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Generalized Riordan arrays

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Abstract

In this paper, we generalize the concept of Riordan array. A generalized Riordan array with respect to c_n is an infinite, lower triangular array determined by the pair $(g(t), f(t))$ and has the generic element $d_{n,k} = [t^n/c_n]g(t)(f(t))^k/c_k$, where c_n is a fixed sequence of non-zero constants with $c_0 = 1$.

We demonstrate that the generalized Riordan arrays have similar properties to those of the classical Riordan arrays. Based on the definition, the iteration matrices related to the Bell polynomials are special cases of the generalized Riordan arrays and the set of iteration matrices is a subgroup of the Riordan group. We also study the relationships between the generalized Riordan arrays and the Sheffer sequences and show that the Riordan group and the group of Sheffer sequences are isomorphic. From the Sheffer sequences, many special Riordan arrays are obtained. Additionally, we investigate the recurrence relations satisfied by the elements of the Riordan arrays. Based on one of the recurrences, some matrix factorizations satisfied by the Riordan arrays are presented. Finally, we give two applications of the Riordan arrays, including the inverse relations problem and the connection constants problem.

Keywords: Riordan arrays; Sheffer sequences; Iteration matrices; Matrix factorizations; Inverse relations; Connection constants problem; Combinatorial identities

1. Introduction

The central concepts in this article are Riordan arrays and Sheffer sequences. Let us first make a brief introduction.

In 1978, Rogers [26] introduced the renewal array, which is a generalization of the Pascal, Catalan and Motzkin triangles. Kettle [16] used the theory of renewal arrays to study other types of combinatorial triangles, especially those found in walk problems. In 1991, Shapiro et al. [32] further generalized the same concept under the name of Riordan array, gave a clear formulation of the theory of Riordan arrays and presented many applications. Sprugnoli [35,36] also investigated the Riordan arrays and showed that they constitute a practical device for solving combinatorial sums by means of the generating functions and the Lagrange inversion formula. In the following days, many works on the Riordan arrays have been done, for example [9,13,14,17–22,24,31,44,46]. From the works referred to above, we can see that the theory of Riordan arrays is indeed a powerful tool to study combinatorial sums and special sequences.

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The Sheffer sequence is a very general concept and includes many polynomial sequences as its special cases. There are also several similar concepts in the literature, such as sequences of Sheffer A-type zero [33,34] and generalized Appell sequences [3,5–7], and in the present paper, we will follow the definitions of Rota and Roman. In [28–30], Rota and Roman et al. developed the theory of modern umbral calculus and studied the Sheffer sequences systematically by the umbral method. In these papers, a polynomial sequence $s_n(x)$ is *Sheffer* if and only if the generating function of $s_n(x)$ has the following form

$$A(t)\mathrm{e}^{xB(t)} = \sum_{k=0}^{\infty} \frac{s_k(x)}{k!} t^k.$$

By the definition, many well-known polynomial sequences are Sheffer, for example, the Hermite polynomials, the generalized Bernoulli and Euler polynomials and the Laguerre polynomials. In [27], Roman further developed the theory of umbral calculus and generalized the concept of Sheffer sequence. In this paper, $s_n(x)$ is a *generalized Sheffer sequence* if and only if the generating function has the form

$$A(t)\varepsilon_x(B(t)) = \sum_{k=0}^{\infty} \frac{s_k(x)}{c_k} t^k, \tag{1.1}$$

where $\varepsilon_x(t) = \sum_{k=0}^{\infty} x^k t^k / c_k$ is a generalization of the exponential series. According to the definition (1.1), more special polynomial sequences are included, such as the Gegenbauer polynomials, the Chebyshev polynomials and the Jacobi polynomials. The reader is referred to [27–30] for more Sheffer sequences and their properties.

The connection between the Riordan arrays and the Sheffer sequences has already been pointed out by Shapiro et al. [32] and Sprugnoli [13,35,36]. In fact, the classical Riordan arrays studied by Shapiro and Sprugnoli et al. are related to the 1-umbral calculus and thus related to the Sheffer sequence defined by (1.1) where $c_k = 1$. In the present paper, we will introduce the concept of generalized Riordan array, and give explicitly the relationships between the generalized Riordan arrays and the generalized Sheffer sequences defined by (1.1). Moreover, we will consider some properties and applications of the generalized Riordan arrays.

This article is organized as follows. In Section 2, we will introduce the concepts of generalized Riordan array and Riordan group. Based on the definitions, some properties will be demonstrated, and it will also be shown that the iteration matrices related to the Bell polynomials are special cases of the generalized Riordan arrays and the set of iteration matrices is a subgroup of the Riordan group. Section 3 is devoted to the relationships between the Riordan arrays and the generalized Sheffer sequences, and we can see that the Riordan group and the group of Sheffer sequences are in fact isomorphic. Based on the studies of these two sections, in Section 4, we give some special Riordan arrays from the Sheffer sequences. In Section 5, we concentrate on the recurrence relations satisfied by the elements of the Riordan arrays, and from one of these recurrences we construct some matrix factorizations for the Riordan arrays. Finally, in Section 6, we present two applications, including the inverse relations problem and the connection constants problem. Some results of the present paper can be obtained by other methods but they are given in order to show the power of the theory of Riordan arrays.

2. Riordan arrays and Riordan group

Since formal series play a predominant role in the present paper, we would like to introduce some basic definitions first. For more details of formal power series, the reader is referred to the paper of Niven [23] and the book of Comtet [12, Section 1.12].

Let $\mathbb{C}$ be a field of characteristic zero. Let $\mathscr{F}$ be the set of all formal power series in the indeterminate t over $\mathbb{C}$. Thus an element of $\mathscr{F}$ has the form

$$f(t) = \sum_{k=0}^{\infty} a_k t^k,$$

where $a_k \in \mathbb{C}$ for all $k \in \mathbb{N}$, and $\mathbb{N} := \{0, 1, 2, \ldots\}$. The *order* $o(f(t))$ of a power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish. The series $f(t)$ has a multiplicative inverse, denoted by $f(t)^{-1}$ or $1/f(t)$, if and only if $o(f(t)) = 0$. We call such a series *invertible*. The series $f(t)$ has a compositional

inverse, denoted by $\bar{f}(t)$ and satisfying $f(\bar{f}(t)) = \bar{f}(f(t)) = t$, if and only if $o(f(t)) = 1$. We call any series with $o(f(t)) = 1$ a *delta series*.

If $(f_n)_{n\in\mathbb{N}}$ is a sequence of real numbers, the formal power series $f(t) = \sum_{k=0}^{\infty} f_k t^k/c_k$ is called the *generating function* of the sequence, where $(c_n)_{n\in\mathbb{N}}$ is a fixed sequence of non-zero constants with $c_0 = 1$, given once and for all. Particularly, $f(t)$ is the ordinary generating function if $c_n = 1$, and $f(t)$ is the exponential generating function if $c_n = n!$.

As usual, the notation $[t^n]$ stands for the "coefficient of" operator, and if $f(t) = \sum_{k=0}^{\infty} f_k t^k$, then $[t^n]f(t) = f_n$. Similarly, if $f(t) = \sum_{k=0}^{\infty} f_k t^k/c_k$, then $[t^n/c_n]f(t) = f_n$. It is easy to see that $[t^n/c_n]f(t) = c_n[t^n]f(t)$.

Definition 2.1. A *generalized Riordan array with respect to the sequence* c_n is a pair $(g(t), f(t))$ of formal power series, where $g(t) = \sum_{k=0}^{\infty} g_k t^k/c_k$ and $f(t) = \sum_{k=1}^{\infty} f_k t^k/c_k$ with $f_1 \neq 0$, i.e., $f(t)$ is a delta series. The Riordan array $(g(t), f(t))$ defines an infinite, lower triangular array $(d_{n,k})_{0\leq k\leq n<\infty}$ according to the rule:

$$d_{n,k} = \left[\frac{t^n}{c_n}\right] g(t)\frac{(f(t))^k}{c_k}, \tag{2.1}$$

where the functions $g(t)(f(t))^k/c_k$ are called the *column generating functions* of the Riordan array.

By the definition, the *classical Riordan arrays* introduced and studied by Shapiro et al. [32] and Sprugnoli [35] correspond to the case of $c_n = 1$, and the *exponential Riordan arrays* presented in [9,46] correspond to the case of $c_n = n!$.

One of the most important applications of the theory of Riordan arrays is to deal with the summation of the form $\sum_{k=0}^{n} d_{n,k}h_k$. To see this, the reader is referred to [35,36]. In the context of the generalized Riordan arrays, for the summation given above, we have the following theorem.

Theorem 2.2. *Let* $D = (g(t), f(t)) = (d_{n,k})_{n,k\in\mathbb{N}}$ *be a Riordan array with respect to* c_n *and let* $h(t) = \sum_{k=0}^{\infty} h_k t^k/c_k$ *be the generating function of the sequence* h_n*. Then we have*

$$\sum_{k=0}^{n} d_{n,k}h_k = \left[\frac{t^n}{c_n}\right] g(t)h(f(t)), \tag{2.2}$$

or equivalently,

$$(g(t), f(t)) * h(t) = g(t)h(f(t)).$$

Proof. Based on the definition, we have

$$\sum_{k=0}^{n} d_{n,k}h_k = \sum_{k=0}^{\infty}\left[\frac{t^n}{c_n}\right] g(t)\frac{(f(t))^k}{c_k}h_k = \left[\frac{t^n}{c_n}\right] g(t)\sum_{k=0}^{\infty} h_k\frac{(f(t))^k}{c_k} = \left[\frac{t^n}{c_n}\right] g(t)h(f(t)).$$

This completes the proof. □

Analogous to the classical case [35, Theorem 1.2], we can prove the next result, which is the inverse of Theorem 2.2.

Theorem 2.3. *Let* $D = (d_{n,k})_{0\leq k\leq n<\infty}$ *be an infinite triangle such that for every sequence* $(h_k)_{k\in\mathbb{N}}$ *we have* $\sum_{k=0}^{n} d_{n,k}h_k = [t^n/c_n]g(t)h(f(t))$*, where* $h(t) = \sum_{k=0}^{\infty} h_k t^k/c_k$ *is the generating function of the sequence* h_k *and* $g(t)$*,* $f(t)$ *are two formal power series not depending on* $h(t)$*. Then the triangle defined by the Riordan array* $(g(t), f(t))$ *coincides with* $(d_{n,k})_{n,k\in\mathbb{N}}$*.*

Proof. It is the same as that of [35, Theorem 1.2]. For any $k \in \mathbb{N}$, take the sequence which is 0 everywhere except in the kth element $h_k = 1$. Then $h(t) = \sum_{i=0}^{\infty} h_i t^i/c_i = t^k/c_k$ and $\sum_{i=0}^{n} d_{n,i}h_i = d_{n,k} = [t^n/c_n]g(t)(f(t))^k/c_k$, which proves the assertion of the theorem. □

With Theorem 2.2, we can further compute the product of two Riordan arrays $(g(t), f(t)) * (h(t), l(t))$. In fact, the column generating function of $(h(t), l(t))$ is $h(t)(l(t))^k/c_k$. Thus, by matrix multiplication, the column generating function of the product $(g(t), f(t)) * (h(t), l(t))$ is

$$g(t)h(f(t))\frac{(l(f(t)))^k}{c_k},$$

which means the product is also a Riordan array, i.e.,

$$(g(t), f(t)) * (h(t), l(t)) = (g(t)h(f(t)), l(f(t))). \tag{2.3}$$

Analogous to the classical case, for a fixed sequence c_n, the set of all Riordan arrays $(g(t), f(t))$ with $g(t)$ an invertible series is a group.

Theorem 2.4. *For any fixed sequence c_n, the set of all Riordan arrays $(g(t), f(t))$ with $g(t)$ an invertible series is a group under matrix multiplication. Moreover, the identity of this group is $(1, t)$ and the inverse of the array $(g(t), f(t))$ is $(1/g(\bar{f}(t)), \bar{f}(t))$, where $\bar{f}(t)$ is the compositional inverse of $f(t)$.*

Proof. Denote the set by R. Then R is closed under matrix multiplication according to (2.3) and the multiplication is associative. The array $(1, t)$ is an element of R and for each array $(g(t), f(t)) \in R$, there exists an array $(1/g(\bar{f}(t)), \bar{f}(t)) \in R$, for which we have

$$(g(t), f(t)) * (1, t) = (g(t), f(t)) = (1, t) * (g(t), f(t)),$$
$$(g(t), f(t)) * \left(\frac{1}{g(\bar{f}(t))}, \bar{f}(t)\right) = (1, t) = \left(\frac{1}{g(\bar{f}(t))}, \bar{f}(t)\right) * (g(t), f(t)).$$

Then R is a group and the proof is complete. □

The group introduced in Theorem 2.4 is called the *Riordan group with respect to c_n*. It should be noticed that, for any fixed sequence c_n, the identity $(1, t)$ of the Riordan group R is the usual infinite identity matrix I. Actually, by Eq. (2.1), the generic element of $(1, t)$ is

$$d_{n,k} = \left[\frac{t^n}{c_n}\right]\frac{t^k}{c_k} = \frac{c_n}{c_k}[t^{n-k}]1 = \delta_{n,k},$$

where $\delta_{n,k}$ is the *Kronecker delta* defined by $\delta_{n,n} = 1$ and $\delta_{n,k} = 0$ for $n \neq k$.

A large number of infinite lower triangular arrays are Riordan arrays. Particularly, the iteration matrices are in this case. With every formal power series $f(t) = \sum_{k=1}^{\infty} f_k t^k/c_k$, we associate the infinite lower *iteration matrix with respect to c_n* [12, p. 145]:

$$B(f(t)) := \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & B_{1,1} & 0 & 0 & \cdots \\ 0 & B_{2,1} & B_{2,2} & 0 & \cdots \\ 0 & B_{3,1} & B_{3,2} & B_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $B_{n,k} = B_{n,k}(f_1, f_2, \ldots)$ is the Bell polynomial with respect to c_n, defined as follows:

$$\frac{1}{c_k}(f(t))^k = \sum_{n=k}^{\infty} B_{n,k}\frac{t^n}{c_n}. \tag{2.4}$$

Therefore, $B_{n,k} = [t^n/c_n](f(t))^k/c_k$ which implies that the iteration matrix $B(f(t))$ is the Riordan array $(1, f(t))$. Now, the following important property of the iteration matrix [12, p. 145, Theorem A]

$$B(f(g(t))) = B(g(t)) * B(f(t))$$

is trivial in the context of the theory of Riordan arrays, i.e.,

$$(1, f(g(t))) = (1, g(t)) * (1, f(t));$$

and the well-known Faà di Bruno formula [12, p. 137, Theorem A] is a specialization of the summation rule (2.2):

$$\sum_{k=1}^{n} B_{n,k}(g_1, g_2, \ldots, g_{n-k+1}) f_k = \left[\frac{t^n}{n!}\right] f(g(t)).$$

Additionally, for any delta series $f(t)$, $(1, \bar{f}(t)) = B(\bar{f}(t))$ is also an iteration matrix. Thus, the set of iteration matrices with respect to c_n, denoted by B, is a nonempty subset of the Riordan group R with respect to c_n, closed under multiplication and taking inverses in R. These indicate that B is a subgroup of R and we call it the *associated subgroup*.

It can be shown that the set of Riordan arrays which have the form $(g(t), t)$, where $g(t)$ is an invertible series, is also a subgroup of R. We call it the *Appell subgroup* and denote it by A. Since

$$(g(t), f(t)) = (g(t), t) * (1, f(t)) = (1, f(t)) * (g(\bar{f}(t)), t),$$

then we have $AB = BA = R$. The reader can see the paper [31] by Shapiro for more subgroups.

3. Riordan arrays and Sheffer sequences

The Riordan arrays determined by an invertible series and a delta series play a very important role in the present paper, and in this section, we will consider the relationships between such Riordan arrays and the Sheffer sequences.

Definition 3.1 (*[27, Theorem 5.3]*). Let $f(t)$ be a delta series and let $g(t)$ be an invertible series; we say that the sequence $s_n(x)$ is *Sheffer for* the pair $(g(t), f(t))$ if and only if

$$\sum_{k=0}^{\infty} s_k(x)\frac{t^k}{c_k} = \frac{1}{g(\bar{f}(t))}\varepsilon_x(\bar{f}(t)), \tag{3.1}$$

where $\varepsilon_x(t) = \sum_{k=0}^{\infty} x^k t^k/c_k$ is the generalized exponential series ($\varepsilon_x(t) = \mathrm{e}^{xt}$ for $c_n = n!$ and $\varepsilon_x(t) = 1/(1-xt)$ for $c_n = 1$). Particularly, the Sheffer sequence for $(1, f(t))$ is called the *associated sequence* for $f(t)$, and the generating function (3.1) reduces to

$$\sum_{k=0}^{\infty} s_k(x)\frac{t^k}{c_k} = \varepsilon_x(\bar{f}(t));$$

the Sheffer sequence for $(g(t), t)$ is called the *Appell sequence* for $g(t)$, and the generating function (3.1) reduces to

$$\sum_{k=0}^{\infty} s_k(x)\frac{t^k}{c_k} = \frac{1}{g(t)}\varepsilon_x(t).$$

By Definitions 2.1 and 3.1, the following theorem can be established.

