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Generalized higher order Bernoulli number pairs and generalized Stirling number pairs

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ABSTRACT

From a delta series $f(t)$ and its compositional inverse $g(t)$, Hsu defined the generalized Stirling number pair $(\hat{S}(n,k), \hat{s}(n,k))$. In this paper, we further define from $f(t)$ and $g(t)$ the generalized higher order Bernoulli number pair $(\hat{B}_n^{(z)}, \hat{b}_n^{(z)})$. Making use of the Bell polynomials, the potential polynomials as well as the Lagrange inversion formula, we give some explicit expressions and recurrences of the generalized higher order Bernoulli numbers, present the relations between the generalized higher order Bernoulli numbers of both kinds and the corresponding generalized Stirling numbers of both kinds, and study the relations between any two generalized higher order Bernoulli numbers. Moreover, we apply the general results to some special number pairs and obtain series of combinatorial identities. It can be found that the introduction of generalized Bernoulli number pair and generalized Stirling number pair provides a unified approach to lots of sequences in mathematics, and as a consequence, many known results are special cases of ours.

1. Introduction

The Stirling numbers of both kinds and the Bernoulli numbers of both kinds are among the most interesting and important sequences in mathematics and have numerous applications in combinatorics, number theory, numerical analysis, and other fields. The Stirling number pair is $(S(n,k), s(n,k))$, where $s(n,k)$ and $S(n,k)$ are the Stirling numbers of the first and second kinds. Analogously, the Bernoulli number pair is (B_n, b_n), where B_n and b_n are the Bernoulli numbers of the first and second kinds. In this paper, we will make a systematical study on various number pairs analogous to the Stirling number pair and the Bernoulli number pair.

Let

$$f(t) = \sum_{i=0}^{\infty} f_i \frac{t^i}{i!}$$

be a formal power series, then the *order* $o(f(t))$ of $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish. As showed in [15, Section 1.12], the series $f(t)$ has a compositional inverse, denoted by $\bar{f}(t)$ and satisfying $f(\bar{f}(t)) = \bar{f}(f(t)) = t$, if and only if $o(f(t)) = 1$. We call any series with $o(f(t)) = 1$ a *delta series*.

Given a delta series $f(t) = \sum_{i=1}^{\infty} f_i t^i/i!$ and its compositional inverse $g(t) = \sum_{i=1}^{\infty} g_i t^i/i!$, Hsu [21,22] introduced the *generalized Stirling number pair* $(\hat{S}(n,k), \hat{s}(n,k))$, where $\hat{S}(n,k)$ and $\hat{s}(n,k)$ are defined by the following generating functions:

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$$\frac{1}{k!}\big(f(t)\big)^k=\sum_{n=k}^{\infty}\hat{S}(n,k)\frac{t^n}{n!}\quad\text{and}\quad\frac{1}{k!}\big(g(t)\big)^k=\sum_{n=k}^{\infty}\hat{s}(n,k)\frac{t^n}{n!}.\tag{1.1}$$

We may call $\hat{s}(n,k)$ and $\hat{S}(n,k)$ the *generalized Stirling numbers of the first and second kinds*.

Let $f(t)=e^t-1$ and $g(t)=\log(1+t)$, then the pair turns to the classical Stirling number pair $(S(n,k),s(n,k))$. Let $f(t)=t/(1-t)$ and $g(t)=t/(1+t)$, then the pair turns to

$$\left(\binom{n-1}{k-1}\frac{n!}{k!},(-1)^{n-k}\binom{n-1}{k-1}\frac{n!}{k!}\right),$$

where $\binom{n-1}{k-1}\frac{n!}{k!}$ are the Lah numbers (see [15, Section 3.3]). Various special generalized Stirling number pairs can be found in Hsu's papers [21,22]. The readers are also referred to [19,37,39,40] for more results on generalized Stirling number pairs. Note that the generalized Stirling numbers defined above, say $\hat{S}(n,k)$, can be viewed as the elements of the exponential Riordan array $(1,f(t))$ or of the iteration matrix associated with $f(t)$. The theory of Riordan arrays can be found in the papers of Shapiro, Sprugnoli, et al. [30,31], and the results on the iteration matrix can be found in Comtet's book [15, Section 3.7]. Close relations between Riordan arrays, iteration matrices, Sheffer sequences, binomial sequences and generalized Stirling number pairs are demonstrated explicitly in [19,37].

On the other hand, for the delta series $f(t)$ and its compositional inverse $g(t)$ given above, we can define the *generalized higher order Bernoulli number pair* $(\hat{B}_n^{(z)},\hat{b}_n^{(z)})$, where $\hat{B}_n^{(z)}$ and $\hat{b}_n^{(z)}$ satisfy the following generating functions:

$$\left(\frac{f_1t}{f(t)}\right)^z=\sum_{n=0}^{\infty}\hat{B}_n^{(z)}\frac{t^n}{n!}\quad\text{and}\quad\left(\frac{g_1t}{g(t)}\right)^z=\sum_{n=0}^{\infty}\hat{b}_n^{(z)}\frac{t^n}{n!}.\tag{1.2}$$

The numbers $\hat{B}_n^{(z)}$ and $\hat{b}_n^{(z)}$ may be called the *generalized higher order Bernoulli numbers of the first and second kinds*, or more explicitly, the *higher order Bernoulli numbers of the first and second kinds associated with the delta series* $f(t)$. Additionally, the numbers $\hat{B}_n:=\hat{B}_n^{(1)}$ and $\hat{b}_n:=\hat{b}_n^{(1)}$ may be called the *generalized Bernoulli numbers of the first and second kinds*.

