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Research Article

Determinant Representations of Polynomial Sequences of Riordan Type

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In this paper, using the production matrix of a Riordan array, we obtain a recurrence relation for polynomial sequence associated with the Riordan array, and we also show that the general term for the sequence can be expressed as the characteristic polynomial of the principal submatrix of the production matrix. As applications, a unified determinant expression for the four kinds of Chebyshev polynomials is given.

1. Introduction

The concept of a Riordan array is very useful in combinatorics. The infinite triangles of Pascal, Catalan, Motzkin, and Schröder are important and meaningful examples of Riordan array, and many others have been proposed and developed (see, e.g., [1–7]). In the recent literature, Riordan arrays have attracted the attention of various authors from many points of view and many examples and generalizations can be found (see, e.g., [8–12]).

A Riordan array denoted by $(g(t), f(t))$ is an infinite lower triangular matrix such that its column k $(k = 0, 1, 2, \ldots)$ has generating function $g(t)f(t)^k$, where $g(t) = \sum_{n=0}^{\infty} g_n t^n$ and $f(t) = \sum_{n=0}^{\infty} f_n t^n$ are formal power series with $g_0 = 1$, $f_0 = 0$, and $f_1 \neq 0$. That is, the general term of matrix $R = (g(t), f(t))$ is $r_{n,k} = [t^n]g(t)f(t)^k$; here $[t^n]h(t)$ denotes the coefficient of t^n in power series $h(t)$. Given a Riordan array $(g(t), f(t))$ and column vector $B = (b_0, b_1, b_2, \ldots)^T$, the product of $(g(t), f(t))$ and B gives a column vector whose generating function is $g(t)\, b(f(t))$, where $b(t) = \sum_{n=0}^{\infty} b_n t^n$. If we identify a vector with its ordinary generating function, the composition rule can be rewritten as

$$\left(g(t), f(t)\right) b(t) = g(t)\, b\left(f(t)\right). \tag{1}$$

This property is called the fundamental theorem for Riordan arrays and this leads to the matrix multiplication for Riordan arrays:

$$\left(g(t), f(t)\right)\left(h(t), l(t)\right) = \left(g(t)\, h\left(f(t)\right), l\left(f(t)\right)\right). \tag{2}$$

The set of all Riordan arrays forms a group under the previos operation of a matrix multiplication. The identity element of the group is $(1, t)$. The inverse element of $(g(t), f(t))$ is

$$\left(g(t), f(t)\right)^{-1} = \left(\frac{1}{g\left(\overline{f}(t)\right)}, \overline{f}(t)\right), \tag{3}$$

where $\overline{f}(t)$ is compositional inverse of $f(t)$.

A Riordan array $R = (g(t), f(t)) = (r_{n,k})_{n,k\geq 0}$ can be characterized by two sequences $A = (a_i)_{i\geq 0}$ and $Z = (z_i)_{i\geq 0}$ such that, for $n, k \geq 0$

$$\begin{aligned} r_{n+1,0} &= z_0 r_{n,0} + z_1 r_{n,1} + z_2 r_{n,2} + \cdots, \\ r_{n+1,k+1} &= a_0 r_{n,k} + a_1 r_{n,k+1} + a_2 r_{n,k+2} + \cdots. \end{aligned} \tag{4}$$

If $A(t)$ and $Z(t)$ are the generating functions for the A- and Z-sequences, respectively, then it follows that [9, 13]

$$g(t) = \frac{1}{1 - tZ\left(f(t)\right)}, \quad f(t) = tA\left(f(t)\right). \tag{5}$$

If the inverse of $R = (g(t), f(t))$ is $R^{-1} = (d(t), h(t))$, then the A- and Z-sequences of R are

$$A(t) = \frac{t}{h(t)}, \qquad Z(t) = \frac{1}{h(t)}(1 - d(t)). \tag{6}$$

For an invertible lower triangular matrix R, its production matrix (also called its Stieltjes matrix; see [11, 14]) is the matrix $P = R^{-1}\overline{R}$, where $\overline{R}$ is the matrix R with its first row removed. The production matrix P can be characterized by the matrix equality $RP = DR$, where $D = (\delta_{i+1,j})_{i,j\geq 0}$ (δ is the usual Kronecker delta).