Theorem 3.2. *For any fixed sequence c_n, if $d_{n,k}$ is the generic element of the Riordan array $(g(t), f(t))$, then the polynomial sequence $\sum_{k=0}^{n} d_{n,k}x^k$ is Sheffer for $(1/g(\bar{f}(t)), \bar{f}(t))$. Conversely, if the sequence $s_n(x) = \sum_{k=0}^{n} s_{n,k}x^k$ is Sheffer for $(g(t), f(t))$, then the coefficient $s_{n,k}$ is the generic element of the Riordan array $(1/g(\bar{f}(t)), \bar{f}(t))$.*

Proof. From the definition, we have $\sum_{n=k}^{\infty} d_{n,k}t^n/c_n = g(t)(f(t))^k/c_k$. Thus

$$\begin{aligned}\sum_{k=0}^{\infty}\left(\sum_{n=k}^{\infty} d_{n,k}\frac{t^n}{c_n}\right)x^k &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} d_{n,k}x^k\right)\frac{t^n}{c_n}\\ &= \sum_{k=0}^{\infty} g(t)\frac{(f(t))^k}{c_k}x^k = g(t)\sum_{k=0}^{\infty}\frac{x^k}{c_k}(f(t))^k = g(t)\varepsilon_x(f(t)).\end{aligned}$$

Therefore, the first statement of the theorem can be proved by means of (3.1). To prove the second statement we observe that

$$\sum_{n=0}^{\infty} s_n(x)\frac{t^n}{c_n} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} s_{n,k}x^k\right)\frac{t^n}{c_n} = \sum_{k=0}^{\infty}\left(\sum_{n=k}^{\infty} s_{n,k}\frac{t^n}{c_n}\right)x^k = \frac{1}{g(\bar{f}(t))}\sum_{k=0}^{\infty}(\bar{f}(t))^k\frac{x^k}{c_k}.$$

By equating the coefficients of x^k in the last equation, we have

$$\sum_{n=k}^{\infty} s_{n,k}\frac{t^n}{c_n} = \frac{1}{g(\bar{f}(t))}\frac{(\bar{f}(t))^k}{c_k}$$

and then $s_{n,k}$ is the generic element of the Riordan array $(1/g(\bar{f}(t)), \bar{f}(t))$. □

With some specializations, we can obtain from Theorem 3.2 the relationship between the iteration matrices and the associated sequences, which has already been indicated by Roman [28, Section 4.1.8] for the case $c_n = n!$.

As an example of Theorem 3.2, consider the classical Riordan array $\left(\frac{1}{1-t}, \frac{t}{1-t}\right)$. The generic element is $[t^n]\frac{1}{1-t}\left(\frac{t}{1-t}\right)^k = \binom{n}{k}$, then $\left(\frac{1}{1-t}, \frac{t}{1-t}\right)$ is the well-known Pascal matrix. The corresponding row generating functions are $(x+1)^n$, which form the Sheffer sequence for $\left(\frac{1}{1+t}, \frac{t}{1+t}\right)$. The inverse of the array $\left(\frac{1}{1-t}, \frac{t}{1-t}\right)$ is $\left(\frac{1}{1+t}, \frac{t}{1+t}\right)$, whose generic element is $(-1)^{n-k}\binom{n}{k}$. Now, the row generating functions are $(x-1)^n$, which form the Sheffer sequence for $\left(\frac{1}{1-t}, \frac{t}{1-t}\right)$.

As another example, let us consider the exponential Riordan array $(1, \log(1+t))$. The generic element is $[t^n/n!](\log(1+t))^k/k! = s(n,k)$, the Stirling number of the first kind [12, p. 212]; and the row generating functions are $\sum_{k=0}^{n} s(n,k)x^k = (x)_n$, which are lower factorial polynomials defined by $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$ and form the sequence associated to $\mathrm{e}^t - 1$ [28, Section 4.1.2]. The inverse of the array $(1, \log(1+t))$ is $(1, \mathrm{e}^t - 1)$, whose generic element is $[t^n/n!](\mathrm{e}^t-1)^k/k! = S(n,k)$, the Stirling number of the second kind [12, p. 206]. $\sum_{k=0}^{n} S(n,k)x^k$ are called the exponential polynomials and denoted by $\phi_n(x)$. The sequence $\phi_n(x)$ is associated to $\log(1+t)$ [28, Section 4.1.3]. More Riordan arrays can be found in the next section.

It is instructive to further study the relationship between the Riordan arrays and the Sheffer sequences, from the group point of view.

If $p_n(x)$ and $q_n(x) = \sum_{k=0}^{n} q_{n,k}x^k$ are two sequences of polynomials, then the *umbral composition* of $q_n(x)$ with $p_n(x)$ is the sequence

$$q_n(x) \circ p_n(x) = q_n(\mathbf{p}(x)) = \sum_{k=0}^{n} q_{n,k}\, p_k(x).$$

Roman [28, Theorem 3.5.5] (see also [30, p. 708, Theorem 7]) shows that in the case of $c_n = n!$, the set of Sheffer sequences is a group under umbral composition. Particularly, if $s_n(x)$ is Sheffer for $(g(t), f(t))$ and $r_n(x)$ is Sheffer for $(h(t), l(t))$, then $r_n(\mathbf{s}(x))$ is Sheffer for $(g(t)h(f(t)), l(f(t)))$. In the general case, the composition rule still holds [27, Theorem 8.4] and following the methods developed in [28, Section 3.5] by Roman, we can prove that the set of Sheffer sequences is also a group with the set of associated sequences and the set of Appell sequences its subgroups. Moreover, in view of (2.3), it can be readily found that the umbral composition and the Riordan array product formally follow the same rule. This fact implies the next result.

Theorem 3.3. *For any fixed sequence c_n, the Riordan group R and the group of Sheffer sequences S are isomorphic.*

Proof. Define $\sigma : R \to S$ by $\sigma(g(t), f(t)) = s_n(x)$, where $s_n(x)$ is Sheffer for $(1/g(\bar{f}(t)), \bar{f}(t))$. Because $\bar{f}(t)$ is uniquely determined by $f(t)$ and $s_n(x)$ is uniquely determined by the pair $(1/g(\bar{f}(t)), \bar{f}(t))$ [27, Theorem 5.1], the map σ is well defined. Now, let us prove that σ is an isomorphism.

In fact, if $s_n(x)$ is Sheffer for $(h(t), l(t))$, then there exists a Riordan array $(1/h(\bar{l}(t)), \bar{l}(t))$ such that $\sigma(1/h(\bar{l}(t)), \bar{l}(t)) = s_n(x)$. This indicates that σ is surjective. Next, suppose

$$\sigma(g_1(t), f_1(t)) = \sigma(g_2(t), f_2(t)) = s_n(x),$$

where $s_n(x)$ is Sheffer for $(h(t), l(t))$. Then $\bar{f}_1(t) = l(t) = \bar{f}_2(t)$ and $f_1(t) = \bar{l}(t) = f_2(t)$. Additionally, we have $g_1(\bar{f}_1(t)) = 1/h(t) = g_2(\bar{f}_2(t))$ which leads us to the fact that

$$g_1(\bar{f}_1(f_1(t))) = g_1(t) = \frac{1}{h(\bar{l}(t))} = g_2(\bar{f}_2(f_2(t))) = g_2(t).$$

Therefore, $(g_1(t), f_1(t)) = (g_2(t), f_2(t))$ and σ is injective.

We have shown that σ is a bijection, and it remains to check that σ preserves the group operation. To do this, suppose $\sigma(g(t), f(t)) = q_n(x)$ where $q_n(x)$ is Sheffer for $(1/g(\bar{f}(t)), \bar{f}(t))$, and $\sigma(h(t), l(t)) = p_n(x)$ where $p_n(x)$ is Sheffer for $(1/h(\bar{l}(t)), \bar{l}(t))$. By the umbral composition, we have

$$\sigma(g(t), f(t)) \circ \sigma(h(t), l(t)) = q_n(x) \circ p_n(x) = q_n(\mathbf{p}(x)),$$

where the sequence $q_n(\mathbf{p}(x))$ is Sheffer for

$$\left(\frac{1}{h(\bar{l}(t))g(\bar{f}(\bar{l}(t)))}, \bar{f}(\bar{l}(t))\right).$$

On the other hand, by Eq. (2.3) we have

$$\sigma((g(t), f(t)) * (h(t), l(t))) = \sigma(g(t)h(f(t)), l(f(t))) = s_n(x),$$

where the sequence $s_n(x)$ is Sheffer for

$$\left(\frac{1}{g(\bar{f}(\bar{l}(t)))h(\bar{l}(t))}, \bar{f}(\bar{l}(t))\right).$$

Thus, $q_n(\mathbf{p}(x)) = s_n(x)$ which shows that

$$\sigma((g(t), f(t)) * (h(t), l(t))) = \sigma(g(t), f(t)) \circ \sigma(h(t), l(t)).$$

Therefore, σ is indeed an isomorphism and $R \cong S$. □

Theorem 3.4. *The associated subgroup of R (i.e., the group of iteration matrices) and the group of associated sequences are isomorphic. The Appell subgroup of R and the group of Appell sequences are isomorphic.*

4. Special Riordan arrays from Sheffer sequences

According to Theorem 3.2, the Riordan arrays can be obtained from the Sheffer sequences, and in this section, we will present some Riordan arrays in this way. The Sheffer sequences used here can be found in the works of Roman [27,28].

4.1. The case of $c_n = n!$

4.1.1. The Hermite polynomials [28, Section 4.2.1]

The Hermite polynomials $H_n^{(v)}(x)$ form the Sheffer sequence for the pair $(\exp(vt^2/2), t)$. By Theorem 3.2, $H_{n,k}^{(v)} := [x^k]H_n^{(v)}(x)$ is the generic element of the Riordan array $(\exp(-vt^2/2), t)$:

$$H_{n,k}^{(v)} = \left[\frac{t^n}{n!}\right] \mathrm{e}^{-\frac{vt^2}{2}} \frac{t^k}{k!} = \frac{n!}{k!}[t^{n-k}]\mathrm{e}^{-\frac{vt^2}{2}} = \begin{cases} 0, & n-k \text{ odd}, \\ \dfrac{n!}{k!}\dfrac{(-\frac{v}{2})^{\frac{n-k}{2}}}{(\frac{n-k}{2})!}, & n-k \text{ even}. \end{cases} \tag{4.1}$$

Therefore, the explicit expression of the Hermite polynomials is

$$H_n^{(v)}(x) = \sum_{\substack{k=0 \\ n-k \text{ even}}}^{n} \frac{n!}{k!}\frac{(-\frac{v}{2})^{\frac{n-k}{2}}}{(\frac{n-k}{2})!}x^k = \sum_{\substack{k=0 \\ k \text{ even}}}^{n} \frac{n!}{(n-k)!}\frac{(-\frac{v}{2})^{\frac{k}{2}}}{(\frac{k}{2})!}x^{n-k} = \sum_{k=0}^{\left[\frac{n}{2}\right]} \left(-\frac{v}{2}\right)^k \frac{(n)_{2k}}{k!}x^{n-2k}.$$

The inverse of the Riordan array $(\exp(-vt^2/2), t)$ is $(\exp(vt^2/2), t)$, whose generic element can be obtained easily from (4.1) by replacing v with $-v$.

4.1.2. The generalized Bernoulli polynomials [28, Section 4.2.2]

The generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ are Sheffer for $\left(\left(\frac{\mathrm{e}^t-1}{t}\right)^\alpha, t\right)$, then the corresponding Riordan array is $\left(\left(\frac{t}{\mathrm{e}^t-1}\right)^\alpha, t\right)$, whose generic element is

$$[x^k]B_n^{(\alpha)}(x) = \left[\frac{t^n}{n!}\right]\left(\frac{t}{\mathrm{e}^t-1}\right)^\alpha \frac{t^k}{k!} = \frac{n!}{k!}[t^{n-k}]\left(\frac{t}{\mathrm{e}^t-1}\right)^\alpha = \binom{n}{k} B_{n-k}^{(\alpha)},$$

where $B_k^{(\alpha)} := B_k^{(\alpha)}(0)$ are the generalized Bernoulli numbers. From the equation above, the generic element of

$$\left(\left(\frac{t}{\mathrm{e}^t-1}\right)^{\alpha},t\right)^{-1}=\left(\left(\frac{\mathrm{e}^t-1}{t}\right)^{\alpha},t\right)=\left(\left(\frac{t}{\mathrm{e}^t-1}\right)^{-\alpha},t\right)$$

is $\binom{n}{k}B_{n-k}^{(-\alpha)}$; while if we compute it by [12, p. 141, Theorem B], we have

$$\left[\frac{t^n}{n!}\right]\left(\frac{\mathrm{e}^t-1}{t}\right)^{\alpha}\frac{t^k}{k!}=\frac{n!}{k!}[t^{n-k}]\left(1+\sum_{i=1}^{\infty}\frac{1}{i+1}\frac{t^i}{i!}\right)^{\alpha}=\binom{n}{k}\sum_{i=0}^{n-k}(\alpha)_i B_{n-k,i}\left(\frac{1}{2},\frac{1}{3},\ldots\right), \tag{4.2}$$

where $B_{n,k}$ are the partial exponential Bell polynomials [12, p. 133]. Thus, the following expressions for the generalized Bernoulli polynomials can be derived:

$$B_n^{(\alpha)}(x)=\sum_{k=0}^{n}\binom{n}{k}B_{n-k}^{(\alpha)}x^k=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{n-k}(-\alpha)_i B_{n-k,i}\left(\frac{1}{2},\frac{1}{3},\ldots\right)x^k.$$

Moreover, by setting $\alpha=1$, we can obtain from (4.2) the Riordan array

$$\left(\frac{\mathrm{e}^t-1}{t},t\right)=\left(\binom{n}{k}\frac{1}{n-k+1}\right)_{n,k\in\mathbb{N}}.$$

4.1.3. The generalized Euler polynomials [28, Section 4.2.3]

The generalized Euler polynomials $E_n^{(\alpha)}(x)$ are Sheffer for $\left(\left(\frac{\mathrm{e}^t+1}{2}\right)^{\alpha},t\right)$, then the corresponding Riordan array is

$$\left(\left(\frac{2}{\mathrm{e}^t+1}\right)^{\alpha},t\right)=\left(\binom{n}{k}E_{n-k}^{(\alpha)}(0)\right)_{n,k\in\mathbb{N}},$$

and the inverse array is

$$\left(\left(\frac{\mathrm{e}^t+1}{2}\right)^{\alpha},t\right)=\left(\binom{n}{k}E_{n-k}^{(-\alpha)}(0)\right)_{n,k\in\mathbb{N}}=\left(\binom{n}{k}\sum_{i=0}^{n-k}(\alpha)_i 2^{-i}S(n-k,i)\right)_{n,k\in\mathbb{N}}. \tag{4.3}$$

In fact,

$$\begin{aligned}\left[\frac{t^n}{n!}\right]\left(\frac{\mathrm{e}^t+1}{2}\right)^{\alpha}\frac{t^k}{k!}&=\frac{n!}{k!}[t^{n-k}]\left(1+\sum_{i=1}^{\infty}\frac{1}{2}\frac{t^i}{i!}\right)^{\alpha}\\&=\binom{n}{k}\sum_{i=0}^{n-k}(\alpha)_i B_{n-k,i}\left(\frac{1}{2},\frac{1}{2},\ldots\right)=\binom{n}{k}\sum_{i=0}^{n-k}(\alpha)_i 2^{-i}S(n-k,i),\end{aligned}$$

where we have made use of the expression of the Bell polynomials [12, p. 134, Eq. (3d)] and the fact that $B_{n,k}(1,1,1,\ldots)=S(n,k)$ [12, p. 135, Eq. (3g)] in the last step. By means of (4.3), we obtain the formulae for the generalized Euler polynomials:

$$E_n^{(\alpha)}(x)=\sum_{k=0}^{n}\binom{n}{k}E_{n-k}^{(\alpha)}(0)x^k=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{n-k}(-\alpha)_i 2^{-i}S(n-k,i)x^k.$$

When $\alpha=1$, the generic element of the Riordan array $\left(\frac{\mathrm{e}^t+1}{2},t\right)$ is $\frac{1}{2}\binom{n}{k}$ for $n\neq k$ and 1 for $n=k$, which can be deduced directly from (4.3).