Within our knowledge, Carlitz [8,9] was the first to seriously consider generalized (higher order) Bernoulli numbers and he was mainly interested in arithmetic properties of them. Clarke [14] and Adelberg [2] introduced respectively the universal Bernoulli numbers $\tilde{B}_n$ and the higher order universal Bernoulli numbers $\tilde{B}_n^{(z)}$. Let the "universal" power series $F(t)$ be defined by $F(t):=t+\sum_{i=1}^{\infty}c_it^{i+1}/(i+1)$ where $c_1,c_2,\dots$ are indeterminates, and let $G(t):=\bar{F}(t)$ be the compositional inverse. Then $\tilde{B}_n$ and $\tilde{B}_n^{(z)}$ are defined by

$$\frac{t}{G(t)}=\sum_{n=0}^{\infty}\tilde{B}_n\frac{t^n}{n!}\quad\text{and}\quad\left(\frac{t}{G(t)}\right)^z=\sum_{n=0}^{\infty}\tilde{B}_n^{(z)}\frac{t^n}{n!}.$$

From the definitions, it is clear that the (higher order) universal Bernoulli numbers are essentially the generalized (higher order) Bernoulli numbers. For more results on universal Bernoulli numbers, the readers may consult [3,4,34,35].

If $f(t)=e^t-1$ and $g(t)=\log(1+t)$, then the corresponding Bernoulli number pair is $(B_n^{(z)},b_n^{(z)})$. The $B_n^{(z)}$ are just the classical higher order Bernoulli numbers, which are also called Nörlund polynomials [27, Chapter 6] (see, e.g., [1,11,25] for various properties). Setting $z=1$ gives the famous Bernoulli numbers (of the first kind). The $b_n^{(z)}$ are called the higher order Bernoulli numbers of the second kind and the $b_n:=b_n^{(1)}$ are the Bernoulli numbers of the second kind. Some care must be taken here because the (higher order) Bernoulli numbers of the second kind are frequently defined as $b_n/n!$ (or correspondingly, $b_n^{(z)}/n!$) (see, e.g., [24, pp. 265–287] and [10,20,25]). Our definition of $\hat{b}_n^{(z)}$ coincides with Roman's [29, Section 3.2] and will bring us more convenience.

Now, given two delta series $f(t)$ and $g(t)$ with $f(g(t))=g(f(t))=t$, we have two pairs

$$\big(\hat{S}(n,k),\hat{s}(n,k)\big)\quad\text{and}\quad\big(\hat{B}_n^{(z)},\hat{b}_n^{(z)}\big).$$

Since there are many studies on generalized Stirling number pairs, in this paper, we give emphasis to generalized higher order Bernoulli number pairs.

In Section 2, we show that the generalized Stirling numbers and the generalized higher order Bernoulli numbers are essentially the Bell polynomials and the potential polynomials. Section 3 gives some special generalized Stirling number pairs and the corresponding generalized higher order Bernoulli number pairs. Sections 4 and 5 are devoted to the expressions and recurrences of the generalized higher order Bernoulli numbers. Sections 6 and 7 present some relations between the generalized higher order Bernoulli numbers of both kinds and the corresponding generalized Stirling numbers of both kinds. Finally, in Section 8, we establish the relations between any two generalized higher order Bernoulli numbers. It can be found that the study of generalized higher order Bernoulli number pairs and generalized Stirling number pairs provides a unified approach to many sequences in combinatorics. As a consequence, many results obtained before are special cases of ours.

2. Bell polynomials and potential polynomials

Essentially, the generalized Stirling numbers and the generalized higher order Bernoulli numbers defined in Section 1 are the Bell polynomials and the potential polynomials, respectively.

The *exponential partial Bell polynomials* [15, pp. 133 and 134] are the polynomials

$$B_{n,k} = B_{n,k}(x_1, x_2, \ldots)$$

in an infinite number of variables $x_1, x_2, \ldots,$ defined by the series expansion:

$$\frac{1}{k!}\left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right)^k = \sum_{n=k}^{\infty} B_{n,k}\frac{t^n}{n!}, \quad k = 0, 1, 2, \ldots.$$

Their exact expression is

$$B_{n,k}(x_1, x_2, \ldots) = \sum_{\sigma(n,k)} \frac{n!}{c_1!c_2!\cdots}\left(\frac{x_1}{1!}\right)^{c_1}\left(\frac{x_2}{2!}\right)^{c_2}\cdots, \tag{2.1}$$

where the summation takes place over the set $\sigma(n,k)$ of all partitions of n into k parts, that is, over all integers $c_1, c_2, c_3, \ldots \geqslant 0$, such that $c_1 + 2c_2 + 3c_3 + \cdots = n$ and $c_1 + c_2 + c_3 + \cdots = k$. Thus, the generalized Stirling numbers, say $\hat{S}(n,k)$, are equal to $B_{n,k}(f_1, f_2, \ldots)$.

Let $f(t) := \sum_{i=1}^{\infty} f_i t^i/i!$ be a generic delta series. For each complex number z, define the *potential polynomials* $P_n^{(z)}$ by

$$1 + \sum_{n=1}^{\infty} P_n^{(z)} \frac{t^n}{n!} = \left(1 + \sum_{i=1}^{\infty} f_i \frac{t^i}{i!}\right)^z.$$

Denote $P_0^{(z)} := 1$, then according to [15, p. 141, Theorem B], we have

$$P_n^{(z)} = P_n^{(z)}(f_1, f_2, \ldots, f_n) = \sum_{k=0}^{n} (z)_k B_{n,k}(f_1, f_2, \ldots). \tag{2.2}$$

By the definition of the potential polynomials, it can be found that

$$\hat{B}_n^{(z)} = \left[\frac{t^n}{n!}\right]\left(\frac{f_1 t}{f(t)}\right)^z = \left[\frac{t^n}{n!}\right]\left(1 + \sum_{i=1}^{\infty} \frac{f_{i+1}}{(i+1)f_1}\frac{t^i}{i!}\right)^{-z} = P_n^{(-z)}\left(\frac{f_2}{2f_1}, \frac{f_3}{3f_1}, \ldots\right), \tag{2.3}$$

where $[t^n/n!]H(t) = n![t^n]H(t)$ and $[t^n]H(t)$ is the coefficient of t^n in the power series $H(t)$.

3. Some special number pairs

In this section, we give some special number pairs. Each of the pairs is related to a delta series $f(t)$ and its compositional inverse $\bar{f}(t)$. For clarity, these pairs are listed in Table 1.

