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An Extension of the Generalized Pascal Matrix and its Algebraic Properties

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ABSTRACT

The extended generalized Pascal matrix can be represented in two different ways: as a lower triangular matrix $\Phi_n[x, y]$ or as a symmetric $\Psi_n[x, y]$. These matrices generalize $P_n[x]$, $Q_n[x]$, and $R_n[x]$, which are defined by Zhang and by Call and Velleman. A product formula for $\Phi_n[x, y]$ has been found which generalizes the result of Call and Velleman. It is shown that not only can $\Phi_n[x, y]$ be factorized by special summation, but also $\Psi_n[x, y]$ as $Q_n[xy]\Phi_s^T[y, 1/x]$ or $\Phi_n[x, y]P_n^T[y/x]$. Finally, the inverse of $\Psi_n[x, y]$ and the values of $\det \Phi_n[x, y]$, $\det \Phi_n^{-1}[x, y]$, $\det \Psi_n[x, y]$, and $\det \Psi_n^{-1}[x, y]$ are given. © 1998 Elsevier Science Inc.

Let x, y be any two nonzero real numbers. The extended generalized Pascal matrix $\Phi_n[x, y]$ is defined as

$$\Phi_n(x, y; i, j) = x^{i-j} y^{i+j} \binom{i}{j}, \qquad i, j = 0, 1, \dots, n,$$

with

$$\binom{i}{j} = 0 \quad \text{if} \quad j > i.$$

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By this definition, then

$$\Phi_n[x, 1] = P_n[x],$$

$$\Phi_n[1, y] = Q_n[y],$$

where $P_n[x]$ and $Q_n[y]$ are defined in [1, 2], respectively.

THEOREM 1. *For any four real numbers x_1, y_1, x_2, y_2, we have*

$$\Phi_n[x_1, y_1]\Phi_n[x_2, y_2] = \Phi_n\left[\frac{x_1}{y_2} + x_2 y_1, y_1 y_2\right].$$

Proof. Let $\Phi_n[x_1, y_1]\Phi_n[x_2, y_2] = (C_n(x_1, y_1, x_2, y_2; i, j))$. Then

$$\begin{aligned} C_n(x_1, y_1, x_2, y_2; i, j) &= \sum_{k=0}^{n} x_1^{i-k} y_1^{i+k} \binom{i}{k} x_2^{k-j} y_2^{k+j} \binom{k}{j} \\ &= \sum_{k=0}^{n} x_1^{i-k} y_1^{i+k} x_2^{k-j} y_2^{k+j} \binom{i}{j}\binom{i-j}{k-j} \\ &= \binom{i}{j}(y_1 y_2)^{i+j} \sum_{k=0}^{n} \binom{i-j}{k-j}\left(\frac{x_1}{y_2}\right)^{i-k} (x_2 y_1)^{k-j} \\ &= \binom{i}{j}(y_1 y_2)^{i+j}\left(\frac{x_1}{y_2} + x_2 y_1\right)^{i-j}. \end{aligned}$$

This completes the proof. ∎

If we take $y_i = y_2 = 1$ in Theorem 1, then we can get the following results of G. S. Call and D. J. Velleman [2]:

COROLLARY. $P_n[x]P_n[y] = P_n[x + y]$.

Now we list several definitions and results found in [1, 2] which will be required in the development of this paper.

The $(n+1)\times(n+1)$ matrices I_n, $S_n[x]$, and $D_n[x]$ are defined by

$$I_n = \operatorname{diag}(1,1,\dots,1),$$

$$S_n(x;i,j) = \begin{cases} x^{i-j} & \text{if } j \leqslant i, \\ 0 & \text{if } j > i, \end{cases}$$

$$D_n(x;i,i) = 1 \qquad \text{for } i = 0,1,\dots,n,$$

$$D_n(x;i+1,i) = -x \qquad \text{for } i = 0,1,\dots,n-1,$$

$$D_n(x;i,j) = 0 \qquad \text{for } j > i \text{ or } j < i-1,$$

and we also define the matrices

$$\bar{P}_k[x] = \begin{bmatrix} 1 & O^T \\ O & P_k[x] \end{bmatrix} \in R^{(k+2)\times(k+2)}, \qquad k \geqslant 0,$$