4.1.4. The Laguerre polynomials [28, Section 4.3.1]

The Laguerre polynomials $L_n^{(\alpha)}(x)$ form the Sheffer sequence for $((1-t)^{-\alpha-1},t/(t-1))$ and the corresponding Riordan array is $((1-t)^{-\alpha-1},t/(t-1))$. It is interesting to notice that the inverse of the Riordan array $((1-t)^{-\alpha-1},$

Table 1
Some exponential Riordan arrays and the corresponding Sheffer sequences

Riordan array	Generic element $d_{n,k}$	Sheffer sequence	Sheffer pair
$(1, \log(1+t))$	$s(n,k)$	Lower factorial $(x)_n$	$(1, \mathrm{e}^t - 1)$
$(1, \mathrm{e}^t - 1)$	$S(n,k)$	Exponential $\phi_n(x)$	$(1, \log(1+t))$
$(1, t\mathrm{e}^{at})$	$\binom{n}{k}(ak)^{n-k}$	$\sum_{k=0}^{n}\binom{n}{k}(ak)^{n-k}x^k$	$\left(1, \sum_{k=1}^{\infty}\frac{(-ak)^{k-1}}{k!}t^k\right)$
$\left(\mathrm{e}^{-\frac{vt^2}{2}}, t\right)$	$0, n-k$ odd $\frac{n!}{k!}\frac{(-\frac{v}{2})^{\frac{n-k}{2}}}{(\frac{n-k}{2})!}, n-k$ even	Hermite $H_n^{(v)}(x)$	$\left(\mathrm{e}^{\frac{vt^2}{2}}, t\right)$
$\left(\left(\frac{t}{\mathrm{e}^t-1}\right)^{\alpha}, t\right)$	$\binom{n}{k}B_{n-k}^{(\alpha)}$	Bernoulli $B_n^{(\alpha)}(x)$	$\left(\left(\frac{\mathrm{e}^t-1}{t}\right)^{\alpha}, t\right)$
$\left(\left(\frac{2}{\mathrm{e}^t+1}\right)^{\alpha}, t\right)$	$\binom{n}{k}E_{n-k}^{(\alpha)}(0)$	Euler $E_n^{(\alpha)}(x)$	$\left(\left(\frac{\mathrm{e}^t+1}{2}\right)^{\alpha}, t\right)$
$\left((1-t)^{-\alpha-1}, \frac{t}{t-1}\right)$	$(-1)^k\frac{n!}{k!}\binom{n+\alpha}{n-k}$	Laguerre $L_n^{(\alpha)}(x)$	$\left((1-t)^{-\alpha-1}, \frac{t}{t-1}\right)$
$\left(\frac{t}{\log(1+t)}, \log(1+t)\right)$	$b_n(0), k=0$ $\frac{n}{k}s(n-1,k-1), k\geq 1$	Bernoulli II $b_n(x)$	$\left(\frac{t}{\mathrm{e}^t-1}, \mathrm{e}^t-1\right)$
$\left(\mathrm{e}^{-t}, \log\left(1+\frac{t}{a}\right)\right)$	$\sum\binom{n}{j}(-1)^{n-j}a^{-j}s(j,k)$	Poisson–Charlier	$\left(\mathrm{e}^{a(\mathrm{e}^t-1)}, a(\mathrm{e}^t-1)\right)$
$\left(\mathrm{e}^{a(\mathrm{e}^t-1)}, a(\mathrm{e}^t-1)\right)$	$a^k\sum\binom{n}{j}\phi_{n-j}(a)S(j,k)$		$\left(\mathrm{e}^{-t}, \log\left(1+\frac{t}{a}\right)\right)$
$\left(\mathrm{e}^{\beta t}, 1-\mathrm{e}^t\right)$	$(-1)^k\sum\binom{n}{j}\beta^{n-j}S(j,k)$	Actuarial $a_n^{(\beta)}(x)$	$\left(\frac{1}{(1-t)^{\beta}}, \log(1-t)\right)$
$\left(\frac{1}{(1-t)^{\beta}}, \log(1-t)\right)$	$(-1)^n\sum\binom{n}{j}(-\beta)_{n-j}s(j,k)$		$\left(\mathrm{e}^{\beta t}, 1-\mathrm{e}^t\right)$

$t/(t-1))$ is just itself, that is, $((1-t)^{-\alpha-1}, t/(t-1))^2 = I$. Furthermore, the generic element is

$$[x^k]L_n^{(\alpha)}(x) = \left[\frac{t^n}{n!}\right]\frac{1}{k!}\left(\frac{-1}{t-1}\right)^{\alpha+1}\left(\frac{t}{t-1}\right)^k$$
$$= (-1)^k\frac{n!}{k!}[t^{n-k}](1-t)^{-(\alpha+k+1)} = (-1)^k\frac{n!}{k!}\binom{n+\alpha}{n-k}.$$

This implies that

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n}\frac{n!}{k!}\binom{n+\alpha}{n-k}(-x)^k.$$

For convenience, we list some exponential Riordan arrays and their corresponding Sheffer sequences in Table 1.

4.2. Other cases

4.2.1. The polynomials of the Gegenbauer case [27, p. 97]

In this case, the sequence c_n is taken as $1/\binom{-\lambda}{n}$. Let $s_n(x)$ be the Sheffer sequence for the pair $(g(t), f(t))$ where

$$g(t) = \left(\frac{2}{1+\sqrt{1-t^2}}\right)^{\lambda_0}, \qquad f(t) = \frac{-t}{1+\sqrt{1-t^2}}.$$

Then $\bar{f}(t) = -2t/(1+t^2)$ and $g(\bar{f}(t)) = (1+t^2)^{\lambda_0}$. The sequence $s_n(x)$ is related to the Gegenbauer polynomials and $[x^k]s_n(x)$ is the generic element of the Riordan array $R = ((1+t^2)^{-\lambda_0}, -2t/(1+t^2))$.

$$[x^k]s_n(x) = \left[\frac{t^n}{c_n}\right]\frac{(1+t^2)^{-\lambda_0}}{c_k}\left(\frac{-2t}{1+t^2}\right)^k$$

$$= \frac{c_n}{c_k}(-2)^k[t^{n-k}]\sum_{j=0}^{\infty}\binom{-\lambda_0-k}{j}t^{2j} = \begin{cases}0, & n-k \text{ odd},\\ \dfrac{c_n}{c_k}(-2)^k\binom{-\lambda_0-k}{\frac{n-k}{2}}, & n-k \text{ even},\end{cases}$$

from which the explicit expression of $s_n(x)$ can be obtained:

$$s_n(x) = \sum_{\substack{k=0\\ n-k \text{ even}}}^{n}\frac{\binom{-\lambda}{k}}{\binom{-\lambda}{n}}\binom{-\lambda_0-k}{\frac{n-k}{2}}(-2)^k x^k = \sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{\binom{-\lambda}{n-2k}\binom{-\lambda_0-n+2k}{k}}{\binom{-\lambda}{n}}(-2x)^{n-2k}. \tag{4.4}$$

The inverse of R is $\left(\left(\frac{2}{1+\sqrt{1-t^2}}\right)^{\lambda_0}, \frac{-t}{1+\sqrt{1-t^2}}\right)$, with the following generic element:

$$\begin{aligned}\left[\frac{t^n}{c_n}\right]\frac{1}{c_k}\left(\frac{2}{1+\sqrt{1-t^2}}\right)^{\lambda_0}\left(\frac{-t}{1+\sqrt{1-t^2}}\right)^{k} &= \frac{c_n}{c_k}2^{\lambda_0}(-1)^k[t^{n-k}]\left(1+\sqrt{1-t^2}\right)^{-\lambda_0-k}\\ &= \frac{c_n}{c_k}2^{\lambda_0}(-1)^k[t^{n-k}]\sum_{j=0}^{\infty}\frac{(\lambda_0+k)(\lambda_0+k+2j-1)_{j-1}}{2^{\lambda_0+k+2j}\,j!}t^{2j}\\ &= \begin{cases}0, & n-k \text{ odd},\\ \dfrac{c_n}{c_k}\dfrac{(-1)^k(\lambda_0+k)(\lambda_0+n-1)_{\frac{n-k}{2}-1}}{2^n\left(\frac{n-k}{2}\right)!}, & n-k \text{ even},\end{cases}\end{aligned}$$

where we have used the formula [27, p. 94, Eq. (9.1)]

$$(1+\sqrt{1-z})^{-\alpha} = \sum_{k=0}^{\infty}\frac{\alpha(\alpha+2k-1)_{k-1}}{2^{\alpha+2k}k!}z^k.$$

4.2.2. The polynomials of the Chebyshev case [27, p. 99]

In this case we take $c_n = (-1)^n$. The Sheffer sequence $T_n(x)$ for the pair

$$(g(t), f(t)) = \left(\frac{1}{\sqrt{1-t^2}}, \frac{-t}{1+\sqrt{1-t^2}}\right)$$

is related to the Chebyshev polynomials of the first kind. Since $g(\bar{f}(t)) = \frac{1+t^2}{1-t^2}$, then the corresponding Riordan array is $\left(\frac{1-t^2}{1+t^2}, \frac{-2t}{1+t^2}\right)$, whose generic element is

$$\begin{aligned}[x^k]T_n(x) &= \left[\frac{t^n}{c_n}\right]\frac{1}{c_k}\frac{1-t^2}{1+t^2}\left(\frac{-2t}{1+t^2}\right)^k\\ &= \frac{c_n}{c_k}(-2)^k[t^{n-k}]\left(\frac{1}{(1+t^2)^{k+1}} - \frac{t^2}{(1+t^2)^{k+1}}\right).\end{aligned}$$

Because

$$\begin{aligned}\frac{1}{(1+t^2)^{k+1}} - \frac{t^2}{(1+t^2)^{k+1}} &= \sum_{i=0}^{\infty}\binom{-k-1}{i}t^{2i} - \sum_{i=0}^{\infty}\binom{-k-1}{i}t^{2i+2}\\ &= 1+\sum_{i=1}^{\infty}\left(\binom{-k-1}{i} - \binom{-k-1}{i-1}\right)t^{2i} = 1+\sum_{i=1}^{\infty}(-1)^i\frac{k+2i}{i}\binom{k+i-1}{i-1}t^{2i},\end{aligned}$$

we have

$$[x^k]T_n(x) = \begin{cases} 0, & n-k \text{ odd}, \\ (-2)^n, & n-k=0, \\ \dfrac{c_n}{c_k}(-2)^k \dfrac{(-1)^{\frac{n-k}{2}} n}{\frac{n-k}{2}} \binom{k+\frac{n-k}{2}-1}{\frac{n-k}{2}-1}, & n-k \text{ even and } n \neq k, \end{cases}$$

which can be used to get the explicit expression for $T_n(x)$:

$$\begin{aligned} T_n(x) &= \sum_{\substack{k=0 \\ n-k \text{ even}}}^{n-2} (-1)^{n-k}(-2)^k \frac{(-1)^{\frac{n-k}{2}} n}{\frac{n-k}{2}} \binom{k+\frac{n-k}{2}-1}{\frac{n-k}{2}-1} x^k + (-2)^n x^n \\ &= \sum_{k=1}^{\left[\frac{n}{2}\right]} \frac{(-1)^k n}{k} \binom{n-k-1}{k-1} (-2x)^{n-2k} + (-2)^n x^n = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k n}{n-k} \binom{n-k}{k} (-2x)^{n-2k}. \end{aligned}$$

The Sheffer sequence $U_n(x)$ for the pair

$$(g(t), f(t)) = \left(\frac{2-2\sqrt{1-t^2}}{t^2}, \frac{-t}{1+\sqrt{1-t^2}} \right)$$

is related to the Chebyshev polynomials of the second kind. Since $g(\bar{f}(t)) = 1+t^2$, then the corresponding Riordan array is $\left(\frac{1}{1+t^2}, \frac{-2t}{1+t^2}\right)$, which has the following generic element:

$$\begin{aligned} [x^k]U_n(x) &= \left[\frac{t^n}{c_n}\right] \frac{1}{c_k} \frac{1}{1+t^2} \left(\frac{-2t}{1+t^2}\right)^k \\ &= \frac{c_n}{c_k}(-2)^k [t^{n-k}] \sum_{i=0}^{\infty} \binom{-k-1}{i} t^{2i} = \begin{cases} 0, & n-k \text{ odd}, \\ \dfrac{c_n}{c_k}(-2)^k \binom{-k-1}{\frac{n-k}{2}}, & n-k \text{ even}. \end{cases} \end{aligned}$$

Therefore, we have

$$U_n(x) = \sum_{\substack{k=0 \\ n-k \text{ even}}}^{n} (-1)^{n-k}(-2)^k \binom{-k-1}{\frac{n-k}{2}} x^k = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{-n+2k-1}{k} (-2x)^{n-2k}.$$

4.2.3. The polynomials of the Jacobi case [27, p. 103]

Let $c_n = \frac{2^{2n}\langle 1+\alpha\rangle_n}{\langle 1+\alpha+\beta\rangle_{2n}}$, where $\langle x\rangle_n$ is the rising factorial defined by $\langle x\rangle_n = x(x+1)(x+2)\cdots(x+n-1)$. Suppose $J_n(x)$ is the Sheffer sequence for the pair $(g(t), f(t))$, where

$$g(t) = \left(\frac{2}{1+\sqrt{1+2t}}\right)^{1+\alpha+\beta}, \qquad f(t) = \frac{1+t-\sqrt{1+2t}}{t}.$$

From [27, p. 104], we know that $J_n(x)$ is related to the Jacobi polynomials. Since $\bar{f}(t) = 2t/(1-t)^2$ and $g(\bar{f}(t)) = (1-t)^{1+\alpha+\beta}$, then $[x^k]J_n(x)$ is the generic element of the Riordan array $((1-t)^{-(1+\alpha+\beta)}, 2t/(1-t)^2)$. By the definition, we have

$$\begin{aligned} [x^k]J_n(x) &= \left[\frac{t^n}{c_n}\right] \frac{1}{c_k} \frac{1}{(1-t)^{1+\alpha+\beta}} \left(\frac{2t}{(1-t)^2}\right)^k = \frac{c_n}{c_k} 2^k \binom{\alpha+\beta+n+k}{n-k} \\ &= 2^k \frac{2^{2n-2k}\langle 1+\alpha+k\rangle_{n-k}}{\langle 1+\alpha+\beta+2k\rangle_{2n-2k}} \binom{\alpha+\beta+n+k}{n-k} \\ &= \binom{\alpha+\beta+n+k}{n-k} \frac{(\alpha+n)_{n-k}}{(\alpha+\beta+2n)_{2n-2k}} 2^{2n-k}. \end{aligned}$$

Thus, the following expression holds:

$$J_n(x) = \sum_{k=0}^{n} \binom{\alpha+\beta+n+k}{n-k} \frac{(\alpha+n)_{n-k}}{(\alpha+\beta+2n)_{2n-2k}} 2^{2n-k} x^k.$$

4.2.4. The polynomials of the q-case [27, p. 107]

In this case, let

$$c_n = \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)^n} = \frac{(q;q)_n}{(1-q)^n} = n!_q,$$

which is the q-analog of $n!$ [1, Section 10.2]. Define the sequence $[x]_{a,n}$ by

$$[x]_{a,0} = 1, \qquad [x]_{a,n} = (x-a)(x-qa)\cdots(x-q^{n-1}a),$$

and let $[x]_i := [x]_{1,i}$. From [27, p. 108] we know that $[x]_{a,n}$ is Sheffer for the pair $(\varepsilon_a(t), t)$. Then the generic element of the Riordan array $(1/\varepsilon_a(t), t)$ is $[x^k][x]_{a,n}$. Since

$$\frac{1}{\varepsilon_a(t)} = \sum_{k=0}^{\infty} \frac{(1-q)^k}{(1-q)(1-q^2)\cdots(1-q^k)} q^{\binom{k}{2}} (-at)^k = \sum_{k=0}^{\infty} q^{\binom{k}{2}} (-at)^k \frac{1}{c_k},$$

we have

$$[x^k][x]_{a,n} = \left[\frac{t^n}{c_n}\right] \frac{1}{\varepsilon_a(t)} \frac{t^k}{c_k} = \frac{c_n}{c_k} [t^{n-k}] \sum_{i=0}^{\infty} q^{\binom{i}{2}} (-at)^i \frac{1}{c_i} = \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} (-a)^{n-k},$$

$$[x]_{a,n} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-a)^{n-k} q^{\binom{n-k}{2}} x^k, \tag{4.5}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{c_n}{c_k c_{n-k}} = \frac{(1-q)\cdots(1-q^n)}{(1-q)\cdots(1-q^k)(1-q)\cdots(1-q^{n-k})} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$$

is the q-binomial coefficient. In addition to these, by the definition, we can easily find that the Riordan array

$$\left(\frac{1}{\varepsilon_a(t)}, t\right)^{-1} = (\varepsilon_a(t), t) = \left(\begin{bmatrix} n \\ k \end{bmatrix}_q a^{n-k} \right)_{n,k\in\mathbb{N}}.$$

The q-Bernoulli polynomials $\mathfrak{B}_n(x)$ [27, p. 111] are Sheffer for the pair $((\varepsilon_1(t)-1)/t, t)$. Therefore, $[x^k]\mathfrak{B}_n(x)$ is the generic element of the Riordan array $(t/(\varepsilon_1(t)-1), t)$. According to Definition 3.1, $\mathfrak{B}_n(x)$ has the following generating function

$$\sum_{k=0}^{\infty} \mathfrak{B}_k(x) \frac{t^k}{c_k} = \frac{t}{\varepsilon_1(t)-1} \varepsilon_x(t).$$