As showed in Section 1, the classical Stirling number pair $(S(n,k), s(n,k))$ and the classical higher order Bernoulli number pair $(B_n^{(z)}, b_n^{(z)})$ construct an example. They correspond to the delta series $e^t - 1$ and $\log(1+t)$, and are given by entries (A1) and (A2) of Table 1.

In [12], Carlitz defined the degenerate Stirling numbers of the second kind $S(n,k|\lambda)$ and the higher order degenerate Bernoulli numbers of the first kind $\beta_n^{(z)}(\lambda)$, by means of

$$\frac{1}{k!}\left((1+\lambda t)^{\frac{1}{\lambda}} - 1\right)^k = \sum_{n=k}^{\infty} S(n,k|\lambda)\frac{t^n}{n!}$$

and

$$\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}\right)^z = \sum_{n=0}^{\infty} \beta_n^{(z)}(\lambda)\frac{t^n}{n!},$$

respectively. These give us entry (B1). The compositional inverse of $(1+\lambda t)^{1/\lambda} - 1$ is $((1+t)^{\lambda} - 1)/\lambda$, from which we can obtain entry (B2). It is obvious that when $\lambda \to 0$, (B1) and (B2) will reduce to (A1) and (A2), respectively.

Entries (E1), (E2), (F1), (F2), (G1) and (G2) are in fact special cases of (B1) and (B2). However, they are of particular interest.

Table 1
Several generalized higher order Bernoulli number pairs and the corresponding generalized Stirling number pairs.

	$f(t)$	$\bar{f}(t)$	$\hat{S}(n,k)$ or $\hat{s}(n,k)$	$\hat{B}_n^{(z)}$ or $\hat{b}_n^{(z)}$
(A1)	e^t-1		$S(n,k)$	$B_n^{(z)}$
(A2)		$\log(1+t)$	$s(n,k)$	$b_n^{(z)}$
(B1)	$(1+\lambda t)^{\frac{1}{\lambda}}-1$		$S(n,k\|\lambda)$	$\beta_n^{(z)}(\lambda)$
(B2)		$\frac{(1+t)^\lambda-1}{\lambda}$	$s(n,k\|\lambda)=\lambda^{n-k}S(n,k\|\frac{1}{\lambda})$	$\alpha_n^{(z)}(\lambda)=\lambda^n\beta_n^{(z)}(\frac{1}{\lambda})$
(C1)	te^t		$\binom{n}{k}k^{n-k}$	$(-z)^n$
(C2)		$(te^t)^{\langle -1\rangle}$	$(-1)^{n-k}\binom{n-1}{k-1}n^{n-k}$	$z(z-n)^{n-1}$
(D1)	$2\sinh\frac{t}{2}$		$T(n,k)=\binom{n}{k}B_{n-k}^{(-k)}(-\frac{k}{2})$	$B_n^{(z)}(\frac{z}{2})$
(D2)		$2\ \mathrm{arcsinh}\frac{t}{2}$	$t(n,k)=\binom{n-1}{k-1}B_{n-k}^{(n)}(\frac{n}{2})$	$\frac{-z}{n-z}B_n^{(n-z)}(\frac{n-z}{2})$ if $z\neq n$; B_n if $z=n$, n is even; 0 if $z=n$, n is odd
(E1)	$\frac{t}{1-t}$		$L(n,k)=\binom{n-1}{k-1}\frac{n!}{k!}$	$(-1)^n(z)_n=\beta_n^{(z)}(-1)$
(E2)		$\frac{t}{1+t}$	$(-1)^{n-k}\binom{n-1}{k-1}\frac{n!}{k!}$	$(z)_n=(-1)^n\beta_n^{(z)}(-1)$
(F1)	$t-t^2$		$(-1)^{n-k}\frac{n!}{k!}\binom{k}{n-k}$	$\langle z\rangle_n=(-4)^n\beta_n^{(z)}(\frac{1}{2})$
(F2)		$\frac{1-\sqrt{1-4t}}{2}$	$\frac{(n-1)!}{(k-1)!}\binom{2n-k-1}{n-1}$	$-z(n-1)!\binom{2n-z-1}{n-1}=(-2)^n\beta_n^{(z)}(2)$
(G1)	$2t+t^2$		$\frac{n!}{k!}\binom{k}{n-k}2^{2k-n}$	$\frac{(-z)_n}{2^n}=2^n\beta_n^{(z)}(\frac{1}{2})$
(G2)		$\sqrt{1+t}-1$	$(-1)^{n-k}\frac{(n-1)!}{(k-1)!}\binom{2n-k-1}{n-1}\frac{1}{2^{2n-k}}$	$\frac{(-1)^{n-1}(n-1)!z}{4^n}\binom{2n-z-1}{n-1}=2^{-n}\beta_n^{(z)}(2)$

With some computation, we can verify entries (E1) and (E2), which are the $\lambda=-1$ case of (B1) and (B2). Note that the generalized Stirling number pair is now $(L(n,k),(-1)^{n-k}L(n,k))$, where $L(n,k)=\binom{n-1}{k-1}\frac{n!}{k!}$ are the Lah numbers. The corresponding generalized higher order Bernoulli number pair is $((-1)^n(z)_n,(z)_n)$, where $(z)_n$ are the falling factorials [15, p. 6, Eq. (4f)] defined by $(z)_0=1$ and $(z)_n=z(z-1)\cdots(z-n+1)$ for $n\geqslant 1$.