$$G_k[x] = \begin{bmatrix} I_{n-k-1} & 0 \\ 0 & S_k[x] \end{bmatrix} \in R^{(n+1)\times(n+1)}, \qquad k = 1,2,\dots,n-1,$$

$$G_n[x] = S_n[x],$$

$$S_n[x] = D_n^{-1}[x].$$

Again, we need the $(n+1)\times(n+1)$ matrices $W_n[x,y]$, $U_n[x,y]$, $J_n[y]$:

$$W_n(x,y;i,j) = \begin{cases} x^{i-j}y^{i+j} & \text{if } j \leqslant i, \\ 0 & \text{if } j > i, \end{cases}$$

$$U_n(x,y;i,i) = y^{-2i} \qquad \text{for } i = 0,1,\dots,n,$$

$$U_n(x,y;i+1,i) = -\frac{x}{y^{2i-1}} \qquad \text{for } i = 0,1,\dots,n-1,$$

$$U_n(x,y;i,j) = 0 \qquad \text{for } j > i \text{ or } j < i-1,$$

$$J_n[y] = \operatorname{diag}\left(1, -\frac{1}{y^2}, \frac{1}{y^4}, -\frac{1}{y^6}, \dots, (-1)^n\frac{1}{y^{2n}}\right).$$

It is easy to see that

THEOREM 2.

$$\Phi_n[-x, y] = \Phi_n[x, -y],$$

$$\Phi_n^{-1}[x, y] = \Phi_n\left[-x, \frac{1}{y}\right] = \Phi_n\left[x, -\frac{1}{y}\right],$$

$$W_n^{-1}[x, y] = U_n[x, y].$$

EXAMPLE.

$$\begin{aligned}
&\Phi_3[x, y]\Phi_3\left[-x, \frac{1}{y}\right] \\
&= \Phi_3[x, y]\Phi_3\left[x, -\frac{1}{y}\right] \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ xy & y^2 & 0 & 0 \\ x^2y^2 & 2xy^3 & y^4 & 0 \\ x^3y^3 & 3x^2y^4 & 3xy^5 & y^6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{x}{y} & \frac{1}{y^2} & 0 & 0 \\ \frac{x^2}{y^2} & -2\frac{x}{y^3} & \frac{1}{y^4} & 0 \\ -\frac{x^3}{y^3} & 3\frac{x^2}{y^4} & -3\frac{x}{y^5} & \frac{1}{y^6} \end{pmatrix} = I_3,
\end{aligned}$$

$$\begin{aligned}
&W_3[x, y]U_3[x, y] \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ xy & y^2 & 0 & 0 \\ x^2y^2 & xy^3 & y^4 & 0 \\ x^3y^3 & x^2y^4 & xy^5 & y^6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{x}{y} & \frac{1}{y^2} & 0 & 0 \\ 0 & -\frac{x}{y^3} & \frac{1}{y^4} & 0 \\ 0 & 0 & -\frac{x}{y^5} & \frac{1}{y^6} \end{pmatrix} = I_3.
\end{aligned}$$

LEMMA 1.

$$W_k[x, y]\overline{P}_{k-1}\left[\frac{x}{y}\right] = \Phi_k[x, y] \qquad \textit{for} \quad k \geqslant 1.$$

Proof. Let $W_k[x, y]\overline{P}_{k-1}[x/y] = (C_k(x, y;\ i, j))$. Obviously, $C_k(x, y;\ i, 0) = x^i y^i$ $(i = 0, 1, \dots, n)$ and $C_k(x, y;\ i, j) = 0$ $(i < j)$. When $i > j$ we have

$$C_k(x, y; i, j) = \sum_{h=0}^{k} x^{i-h} y^{i+h} \binom{h-1}{j-1} \left(\frac{x}{y}\right)^{h-j}$$

$$= x^i y^i \sum_{h=0}^{i} \binom{h-1}{j-1} x^{-h+h-j} y^{h-h+j}$$

$$= x^{i-j} y^{i+j} \binom{i}{j}.$$

This completes the proof. ∎

EXAMPLE.