Letting $x = 1$ gives

$$\begin{aligned} \frac{t}{\varepsilon_1(t)-1} &= \frac{1}{\varepsilon_1(t)} \sum_{k=0}^{\infty} \frac{\mathfrak{B}_k(1) t^k}{c_k} = \sum_{i=0}^{\infty} \frac{q^{\binom{i}{2}} (-t)^i}{c_i} \sum_{j=0}^{\infty} \frac{\mathfrak{B}_j(1) t^j}{c_j} \\ &= \sum_{n=0}^{\infty} \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q q^{\binom{i}{2}} (-1)^i \mathfrak{B}_{n-i}(1) \frac{t^n}{c_n}. \end{aligned}$$

Thus, we have

$$[x^k]\mathfrak{B}_n(x) = \left[\frac{t^n}{c_n}\right] \frac{t}{\varepsilon_1(t)-1} \frac{t^k}{c_k} = \begin{bmatrix} n \\ k \end{bmatrix}_q \sum_{i=0}^{n-k} \begin{bmatrix} n-k \\ i \end{bmatrix}_q q^{\binom{i}{2}} (-1)^i \mathfrak{B}_{n-k-i}(1),$$

Table 2
Some Riordan arrays in various cases of c_n

c_n	Riordan array	Generic element $d_{n,k}$	Sheffer Seq.
$\frac{1}{\binom{-\lambda}{n}}$	$\left((1+t^2)^{-\lambda_0}, \frac{-2t}{1+t^2}\right)$	$0, n-k$ odd $\frac{c_n}{c_k}(-2)^k\binom{-\lambda_0-k}{\frac{n-k}{2}}, n-k$ even	Gegenbauer
$\frac{1}{\binom{-\lambda}{n}}$	$\left(\left(\frac{2}{1+\sqrt{1-t^2}}\right)^{\lambda_0}, \frac{-t}{1+\sqrt{1-t^2}}\right)$	$\frac{c_n}{c_k}\frac{(-1)^k(\lambda_0+k)(\lambda_0+n-1)_{\frac{n-k}{2}-1}}{2^n\left(\frac{n-k}{2}\right)!}$ for $n-k$ even & 0 for $n-k$ odd	
$(-1)^n$	$\left(\frac{1-t^2}{1+t^2}, \frac{-2t}{1+t^2}\right)$	0 for $n-k$ odd $(-2)^n$ for $n=k$ & $\frac{c_n}{c_k}(-2)^k\frac{(-1)^{\frac{n-k}{2}}n}{\frac{n-k}{2}}\binom{k+\frac{n-k}{2}-1}{\frac{n-k}{2}-1}$ for $n-k$ even and $n\neq k$	Chebyshev I
$(-1)^n$	$\left(\frac{1}{1+t^2}, \frac{-2t}{1+t^2}\right)$	$0, n-k$ odd $\frac{c_n}{c_k}(-2)^k\binom{-k-1}{\frac{n-k}{2}}, n-k$ even	Chebyshev II
$\frac{2^{2n}\langle 1+\alpha\rangle_n}{\langle 1+\alpha+\beta\rangle_{2n}}$	$\left(\frac{1}{(1-t)^{1+\alpha+\beta}}, \frac{2t}{(1-t)^2}\right)$	$\frac{c_n}{c_k}2^k\binom{\alpha+\beta+n+k}{n-k}$	Jacobi
$\frac{(q;q)_n}{(1-q)^n}$	$\left(\frac{1}{\varepsilon_a(t)}, t\right)$	$\begin{bmatrix}n\\k\end{bmatrix}_q q^{\binom{n-k}{2}}(-a)^{n-k}$	$[x]_{a,n}$
$\frac{(q;q)_n}{(1-q)^n}$	$(\varepsilon_a(t), t)$	$\begin{bmatrix}n\\k\end{bmatrix}_q a^{n-k}$	
$\frac{(q;q)_n}{(1-q)^n}$	$\left(\frac{t}{\varepsilon_1(t)-1}, t\right)$	$\begin{bmatrix}n\\k\end{bmatrix}_q \sum_{i=0}^{n-k}\begin{bmatrix}n-k\\i\end{bmatrix}_q q^{\binom{i}{2}}(-1)^i\mathfrak{B}_{n-k-i}(1)$	q-Bernoulli

which implies that

$$\begin{aligned}\mathfrak{B}_n(x) &= \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\sum_{i=0}^{n-k}\begin{bmatrix}n-k\\i\end{bmatrix}_q q^{\binom{i}{2}}(-1)^i\mathfrak{B}_{n-k-i}(1)x^k\\ &= \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\sum_{i=k}^{n}\begin{bmatrix}n-k\\i-k\end{bmatrix}_q q^{\binom{i-k}{2}}(-1)^{i-k}\mathfrak{B}_{n-i}(1)x^k\\ &= \sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}_q\mathfrak{B}_{n-i}(1)\sum_{k=0}^{i}\begin{bmatrix}i\\k\end{bmatrix}_q(-1)^{i-k}q^{\binom{i-k}{2}}x^k = \sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}_q\mathfrak{B}_{n-i}(1)[x]_i,\end{aligned}$$

where the last equation comes from (4.5).

The Riordan arrays presented in this subsection are listed in Table 2.

5. Recurrence relations and matrix factorizations

In this section, we study the recurrence relations satisfied by the elements of the Riordan arrays. Additionally, based on one of the recurrences, we also give some matrix factorizations for the Riordan arrays.

5.1. Recurrence relations

Let us first consider the A-sequence of a Riordan array.

For the classical Riordan array $(g(t), f(t))$, Rogers [26] found that every element $d_{n+1,k+1}$, $n, k \in \mathbb{N}$, can be expressed as a linear combination of the elements in the preceding row, i.e.,

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \cdots = \sum_{j=0}^{\infty} a_j d_{n,k+j}.$$

The sequence $A = (a_k)_{k\in\mathbb{N}}$ is fixed and we call it the *A-sequence* of the Riordan array. Rogers has shown that the A-sequence only depends on $f(t)$: let $A(t) = \sum_{k=0}^{\infty} a_k t^k$ be the generating function of the sequence A, then $f(t) = tA(f(t))$. Moreover, Rogers has also shown that the A-sequence determines an infinite triangle as a Riordan array. The reader is referred to the papers of Rogers [26] and Sprugnoli [35] for more details of this concept.

Based on the theory of A-sequences for the classical Riordan arrays, we can further develop the corresponding theory for the generalized Riordan arrays.

Theorem 5.1. *The quantity $d_{n,k}$ is the generic element of the generalized Riordan array $(g(t), f(t))$ with respect to c_n if and only if $c_k d_{n,k}/c_n$ is the generic element of the classical Riordan array $(g(t), f(t))$.*

Proof. By Definition 2.1, we have

$$d_{n,k} = \left[\frac{t^n}{c_n}\right] g(t)\frac{(f(t))^k}{c_k} = \frac{c_n}{c_k}[t^n]g(t)(f(t))^k,$$

which is equivalent to the fact that $c_k d_{n,k}/c_n = [t^n]g(t)(f(t))^k$. This completes the proof. □

Despite its simple proof, Theorem 5.1 is an important result, because it shows that the generalized Riordan arrays can always be reduced to the classical case. From Theorem 5.1, the next two theorems can be obtained without difficulty.

Theorem 5.2. *For any generalized Riordan array $(g(t), f(t)) = (d_{n,k})_{n,k\in\mathbb{N}}$, every element $d_{n+1,k+1}$, $n, k \in \mathbb{N}$, can be expressed as follows:*

$$d_{n+1,k+1} = \sum_{j=0}^{\infty} \frac{c_{n+1}c_{k+j}}{c_{k+1}c_n} a_j d_{n,k+j}, \tag{5.1}$$

where the sum is actually finite and the sequence $A = (a_k)_{k\in\mathbb{N}}$ is fixed. It is called the A-sequence of the generalized Riordan array and it only depends on $f(t)$. That is, let $A(t) = \sum_{k=0}^{\infty} a_k t^k$, then

$$f(t) = tA(f(t)), \qquad A(t) = \frac{t}{\bar{f}(t)}.$$

Proof. According to the results of Rogers [26], for the classical Riordan array $(g(t), f(t)) = (d^*_{n,k})_{n,k\in\mathbb{N}}$, there exists a unique sequence $A = (a_k)_{k\in\mathbb{N}}$ that satisfies the statements of the theorem. Then $d^*_{n+1,k+1} = \sum_{j=0}^{\infty} a_j d^*_{n,k+j}$ and $f(t) = tA(f(t))$. By Theorem 5.1, $d^*_{n,k} = c_k d_{n,k}/c_n$, so we have

$$\frac{c_{k+1}}{c_{n+1}} d_{n+1,k+1} = \sum_{j=0}^{\infty} a_j \frac{c_{k+j}}{c_n} d_{n,k+j},$$

which leads us to the recurrence (5.1) at once. □

Theorem 5.3. *Let c_n be a sequence of non-zero constants with $c_0 = 1$, and let $D := (d_{n,k})_{0\le k\le n<\infty}$ be an infinite triangle such that $d_{n,n} \neq 0$, $\forall n \in \mathbb{N}$ and for which the relation (5.1) holds true for some sequence $A = (a_k)_{k\in\mathbb{N}}$, $a_0 \neq 0$. Then D is a generalized Riordan array $(g(t), f(t))$ with respect to c_n, where $g(t) = \sum_{k=0}^{\infty} d_{k,0}t^k/c_k$ and $f(t)$ is the unique solution of $f(t) = tA(f(t))$ with $A(t) = \sum_{k=0}^{\infty} a_k t^k$.*

Proof. Define $d^*_{n,k} = c_k d_{n,k}/c_n$, then $d^*_{n,n} \neq 0$, $\forall n \in \mathbb{N}$ and $d^*_{n+1,k+1} = \sum_{j=0}^{\infty} a_j d^*_{n,k+j}$. In view of [26,35], the infinite triangle $D^* = (d^*_{n,k})_{n,k\in\mathbb{N}}$ is the classical Riordan array $(g(t), f(t))$, where $g(t) = \sum_{k=0}^{\infty} d^*_{k,0}t^k$ and $f(t)$ is the unique solution of $f(t) = tA(f(t))$. Thus, $g(t) = \sum_{k=0}^{\infty} c_0 d_{k,0}t^k/c_k = \sum_{k=0}^{\infty} d_{k,0}t^k/c_k$, and by Theorem 5.1, $D = (d_{n,k})_{n,k\in\mathbb{N}}$ is the generalized Riordan array $(g(t), f(t))$ with respect to c_n. □

Next, we will demonstrate another two recurrences related to the elements of the generalized Riordan arrays.

Theorem 5.4. *For any generalized Riordan array* $(g(t), f(t)) = (d_{n,k})_{n,k\in\mathbb{N}}$, *we have*

$$d_{n,k} - \frac{c_n}{nc_{n-1}}\tilde{d}_{n-1,k} = \sum_{l=k}^{n} \frac{c_n}{c_{l-1}c_{n-l+1}} \frac{n-l+1}{n} f_{n-l+1} \frac{kc_{k-1}}{c_k} d_{l-1,k-1}, \quad n,k \geq 1, \tag{5.2}$$

where $\tilde{d}_{n,k}$ *is the generic element of the Riordan array* $(g'(t), f(t))$, $g'(t)$ *is the derivative of* $g(t)$, *and* f_k *are the coefficients of the delta series* $f(t) = \sum_{k=1}^{\infty} f_k t^k/c_k$.

Proof. The column generating function is

$$\sum_{n=k}^{\infty} d_{n,k} \frac{t^n}{c_n} = g(t) \frac{(f(t))^k}{c_k}. \tag{5.3}$$

Differentiate (5.3) with respect to t,

$$\begin{aligned}
\sum_{n=k}^{\infty} d_{n,k} \frac{nt^{n-1}}{c_n} &= g'(t) \frac{(f(t))^k}{c_k} + \frac{kc_{k-1}}{c_k} g(t) f'(t) \frac{(f(t))^{k-1}}{c_{k-1}} \\
&= \sum_{n=k}^{\infty} \tilde{d}_{n,k} \frac{t^n}{c_n} + \frac{kc_{k-1}}{c_k} \sum_{i=k-1}^{\infty} d_{i,k-1} \frac{t^i}{c_i} \sum_{j=0}^{\infty} (j+1) f_{j+1} \frac{t^j}{c_{j+1}} \\
&= \sum_{n=k}^{\infty} \tilde{d}_{n,k} \frac{t^n}{c_n} + \frac{kc_{k-1}}{c_k} \sum_{n=k-1}^{\infty} \sum_{i=k-1}^{n} \frac{c_n}{c_i c_{n-i+1}} (n-i+1) f_{n-i+1} d_{i,k-1} \frac{t^n}{c_n},
\end{aligned}$$

and identify the coefficients of t^{n-1}/c_{n-1} in the equation above, then we have

$$\frac{nc_{n-1}}{c_n} d_{n,k} = \tilde{d}_{n-1,k} + \sum_{i=k-1}^{n-1} \frac{c_{n-1}}{c_i c_{n-i}} (n-i) f_{n-i} \frac{kc_{k-1}}{c_k} d_{i,k-1},$$

which, after some transformations, leads us to (5.2) finally. □

For the iteration matrix with respect to c_n, $\tilde{d}_{n-1,k} = 0$ because of the fact that $g'(t) = 0$. Thus, (5.2) reduces to

$$d_{n,k} = \sum_{l=k}^{n} \frac{c_n}{c_{l-1}c_{n-l+1}} \frac{n-l+1}{n} f_{n-l+1} \frac{kc_{k-1}}{c_k} d_{l-1,k-1}, \tag{5.4}$$

which has been given in [39, Lemma 3.1].

Theorem 5.5. *For any generalized Riordan array* $(g(t), f(t)) = (d_{n,k})_{n,k\in\mathbb{N}}$, *we have*

$$\frac{c_k}{c_{k-1}} d_{n,k} = \sum_{l=k}^{n} \frac{c_n}{c_{l-1}c_{n-l+1}} f_{n-l+1} d_{l-1,k-1}, \quad n,k \geq 1, \tag{5.5}$$

where f_k *are the coefficients of the delta series* $f(t) = \sum_{k=1}^{\infty} f_k t^k/c_k$.

Proof. From (5.3), we have

$$\begin{aligned}
\sum_{n=k}^{\infty} \frac{c_k}{c_{k-1}} d_{n,k} \frac{t^n}{c_n} &= f(t) g(t) \frac{(f(t))^{k-1}}{c_{k-1}} = \sum_{i=1}^{\infty} f_i \frac{t^i}{c_i} \sum_{j=k-1}^{\infty} d_{j,k-1} \frac{t^j}{c_j} \\
&= \sum_{n=k}^{\infty} \sum_{j=k-1}^{n-1} f_{n-j} d_{j,k-1} \frac{c_n}{c_j c_{n-j}} \frac{t^n}{c_n}.
\end{aligned}$$

By equating the coefficients of t^n/c_n, we can obtain the desired result. □

For convenience, let us give the specializations of the recurrences (5.1), (5.2) and (5.5) for the cases $c_n = 1$ and $c_n = n!$, respectively.

Corollary 5.6. *For any classical Riordan array* $(g(t), f(t)) = (d_{n,k})_{n,k\in\mathbb{N}}$, *we have*

$$d_{n+1,k+1} = \sum_{j=0}^{\infty} a_j d_{n,k+j}, \tag{5.6}$$

$$d_{n,k} - \frac{1}{n}\tilde{d}_{n-1,k} = \sum_{l=k}^{n} \frac{k}{n}(n-l+1) f_{n-l+1} d_{l-1,k-1}, \tag{5.7}$$

$$d_{n,k} = \sum_{l=k}^{n} f_{n-l+1} d_{l-1,k-1}. \tag{5.8}$$

Corollary 5.7. *For any exponential Riordan array* $(g(t), f(t)) = (d_{n,k})_{n,k\in\mathbb{N}}$, *we have*

$$d_{n+1,k+1} = \sum_{j=0}^{\infty} \frac{n+1}{k+1}\binom{k+j}{j} j! a_j d_{n,k+j}, \tag{5.9}$$

$$d_{n,k} - \tilde{d}_{n-1,k} = \sum_{l=k}^{n} \binom{n-1}{l-1} f_{n-l+1} d_{l-1,k-1}, \tag{5.10}$$

$$k d_{n,k} = \sum_{l=k}^{n} \binom{n}{l-1} f_{n-l+1} d_{l-1,k-1}. \tag{5.11}$$

It should be noticed that, according to Theorem 5.1, the recurrences presented in Corollaries 5.6 and 5.7 are in fact equivalent to the general ones (i.e., recurrences (5.1), (5.2) and (5.5)).

5.2. Matrix factorizations

The lower triangular matrices and matrix factorizations problem have catalyzed many investigations in recent years. The Pascal matrix and several generalized Pascal matrices first received wide concern [2,4,42,43,45]; some other lower triangular matrices were also studied systematically, for example, the Stirling matrices of the first kind and of the second kind [10,11], the Lah matrix [38], and the matrix related to the idempotent numbers [41]. From [12, Sections 3.3 and 3.7], we know that the matrices referred to above are all special iteration matrices. Based on this fact, we have presented in [39] a unified approach to the matrix factorizations problem.