Lemma 1 (see [14]). *Assume that $R = (r_{n,k})$ is an infinite lower triangular matrix with $r_{n,n} \neq 0$. Then R is a Riordan array if and only if its production matrix P is of the form*

$$P = \begin{pmatrix} z_0 & a_0 & 0 & 0 & 0 & 0 & \cdots \\ z_1 & a_1 & a_0 & 0 & 0 & 0 & \cdots \\ z_2 & a_2 & a_1 & a_0 & 0 & 0 & \cdots \\ z_3 & a_3 & a_2 & a_1 & a_0 & 0 & \cdots \\ z_4 & a_4 & a_3 & a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{7}$$

where $(a_0, a_1, a_2, \ldots)$ is the A-sequence and $(z_0, z_1, z_2, \ldots)$ is the Z-sequence of the Riordan array R.

Definition 2. Let $(r_n(x))_{n\geq 0}$ be a sequence of polynomials where $r_n(x)$ is of degree n and $r_n(x) = \sum_{k=0}^{n} r_{n,k}x^k$. We say that $(r_n(x))_{n\geq 0}$ is a polynomial sequence of Riordan type if the coefficient matrix $(r_{n,k})_{n,k\geq 0}$ is an element of the Riordan group; that is, there exists a Riordan array $(g(t), f(t))$ such that $(r_{n,k})_{n,k\geq 0} = (g(t), f(t))$. In this case, we say that $(r_n(x))_{n\geq 0}$ is the polynomial sequence associated with the Riordan array $(g(t), f(t))$.

Letting $r_n(x) = \sum_{k=0}^{n} r_{n,k}x^k$, $n \geq 0$, then in matrix form we have

$$\begin{pmatrix} r_{0,0} & 0 & 0 & 0 & \cdots \\ r_{1,0} & r_{1,1} & 0 & 0 & \cdots \\ r_{2,0} & r_{2,1} & r_{2,2} & 0 & \cdots \\ r_{3,0} & r_{3,1} & r_{3,2} & r_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ \vdots \end{pmatrix} = \begin{pmatrix} r_0(x) \\ r_1(x) \\ r_2(x) \\ r_3(x) \\ \vdots \end{pmatrix}. \tag{8}$$

Hence, by using (1), we have the following lemma.

Lemma 3. *Let $(r_n(x))_{n\geq 0}$ be the polynomial sequence associated with a Riordan array $(g(t), f(t))$, and let $r(t, x) = \sum_{n=0}^{\infty} r_n(x)t^n$ be its generating function. Then*

$$r(t, x) = \frac{g(t)}{1 - xf(t)}. \tag{9}$$

In [15], Luzón introduced a new notation $T(f \mid g)$ to represent the Riordan arrays and gave a recurrence relation for the family of polynomials associated to Riordan arrays. In recent works [16, 17], a new definition by means of a determinant form for Appell polynomials is given. Sequences of Appell polynomials are special of the Sheffer sequences [18]. In [19], the author obtains a determinant representation for the Sheffer sequence. The aim of this work is to propose a similar approach for polynomial sequences of Riordan type, which are special of the generalized Sheffer sequences [12, 18]. A determinant representation for polynomial sequences of Riordan type is obtained by using production matrix of Riordan array. In fact, we will show that the general formula for the polynomial sequences of Riordan type can be expressed as the characteristic polynomial of the principal submatrix of the production matrix. As applications, determinant expressions for some classical polynomial sequences such as Fibonacci, Pell, and Chebyshev are derived, and a unified determinant expression for the four kinds of Chebyshev polynomials [20, 21] is established.

2. Main Theorem

In this section we are going to develop our main theorem.