Entry (F1) can be obtained directly. Note that $\langle z\rangle_n$ are the rising factorials [15, p. 6, Eq. (4g)] defined by $\langle z\rangle_0=1$ and $\langle z\rangle_n=z(z+1)\cdots(z+n-1)$ for $n\geqslant 1$. To obtain entry (F2), we may use the Lagrange inversion formula [15, Section 3.8, Theorems A and B]:

Lemma 3.1 *(The Lagrange inversion formula). Let $f(t)$ be a delta series and $\bar{f}(t)$ be the corresponding compositional inverse, then for any formal power series $\Phi(t)$ we have*

$$\Phi\big(\bar{f}(t)\big)=\Phi(0)+\sum_{n=1}^{\infty}\frac{t^n}{n}\big[t^{n-1}\big]\Phi'(t)\left(\frac{f(t)}{t}\right)^{-n}.$$

Equivalently, for positive integer n we have

$$\big[t^n\big]\Phi\big(\bar{f}(t)\big)=\frac{1}{n}\big[t^{n-1}\big]\Phi'(t)\left(\frac{f(t)}{t}\right)^{-n}. \tag{3.1}$$

Particularly, if $\Phi(t)=t^k$, where k is a nonnegative integer, then (3.1) *reduces to*

$$\big[t^n\big]\big(\bar{f}(t)\big)^k=\frac{k}{n}\big[t^{n-k}\big]\left(\frac{f(t)}{t}\right)^{-n}. \tag{3.2}$$

By appealing to (3.1) and (3.2), we derive for $n \geqslant k \geqslant 1$ that

$$\hat{s}(n,k) = \left[\frac{t^n}{n!}\right]\frac{1}{k!}\left(\frac{1-\sqrt{1-4t}}{2}\right)^k = \frac{n!}{k!}\frac{k}{n}\left[t^{n-k}\right]\left(\frac{t-t^2}{t}\right)^{-n} = \frac{(n-1)!}{(k-1)!}\binom{2n-k-1}{n-1},$$

and for $n \geqslant 1$ that

$$\hat{b}_n^{(z)} = \left[\frac{t^n}{n!}\right]\left(\frac{2t}{1-\sqrt{1-4t}}\right)^z = -\frac{z}{n}n!\left[t^{n-1}\right](1-t)^{z-1-n} = -\frac{z}{n}n!\binom{2n-z-1}{n-1}.$$

Thus, entry (F2) is obtained (see also [15, Exercise 3.21(2)]). As a by-product, we have

$$\frac{1}{n!}\hat{b}_n^{(-1)} = \frac{1}{n+1}\binom{2n}{n} = C_n,$$

where C_n are the famous Catalan numbers [18, p. 203].

In a similar way, entries (G1) and (G2) can be verified. Note that the generalized Stirling numbers given by (G1) and (G2) can be found in [15, Exercise 3.7].

Entry (C1) can be obtained directly, and entry (C2) can be verified by means of the Lagrange inversion formula. Note that $\hat{S}(n,k) = \binom{n}{k}k^{n-k}$ in (C1) are the idempotent numbers (see [15, p. 135, Theorem B]), and $|\hat{s}(n,k)| = \binom{n-1}{k-1}n^{n-k}$ in (C2) are the numbers of planted forests with k components on the vertex set $[n]$ (see [33, Section 5.3]).

Finally, let us verify entries (D1) and (D2). The generalized Stirling numbers $T(n,k)$ and $t(n,k)$ are called central factorial numbers, which were studied systematically by Butzer and his cooperators [6,7]. As evaluated by Mathematical Reviews, "the central factorial numbers are at least as important as Stirling's numbers, if judged by their performance in expansions". To obtain the explicit expression of $T(n,k)$, note that $\sinh t = (e^t - e^{-t})/2$, then

$$\begin{aligned} T(n,k) &= \left[\frac{t^n}{n!}\right]\frac{1}{k!}\left(2\sinh\frac{t}{2}\right)^k = \left[\frac{t^n}{n!}\right]\frac{1}{k!}\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^k \\ &= \frac{n!}{k!}\left[t^{n-k}\right]\left(\frac{t}{e^t-1}\right)^{-k}e^{-\frac{k}{2}t} = \binom{n}{k}B_{n-k}^{(-k)}\left(-\frac{k}{2}\right), \end{aligned}$$

where $B_n^{(z)}(x)$ are the higher order Bernoulli polynomials defined by (for details, see [26, Section 2.8] and [32, Section 1.6])

$$\left(\frac{t}{e^t-1}\right)^z e^{xt} = \sum_{n=0}^{\infty} B_n^{(z)}(x)\frac{t^n}{n!} \quad \left(|t| < 2\pi\right).$$

Similarly, we have $\hat{B}_n^{(z)} = B_n^{(z)}(z/2)$. By means of the Lagrange inversion formula, we can also find that

$$t(n,k) = \binom{n-1}{k-1}B_{n-k}^{(n)}\left(\frac{n}{2}\right).$$

To obtain $\hat{b}_n^{(z)}$, more computation is required. According to the Lagrange inversion formula and the expansion

$$\coth t = \frac{\cosh t}{\sinh t} = \frac{1}{t} + \sum_{n=1}^{\infty}\frac{2^{2n}B_{2n}}{(2n)!}t^{2n-1} \quad \left(0 < |t| < \pi\right)$$

of the hyperbolic cotangent [38], for $n \geqslant 1$ we obtain

$$\begin{aligned} \hat{b}_n^{(z)} &= \left[\frac{t^n}{n!}\right]\left(\frac{t}{2\,\operatorname{arcsinh}\frac{t}{2}}\right)^z = (n-1)!\left[t^{n-1}\right]\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2\sinh\frac{t}{2}}{t}\right)^z\right)\left(\frac{2\sinh\frac{t}{2}}{t}\right)^{-n} \\ &= z\left[\frac{t^{n-1}}{(n-1)!}\right]\left(\frac{2\sinh\frac{t}{2}}{t}\right)^{z-n}\left(\frac{1}{2}\coth\frac{t}{2} - \frac{1}{t}\right) \\ &= z\left[\frac{t^{n-1}}{(n-1)!}\right]\left(\sum_{i=0}^{\infty}B_i^{(n-z)}\left(\frac{n-z}{2}\right)\frac{t^i}{i!}\right)\left(\sum_{j=1}^{\infty}\frac{B_{2j}t^{2j-1}}{(2j)!}\right) \\ &= z\left[\frac{t^{n-1}}{(n-1)!}\right]\sum_{l=2}^{\infty}\left(\sum_{i=0}^{l-2}\binom{l}{i}B_i^{(n-z)}\left(\frac{n-z}{2}\right)B_{l-i}\right)\frac{t^{l-1}}{l!} \\ &= \frac{z}{n}\left[\frac{t^n}{n!}\right]\sum_{l=2}^{\infty}\left(B_l^{(n-z+1)}\left(\frac{n-z}{2}\right) + \frac{l}{2}B_{l-1}^{(n-z)}\left(\frac{n-z}{2}\right) - B_l^{(n-z)}\left(\frac{n-z}{2}\right)\right)\frac{t^l}{l!}. \end{aligned}$$