$$W_3[x, y]\overline{P}_2\left[\frac{x}{y}\right] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ xy & y^2 & 0 & 0 \\ x^2y^2 & xy^3 & y^4 & 0 \\ x^3y^3 & x^2y^4 & xy^5 & y^6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{x}{y} & 1 & 0 \\ 0 & \frac{x^2}{y^2} & 2\frac{x}{y} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ xy & y^2 & 0 & 0 \\ x^2y^2 & 2xy^3 & y^4 & 0 \\ x^3y^3 & 3x^2y^4 & 3xy^5 & y^6 \end{pmatrix} = \Phi_3[x, y].$$

By Lemma 1 and the definition of $G_k[x]$, we get the following result:

THEOREM 3. *The extended generalized Pascal matrix* $\Phi_n[x, y]$ *can be factorized by the summations* $G_k[x/y]$ *and* $W_n[x, y]$:

$$\Phi_n[x, y] = W_n[x, y]G_{n-1}\left[\frac{x}{y}\right]G_{n-2}\left[\frac{x}{y}\right]\cdots G_1\left[\frac{x}{y}\right].$$

EXAMPLE.

$$\Phi_3[x, y]$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ xy & y^2 & 0 & 0 \\ x^2y^2 & 2xy^3 & y^4 & 0 \\ x^3y^3 & 3x^2y^4 & 3xy^5 & y^6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ xy & y^2 & 0 & 0 \\ x^2y^2 & xy^3 & y^4 & 0 \\ x^3y^3 & x^2y^4 & xy^5 & y^6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{x}{y} & 1 & 0 \\ 0 & \frac{x^2}{y^2} & \frac{x}{y} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{x}{y} & 1 \end{pmatrix}.$$

For the inverse of the extended generalized Pascal matrix $\Phi_n[x, y]$, by applying Theorem 2 and 3, we get

THEOREM 4.

$$\Phi_n^{-1}[x, y] = \Phi_n\left[-x, \frac{1}{y}\right] = \Phi_n\left[x, -\frac{1}{y}\right]$$
$$= F_1\left[\frac{x}{y}\right]F_2\left[\frac{x}{y}\right]\cdots F_{n-1}\left[\frac{x}{y}\right]U_n[x, y],$$

where

$$F_k[x] = G_k^{-1}[x] = \begin{pmatrix} I_{n-k-1} & 0 \\ 0 & D_k[x] \end{pmatrix}, \qquad k = 1, 2, \ldots, n-1,$$

and $F_n[x] = G_n^{-1}[x] = D_n[x]$.

In particular

$$\Phi_n^{-1}[x, y] = J_n[y]\Phi_n[x, y]J_n[y].$$

We define the extended generalized symmetric Pascal matrix $\Psi_n[x, y]$ as

$$\Psi_n(x, y; i, j) = x^{i-j}y^{i+j}\binom{i+j}{j}.$$

THEOREM 5. *One has*

$$F_1\left[\frac{x}{y}\right]F_2\left[\frac{x}{y}\right]\cdots F_{n-1}\left[\frac{x}{y}\right]U_n[x, y]\Psi_n[x, y] = P_n^T\left[\frac{y}{x}\right],$$

$$F_1\left[\frac{x}{y}\right]F_2\left[\frac{x}{y}\right]\cdots F_{n-1}\left[\frac{x}{y}\right]W_n\left[\frac{1}{x}, y\right]\Psi_n\left[\frac{1}{x}, y\right] = \Phi_n^T[y, x],$$

and the Cholesky factorization [4] *of* $\Psi_n[x, y]$ *is given by*

$$\Psi_n[x, y] = Q_n[xy]\Phi_n^T\left[y, \frac{1}{x}\right]$$

$$= \Phi_n[x, y]P_n^T\left[\frac{y}{x}\right].$$

Proof. Let $Q_n[xy]\Phi_n^T[y, 1/x] = (C_n(x, y; i, j))$. Then

$$C_n(x, y; i, j) = \begin{cases} \sum_{k=0}^{j}\binom{i}{k}\binom{j}{k}x^{i-j}y^{i+j}, & i \geqslant j, \\ \sum_{k=0}^{i}\binom{i}{k}\binom{j}{k}x^{i-j}y^{i+j}, & i < j, \end{cases}$$