Theorem 4. *Let $R = (g(t), f(t))$ be a Riordan array with the Z-sequence $(z_i)_{i\geq 0}$ and the A-sequence $(a_i)_{i\geq 0}$. Let $(p_n(x))_{n\geq 0}$ be the polynomial sequence associated with R^{-1}. Then $(p_n(x))_{n\geq 0}$ satisfies the recurrence relation:*

$$\begin{aligned} a_0 p_n(x) = {} & x p_{n-1}(x) - a_1 p_{n-1}(x) - a_2 p_{n-2}(x) \\ & - \cdots - a_{n-1} p_1(x) - z_{n-1} p_0(x), \quad n > 1, \end{aligned} \tag{10}$$

with initial condition $p_0(x) = 1$, and $a_0 p_1(x) = x - z_0$. In general, for all $n \geq 1$, $p_n(x)$ is given by the following Hessenberg determinant:

$$\begin{aligned} & p_n(x) \\ & = (-1)^n a_0^{-n} \\ & \times \begin{vmatrix} z_0 - x & a_0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ z_1 & a_1 - x & a_0 & 0 & 0 & \cdots & 0 & 0 \\ z_2 & a_2 & a_1 - x & a_0 & 0 & \cdots & 0 & 0 \\ z_3 & a_3 & a_2 & a_1 - x & a_0 & \cdots & 0 & 0 \\ z_4 & a_4 & a_3 & a_2 & a_1 - x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ z_{n-2} & a_{n-2} & a_{n-3} & a_{n-4} & a_{n-5} & \cdots & a_1 - x & a_0 \\ z_{n-1} & a_{n-1} & a_{n-2} & a_{n-3} & a_{n-4} & \cdots & a_2 & a_1 - x \end{vmatrix}. \end{aligned} \tag{11}$$

Proof. Let $R = (g(t), f(t))$ and $R^{-1} = (g(t), f(t))^{-1} = (d(t), h(t))$. Then from definition and (3), we have $g(0) = 1$ and $d(t) = 1/g(\overline{f}(t))$. Hence $d(0) = 1$ and $p_0(x) = 1$. Letting $E = (1, x, x^2, \ldots)^T$, then $R^{-1}E = (p_0(x), p_1(x), p_2(x), \ldots)^T$ and $DE = (x, x^2, x^3, \ldots)^T$, where $D = (\delta_{i+1,j})$. Letting P be the production matrix of R, then $RP = DR$, and $PR^{-1} = R^{-1}D$. Thus $PR^{-1}E = R^{-1}DE$, and $PR^{-1}E = xR^{-1}E$. In matrix form, we have

$$\begin{pmatrix} z_0 & a_0 & 0 & 0 & 0 & \cdots \\ z_1 & a_1 & a_0 & 0 & 0 & \cdots \\ z_2 & a_2 & a_1 & a_0 & 0 & \cdots \\ z_3 & a_3 & a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ p_3(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} x p_0(x) \\ x p_1(x) \\ x p_2(x) \\ x p_3(x) \\ \vdots \end{pmatrix}. \tag{12}$$

Using the block matrix method, we get

$$p_0(x)\begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ \vdots \end{pmatrix} + \begin{pmatrix} a_0 & 0 & 0 & 0 & \cdots \\ a_1 & a_0 & 0 & 0 & \cdots \\ a_2 & a_1 & a_0 & 0 & \cdots \\ a_3 & a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \times \begin{pmatrix} p_1(x) \\ p_2(x) \\ p_3(x) \\ p_4(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} xp_0(x) \\ xp_1(x) \\ xp_2(x) \\ xp_3(x) \\ \vdots \end{pmatrix}. \tag{13}$$

Since

$$\begin{pmatrix} xp_0(x) \\ xp_1(x) \\ xp_2(x) \\ xp_3(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} xp_0(x) \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} + \begin{pmatrix} 0 \\ xp_1(x) \\ xp_2(x) \\ xp_3(x) \\ \vdots \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ xp_1(x) \\ xp_2(x) \\ xp_3(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ x & 0 & 0 & 0 & \cdots \\ 0 & x & 0 & 0 & \cdots \\ 0 & 0 & x & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_1(x) \\ p_2(x) \\ p_3(x) \\ p_4(x) \\ \vdots \end{pmatrix}. \tag{14}$$