Therefore, $\hat{b}_1^{(z)} = 0$ and

$$\hat{b}_n^{(z)} = \frac{z}{n}\left(B_n^{(n-z+1)}\left(\frac{n-z}{2}\right) + \frac{n}{2}B_{n-1}^{(n-z)}\left(\frac{n-z}{2}\right) - B_n^{(n-z)}\left(\frac{n-z}{2}\right)\right)$$

for $n \geqslant 2$. Since (see [29, Eq. (4.2.7)])

$$B_n^{(a+1)}(x) = \left(1 - \frac{n}{a}\right)B_n^{(a)}(x) + n\left(\frac{x}{a} - 1\right)B_{n-1}^{(a)}(x) \quad (a \neq 0),$$

then for $n \geqslant 2$, we have $\hat{b}_n^{(z)} = \frac{-z}{n-z}B_n^{(n-z)}(\frac{n-z}{2})$ if $z \neq n$ and $\hat{b}_n^{(z)} = B_n$ if $z = n$. Making use of the facts that $B_1^{(z)}(\frac{z}{2}) = 0$ and $B_n = 0$ when $n = 3, 5, 7, \ldots,$ we can obtain the final result.

In the following sections, we will study the properties of the generalized higher order Bernoulli numbers and the relations between the generalized higher order Bernoulli numbers and the corresponding generalized Stirling numbers. We will also apply the general results obtained later to the special number pairs given in this section. All of these general results have dual forms by the substitutions $\hat{B}_n^{(z)} \rightleftharpoons \hat{b}_n^{(z)}$, $\hat{S}(n,k) \rightleftharpoons \hat{s}(n,k)$ and $f_n \rightleftharpoons g_n$. However, for simplicity, most of the dual results will not be presented explicitly.

4. Expressions and recurrences

Theorem 4.1. *For integer $n \geqslant 1$, we have the expression*

$$\hat{B}_n^{(z)} = n! \sum_{k=0}^{n} \sum_{\sigma(n,k)} \binom{-z}{c_1, c_2, \ldots, c_n} f_1^{-k} \left(\frac{f_2}{2!}\right)^{c_1} \left(\frac{f_3}{3!}\right)^{c_2} \cdots \left(\frac{f_{n+1}}{(n+1)!}\right)^{c_n}, \tag{4.1}$$

where $\binom{-z}{c_1,c_2,\ldots,c_n}$ are the multinomial coefficients defined by

$$\binom{-z}{c_1, c_2, \ldots, c_n} = \frac{(-z)_{c_1+c_2+\cdots+c_n}}{c_1!c_2!\cdots c_n!}.$$

Proof. By appealing to Eqs. (2.2) and (2.3), we have

$$\begin{aligned}
\hat{B}_n^{(z)} &= \sum_{k=0}^{n} (-z)_k B_{n,k}\left(\frac{f_2}{2f_1}, \frac{f_3}{3f_1}, \ldots\right) = \sum_{k=0}^{n} (-z)_k f_1^{-k} B_{n,k}\left(\frac{f_2}{2}, \frac{f_3}{3}, \ldots\right) \\
&= \sum_{k=0}^{n} \sum_{\sigma(n,k)} \frac{(-z)_k f_1^{-k} n!}{c_1!c_2!\cdots c_n!} \left(\frac{f_2}{2!}\right)^{c_1} \left(\frac{f_3}{3!}\right)^{c_2} \cdots \left(\frac{f_{n+1}}{(n+1)!}\right)^{c_n},
\end{aligned}$$

which is just (4.1). □

For example, the classical Bernoulli numbers B_n satisfy

$$B_n = n! \sum_{k=0}^{n} \sum_{\sigma(n,k)} (-1)^k \binom{k}{c_1, c_2, \ldots, c_n} \frac{1}{2!^{c_1} 3!^{c_2} \cdots (n+1)!^{c_n}}$$

and the Bernoulli numbers of the second kind b_n satisfy

$$b_n = n! \sum_{k=0}^{n} \sum_{\sigma(n,k)} (-1)^{n-k} \binom{k}{c_1, c_2, \ldots, c_n} \frac{1}{2^{c_1} 3^{c_2} \cdots (n+1)^{c_n}}.$$

Based on Theorem 4.1, the following special values of the generalized higher order Bernoulli numbers can be obtained without difficulty:

$$\begin{aligned}
&\hat{B}_0^{(z)} = 1, \qquad \hat{B}_1^{(z)} = -\frac{f_2}{2f_1}z, \qquad \hat{B}_2^{(z)} = \frac{f_2^2}{4f_1^2}z^2 + \left(\frac{f_2^2}{4f_1^2} - \frac{f_3}{3f_1}\right)z, \\
&\hat{B}_3^{(z)} = -\frac{f_2^3}{8f_1^3}z^3 + \left(-\frac{3f_2^3}{8f_1^3} + \frac{f_2 f_3}{2f_1^2}\right)z^2 + \left(-\frac{f_2^3}{4f_1^3} + \frac{f_2 f_3}{2f_1^2} - \frac{f_4}{4f_1}\right)z.
\end{aligned}$$

Setting $f_n = 1$ and $f_n = (-1)^{n-1}(n-1)!$ respectively, we can obtain some values of the classical higher order Bernoulli numbers of both kinds (see Liu and Srivastava's paper [25]).