Peart and Woodson [24] did some researches on this problem with a different method. They showed that some classical Riordan arrays have triple factorization of the form $R = PCF$, where P, C, F are also Riordan arrays. Particularly, $P = \left(\frac{1}{1-bt}, \frac{t}{1-bt}\right)$ is a Pascal-type matrix, $C = (c(\lambda t^2), tc(\lambda t^2))$ involves the generating function $c(t)$ for the Catalan numbers, and $F = \left(\frac{1}{1-\varepsilon t-\delta t^2}, t\right)$ involves the Fibonacci generating function.

Here, we will give some factorizations satisfied by the Riordan arrays. Our results are based on the third recurrence (5.5) obtained in Section 5.1.

Actually, from (5.5), we have

$$c_k d_{n,k} = \sum_{l=k}^{n} \frac{c_n}{c_{l-1}c_{n-l+1}} f_{n-l+1} c_{k-1} d_{l-1,k-1}. \tag{5.12}$$

Now, defining $\hat{R}_n$ and P_n to be the $n \times n$ matrices by

$$(\hat{R}_n)_{i,j} = c_j d_{i,j}, \qquad (P_n)_{i,j} = \begin{cases} \dfrac{c_i}{c_{j-1}c_{i-j+1}} f_{i-j+1}, & \text{if } j \geq 1, \\ d_{i,0}, & \text{if } j = 0, \end{cases} \quad \text{for } i, j \in \mathbb{N},$$

and using the notation $\oplus$ for the direct sum of two matrices, we can obtain a matrix factorization from Eq. (5.12), i.e., $\hat{R}_n = P_n(1 \oplus \hat{R}_{n-1})$. Moreover, let R_n and D_n be the $n \times n$ matrices satisfying

$$(R_n)_{i,j} = d_{i,j}, \qquad D_n = \operatorname{diag}\{c_0, c_1, \ldots, c_{n-1}\}, \quad \text{for } i, j \in \mathbb{N},$$

and analogous to [4,10,39,42,43], for any $k \times k$ matrix P_k, define the $n \times n$ matrix $\bar{P}_k$ by

$$\bar{P}_k = \begin{pmatrix} I_{n-k} & O \\ O & P_k \end{pmatrix}.$$

Then $\hat{R}_n = R_n D_n$, $\bar{P}_n = P_n$, and we have the next theorem.

Theorem 5.8. *The following factorizations hold:*

$$\hat{R}_n = R_n D_n = P_n([1] \oplus \hat{R}_{n-1}) = \bar{P}_n \bar{P}_{n-1} \cdots \bar{P}_1.$$

Because $c_n \neq 0$, the matrix D_n has the inverse $D_n^{-1} = \operatorname{diag}\left\{\frac{1}{c_0}, \frac{1}{c_1}, \ldots, \frac{1}{c_{n-1}}\right\}$, by which we obtain the factorizations for all Riordan arrays.

Theorem 5.9. *The Riordan array R has the following factorizations:*

$$R_n = P_n([1] \oplus \hat{R}_{n-1}) D_n^{-1} = \bar{P}_n \bar{P}_{n-1} \cdots \bar{P}_1 D_n^{-1}. \tag{5.13}$$

If the Riordan array R has the inverse R^{-1}, then

$$R_n^{-1} = D_n([1] \oplus D_{n-1}^{-1} R_{n-1}^{-1}) P_n^{-1} = D_n \bar{P}_1^{-1} \bar{P}_2^{-1} \cdots \bar{P}_n^{-1}.$$

5.3. Examples

Now, let us give some examples for the recurrences and matrix factorizations.

5.3.1. The Pascal matrix

The classical Riordan array $R = (1/(1-t), t/(1-t))$ is just the Pascal matrix $\left(\binom{n}{k}\right)_{n,k\in\mathbb{N}}$. Because $g'(t) = 1/(1-t)^2$, we have

$$\tilde{d}_{n-1,k} = [t^{n-1}]g'(t)(f(t))^k = [t^{n-1-k}](1-t)^{-k-2} = \binom{n}{k+1}.$$

Thus, in view of $f_k = 1$, we deduce from (5.7) that

$$\binom{n+1}{k+1} = \sum_{l=k}^{n} (n-l+1)\binom{l-1}{k-1}.$$

The other two recurrence relations for R given by Corollary 5.6 are trivial. Since $\bar{f}(t) = t/(1+t)$, then $A(t) = t/\bar{f}(t) = 1+t$ and (5.6) gives $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$. Next, (5.8) gives $\binom{n}{k} = \sum_{l=k}^{n}\binom{l-1}{k-1}$. Finally, we consider the factorizations of R. Since $D_n = I_n$, then $\hat{R}_n = R_n$. Moreover, $(P_n)_{i,j} = f_{i-j+1} = 1$. Thus, we have

$$R_n = P_n([1] \oplus R_{n-1}) = \bar{P}_n \bar{P}_{n-1} \cdots \bar{P}_1. \tag{5.14}$$

When $n = 4$, (5.14) turns to

$$\begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & 1 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{pmatrix}.$$

5.3.2. The Stirling matrices of both kinds

The Riordan array $(1, \log(1+t)) = (s(n,k))_{n,k\in\mathbb{N}}$ is the Stirling matrix of the first kind and the generating function of the A-sequence is

$$A(t) = \frac{t}{\bar{f}(t)} = \frac{t}{\mathrm{e}^t - 1} = \sum_{j=0}^{\infty} B_j \frac{t^j}{j!},$$

where B_j are the Bernoulli numbers (see, e.g., [12, p. 48]). Then (5.9) reduces to

$$s(n+1,k+1) = \sum_{j=0}^{\infty} \frac{n+1}{k+1} \binom{k+j}{j} B_j s(n,k+j).$$

Additionally, because $f_k = (-1)^{k-1}(k-1)!$, we obtain from (5.10) and (5.11) that

$$s(n,k) = \sum_{l=k}^{n} \binom{n-1}{l-1} (-1)^{n-l}(n-l)!s(l-1,k-1),$$

$$ks(n,k) = \sum_{l=k}^{n} \binom{n}{l-1} (-1)^{n-l}(n-l)!s(l-1,k-1).$$

Let us turn to the matrix factorization. Now $(\hat{R}_n)_{i,j} = j!s(i,j)$ and $(P_n)_{i,j} = \binom{i}{j-1}(-1)^{i-j}(i-j)!$ for $j \geq 1$. When $n = 5$, (5.13) gives

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & -1 & 1 & & \\ & 2 & -3 & 1 & \\ & -6 & 11 & -6 & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & -1 & 2 & & \\ & 2 & -3 & 3 & \\ & -6 & 8 & -6 & 4 \end{pmatrix} \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & -1 & 2 & \\ & & 2 & -6 & 6 \end{pmatrix} \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \frac{1}{2} & & \\ & & & \frac{1}{6} & \\ & & & & \frac{1}{24} \end{pmatrix}.$$

The Riordan array $(1, \mathrm{e}^t - 1) = (S(n,k))_{n,k\in\mathbb{N}}$ is the Stirling matrix of the second kind. For this array,

$$A(t) = \frac{t}{\bar{f}(t)} = \frac{t}{\log(1+t)} = \sum_{j=0}^{\infty} b_j(0) \frac{t^j}{j!},$$

where $b_j(0)$ are the Bernoulli numbers of the second kind [28, p. 114] and they are also called the Cauchy numbers of the first kind (see [12, p. 294] and [21]). Then we have

$$S(n+1,k+1) = \sum_{j=0}^{\infty} \frac{n+1}{k+1} \binom{k+j}{j} b_j(0) S(n,k+j).$$

Next, because $f_k = 1$, (5.10) and (5.11) lead us at once to

$$S(n,k) = \sum_{l=k}^{n} \binom{n-1}{l-1} S(l-1,k-1),$$

$$kS(n,k) = \sum_{l=k}^{n} \binom{n}{l-1} S(l-1,k-1).$$

By means of [12, p. 209, Eq. (3f)], we obtain

$$k!S(n,k)=\sum_{l=k}^{n}\binom{n}{l-1}(k-1)!S(l-1,k-1)=k^n-\sum_{j=1}^{k-1}(k)_j S(n,j).$$

Finally, from the fact that $(\hat{R}_n)_{i,j}=j!S(i,j)$ and $(P_n)_{i,j}=\binom{i}{j-1}$, we can give the factorization of the array. When $n=5$ it is

$$\begin{pmatrix}1&&&&\\1&1&&&\\1&3&1&&\\1&7&6&1&\end{pmatrix}=\begin{pmatrix}1&&&&\\1&1&&&\\1&3&3&&\\1&4&6&4&\end{pmatrix}\begin{pmatrix}1&&&&\\&1&&&\\&1&2&&\\&1&6&6&\end{pmatrix}\begin{pmatrix}1&&&&\\&1&&&\\&&\frac{1}{2}&&\\&&&\frac{1}{6}&\\&&&&\frac{1}{24}\end{pmatrix}.$$

5.3.3. The Riordan array related to the polynomials of the Gegenbauer case

According to the discussion of Section 4.2.1, we know that

$$\left(\frac{1}{(1+t^2)^{\lambda_0}},\frac{-2t}{1+t^2}\right)_{n,k}=\begin{cases}0, & n-k \text{ odd},\\ \dfrac{c_n}{c_k}(-2)^k\binom{-\lambda_0-k}{\frac{n-k}{2}}, & n-k \text{ even}.\end{cases}$$

Because $\bar{f}(t)=\frac{-t}{1+\sqrt{1-t^2}}$, the generating function of the A-sequence is

$$A(t)=\frac{t}{\bar{f}(t)}=-1-\sqrt{1-t^2}=-2-\sum_{i=1}^{\infty}\frac{(\frac{1}{2})_i}{i!}(-1)^i t^{2i}=-2+\sum_{i=1}^{\infty}\frac{(2i-2)!}{2^{2i-1}i!(i-1)!}t^{2i}.$$

Therefore, when $n-k$ is even, (5.1) gives

$$\binom{-\lambda_0-k-1}{\frac{n-k}{2}-1}=\sum_{i=1}^{\infty}\frac{1}{i}\binom{2i-2}{i-1}\binom{-\lambda_0-k-2i}{\frac{n-k}{2}-i}.$$

Upon setting $n\to 2n+k$, this reduces to

$$\binom{-\lambda_0-k-1}{n-1}=\sum_{i=1}^{\infty}\frac{1}{i}\binom{2i-2}{i-1}\binom{-\lambda_0-k-2i}{n-i}.$$

It is interesting to notice that the quantity $\frac{1}{i}\binom{2i-2}{i-1}$ is a Catalan number [12, p. 53]. Next, by computation, $g'(t)=-2\lambda_0 t(1+t^2)^{-\lambda_0-1}$. Thus, when $n-k\geq 2$ and $n-k$ is even, we have

$$\begin{aligned}\tilde{d}_{n-1,k}&=\left[\frac{t^{n-1}}{c_{n-1}}\right]g'(t)\frac{(f(t))^k}{c_k}=\frac{c_{n-1}}{c_k}(-2)^{k+1}\lambda_0[t^{n-k-2}](1+t^2)^{-\lambda_0-1-k}\\&=\frac{c_{n-1}}{c_k}(-2)^{k+1}\lambda_0[t^{n-k-2}]\sum_{i=0}^{\infty}\binom{-\lambda_0-1-k}{i}t^{2i}=\frac{c_{n-1}}{c_k}(-2)^{k+1}\lambda_0\binom{-\lambda_0-1-k}{\frac{n-k}{2}-1},\end{aligned}$$

by which, we can compute the left-hand side of (5.2):

$$\frac{c_n}{c_k}(-2)^k\binom{-\lambda_0-k}{\frac{n-k}{2}}-\frac{c_n}{nc_{n-1}}\frac{c_{n-1}}{c_k}(-2)^{k+1}\lambda_0\binom{-\lambda_0-1-k}{\frac{n-k}{2}-1}=\frac{c_n}{c_k}(-2)^k\frac{k(\lambda_0+n)}{n(\lambda_0+k)}\binom{-\lambda_0-k}{\frac{n-k}{2}}.$$

From

$$f(t)=\frac{-2t}{1+t^2}=-2\sum_{i=0}^{\infty}(-1)^i t^{2i+1},$$

we deduce that $f_{n-l+1}=-2(-1)^{\frac{n-l}{2}}c_{n-l+1}$ if $n-l$ is even and 0 if $n-l$ is odd. Furthermore, if $n-k$ and $n-l$ are even, then $l-k$ is even, so the right-hand side of (5.2) yields

$$\frac{c_n}{c_k}(-2)^k\frac{k}{n}\sum_{\substack{l=k\\ l-k \text{ even}}}^{n}(n-l+1)(-1)^{\frac{n-l}{2}}\binom{-\lambda_0-k+1}{\frac{l-k}{2}}.$$

Combining with the left-hand side and doing some computation, we have

$$\frac{\lambda_0+n}{\lambda_0+k}\binom{-\lambda_0-k}{\frac{n-k}{2}}=\sum_{j=0}^{\frac{n-k}{2}}(n-k-2j+1)(-1)^{\frac{n-k}{2}-j}\binom{-\lambda_0-k+1}{j}\quad\text{for } n-k \text{ even}.$$

Once again, let $n\to 2n+k$, then the identity above turns to

$$\frac{\lambda_0+2n+k}{\lambda_0+k}\binom{-\lambda_0-k}{n}=\sum_{j=0}^{n}(2n-2j+1)(-1)^{n-j}\binom{-\lambda_0-k+1}{j}.$$

Finally, when $n-k$ is even, the recurrence (5.5) gives

$$\binom{-\lambda_0-k}{\frac{n-k}{2}}=\sum_{\substack{j=0\\ j \text{ even}}}^{n-k}\binom{-\lambda_0-k+1}{\frac{j}{2}}(-1)^{\frac{n-j-k}{2}}=\sum_{j=0}^{\frac{n-k}{2}}\binom{-\lambda_0-k+1}{j}(-1)^{\frac{n-k}{2}-j},$$

which is equivalent to [12, p. 10, Eq. (5h)]

$$\binom{-\lambda_0-k}{n}=\sum_{j=0}^{n}(-1)^{n-j}\binom{-\lambda_0-k+1}{j}.$$

Let us consider the factorization of the Riordan array. If $i-j$ is even,

$$(P_n)_{i,j}=\frac{c_i}{c_{j-1}c_{i-j+1}}f_{i-j+1}=\frac{c_i}{c_{j-1}c_{i-j+1}}(-2)(-1)^{\frac{i-j}{2}}c_{i-j+1}=\frac{c_i}{c_{j-1}}(-2)(-1)^{\frac{i-j}{2}},$$

and if $i-j$ is odd, $(P_n)_{i,j}=0$. Then for $n=4$ we have

$$\begin{pmatrix}1&&&\\0&-2&&\\-c_2\lambda_0&0&4&\\0&2\frac{c_3}{c_1}(\lambda_0+1)&0&-8\end{pmatrix}=\begin{pmatrix}1&&&\\0&-2c_1&&\\-c_2\lambda_0&0&-2\frac{c_2}{c_1}&\\0&2c_3&0&-2\frac{c_3}{c_2}\end{pmatrix}$$
$$\times\begin{pmatrix}1&&&\\&1&&\\&0&-2c_1&\\&-c_2\lambda_0&0&4c_2\end{pmatrix}\begin{pmatrix}1&&&\\&\frac{1}{c_1}&&\\&&\frac{1}{c_2}&\\&&&\frac{1}{c_3}\end{pmatrix}.$$

5.3.4. The Riordan array related to the polynomials of the Jacobi case

From Section 4.2.3, we know that

$$\left(\frac{1}{(1-t)^{1+\alpha+\beta}}, \frac{2t}{(1-t)^2}\right) = \left(\frac{c_n}{c_k} 2^k \binom{\alpha+\beta+n+k}{n-k}\right)_{n,k\in\mathbb{N}}.$$

Since $\bar{f}(t) = \frac{1+t-\sqrt{1+2t}}{t}$, then

$$\begin{aligned} A(t) &= \frac{t}{\bar{f}(t)} = \frac{t^2}{1+t-\sqrt{1+2t}} = 1+t+\sqrt{1+2t} \\ &= 2+2t+\sum_{i=2}^{\infty} \frac{(-1)^{i-1}(2i-2)!}{i!(i-1)!2^{i-1}} t^i = 2+2t+\sum_{i=2}^{\infty} \left(-\frac{1}{2}\right)^{i-1} \frac{1}{i} \binom{2i-2}{i-1} t^i, \end{aligned}$$

where the Catalan numbers can be found again. With some computation, we have

$$\binom{\alpha+\beta+n+k+2}{n-k} - \binom{\alpha+\beta+n+k}{n-k} - 2\binom{\alpha+\beta+n+k+1}{n-k-1} = -\binom{\alpha+\beta+n+k}{n-k-2}.$$

Thus, the recurrence (5.1) gives

$$\binom{\alpha+\beta+n+k}{n-k-2} = \sum_{i=2}^{n-k} (-1)^i \frac{1}{i} \binom{2i-2}{i-1} \binom{\alpha+\beta+n+k+i}{n-k-i}.$$