The previous matrix equation can be rewritten as

$$\begin{pmatrix} a_0 & 0 & 0 & 0 & \cdots \\ a_1 - x & a_0 & 0 & 0 & \cdots \\ a_2 & a_1 - x & a_0 & 0 & \cdots \\ a_3 & a_2 & a_1 - x & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_1(x) \\ p_2(x) \\ p_3(x) \\ p_4(x) \\ \vdots \end{pmatrix} = p_0(x) \begin{pmatrix} x - z_0 \\ -z_1 \\ -z_2 \\ -z_3 \\ \vdots \end{pmatrix}. \tag{15}$$

Therefore, $a_0 p_1(x) = x - z_0$, and for $n > 1$, we have

$$a_{n-1}p_1(x) + \cdots + a_2 p_{n-2}(x) + (a_1 - x)\, p_{n-1}(x) + a_0 p_n(x) = -z_{n-1}p_0(x), \tag{16}$$

or equivalently

$$a_0 p_n(x) = (x - a_1)\, p_{n-1}(x) - a_2 p_{n-2}(x) - \cdots - a_{n-1}p_1(x) - z_{n-1}p_0(x). \tag{17}$$

By applying the Cramer's rule, we can work out the unknown $p_n(x)$ operating with the first n equations in (15):

$$p_n(x) = a_0^{-n} \begin{vmatrix} a_0 & 0 & 0 & 0 & 0 & \cdots & 0 & (x - z_0)\, p_0(x) \\ a_1 - x & a_0 & 0 & 0 & 0 & \cdots & 0 & -z_1 p_0(x) \\ a_2 & a_1 - x & a_0 & 0 & 0 & \cdots & 0 & -z_2 p_0(x) \\ a_3 & a_2 & a_1 - x & a_0 & 0 & \cdots & 0 & -z_3 p_0(x) \\ a_4 & a_3 & a_2 & a_1 - x & a_0 & \cdots & 0 & -z_4 p_0(x) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-2} & a_{n-3} & a_{n-4} & a_{n-5} & a_{n-6} & \cdots & a_0 & -z_{n-2} p_0(x) \\ a_{n-1} & a_{n-2} & a_{n-3} & a_{n-4} & a_{n-5} & \cdots & a_1 - x & -z_{n-1} p_0(x) \end{vmatrix} \tag{18}$$
$$= p_0(x)\, a_0^{-n} \begin{vmatrix} a_0 & 0 & 0 & 0 & 0 & \cdots & 0 & x - z_0 \\ a_1 - x & a_0 & 0 & 0 & 0 & \cdots & 0 & -z_1 \\ a_2 & a_1 - x & a_0 & 0 & 0 & \cdots & 0 & -z_2 \\ a_3 & a_2 & a_1 - x & a_0 & 0 & \cdots & 0 & -z_3 \\ a_4 & a_3 & a_2 & a_1 - x & a_0 & \cdots & 0 & -z_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-2} & a_{n-3} & a_{n-4} & a_{n-5} & a_{n-6} & \cdots & a_0 & -z_{n-2} \\ a_{n-1} & a_{n-2} & a_{n-3} & a_{n-4} & a_{n-5} & \cdots & a_1 - x & -z_{n-1} \end{vmatrix}.$$

After transferring the last column to the first position, an operation which introduces the factor $(-1)^{n-1}$, the theorem follows. □

Corollary 5. *Let $R = (g(t), f(t))$ be a Riordan array with production matrix P. Let $(p_n(x))_{n\geq 0}$ be the polynomial sequence associated with $R^{-1} = (g(t), f(t))^{-1}$. Then $p_0(x) = 1$, and for all $n \geq 1$,*

$$p_n(x) = a_0^{-n} \det(xI_n - P_n), \tag{19}$$

where $a_0 = p_{0,1}$, P_n is the principal submatrix of order n of the production matrix P and I_n is the identity matrix of order n.

3. Applications

A useful application of Theorem 4 is to find the determinant expression of a well-known sequence. We illustrate the ideal in the following examples. In the final paragraph, we will give a unified determinant expression for the four kinds of Chebyshev polynomials.

