpha}{ab}\right)^{2n-2} + O\left(\frac{1}{\alpha^n |\beta|^n}\right). \end{aligned} \quad (3)$$

On the other hand,

$$\begin{aligned} \frac{u_n u_{n+2c+1}}{a^n b^{n+2c+1}} - \frac{u_{n-1} u_{n+2c}}{a^{n-1} b^{n+2c}} \\ &= \frac{\alpha^{2c+1}}{a^{c+1} b^{3c+3} + 4a^c b^{3c+2}} \cdot \left(\frac{\alpha}{ab}\right)^{2n} - \frac{\alpha^{2c+1}}{a^{c+1} b^{3c+3} + 4a^c b^{3c+2}} \cdot \left(\frac{\alpha}{ab}\right)^{2n-2} + O\left(\frac{1}{a^n b^n}\right) \\ &= \frac{1}{B} \cdot \left(\frac{\alpha}{ab}\right)^{2n} - \frac{1}{B} \cdot \left(\frac{\alpha}{ab}\right)^{2n-2} + O\left(\frac{1}{a^n b^n}\right). \end{aligned} \quad (4)$$

Combining (3) and (4), finally we have

$$\left(\sum_{k=n}^{\infty} \frac{a^k b^{k+2c+1}}{u_k u_{k+2c+1}} \right)^{-1} - \left(\frac{u_n u_{n+2c+1}}{a^n b^{n+2c+1}} - \frac{u_{n-1} u_{n+2c}}{a^{n-1} b^{n+2c}} \right) = O\left(\frac{1}{a^n b^n}\right). \quad (5)$$

It follows that for any integer $c \geq 0$, there exists $n \geq n_2$ sufficiently large such that the modulus of the last error term of identity (5) becomes less than 1/2. This completes the proof of Theorem 2. \square

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

HZ obtained the theorems and completed the proof. ZW corrected and improved the final version. Both authors read and approved the final manuscript.

Acknowledgements

The authors express their gratitude to the referee for very helpful and detailed comments. This work is supported by the N.S.F. (11371291), the S.R.F.D.P. (20136101110014), the N.S.F. (2013JZ001) of Shaanxi Province, and the G.I.C.F. (YZZ12062) of NWU, P.R. China.

Received: 16 October 2013 Accepted: 1 December 2013 Published: 23 Dec 2013

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10.1186/1687-1847-2013-377

Cite this article as: Zhang and Wu: On the reciprocal sums of the generalized Fibonacci sequences. *Advances in Difference Equations* 2013, 2013:377

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