From the special values, it can be seen that $\hat{B}_n^{(z)}$ are polynomials in z. In fact, we have

$$\sum_{n=0}^{\infty}\hat{B}_n^{(z)}\frac{t^n}{n!}=\left(\frac{f(t)}{f_1 t}\right)^{-z}=\left(1+\sum_{i=1}^{\infty}\frac{f_{i+1}}{(i+1)f_1}\frac{t^i}{i!}\right)^{-z}=\sum_{j=0}^{\infty}\binom{-z}{j}\left(\sum_{i=1}^{\infty}\frac{f_{i+1}}{(i+1)f_1}\frac{t^i}{i!}\right)^j.$$

Let

$$F(t)=\sum_{i=1}^{\infty}\frac{f_{i+1}}{(i+1)f_1}\frac{t^i}{i!},$$

then $\deg_t F(t)^j \geqslant j$, which indicates that

$$\hat{B}_n^{(z)}=\left[\frac{t^n}{n!}\right]\sum_{j=0}^{\infty}\binom{-z}{j}F(t)^j=\left[\frac{t^n}{n!}\right]\sum_{j=0}^{n}\binom{-z}{j}F(t)^j.$$

Therefore, $\hat{B}_n^{(z)}$ are indeed polynomials in z of degree not greater than n. Moreover, since

$$\left[z^n\right]\hat{B}_n^{(z)}=\left[z^n\right]\left[\frac{t^n}{n!}\right]\binom{-z}{n}F(t)^n=\left[z^n\right]\left[\frac{t^n}{n!}\right]\frac{(-z)_n}{n!}\left(\frac{f_2}{2f_1}\right)^n t^n=\left(-\frac{f_2}{2f_1}\right)^n,$$

then we have $\deg_z \hat{B}_n^{(z)}=n$ and $[z^n]\hat{B}_n^{(z)}=(-f_2/(2f_1))^n$ when $f_2\neq 0$.

We now determine the general coefficients of $\hat{B}_n^{(z)}$. Let $\sigma(n,k)=[z^k]\hat{B}_n^{(z)}$ for $0\leqslant k\leqslant n$ and define the associated Stirling numbers $A(n,k)$ related to the delta series $f(t)$ by

$$\frac{1}{k!}\big(f(t)-f_1 t\big)^k=\sum_{n=2k}^{\infty}A(n,k)\frac{t^n}{n!}, \tag{4.2}$$

then the next theorem can be established.

Theorem 4.2. *We have*

$$\hat{B}_n^{(z)}=\sum_{k=0}^{n}\sigma(n,k)z^k$$

and

$$\sigma(n,k)=(-1)^k\sum_{j=k}^{n}f_1^{-j}\frac{n!}{(n+j)!}s(j,k)A(n+j,j), \tag{4.3}$$

where $s(j,k)$ are the classical Stirling numbers of the first kind and $A(n,j)$ are the associated Stirling numbers defined in (4.2).

Proof. Based on the generating functions of $\hat{B}_n^{(z)}$ and $A(n,j)$, it can be found that

$$\begin{aligned}
\hat{B}_n^{(z)}&=\left[\frac{t^n}{n!}\right]\left(\frac{f_1 t}{f(t)}\right)^z=\left[\frac{t^n}{n!}\right]\left(\frac{1}{1+\frac{1}{f_1 t}(f(t)-f_1 t)}\right)^z\\
&=\left[\frac{t^n}{n!}\right]\sum_{j=0}^{\infty}(-1)^j\binom{z+j-1}{j}\big(f(t)-f_1 t\big)^j f_1^{-j}t^{-j}\\
&=\left[\frac{t^n}{n!}\right]\sum_{j=0}^{\infty}\left(-\frac{1}{f_1}\right)^j(z+j-1)_j\sum_{n=2j}^{\infty}A(n,j)\frac{t^{n-j}}{n!}\\
&=\sum_{j=0}^{n}\left(-\frac{1}{f_1}\right)^j\frac{\langle z\rangle_j n!}{(n+j)!}A(n+j,j).
\end{aligned}$$

Because (see [15, p. 213, Eq. (5f)])

$$\langle z\rangle_j=\sum_{k=0}^{j}(-1)^{j-k}s(j,k)z^k,$$

we have

$$\hat{B}_n^{(z)} = \sum_{j=0}^{n}\left(-\frac{1}{f_1}\right)^j \frac{n!}{(n+j)!} A(n+j,j) \sum_{k=0}^{j} (-1)^{j-k} s(j,k) z^k$$
$$= \sum_{k=0}^{n}\left((-1)^k \sum_{j=k}^{n} f_1^{-j} \frac{n!}{(n+j)!} s(j,k) A(n+j,j)\right) z^k.$$

This completes the proof. □

Similarly, let $\tau(n,k) = [z^k]\hat{b}_n^{(z)}$ for $0 \leqslant k \leqslant n$ and define the associated Stirling numbers $a(n,k)$ related to the delta series $g(t)$ by

$$\frac{1}{k!}\big(g(t) - g_1 t\big)^k = \sum_{n=2k}^{\infty} a(n,k)\frac{t^n}{n!}.$$

Then

$$\hat{b}_n^{(z)} = \sum_{k=0}^{n} \tau(n,k) z^k = \sum_{k=0}^{n}\left((-1)^k \sum_{j=k}^{n} g_1^{-j} \frac{n!}{(n+j)!} s(j,k) a(n+j,j)\right) z^k. \tag{4.4}$$

When $f(t) = e^t - 1$ and $g(t) = \log(1+t)$, the above results (4.3) and (4.4) will reduce to the ones due to Liu and Srivastava [25, Theorems 1 and 3].

Next, let us establish some recurrence relations for the generalized higher order Bernoulli numbers $\hat{B}_n^{(z)}$.