Example 6. Considering the Riordan array $\mathscr{A} = (1/(1 + rt^2), at/(1 + rt^2))$, we have $(1/(1 + rt^2), at/(1 + rt^2))(1/(1 - xt)) = 1/(1 - axt + rt^2)$. The generating functions of the A- and Z-sequences of $\mathscr{A}^{-1}$ are

$$A(t) = \frac{1 + rt^2}{a}, \qquad Z(t) = \frac{rt}{a}. \tag{20}$$

Let $(p_n(x))_{n\geq 0}$ be the polynomial sequence associated with $\mathscr{A} = (1/(1 + rt^2), at/(1 + rt^2))$. Then $(p_n(x))_{n\geq 0}$ satisfies the recurrence relation:

$$p_n(x) = axp_{n-1}(x) - rp_{n-2}(x), \quad n \geq 2, \tag{21}$$

with initial condition $p_0(x) = 1$, and $p_1(x) = ax$. In general, $p_n(x)$ is also given by the following Hessenberg determinant:

$$p_n(x) = (-1)^n \begin{vmatrix} -ax & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ r & -ax & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & r & -ax & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & r & -ax & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & r & -ax & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -ax \end{vmatrix}. \tag{22}$$

If $a = 1$, $r = -1$, then $p_n(x)$ becomes the Fibonacci polynomials:

$$F_n(x) = (-1)^n \begin{vmatrix} -x & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -1 & -x & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -x & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & -x & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -1 & -x & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -x \end{vmatrix}. \tag{23}$$

If $a = 2, r = -1$, then $p_n(x)$ gives the Pell polynomials:

$$P_n(x) = F_n(2x)$$

$$= (-1)^n \begin{vmatrix} -2x & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -1 & -2x & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -2x & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & -2x & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -1 & -2x & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -2x \end{vmatrix}. \tag{24}$$

In case $a = 2$, $r = 1$, $p_n(x)$ becomes the Chebyshev polynomials of the second kind:

$$U_n(x) = (-1)^n \begin{vmatrix} -2x & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -2x & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -2x & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -2x & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & -2x & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -2x \end{vmatrix}. \tag{25}$$

Example 7. Considering the Riordan array $\mathscr{B} = ((1-bt^2)/(1+rt^2), at/(1 + rt^2))$, we have $((1 - bt^2)/(1 + rt^2), at/(1 + rt^2))(1/(1-xt)) = (1-bt^2)/(1-axt+rt^2)$. Then the generating functions of the A- and Z-sequences of $\mathscr{B}^{-1}$ are

$$A(t) = \frac{1 + rt^2}{a}, \qquad Z(t) = \frac{(b + r)t}{a}. \tag{26}$$

Let $(p_n(x))_{n\geq 0}$ be the polynomial sequence associated with $\mathscr{B} = ((1 - bt^2)/(1 + rt^2), at/(1 + rt^2))$. Then $(p_n(x))_{n\geq 0}$ satisfies the recurrence relation:

$$p_n(x) = axp_{n-1}(x) - rp_{n-2}(x), \quad n \geq 3, \tag{27}$$

with initial condition $p_0(x) = 1$, and $p_1(x) = ax$, $p_2(x) = a^2x^2 - b - r$.

In general, $p_n(x)$ is also given by the following Hessenberg determinant:

$$p_n(x) = (-1)^n \begin{vmatrix} -ax & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ b + r & -ax & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & r & -ax & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & r & -ax & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & r & -ax & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -ax \end{vmatrix}. \tag{28}$$

If $a = 2$, $b = r = 1$, then $p_n(x)$ become the Chebyshev polynomials of the first kind $2T_n(x) - 0^n$:

$$2T_n(x) = (-1)^n \begin{vmatrix} -2x & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 2 & -2x & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -2x & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -2x & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & -2x & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -2x \end{vmatrix}. \tag{29}$$

In case $a = 3$, $b = 0$, $r = 2$, $p_n(x)$ give the Fermat polynomials (see [15]):

$$f_n(x) = (-1)^n \begin{vmatrix} -3x & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 2 & -3x & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 2 & -3x & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & -3x & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 2 & -3x & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -3x \end{vmatrix}. \tag{30}$$