$$\sum_{k=0}^{i}\binom{i}{k}\binom{j}{k} = \sum_{k=0}^{i}\binom{i}{k}\binom{j}{j-k} = \binom{i+j}{j},$$

$$\sum_{k=0}^{j}\binom{i}{k}\binom{j}{k} = \sum_{k=0}^{j}\binom{i}{i-k}\binom{j}{k} = \binom{i+j}{j},$$

(Vandermonde identities). Thus, we have

$$\Psi_n[x, y] = Q_n[xy]\Phi_n^T\left[y, \frac{1}{x}\right].$$

Similarly

$$\Psi_n[x, y] = \Phi_n[x, y]P_n^T\left[\frac{y}{x}\right]. \qquad \blacksquare$$

EXAMPLE.

$$\Psi_3[x, y] = \begin{pmatrix} 1 & \frac{y}{x} & \frac{y^2}{x^2} & \frac{y^3}{x^3} \\ xy & 2y^2 & 3\frac{y^3}{x} & 4\frac{y^4}{x^2} \\ x^2y^2 & 3xy^3 & 6y^4 & 10\frac{y^5}{x} \\ x^3y^3 & 4x^2y^4 & 10xy^5 & 20y^6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ xy & x^2y^2 & 0 & 0 \\ x^2y^2 & 2x^3y^3 & x^4y^4 & 0 \\ x^3y^3 & 3x^4y^4 & 3x^5y^5 & x^6y^6 \end{pmatrix} \begin{pmatrix} 1 & \frac{y}{x} & \frac{y^2}{x^2} & \frac{y^3}{x^3} \\ 0 & \frac{1}{x^2} & 2\frac{y}{x^3} & 3\frac{y^2}{x^4} \\ 0 & 0 & \frac{1}{x^4} & 3\frac{y}{x^5} \\ 0 & 0 & 0 & \frac{1}{x^6} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ xy & y^2 & 0 & 0 \\ x^2y^2 & 2xy^3 & y^4 & 0 \\ x^3y^3 & 3x^2y^4 & 2xy^5 & y^6 \end{pmatrix} \begin{pmatrix} 1 & \frac{y}{x} & \frac{y^2}{x^2} & \frac{y^3}{x^3} \\ 0 & 1 & 2\frac{y}{x} & 3\frac{y^2}{x^2} \\ 0 & 0 & 1 & 3\frac{y}{x} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By using Theorems 2 and 5, we have

THEOREM 6.

$$\Psi_n^{-1}[x, y] = P_n^T\left[-\frac{y}{x}\right]\Phi_n\left[x, -\frac{1}{y}\right] = \Phi_n^T[y, -x]Q_n\left[-\frac{1}{xy}\right].$$

Applying Theorem 4 and 5, we get

THEOREM 7.

$$\begin{aligned}\Psi_n^{-1}[x, y] &= J_n[1]P_n^T\left[\frac{y}{x}\right]J_n[1]J_n[y]\Phi_n[x, y]J_n[y]\\ &= J_n\left[\frac{1}{x}\right]\Phi_n\left[y, \frac{1}{x}\right]J_n\left[\frac{1}{x}\right]J_n[y]Q_n^T[xy]J_n[y].\end{aligned}$$

For the previous two kinds of extended generalized Pascal matrices, we also can get

THEOREM 8.

$$\begin{aligned}\det \Phi_n[x, y] &= y^{n(n+1)},\\ \det \Phi_n^{-1}[x, y] &= y^{-n(n+1)},\\ \det \Psi_n[x, y] &= y^{n(n+1)},\\ \det \Psi_n^{-1}[x, y] &= y^{-n(n+1)}.\end{aligned}$$

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