Next, because $g'(t) = (1+\alpha+\beta)(1-t)^{-(2+\alpha+\beta)}$, we have

$$\tilde{d}_{n-1,k} = \left[\frac{t^{n-1}}{c_{n-1}}\right] g'(t) \frac{(f(t))^k}{c_k} = \frac{c_{n-1}}{c_k} (1+\alpha+\beta) 2^k \binom{\alpha+\beta+n+k}{n-k-1}.$$

Making use of the fact that $f_k = 2kc_k$, we obtain from (5.2) that

$$\frac{\alpha+\beta+2n+1}{\alpha+\beta+2k+1} \binom{\alpha+\beta+n+k}{n-k} = \sum_{l=k}^{n} (n-l+1)^2 \binom{\alpha+\beta+l+k-2}{l-k}.$$

Also, (5.5) gives

$$\binom{\alpha+\beta+n+k}{n-k} = \sum_{l=k}^{n} (n-l+1) \binom{\alpha+\beta+l+k-2}{l-k}.$$

Additionally, since $(P_n)_{i,j} = 2\frac{c_i}{c_{j-1}}(i-j+1)$, then the following matrix factorization holds:

$$\begin{pmatrix} 1 & & & \\ c_1\binom{\alpha+\beta+1}{1} & 2 & & \\ c_2\binom{\alpha+\beta+2}{2} & 2\frac{c_2}{c_1}\binom{\alpha+\beta+3}{1} & 4 & \\ c_3\binom{\alpha+\beta+3}{3} & 2\frac{c_3}{c_1}\binom{\alpha+\beta+4}{2} & 4\frac{c_3}{c_2}\binom{\alpha+\beta+5}{1} & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & \\ c_1\binom{\alpha+\beta+1}{1} & 2c_1 & & \\ c_2\binom{\alpha+\beta+2}{2} & 4c_2 & 2\frac{c_2}{c_1} & \\ c_3\binom{\alpha+\beta+3}{3} & 6c_3 & 4\frac{c_3}{c_1} & 2\frac{c_3}{c_2} \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & & & \\ & 1 & & \\ & c_1\binom{\alpha+\beta+1}{1} & 2c_1 & \\ & c_2\binom{\alpha+\beta+2}{2} & 2c_2\binom{\alpha+\beta+3}{1} & 4c_2 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & \frac{1}{c_1} & & \\ & & \frac{1}{c_2} & \\ & & & \frac{1}{c_3} \end{pmatrix}.$$

Finally, we would like to give the factorization of the Riordan array $(1/\varepsilon_a(t), t)$, whose generic element is $\begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}}(-a)^{n-k}$. Now $(P_n)_{i,j} = \frac{1-q^i}{1-q}$ for $i = j$ and 0 for $i \neq j$. Therefore, we have

$$\begin{pmatrix} 1 & & & \\ -a & 1 & & \\ qa^2 & -\frac{1-q^2}{1-q}a & 1 & \\ -q^3a^3 & \frac{1-q^3}{1-q}qa^2 & -\frac{1-q^3}{1-q}a & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ -a & 1 & & \\ qa^2 & 0 & \frac{1-q^2}{1-q} & \\ -q^3a^3 & 0 & 0 & \frac{1-q^3}{1-q} \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & & & \\ & 1 & & \\ & -a & 1 & \\ & qa^2 & -\frac{1-q^2}{1-q}a & \frac{1-q^2}{1-q} \end{pmatrix} \begin{pmatrix} 1 & & & \\ & \frac{1}{c_1} & & \\ & & \frac{1}{c_2} & \\ & & & \frac{1}{c_3} \end{pmatrix}.$$

In the following examples, we will concentrate on the three recurrence relations.

5.3.5. The Riordan arrays related to the Poisson–Charlier polynomials

According to Table 1, the Riordan array $R = \left(\mathrm{e}^{-t}, \log\left(1 + \frac{t}{a}\right)\right)$ with the generic element

$$d_{n,k} = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} a^{-j} s(j,k)$$

corresponds to the Poisson–Charlier polynomials [28, Section 4.3.3]. Since $\bar{f}(t) = a(\mathrm{e}^t - 1)$, then

$$A(t) = \frac{t}{a(\mathrm{e}^t - 1)} = \frac{1}{a} \sum_{i=0}^{\infty} B_i \frac{t^i}{i!}$$

and (5.9) turns to

$$d_{n+1,k+1} = \sum_{j=0}^{\infty} \frac{n+1}{k+1} \binom{k+j}{j} \frac{B_j}{a} d_{n,k+j},$$

where B_j are the Bernoulli numbers. Next, from $g'(t) = -\mathrm{e}^{-t}$, we have

$$\tilde{d}_{n-1,k} = -\sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^{n-1-j} a^{-j} s(j,k).$$

Then the left-hand side of (5.10) gives

$$\sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} a^{-j} s(j,k) + \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^{n-1-j} a^{-j} s(j,k)$$

$$= \sum_{j=0}^{n-1} \binom{n-1}{j-1} (-1)^{n-j} a^{-j} s(j,k) + a^{-n} s(n,k) = \sum_{j=0}^{n} \binom{n-1}{j-1} (-1)^{n-j} a^{-j} s(j,k).$$

Because $f_k = (-1)^{k-1}(k-1)!/a^k$, by series rearrangement, the right-hand side of (5.10) yields

$$\sum_{j=k}^{n} \binom{n-1}{j-1} (-1)^{n-j} (n-j)! a^{-n} \left(\sum_{l=j}^{n} \frac{a^{l-j}}{(l-j)!} \right) s(j-1,k-1).$$

Thus, the following identity holds:

$$\sum_{j=k}^{n} \binom{n-1}{j-1} (-a)^{n-j} s(j,k) = \sum_{j=k}^{n} \frac{(n-1)!}{(j-1)!} (-1)^{n-j} \left(\sum_{l=j}^{n} \frac{a^{l-j}}{(l-j)!} \right) s(j-1,k-1).$$

Finally, with some computation, we deduce from (5.11) that

$$k \sum_{j=k}^{n} \binom{n}{j} (-a)^{n-j} s(j,k) = \sum_{j=k}^{n} \frac{n!}{(j-1)!} (-1)^{n-j} \left(\sum_{l=j}^{n} \frac{a^{l-j}}{(l-j)!(n-l+1)} \right) s(j-1,k-1).$$

Recall that $\phi_n(x)$ are the exponential polynomials (see the second example given below Theorem 3.2 and [28, Section 4.1.3]). From Table 1, we know that the Riordan array

$$\left(\mathrm{e}^{a(\mathrm{e}^t-1)}, a(\mathrm{e}^t-1) \right) = \left(a^k \sum_{j=0}^{n} \binom{n}{j} \phi_{n-j}(a) S(j,k) \right)_{n,k\in\mathbb{N}}$$

is the inverse of R. Because the generating function of the A-sequence is

$$A(t) = \frac{t}{\log(1+\frac{t}{a})} = a \sum_{k=0}^{\infty} \frac{b_k(0)}{k! a^k} t^k,$$

we obtain the following recurrence:

$$d_{n+1,k+1} = \sum_{j=0}^{\infty} \frac{n+1}{k+1} \binom{k+j}{j} a^{1-j} b_j(0) d_{n,k+j},$$

where $b_j(0)$ are the Bernoulli numbers of the second kind (see Section 5.3.2 and [28, p. 114]). Additionally, since $g'(t) = \mathrm{e}^{a(\mathrm{e}^t-1)} a \mathrm{e}^t$, then

$$\begin{aligned}
\tilde{d}_{n-1,k} &= \left[\frac{t^{n-1}}{(n-1)!} \right] \mathrm{e}^{a(\mathrm{e}^t-1)} a \mathrm{e}^t \frac{a^k (\mathrm{e}^t-1)^k}{k!} \\
&= \left[\frac{t^{n-1}}{(n-1)!} \right] \mathrm{e}^{a(\mathrm{e}^t-1)} \left((k+1) \frac{a^{k+1}(\mathrm{e}^t-1)^{k+1}}{(k+1)!} + a \frac{a^k (\mathrm{e}^t-1)^k}{k!} \right) \\
&= a^{k+1}(k+1) \sum_{j=0}^{n-1} \binom{n-1}{j} \phi_{n-1-j}(a) S(j,k+1) + a^{k+1} \sum_{j=0}^{n-1} \binom{n-1}{j} \phi_{n-1-j}(a) S(j,k) \\
&= a^{k+1} \sum_{j=0}^{n-1} \binom{n-1}{j} \phi_{n-1-j}(a) S(j+1,k+1).
\end{aligned}$$

Based on the fact that [28, p. 64, Eq. (4.1.5)] $\phi_{n+1}(x) = x \sum_{k=0}^{n} \binom{n}{k} \phi_k(x)$ and by the method of series rearrangement, the right-hand side of (5.10) gives

$$a^{k-1} \sum_{j=k}^{n} \binom{n-1}{j-1} \phi_{n-j+1}(a) S(j-1,k-1).$$

Then from (5.10) we have

$$a\sum_{j=k}^{n}\binom{n}{j}\phi_{n-j}(a)S(j,k)-a^2\sum_{j=k}^{n-1}\binom{n-1}{j}\phi_{n-1-j}(a)S(j+1,k+1)$$
$$=\sum_{j=k}^{n}\binom{n-1}{j-1}\phi_{n-j+1}(a)S(j-1,k-1). \tag{5.15}$$

With a similar method, we can obtain from (5.11) that

$$k\sum_{j=k}^{n}\binom{n}{j}\phi_{n-j}(a)S(j,k)=\sum_{j=k}^{n}\binom{n}{j-1}\left(\frac{\phi_{n-j+2}(a)}{a}-\phi_{n-j+1}(a)\right)S(j-1,k-1). \tag{5.16}$$

In particular, because $\phi_n(1)=\mathscr{B}_n$, where $\mathscr{B}_n$ are the Bell numbers [12, p. 210], (5.15) and (5.16) will further give the next two identities:

$$\sum_{j=k}^{n}\binom{n}{j}\mathscr{B}_{n-j}S(j,k)-\sum_{j=k}^{n-1}\binom{n-1}{j}\mathscr{B}_{n-1-j}S(j+1,k+1)$$
$$=\sum_{j=k}^{n}\binom{n-1}{j-1}\mathscr{B}_{n-j+1}S(j-1,k-1),$$
$$k\sum_{j=k}^{n}\binom{n}{j}\mathscr{B}_{n-j}S(j,k)=\sum_{j=k}^{n}\binom{n}{j-1}(\mathscr{B}_{n-j+2}-\mathscr{B}_{n-j+1})S(j-1,k-1).$$

5.3.6. The Riordan arrays related to the Actuarial polynomials

From Table 1, it can be seen that the Riordan array $R=(\mathrm{e}^{\beta t},1-\mathrm{e}^{t})$ with the generic element

$$(-1)^k\sum_{j=0}^{n}\binom{n}{j}\beta^{n-j}S(j,k)$$

corresponds to the Actuarial polynomials [28, Section 4.3.4]. In the light of

$$A(t)=\frac{t}{\log(1-t)}=\sum_{j=0}^{\infty}(-1)^{j+1}b_j(0)\frac{t^j}{j!},$$

we have

$$d_{n+1,k+1}=\sum_{j=0}^{\infty}\frac{n+1}{k+1}\binom{k+j}{j}(-1)^{j+1}b_j(0)d_{n,k+j}.$$

Next, since $g'(t)=\beta\mathrm{e}^{\beta t}$, then

$$\tilde{d}_{n-1,k}=(-1)^k\sum_{j=0}^{n-1}\binom{n-1}{j}\beta^{n-j}S(j,k).$$

Based on (5.10) and (5.11), the following two identities hold:

$$\sum_{j=k}^{n}\binom{n-1}{j-1}\beta^{n-j}S(j,k)=\sum_{j=k}^{n}\binom{n-1}{j-1}(1+\beta)^{n-j}S(j-1,k-1),$$
$$k\sum_{j=k}^{n}\binom{n}{j}\beta^{n-j}S(j,k)=\sum_{j=k}^{n}\binom{n}{j-1}((1+\beta)^{n-j+1}-\beta^{n-j+1})S(j-1,k-1).$$

The inverse of R is

$$((1-t)^{-\beta}, \log(1-t)) = \left((-1)^n \sum_{j=0}^{n} \binom{n}{j} (-\beta)_{n-j} s(j,k) \right)_{n,k\in\mathbb{N}}.$$

For this array,

$$A(t) = \frac{t}{1-\mathrm{e}^t} = -\sum_{j=0}^{\infty} B_j \frac{t^j}{j!},$$

which leads us to the recurrence

$$d_{n+1,k+1} = -\sum_{j=0}^{\infty} \frac{n+1}{k+1} \binom{k+j}{j} B_j d_{n,k+j}.$$

Since $g'(t) = \beta(1-t)^{-\beta-1}$, then

$$\tilde{d}_{n-1,k} = (-1)^n \sum_{j=0}^{n-1} \binom{n-1}{j} (-\beta)_{n-j} s(j,k).$$

With some computation, we deduce from (5.10) and (5.11) that

$$\sum_{j=k}^{n} \binom{n-1}{j-1} (-\beta)_{n-j} s(j,k) = \sum_{j=k}^{n} \frac{(n-1)!}{(j-1)!} (-1)^{n-j} \left(\sum_{l=0}^{n-j} \binom{\beta+l-1}{l} \right) s(j-1,k-1),$$

$$k\sum_{j=k}^{n} \binom{n}{j} (-\beta)_{n-j} s(j,k) = \sum_{j=k}^{n} \frac{n!}{(j-1)!} (-1)^{n-j} \left(\sum_{l=0}^{n-j} \binom{\beta+l-1}{l} \frac{1}{n-j+1-l} \right) s(j-1,k-1).$$

5.3.7. Some classical Riordan arrays

Let us consider the classical Riordan array [36, p. 227]

$$\left(\frac{1}{(1-t)^{p+1}} \log \frac{1}{1-t}, \frac{t}{(1-t)^q} \right) = \left(\left(H_{p+n+(q-1)k} - H_{p+qk} \right) \binom{p+n+(q-1)k}{n-k} \right)_{n,k\in\mathbb{N}},$$

where $H_n = \sum_{k=1}^{n} 1/k$ are the Harmonic numbers. From

$$g'(t) = \frac{p+1}{(1-t)^{p+2}} \log \frac{1}{1-t} + \frac{1}{(1-t)^{p+2}},$$

we obtain that

$$\tilde{d}_{n,k} = (p+1)\left(H_{p+(q-1)k+n+1} - H_{p+qk+1} \right) \binom{p+(q-1)k+n+1}{p+qk+1} + \binom{p+(q-1)k+n+1}{p+qk+1}.$$

Additionally, $f_k = \binom{q+k-2}{k-1}$. Therefore, (5.7) and (5.8) give

$$\left(\frac{1}{p+qn+1} + H_{p+n+(q-1)k} - H_{p+qk+1} \right) \frac{p+qn+1}{p+qk+1} \binom{p+n+(q-1)k}{n-k}$$
$$= \sum_{l=k}^{n} (n-l+1) \binom{q+n-l-1}{n-l} \left(H_{p+l-k+q(k-1)} - H_{p+q(k-1)} \right) \binom{p+l-k+q(k-1)}{l-k},$$

$$\left(H_{p+n+(q-1)k} - H_{p+qk} \right) \binom{p+n+(q-1)k}{n-k}$$
$$= \sum_{l=k}^{n} \binom{q+n-l-1}{n-l} \left(H_{p+l-k+q(k-1)} - H_{p+q(k-1)} \right) \binom{p+l-k+q(k-1)}{l-k}.$$

When $p = 0$ and $q = 1$, the above two identities reduce to

$$(H_{n+1} - H_{k+1})\binom{n+1}{k+1} = \sum_{l=k}^{n}(n-l+1)\,(H_{l-1} - H_{k-1})\binom{l-1}{k-1},$$
$$(H_n - H_k)\binom{n}{k} = \sum_{l=k}^{n}(H_{l-1} - H_{k-1})\binom{l-1}{k-1}.$$

The classical Riordan array $(\mathrm{e}^{pt}, t\mathrm{e}^{qt})$ has $d_{n,k} = \frac{(p+qk)^{n-k}}{(n-k)!}$ as its generic element [36, p. 218]. By computation $\tilde{d}_{n-1,k} = p\frac{(p+qk)^{n-k-1}}{(n-k-1)!}$ and $f_k = \frac{q^{k-1}}{(k-1)!}$, then from (5.7) and (5.8) we have

$$(p+qn)(p+qk)^{n-k-1} = \sum_{l=k}^{n}\binom{n-k}{l-k}(n-l+1)q^{n-l}(p+qk-q)^{l-k},$$
$$(p+qk)^{n-k} = \sum_{l=k}^{n}\binom{n-k}{l-k}q^{n-l}(p+qk-q)^{l-k}.$$

The second identity is trivial. Let $l \to l+k$ and $n \to n+k$, then the first identity turns to

$$(p+q(n+k))(p+qk)^{n-1} = \sum_{l=0}^{n}\binom{n}{l}(n-l+1)q^{n-l}(p+qk-q)^{l}.$$

Finally, for the classical Riordan array $((1+\alpha t)^p, t(1+\alpha t)^q) = \left(\binom{p+qk}{n-k}\alpha^{n-k}\right)_{n,k\in\mathbb{N}}$ (see [36, p. 224]), we have $\tilde{d}_{n-1,k} = p\binom{p+qk-1}{n-k-1}\alpha^{n-k}$ and $f_k = \binom{q}{k-1}\alpha^{k-1}$, then (5.7) and (5.8) yield

$$\frac{p+qn}{p+qk}\binom{p+qk}{n-k} = \sum_{l=k}^{n}\binom{q}{n-l}\binom{p+qk-q}{l-k}(n-l+1),$$
$$\binom{p+qk}{n-k} = \sum_{l=k}^{n}\binom{q}{n-l}\binom{p+qk-q}{l-k},$$

where the second identity is also a trivial one.