Theorem 4.3. *For $n \geqslant 0$, we have*

$$(z-n) f_1 \hat{B}_n^{(z)} = z \sum_{k=0}^{n} \binom{n}{k} f_{n+1-k} \hat{B}_k^{(z+1)}. \tag{4.5}$$

Proof. Differentiate with respect to t the generating function of $\hat{B}_n^{(z)}$ and then identify the corresponding coefficients. □

Theorem 4.4. *For integers $n \geqslant 0$ and $l \geqslant k \geqslant 1$, we have*

$$\sum_{j=0}^{n} \binom{n+l}{j} \hat{S}(n+l-j,l) \hat{B}_j^{(k)} = f_1^k \binom{n+l}{k} \binom{l}{k}^{-1} \hat{S}(n+l-k,l-k). \tag{4.6}$$

Proof. Consider the equation

$$\left(\frac{f(t)}{f_1 t}\right)^l \left(\frac{f_1 t}{f(t)}\right)^k = \left(\frac{f(t)}{f_1 t}\right)^{l-k}. \tag{4.7}$$

Since

$$\left(\frac{f(t)}{f_1 t}\right)^l = \frac{l!}{f_1^l t^l} \sum_{n=l}^{\infty} \hat{S}(n,l)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{\hat{S}(n+l,l)}{f_1^l \binom{n+l}{l}} \frac{t^n}{n!},$$

then by extracting the coefficients of $t^n/n!$, we obtain from (4.7) that

$$\sum_{j=0}^{n} \binom{n}{j} \frac{\hat{S}(n-j+l,l)}{f_1^l \binom{n-j+l}{l}} \hat{B}_j^{(k)} = \frac{\hat{S}(n+l-k,l-k)}{f_1^{l-k}\binom{n+l-k}{l-k}},$$

which is equivalent to (4.6). □

Corollary 4.5. *For integers $n \geqslant 0$ and $l \geqslant 1$, we have*

$$\sum_{j=0}^{n} \binom{n+l}{j} \hat{S}(n+l-j,l) \hat{B}_j = f_1 \frac{n+l}{l} \hat{S}(n+l-1,l-1). \tag{4.8}$$

For integers $n \geqslant 1$ and $l \geqslant 1$, we have

$$\sum_{j=0}^{n} \binom{n+l}{j} \hat{S}(n+l-j,l)\hat{B}_j^{(l)} = 0, \tag{4.9}$$

$$\sum_{j=0}^{n} \binom{n+1}{j} f_{n+1-j}\hat{B}_j = 0. \tag{4.10}$$

Proof. The substitution $k=1$ in (4.6) gives (4.8). Since $\hat{S}(n,0)=\delta_{n,0}$, then the substitution $k=l$ in (4.6) gives (4.9). Setting further $l=1$ in (4.9) and taking into account that $\hat{S}(n,1)=f_n$, we finally obtain (4.10). $\square$

5. Applications of the results in Section 4

Example 5.1. For entries (A1) and (A2), Theorem 4.3 gives

$$(z-n)B_n^{(z)} = z\sum_{k=0}^{n}\binom{n}{k}B_k^{(z+1)}, \qquad (z-n)b_n^{(z)} = z\sum_{k=0}^{n}(-1)^{n-k}\frac{n!}{k!}b_k^{(z+1)},$$

and Theorem 4.4 gives

$$\sum_{j=0}^{n}\binom{n+l}{j}S(n+l-j,l)B_j^{(k)} = \binom{n+l}{k}\binom{l}{k}^{-1}S(n+l-k,l-k), \tag{5.1}$$

$$\sum_{j=0}^{n}\binom{n+l}{j}s(n+l-j,l)b_j^{(k)} = \binom{n+l}{k}\binom{l}{k}^{-1}s(n+l-k,l-k), \tag{5.2}$$

where $n \geqslant 0$ and $l \geqslant k \geqslant 1$. The following are special cases of (5.1) and (5.2). For integers $n \geqslant 0$ and $l \geqslant 1$, we have

$$\sum_{j=0}^{n}\binom{n+l}{j}S(n+l-j,l)B_j = \frac{n+l}{l}S(n+l-1,l-1),$$

$$\sum_{j=0}^{n}\binom{n+l}{j}s(n+l-j,l)b_j = \frac{n+l}{l}s(n+l-1,l-1). \tag{5.3}$$

For integers $n \geqslant 1$ and $l \geqslant 1$, we have

$$\sum_{j=0}^{n}\binom{n+l}{j}S(n+l-j,l)B_j^{(l)} = 0, \qquad \sum_{j=0}^{n}\binom{n+l}{j}s(n+l-j,l)b_j^{(l)} = 0,$$

$$\sum_{j=0}^{n}\binom{n+1}{j}B_j = 0, \qquad \sum_{j=0}^{n}\binom{n+1}{j}(-1)^{n-j}(n-j)!b_j = 0. \tag{5.4}$$

Identity (5.3) was given by Agoh and Dilcher [5, Theorem 5.1]. The first identity of (5.4) is a well-known recurrence of Bernoulli numbers (for example, see [18, p. 284, Eq. (6.79)] and [24, p. 233, Eq. (7)]). Similar results can be obtained for entries (B1) and (B2); however, we chose not to present them here.

Example 5.2. For entry (C1), applying Theorems 4.3 and 4.4, we have

$$(z-n)z^{n-1} = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(n+1-k)(z+1)^k,$$

$$\sum_{j=0}^{n}\binom{n}{j}l^{n-j}(-k)^j = (l-k)^n.$$

It is obvious that these two equations can be derived immediately from the binomial identity. For entry (C2), Theorem 4.3 gives

$$(z-n)^n = (z+1)\sum_{k=0}^{n}\binom{n}{k}(k-n-1)^{n-k}(z+1-k)^{k-1},$$

and Theorem 4.4 gives

$$\sum_{j=0}^{n}\binom{n}{j}(n+l-j)^{n-j-1}(j-k)^{j-1} = -\frac{l-k}{lk}(n+l-k)^{n-1},$$

where $n \geqslant 0$ and $l \geqslant k \geqslant 1$. These two equations can also be obtained from the Abel identities [28, Section 1.5, Eqs. (13) and (20)].