Example 8. Considering the Riordan array $\mathscr{C} = ((1-bt)/(1+rt^2), at/(1+rt^2))$, we have $((1-bt)/(1+rt^2), at/(1+rt^2))(1/(1-xt)) = (1-bt)/(1-axt+rt^2)$. The generating functions of the A- and Z-sequences of $\mathscr{C}^{-1}$ are

$$A(t) = \frac{1+rt^2}{a}, \qquad Z(t) = \frac{b+rt}{a}. \tag{31}$$

Let $(p_n(x))_{n\geq 0}$ be the polynomial sequence associated with $\mathscr{C} = ((1-bt)/(1+rt^2), at/(1+rt^2))$. Then $(p_n(x))_{n\geq 0}$ satisfies the recurrence relation:

$$p_n(x) = axp_{n-1}(x) - rp_{n-2}(x), \quad n \geq 3, \tag{32}$$

with initial condition $p_0(x) = 1$, and $p_1(x) = ax - b$, $p_2(x) = a^2x^2 + abx - r$.

In general, $p_n(x)$ is also given by the following Hessenberg determinant:

$$p_n(x) = (-1)^n \begin{vmatrix} b-ax & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ r & -ax & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & r & -ax & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & r & -ax & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & r & -ax & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -ax \end{vmatrix}. \tag{33}$$

If $a = 2$, $b = 1$, $r = 1$, then $p_n(x)$ becomes the Chebyshev polynomials of the third kind $V_n(x)$:

$$V_n(x) = (-1)^n \begin{vmatrix} 1-2x & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -2x & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -2x & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -2x & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & -2x & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -2x \end{vmatrix}. \tag{34}$$

If $a = 2$, $b = -1$, $r = 1$, then $p_n(x)$ gives the Chebyshev polynomials of the fourth kind $W_n(x)$:

$$W_n(x) = (-1)^n \begin{vmatrix} -1-2x & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -2x & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -2x & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -2x & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & -2x & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -2x \end{vmatrix}. \tag{35}$$

Finally, considering the Riordan array $\mathscr{D} = ((1-bt-ct^2)/(1+t^2), at/(1+t^2))$, we have $((1-bt-ct^2)/(1+t^2), at/(1+t^2))(1/(1-xt)) = (1-bt-ct^2)/(1-axt+t^2)$. Then the generating functions of the A- and Z-sequences of $\mathscr{D}^{-1}$ are

$$A(t) = \frac{1+t^2}{a}, \qquad Z(t) = \frac{b+(c+1)t}{a}. \tag{36}$$

Let $(p_n(x))_{n\geq 0}$ be the polynomial sequence associated with $\mathscr{D} = ((1-bt-ct^2)/(1+t^2), at/(1+t^2))$. Then $(p_n(x))_{n\geq 0}$ satisfies the recurrence relation:

$$p_n(x) = axp_{n-1}(x) - p_{n-2}(x), \quad n \geq 3, \tag{37}$$

with initial condition $p_0(x) = 1$, and $p_1(x) = ax - b$, $p_2(x) = axp_1(x) - (c+1)p_0(x) = a^2x^2 - abx - c - 1$. For $n \geq 1$, we have

$$p_n(x) = (-1)^n \begin{vmatrix} b-ax & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ c+1 & -ax & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -ax & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -ax & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & -ax & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -ax \end{vmatrix}. \tag{38}$$

Therefore we can give, now, the following.

Definition 9. The Chebyshev polynomial of degree n, denoted by $C_n(x, a, b, c)$, is defined by

$$C_0(x, a, b, c) = 1,$$

$$C_n(x, a, b, c) = (-1)^n \begin{vmatrix} b-ax & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ c+1 & -ax & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -ax & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -ax & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & -ax & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -ax \end{vmatrix}, \quad n \geq 1, \tag{39}$$

where $C_n(x, a, b, c)$ is represented by a Hessenberg determinant of order n.

Note that $C_n(x, 2, 0, 1) = 2T_n(x)$, $C_n(x, 2, 0, 0) = U_n(x)$, $C_n(x, 2, 1, 0) = V_n(x)$, and $C_n(x, 2, -1, 0) = W_n(x)$. Hence, Definition 9 can be considered as a unified form for the four kinds of Chebyshev polynomials.

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