5.3.8. The Riordan arrays related to the generalized Bernoulli and Euler polynomials

According to Section 4.1.2, we know that the exponential Riordan array $\left(\left(\frac{t}{\mathrm{e}^t-1}\right)^{\alpha}, t\right) = \left(\binom{n}{k}B_{n-k}^{(\alpha)}\right)_{n,k\in\mathbb{N}}$ corresponds to the generalized Bernoulli polynomials. Since

$$g'(t) = \alpha\left(\frac{t}{\mathrm{e}^t-1}\right)^{\alpha-1}\frac{\mathrm{e}^t-1-t\mathrm{e}^t}{(\mathrm{e}^t-1)^2} = \alpha\left(\frac{t}{\mathrm{e}^t-1}\right)^{\alpha-1}\left(\frac{1}{\mathrm{e}^t-1} - \frac{t}{\mathrm{e}^t-1} - \frac{t}{(\mathrm{e}^t-1)^2}\right),$$

then

$$\begin{aligned}\tilde{d}_{n-1,k} &= \left[\frac{t^{n-1}}{(n-1)!}\right]g'(t)\frac{t^k}{k!}\\ &= \alpha\frac{(n-1)!}{k!}[t^{n-1-k}]\left(\frac{1}{t}\left(\frac{t}{\mathrm{e}^t-1}\right)^{\alpha} - \left(\frac{t}{\mathrm{e}^t-1}\right)^{\alpha} - \frac{1}{t}\left(\frac{t}{\mathrm{e}^t-1}\right)^{\alpha+1}\right)\\ &= \frac{\alpha}{n-k}\binom{n-1}{k}\left(B_{n-k}^{(\alpha)} - (n-k)B_{n-k-1}^{(\alpha)} - B_{n-k}^{(\alpha+1)}\right)\\ &= \frac{\alpha}{n-k}\binom{n-1}{k}\left(B_{n-k}^{(\alpha)} - B_{n-k}^{(\alpha+1)}(1)\right),\end{aligned}$$

where the last step comes from the fact that [28, p. 95, Eq. (4.2.6)] $B_n^{(\alpha)}(x+1) = B_n^{(\alpha)}(x) + nB_{n-1}^{(\alpha-1)}(x)$. Now, the recurrence (5.10) gives

$$\binom{n}{k} B_{n-k}^{(\alpha)} - \frac{\alpha}{n-k}\binom{n-1}{k}\left(B_{n-k}^{(\alpha)} - B_{n-k}^{(\alpha+1)}(1)\right) = \binom{n-1}{k-1} B_{n-k}^{(\alpha)}.$$

Replacing k by $n-k$ and doing some simplification, we have

$$B_k^{(\alpha+1)}(1) = \left(1 - \frac{k}{\alpha}\right) B_k^{(\alpha)}(0).$$

Analogously, for the Riordan array $\left(\left(\frac{2}{e^t+1}\right)^{\alpha}, t\right) = \left(\binom{n}{k} E_{n-k}^{(\alpha)}(0)\right)_{n,k\in\mathbb{N}}$, we have

$$g'(t) = -\alpha\left(\left(\frac{2}{e^t+1}\right)^{\alpha} - \frac{1}{2}\left(\frac{2}{e^t+1}\right)^{\alpha+1}\right)$$

and then

$$\tilde{d}_{n-1,k} = -\alpha\binom{n-1}{k}\left(E_{n-1-k}^{(\alpha)}(0) - \frac{1}{2}E_{n-1-k}^{(\alpha+1)}(0)\right).$$

Therefore, (5.10) yields

$$\binom{n}{k} E_{n-k}^{(\alpha)}(0) + \alpha\binom{n-1}{k}\left(E_{n-1-k}^{(\alpha)}(0) - \frac{1}{2}E_{n-1-k}^{(\alpha+1)}(0)\right) = \binom{n-1}{k-1} E_{n-k}^{(\alpha)}(0).$$

Making use of the formula [28, p. 103, Eq. (4.2.11)] $E_n^{(\alpha)}(x+1) = 2E_n^{(\alpha-1)}(x) - E_n^{(\alpha)}(x)$, we get

$$2E_k^{(\alpha)}(0) = -\alpha E_{k-1}^{(\alpha+1)}(1).$$

6. Some applications

6.1. Inverse relations

The problem of inverse relations is one of the most interesting subjects in combinatorics and there is a vast literature on it. For a fundamental discussion, the reader is referred to the famous book [25] by Riordan.

The theory of matrices is an important tool to study the inverse relations. In fact, let $A = (a_{n,k})_{0\le k\le n<\infty}$ and $B = (b_{n,k})_{0\le k\le n<\infty}$ be two infinite lower triangular matrices. Then A and B are inverse to each other (i.e., $\sum_{l=k}^{n} a_{n,l}b_{l,k} = \delta_{n,k}$) is equivalent to the existence of the following inverse relation

$$y_n = \sum_{k=0}^{n} a_{n,k}x_k, \qquad x_n = \sum_{k=0}^{n} b_{n,k}y_k. \tag{6.1}$$

According to the rule explained above, it is easy to see that the Riordan arrays can be used to study the inverse relations. In particular, given a Sheffer sequence, by Theorem 3.2, we can get the corresponding Riordan array R and then the inverse array R^{-1}. From R and R^{-1}, an inverse relation can be established. Thus, we have a method to find the inverse relations systematically and we can further get the inversion of a given combinatorial sum. The reader can read the paper [32] by Shapiro et al. to see some examples.

It should be noticed that Corsani, Merlini and Sprugnoli [13] have also investigated the problem of inverting combinatorial sums by the theory of Riordan arrays. However, the combinatorial sums they dealt with, such as $a_n = \sum_k d_{n,k}b_k$, could not be inverted in terms of the orthogonality relation because the infinite, lower triangular array $P = (d_{n,k})_{n,k\in\mathbb{N}}$'s diagonal elements were equal to zero (except $d_{0,0}$). They presented that for these sums, a left-inverse $\bar{P}$ such that $\bar{P}P = I$ could be found and therefore they could left-invert the original combinatorial sum and obtain $b_n = \sum_k \bar{d}_{n,k}a_k$. More results on both the problem of inverse relations and the theory of Riordan arrays can be found in [14,20,22].

Additionally, in the book [28], Roman presented an interesting approach to the inverse relations based on the theories of Sheffer sequences and umbral calculus (see [28, Theorem 5.5.1]). In this section, we will also show that Roman's approach is trivial from the theory of Riordan arrays point of view; it is equivalent to giving a Riordan array R and its inverse R^{-1}.

Let us go into the details. All the Riordan arrays presented from now on have inverse, in other words, they are all determined by an invertible series and a delta series. As just introduced, from such a Riordan array $(g(t), f(t)) = (a_{n,k})_{n,k\in\mathbb{N}}$ and its inverse $(1/g(\bar{f}(t)), \bar{f}(t)) = (b_{n,k})_{n,k\in\mathbb{N}}$, an inverse relation which has the form (6.1) can be obtained, and we can deduce the inversion of a combinatorial sum by the corresponding inverse relation.

A particular instance is the generalized Stirling number pair introduced and studied by Hsu [15]. Let $f(t)$ be a delta series and let

$$\frac{1}{k!}(f(t))^k = \sum_{n=0}^{\infty} A_1(n,k)\frac{t^n}{n!}, \qquad \frac{1}{k!}(\bar{f}(t))^k = \sum_{n=0}^{\infty} A_2(n,k)\frac{t^n}{n!}.$$

Then $A_1(n,k)$ and $A_2(n,k)$ are called a *generalized Stirling number pair*. In the context of the theory of Riordan arrays, $A_1(n,k)$ and $A_2(n,k)$ are generic elements of the Riordan arrays $(1, f(t))$ and $(1, \bar{f}(t))$, respectively. So we have

$$y_n = \sum_{k=0}^{n} A_1(n,k)x_k, \qquad x_n = \sum_{k=0}^{n} A_2(n,k)y_k.$$

The interested readers can see the relative papers, e.g., [15,46], for the applications of the generalized Stirling number pairs.

From Section 4.1.2, we can see that the Riordan array $\left(\left(\frac{t}{e^t-1}\right)^{\alpha}, t\right) = \left(\binom{n}{k} B_{n-k}^{(\alpha)}\right)_{n,k\in\mathbb{N}}$ has the inverse $\left(\left(\frac{e^t-1}{t}\right)^{\alpha}, t\right) = \left(\binom{n}{k} B_{n-k}^{(-\alpha)}\right)_{n,k\in\mathbb{N}}$. Thus, we have

$$y_n = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}^{(\alpha)} x_k, \qquad x_n = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}^{(-\alpha)} y_k.$$

Particularly, when $\alpha = 1$, the inverse relation above will reduce to

$$y_n = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}x_k, \qquad x_n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n-k+1} y_k. \tag{6.2}$$

Since

$$\begin{aligned}\sum_{k=m}^{n} \binom{n}{k} B_{n-k}S(k,m) &= \left[\frac{t^n}{n!}\right] \frac{t}{e^t-1}\frac{1}{m!}(e^t-1)^m \\ &= \frac{n!}{m!}[t^{n-1}](e^t-1)^{m-1} = \frac{n}{m}S(n-1,m-1),\end{aligned}$$

then the inverse relation (6.2) gives

$$\sum_{k=m}^{n} \binom{n}{k} \frac{1}{n-k+1}\frac{k}{m}S(k-1,m-1) = \frac{1}{m}\sum_{k=m}^{n} \binom{n}{k-1} S(k-1,m-1) = S(n,m).$$

Also by (6.2), from

$$\sum_{k=0}^{n} \binom{n}{k} B_{n-k}x^k = B_n(x),$$

we have

$$\sum_{k=0}^{n} \binom{n}{k} \frac{1}{n-k+1} B_k(x) = x^n.$$

From Section 4.1.3, we know that the Riordan array $\left(\frac{2}{e^t+1}, t\right) = \left(\binom{n}{k} E_{n-k}(0)\right)_{n,k\in\mathbb{N}}$ has the inverse $\left(\frac{e^t+1}{2}, t\right)$, whose generic element $d_{n,k}$ is $\frac{1}{2}\binom{n}{k}$ for $n \neq k$ and 1 for $n = k$. Thus, from

$$\sum_{k=0}^{n} \binom{n}{k} E_{n-k}(0)x^k = E_n(x),$$

we have

$$\sum_{k=0}^{n-1} \frac{1}{2}\binom{n}{k} E_k(x) + E_n(x) = x^n.$$

Next, from Section 4.2.1, we know that the inverse of the Riordan array $((1+t^2)^{-\lambda_0}, -2t/(1+t^2))$ is $\left(\left(\frac{2}{1+\sqrt{1-t^2}}\right)^{\lambda_0}, \frac{-t}{1+\sqrt{1-t^2}}\right)$. In view of the generic elements of these two arrays, we have the following inverse pair:

$$y_n = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{\binom{-\lambda}{n-2k}\binom{-\lambda_0-n+2k}{k}}{\binom{-\lambda}{n}} (-2)^{n-2k} x_{n-2k}, \tag{6.3}$$

$$x_n = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{\binom{-\lambda}{n-2k}(-1)^n(\lambda_0+n-2k)(\lambda_0+n-1)_{k-1}}{\binom{-\lambda}{n}2^n k!} y_{n-2k}. \tag{6.4}$$

For instance, since the polynomials $s_n(x)$ of the Gegenbauer case have the expression (4.4), then x^n can be expressed as a linear combination of $s_n(x)$ by (6.4).

Finally, from Section 4.2.4, the inverse of the Riordan array $(1/\varepsilon_a(t), t)$ is $(\varepsilon_a(t), t)$. Thus, based on their generic elements, the following inverse relation holds:

$$y_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-a)^{n-k} q^{\binom{n-k}{2}} x_k, \qquad x_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^{n-k} y_k.$$

As an example, from (4.5), we have

$$x^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^{n-k} [x]_{a,k}.$$

We could give more inverse relations by this way; instead in the following we will focus on the theorem below.

Theorem 6.1. *For any invertible series $h(t)$, and any delta series $f(t)$ and $l(t)$, we can construct two Riordan arrays which are inverse to each other:*

$$R = (h(\bar{f}(t)), l(\bar{f}(t))), \qquad R^{-1} = \left(\frac{1}{h(\bar{l}(t))}, f(\bar{l}(t))\right).$$

Suppose the generic elements are $a_{n,k}$ and $b_{n,k}$, respectively, then we have the following inverse pair:

$$y_n = \sum_{k=0}^{n} a_{n,k} x_k, \qquad x_n = \sum_{k=0}^{n} b_{n,k} y_k.$$

From the theory of Riordan arrays point of view, this theorem is trivial. But it is actually equivalent to the following theorem, which generalizes the result of Roman [28, Theorem 5.5.1] and is not trivial.

Theorem 6.2. *Let $p_n(x)$ be associated to $f(t)$ and let $q_n(x)$ be associated to $l(t)$. Then for any invertible series $h(t)$ we have the inverse pair*

$$y_n = \sum_{k=0}^{n} \frac{\langle h(t)(l(t))^k | p_n(x)\rangle}{c_k} x_k, \qquad x_n = \sum_{k=0}^{n} \frac{\langle h(t)^{-1}(f(t))^k | q_n(x)\rangle}{c_k} y_k.$$

The proof of Theorem 6.2 is the same as that of [28, Theorem 5.5.1], and in this theorem, the notation $\langle L|p(x)\rangle$ denotes the action of a linear functional L on a polynomial $p(x)$ and if $f(t) = \sum_{k=0}^{\infty} a_k t^k$, then $\langle f(t)|x^n\rangle = [t^n/c_n]f(t) = c_n a_n$. For more details, the reader is referred to [27,28]. Now, let us verify the equivalence of Theorems 6.1 and 6.2.

In fact, let $p_n(x) = \sum_{j=0}^{n} a_{n,j} x^j$, then

$$\begin{aligned}\frac{1}{c_k}\langle h(t)(l(t))^k | p_n(x)\rangle &= \frac{1}{c_k}\left\langle h(t)(l(t))^k \left| \sum_{j=0}^{n} a_{n,j} x^j \right.\right\rangle \\ &= \frac{1}{c_k}\sum_{j=0}^{n} a_{n,j}\langle h(t)(l(t))^k | x^j\rangle = \frac{1}{c_k}\sum_{j=0}^{n} a_{n,j}\left[\frac{t^j}{c_j}\right]\left(h(t)(l(t))^k\right). \end{aligned} \tag{6.5}$$

Since $p_n(x)$ is the Sheffer sequence for the pair $(1, f(t))$, then according to Theorem 3.2, $a_{n,j}$ is the generic element of the Riordan array $(1, \bar{f}(t))$. By the summation rule (2.2), the right-hand side of (6.5) equals

$$\frac{1}{c_k}\left[\frac{t^n}{c_n}\right] h(\bar{f}(t))(l(\bar{f}(t)))^k = \left[\frac{t^n}{c_n}\right] h(\bar{f}(t))\frac{(l(\bar{f}(t)))^k}{c_k},$$

which means $\frac{1}{c_k}\langle h(t)(l(t))^k | p_n(x)\rangle$ is the generic element of the Riordan array $(h(\bar{f}(t)), l(\bar{f}(t)))$. Analogously, we can prove that $\frac{1}{c_k}\langle h(t)^{-1}(f(t))^k | q_n(x)\rangle$ is the generic element of $\left(\frac{1}{h(\bar{l}(t))}, f(\bar{l}(t))\right)$. Therefore, the equivalence of these two theorems is verified.

Because the examples presented in [28, Section 5.5] are all special cases of Theorem 6.2, they can also be deduced from Theorem 6.1. To show the power of the theorem, we would like to give two instances.

6.1.1. $f(t) = \frac{-t}{1+\sqrt{1-t^2}}$, $l(t) = \frac{2-2\sqrt{1-t^2}}{t}$, $h(t) = \frac{2-2\sqrt{1-t^2}}{t^2}$

These series come from the pair $\left(\frac{2-2\sqrt{1-t^2}}{t^2}, \frac{-t}{1+\sqrt{1-t^2}}\right)$ which corresponds to the Sheffer sequence $U_n(x)$ introduced in Section 4.2.2. From these series, we have $\bar{f}(t) = \frac{-2t}{1+t^2}$, $\bar{l}(t) = \frac{4t}{4+t^2}$ and then

$$h(\bar{f}(t)) = 1 + t^2, \qquad l(\bar{f}(t)) = -2t; \qquad \frac{1}{h(\bar{l}(t))} = \frac{4}{4+t^2}, \qquad f(\bar{l}(t)) = -\frac{1}{2}t.$$

The generic element of $(h(\bar{f}(t)), l(\bar{f}(t)))$ is

$$a_{n,k} = \left[\frac{t^n}{c_n}\right](1+t^2)\frac{(-2)^k t^k}{c_k} = \frac{c_n}{c_k}(-2)^k[t^{n-k}](1+t^2) = \frac{c_n}{c_k}(-2)^k(\delta_{n,k} + \delta_{n,k+2}),$$

that is,

$$a_{n,k} = \begin{cases} (-2)^n, & \text{if } n = k, \\ \dfrac{c_n}{c_k}(-2)^k, & \text{if } n = k+2, \\ 0, & \text{else.} \end{cases}$$

The generic element of $\left(\frac{1}{h(\bar{l}(t))}, f(\bar{l}(t))\right)$ is

$$b_{n,k} = \left[\frac{t^n}{c_n}\right]\frac{4}{4+t^2}\left(-\frac{1}{2}\right)^k\frac{t^k}{c_k} = \frac{c_n}{c_k}\left(-\frac{1}{2}\right)^k[t^{n-k}]\sum_{i=0}^{\infty}\frac{(-1)^i}{2^{2i}}t^{2i}$$

$$= \begin{cases} 0, & n-k \text{ odd}, \\ \dfrac{c_n}{c_k}(-1)^k(-1)^{\frac{n-k}{2}}\dfrac{1}{2^n}, & n-k \text{ even}. \end{cases}$$

Because these two Riordan arrays are inverse to each other, we have

$$\begin{pmatrix} 1 & & & & \\ 0 & -2 & & & \\ c_2 & 0 & 4 & & \\ 0 & -2\dfrac{c_3}{c_1} & 0 & -8 & \\ 0 & 0 & 4\dfrac{c_4}{c_2} & 0 & 16 \end{pmatrix} \begin{pmatrix} 1 & & & & \\ 0 & -\dfrac{1}{2} & & & \\ -\dfrac{1}{4}c_2 & 0 & \dfrac{1}{4} & & \\ 0 & \dfrac{1}{8}\dfrac{c_3}{c_1} & 0 & -\dfrac{1}{8} & \\ \dfrac{1}{16}c_4 & 0 & -\dfrac{1}{16}\dfrac{c_4}{c_2} & 0 & \dfrac{1}{16} \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}.$$

By Theorem 6.1, the following inverse relation holds:

$$y_n = (-2)^n x_n + \frac{c_n}{c_{n-2}}(-2)^{n-2}x_{n-2},$$

$$x_n = \sum_{\substack{k=0 \\ n-k \text{ even}}}^{n} \frac{c_n}{c_k}(-1)^k(-1)^{\frac{n-k}{2}}\frac{1}{2^n}y_k = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{c_n}{c_{n-2k}}(-1)^{n-k}\frac{1}{2^n}y_{n-2k},$$

which, by letting $y_n/c_n \to y_n$ and $(-2)^n x_n/c_n \to x_n$, reduces to

$$y_n = x_n + x_{n-2}, \qquad x_n = \sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k y_{n-2k}.$$

6.1.2. $f(t) = \frac{1+t-\sqrt{1+2t}}{t}$, $l(t) = \frac{2t}{1+\sqrt{1+2t}}$, $h(t) = \left(\frac{2}{1+\sqrt{1+2t}}\right)^{1+\alpha+\beta}$

These series come from the pair $\left(\left(\frac{2}{1+\sqrt{1+2t}}\right)^{1+\alpha+\beta}, \frac{1+t-\sqrt{1+2t}}{t}\right)$ which corresponds to the Sheffer sequence $J_n(x)$ introduced in Section 4.2.3. By computation, we have $\bar{f}(t) = \frac{2t}{(1-t)^2}$, $\bar{l}(t) = \frac{t^2+2t}{2}$ and then

$$h(\bar{f}(t)) = (1-t)^{1+\alpha+\beta}, \qquad l(\bar{f}(t)) = \frac{2t}{1-t}; \qquad \frac{1}{h(\bar{l}(t))} = \left(\frac{2+t}{2}\right)^{1+\alpha+\beta}, \qquad f(\bar{l}(t)) = \frac{t}{2+t}.$$

The generic element of $(h(\bar{f}(t)), l(\bar{f}(t)))$ is

$$a_{n,k} = \left[\frac{t^n}{c_n}\right]\frac{1}{c_k}(1-t)^{1+\alpha+\beta}\left(\frac{2t}{1-t}\right)^k = 2^k\frac{c_n}{c_k}\binom{1+\alpha+\beta-k}{n-k}(-1)^{n-k};$$

and the generic element of $\left(\frac{1}{h(\bar{l}(t))}, f(\bar{l}(t))\right)$ is

$$b_{n,k} = \left[\frac{t^n}{c_n}\right]\frac{1}{c_k}\left(\frac{2+t}{2}\right)^{1+\alpha+\beta}\left(\frac{t}{2+t}\right)^k = \frac{c_n}{c_k}\binom{1+\alpha+\beta-k}{n-k}\frac{1}{2^n}.$$

Therefore, we have the inverse pair

$$y_n = \sum_{k=0}^{n}\frac{c_n}{c_k}\binom{1+\alpha+\beta-k}{n-k}(-1)^{n-k}2^k x_k,$$

$$x_n = \sum_{k=0}^{n}\frac{c_n}{c_k}\binom{1+\alpha+\beta-k}{n-k}\frac{1}{2^n}y_k.$$

Upon letting $y_n/c_n \to y_n$, $2^n x_n/c_n \to x_n$ and $-2-\alpha-\beta \to p$, this inverse pair will reduce to

$$y_n = \sum_{k=0}^{n} \binom{n+p}{n-k} x_k, \qquad x_n = \sum_{k=0}^{n} \binom{n+p}{n-k} (-1)^{n-k} y_k,$$

which is the $c = 1$ case of [25, p. 69, Table 2.6, Entry 5].

Now, we briefly introduce the Riordan inverse chain. Let A, B, C be three Riordan arrays defined by

$$A = (g(t), f(t)) = (a_{n,k})_{n,k\in\mathbb{N}}, \qquad B = (h_1(t), l_1(t)) = (b_{n,k})_{n,k\in\mathbb{N}},$$
$$C = (h_2(t), l_2(t)) = (c_{n,k})_{n,k\in\mathbb{N}}.$$

Suppose the sequences x_n, y_n satisfy the relation $y_n = \sum_{k=0}^{n} a_{n,k} x_k$ and construct two sequences x_n^*, y_n^* as follows:

$$x_n^* = \sum_{k=0}^{n} b_{n,k} x_k, \qquad y_n^* = \sum_{k=0}^{n} c_{n,k} y_k.$$

If $y_n^* = \sum_{k=0}^{n} a_{n,k} x_k^*$, then $(A; B, C)$ is called a *Riordan inverse chain*. Additionally, we say that the sequences x_n, y_n form a *Riordan pair* of A and we denote it by $(A; x_n, y_n)$.

Ma [17] investigated the Riordan inverse chains for the classical case $c_n = 1$ and gave some applications to combinatorial sums. It can be verified that the results established for the classical case in [17] are valid for the general case. We present here the main theorem of [17].

Theorem 6.3. *Let A, B, C be the three Riordan arrays defined above, then the Riordan inverse chain $(A; B, C)$ exists if and only if*

$$h_1(t) g(\bar{f}(t)) = h_2(\bar{f}(t)) g(l_2(\bar{f}(t))), \qquad l_1(t) = f(l_2(\bar{f}(t))),$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$.

6.2. Connection constants problem

The connection constants problem is to determine the connection constants $a_{n,k}$ in the expression $r_n(x) = \sum_{k=0}^{n} a_{n,k} s_k(x)$, where $r_n(x)$ and $s_n(x)$ are sequences of polynomials. Roman shows that the umbral methods give an explicit solution to this problem when the sequences involved are Sheffer (see [27, Theorem 8.5] and [28, Section 5.1]). In this section, we will consider this problem in the context of the theory of Riordan arrays. Our result (Theorem 6.4) is equivalent to Roman's one [27, Theorem 8.5], but easier to follow.

Theorem 6.4. *Let $s_n(x)$ be Sheffer for $(g(t), f(t))$ and let $r_n(x)$ be Sheffer for $(h(t), l(t))$. Suppose $r_n(x) = \sum_{k=0}^{n} a_{n,k} s_k(x)$, then $a_{n,k}$ is the generic element of the Riordan array*

$$\left(\frac{g(\bar{l}(t))}{h(\bar{l}(t))}, f(\bar{l}(t)) \right).$$

Proof. We prove the theorem by the technique of matrix representation. Let

$$S[x] = (s_0(x), s_1(x), \ldots)^{\mathrm{T}}, \qquad R[x] = (r_0(x), r_1(x), \ldots)^{\mathrm{T}}, \quad X = (1, x, x^2, \ldots)^{\mathrm{T}}.$$

According to Theorem 3.2,

$$S[x] = \left(\frac{1}{g(\bar{f}(t))}, \bar{f}(t) \right) * X, \qquad R[x] = \left(\frac{1}{h(\bar{l}(t))}, \bar{l}(t) \right) * X.$$

Then $X = (g(t), f(t)) * S[x]$ and by (2.3), we have

$$R[x] = \left(\frac{1}{h(\bar{l}(t))}, \bar{l}(t) \right) * (g(t), f(t)) * S[x] = \left(\frac{g(\bar{l}(t))}{h(\bar{l}(t))}, f(\bar{l}(t)) \right) * S[x].$$

This completes the proof. □

Corollary 6.5. *Let $s_n(x)$ be Sheffer for $(g(t), f(t))$ and suppose $x^n = \sum_{k=0}^{n} a_{n,k} s_k(x)$, then $a_{n,k}$ is the generic element of the Riordan array $(g(t), f(t))$.*

Let $s_n(x) = \sum_{k=0}^{n} s_{n,k} x^k$. According to Theorem 3.2, we know that $(s_{n,k})_{n,k\in\mathbb{N}}$ is just the inverse of the Riordan array $(g(t), f(t))$. This fact also leads us to Corollary 6.5. The reader can find some examples in Section 6.1. Now let us give some applications of Theorem 6.4.

6.2.1. The polynomials of the Chebyshev case

From Section 4.2.2 or [27, p. 99], we know that $T_n(x)$ form the Sheffer sequence for the pair $\left(\frac{1}{\sqrt{1-t^2}}, \frac{-t}{1+\sqrt{1-t^2}}\right)$ and $U_n(x)$ form the Sheffer sequence for the pair $\left(\frac{2-2\sqrt{1-t^2}}{t^2}, \frac{-t}{1+\sqrt{1-t^2}}\right)$. Suppose $U_n(x) = \sum_{k=0}^{n} a_{n,k} T_k(x)$, then by Theorem 6.4, $a_{n,k}$ is the generic element of the Riordan array $(1/(1-t^2), t)$ with respect to $c_n = (-1)^n$, so $a_{n,k} = 1$ if $n-k$ is even and 0 if $n-k$ is odd, and we have

$$U_n(x) = \sum_{k=0}^{n} a_{n,k} T_k(x) = \sum_{\substack{k=0 \\ n-k \text{ even}}}^{n} T_k(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} T_{n-2k}(x).$$

Next, suppose $T_n(x) = \sum_{k=0}^{n} b_{n,k} U_k(x)$, then $b_{n,k}$ is the generic element of the Riordan array $(1-t^2, t)$. By computation, we have $b_{n,k} = \delta_{n,k} - \delta_{n,k+2}$, then

$$T_n(x) = U_n(x) - U_{n-2}(x).$$

6.2.2. The generalized Bernoulli and Euler polynomials

In 2003, Cheon [8] studied the classical Bernoulli polynomials $B_n(x)$ and the classical Euler polynomials $E_n(x)$, by making use of the technique of matrix representation. Srivastava and Pintér [37] followed Cheon's work and established two relations involving the generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ and the generalized Euler polynomials $E_n^{(\alpha)}(x)$.

We also presented two relations between $B_n^{(\alpha)}(x)$ and $E_n^{(\alpha)}(x)$ with matrix representation [40], which are in fact the generalizations of the results of [8,37], and in this section we will give them again by the theory of Riordan arrays.

From Sections 4.1.2 and 4.1.3, we know that $B_n^{(\alpha)}(x)$ are Sheffer for the pair $\left(\left(\frac{e^t-1}{t}\right)^{\alpha}, t\right)$ and $E_n^{(\alpha)}(x)$ are Sheffer for the pair $\left(\left(\frac{e^t+1}{2}\right)^{\alpha}, t\right)$. Suppose $B_n^{(\alpha)}(x) = \sum_{k=0}^{n} a_{n,k} E_k^{(\beta)}(x)$, then $a_{n,k}$ is the generic element of the Riordan array $\left(\left(\frac{2}{e^t+1}\right)^{-\beta}\left(\frac{t}{e^t-1}\right)^{\alpha}, t\right)$. By computation, we have

$$\begin{aligned} a_{n,k} &= \left[\frac{t^n}{n!}\right]\left(\frac{2}{e^t+1}\right)^{-\beta}\left(\frac{t}{e^t-1}\right)^{\alpha}\frac{t^k}{k!} = \frac{n!}{k!}[t^{n-k}]\sum_{i=0}^{\infty} E_i^{(-\beta)}(0)\frac{t^i}{i!}\sum_{j=0}^{\infty} B_j^{(\alpha)}(0)\frac{t^j}{j!} \\ &= \binom{n}{k}\sum_{i=0}^{n-k}\binom{n-k}{i} E_i^{(-\beta)}(0) B_{n-k-i}^{(\alpha)}(0). \end{aligned}$$

Since

$$\begin{aligned} \sum_{i=0}^{\infty} E_i^{(-\beta)}(0)\frac{t^i}{i!} &= \left(\frac{2}{e^t+1}\right)^{-\beta} = \frac{1}{2^{\beta}}(e^t+1)^{\beta} = \frac{1}{2^{\beta}}\sum_{m=0}^{\infty}\binom{\beta}{m} e^{mt} \\ &= \frac{1}{2^{\beta}}\sum_{m=0}^{\infty}\binom{\beta}{m}\sum_{i=0}^{\infty}\frac{m^i t^i}{i!} = \sum_{i=0}^{\infty}\left(\frac{1}{2^{\beta}}\sum_{m=0}^{\infty}\binom{\beta}{m} m^i\right)\frac{t^i}{i!}, \end{aligned}$$

then $E_i^{(-\beta)}(0) = \frac{1}{2^\beta}\sum_{m=0}^{\infty}\binom{\beta}{m}m^i$ and

$$a_{n,k} = \binom{n}{k}\sum_{i=0}^{n-k}\binom{n-k}{i}\frac{1}{2^\beta}\sum_{m=0}^{\infty}\binom{\beta}{m}m^i B_{n-k-i}^{(\alpha)}(0) = \frac{1}{2^\beta}\binom{n}{k}\sum_{m=0}^{\infty}\binom{\beta}{m}B_{n-k}^{(\alpha)}(m),$$

$$B_n^{(\alpha)}(x) = \frac{1}{2^\beta}\sum_{k=0}^{n}\binom{n}{k}\sum_{m=0}^{\infty}\binom{\beta}{m}B_{n-k}^{(\alpha)}(m)E_k^{(\beta)}(x).$$

Next, suppose $E_n^{(\alpha)}(x) = \sum_{k=0}^{n} b_{n,k}B_k^{(\beta)}(x)$, where $\beta \in \mathbb{N}$, then $b_{n,k}$ is the generic element of the Riordan array $\left(\left(\frac{t}{e^t-1}\right)^{-\beta}\left(\frac{2}{e^t+1}\right)^{\alpha}, t\right)$. By computation,

$$b_{n,k} = \left[\frac{t^n}{n!}\right]\left(\frac{t}{e^t-1}\right)^{-\beta}\left(\frac{2}{e^t+1}\right)^{\alpha}\frac{t^k}{k!} = \binom{n}{k}\sum_{i=0}^{n-k}\binom{n-k}{i}B_i^{(-\beta)}(0)E_{n-k-i}^{(\alpha)}(0).$$

Since [28, p. 99] $S(n,k) = \binom{n}{k}B_{n-k}^{(-k)}(0)$, where $S(n,k)$ are the Stirling numbers of the second kind, then $S(i+\beta,\beta) = \binom{i+\beta}{\beta}B_i^{(-\beta)}(0)$ and we have

$$b_{n,k} = \binom{n}{k}\sum_{i=0}^{n-k}\binom{n-k}{i}\binom{i+\beta}{\beta}^{-1}S(i+\beta,\beta)E_{n-k-i}^{(\alpha)}(0),$$

$$E_n^{(\alpha)}(x) = \sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{n-k}\binom{n-k}{i}\binom{i+\beta}{\beta}^{-1}S(i+\beta,\beta)E_{n-k-i}^{(\alpha)}(0)B_k^{(\beta)}(x).$$

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