Example 5.3. For entries (D1) and (D2), Theorem 4.3 yields

$$(z-n)B_n^{(z)}\left(\frac{z}{2}\right) = z\sum_{\substack{k=0\\2|n-k}}^{n}\binom{n}{k}\frac{1}{2^{n-k}}B_k^{(z+1)}\left(\frac{z+1}{2}\right)$$

and

$$B_n^{(n-z)}\left(\frac{n-z}{2}\right) = \sum_{\substack{k=0\\2|n-k}}^{n}(-1)^{\frac{n-k}{2}}\binom{n}{k}\binom{n-k}{\frac{n-k}{2}}\frac{(n-k)!(z+1)}{4^{n-k}(z+1-k)}B_k^{(k-z-1)}\left(\frac{k-z-1}{2}\right).$$

For entry (D1), Theorem 4.4 gives

$$\sum_{j=0}^{n}\binom{n+l}{j}T(n+l-j,l)B_j^{(k)}\left(\frac{k}{2}\right) = \binom{n+l}{k}\binom{l}{k}^{-1}T(n+l-k,l-k).$$

Since $T(n,k) = \binom{n}{k}B_{n-k}^{(-k)}(-\frac{k}{2})$, then the above identity turns into

$$\sum_{j=0}^{n}\binom{n}{j}B_j^{(k)}\left(\frac{k}{2}\right)B_{n-j}^{(-l)}\left(-\frac{l}{2}\right) = B_n^{(k-l)}\left(\frac{k-l}{2}\right),$$

which is a well-known property of the higher order Bernoulli polynomials. For entry (D2), Theorem 4.4 gives

$$\sum_{j=0}^{n}\binom{n+l}{j}\frac{k}{k-j}t(n+l-j,l)B_j^{(j-k)}\left(\frac{j-k}{2}\right) = \binom{n+l}{k}\binom{l}{k}^{-1}t(n+l-k,l-k),$$

where $n \geqslant 0$, $l \geqslant k \geqslant n+1$. Replacing $t(n,k)$ by $\binom{n-1}{k-1}B_{n-k}^{(n)}(\frac{n}{2})$, we have

$$\frac{l-k}{n+l-k}B_n^{(n+l-k)}\left(\frac{n+l-k}{2}\right) = \sum_{j=0}^{n}\binom{n}{j}\frac{kl}{(n+l-j)(k-j)}B_{n-j}^{(n+l-j)}\left(\frac{n+l-j}{2}\right)B_j^{(j-k)}\left(\frac{j-k}{2}\right).$$

Example 5.4. For entries (E1) and (E2), we can obtain from Theorems 4.3 and 4.4 the following two identities:

$$\binom{z-1}{n} = \sum_{k=0}^{n}(-1)^{n-k}(n-k+1)\binom{z+1}{k},$$

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{k}{j}\binom{n+l-1}{j}^{-1} = \binom{n+l-k-1}{n}\binom{n+l-1}{n}^{-1},$$

where the first one can be verified by the Vandermonde convolution formula.

Example 5.5. For entry (F1), Theorem 4.3 gives a trivial result, because now the coefficients f_i are all zero for $i \geqslant 3$. Additionally, Theorem 4.4 yields

$$\sum_{j=0}^{n}\binom{l}{n-j}\binom{-k}{j} = \binom{l-k}{n},$$

which is the Vandermonde convolution formula. For entry (F2), we can obtain from Theorems 4.3 and 4.4 the next two identities:

$$\frac{z-n}{n}\binom{2n-z-1}{n-1} = -\binom{2n}{n} + \sum_{k=1}^{n}\binom{2n-2k}{n-k}\binom{2k-z-2}{k-1}\frac{z+1}{k},$$

$$\frac{1}{2n+l}\binom{2n+l}{n} - \frac{l-k}{(2n+l-k)l}\binom{2n+l-k}{n} = \sum_{j=1}^{n}\binom{2j-k-1}{j-1}\binom{2n+l-2j}{n+l-j}\frac{k}{j(2n+l-2j)},$$

where $n \geqslant 1$ and $l \geqslant k \geqslant 1$. It can be verified that applying Theorems 4.3 and 4.4 (and, in fact, most of the results in Sections 6 and 8) to entries (G1) and (G2) gives us the same results.

6. Relations between Bernoulli numbers and corresponding Stirling numbers

We now present some relations between the generalized higher order Bernoulli numbers of both kinds and the corresponding generalized Stirling numbers of both kinds.

Theorem 6.1. *For integers n and k with $0 \leqslant n < k$, the following relations hold*:

$$\hat{B}_n^{(k)} = f_1^k \frac{\hat{s}(k,k-n)}{\binom{k-1}{n}} = g_1^{-n}\frac{k}{k-n}\hat{b}_n^{(n-k)} \tag{6.1}$$

$$= \frac{f_1^k}{\binom{k-1}{n}}\sum_{i=0}^{n}\binom{k-1}{i} g_{i+1}\hat{s}(k-1-i,k-1-n) = g_1^{-n-1}\sum_{i=0}^{n}\binom{n}{i} g_{i+1}\hat{b}_{n-i}^{(n+1-k)}. \tag{6.2}$$

Proof. According to the Lagrange inversion formula, we have

$$\hat{B}_n^{(k)} = \left[\frac{t^n}{n!}\right]\left(\frac{f_1 t}{f(t)}\right)^k = n! f_1^k [t^n]\left(\frac{f(t)}{t}\right)^{-k} = n! f_1^k \frac{k}{k-n}[t^k]\big(g(t)\big)^{k-n}. \tag{6.3}$$

Thus, combining the generating function of $\hat{s}(n,k)$ with (6.3) gives

$$\hat{B}_n^{(k)} = n! f_1^k \frac{k(k-n)!}{k-n}[t^k]\sum_{j=k-n}^{\infty}\hat{s}(j,k-n)\frac{t^j}{j!} = f_1^k\frac{\hat{s}(k,k-n)}{\binom{k-1}{n}}.$$

Combining the generating function of $\hat{b}_n^{(z)}$ with (6.3) gives

$$\hat{B}_n^{(k)} = n! f_1^k \frac{k}{k-n}[t^n]\left(\frac{g(t)}{t}\right)^{k-n} = g_1^{-n}\frac{k}{k-n}\left[\frac{t^n}{n!}\right]\left(\frac{g_1 t}{g(t)}\right)^{n-k} = g_1^{-n}\frac{k}{k-n}\hat{b}_n^{(n-k)}.$$

Now, let us define

$$\left(\frac{g(t)}{t}\right)^{k-n} = \sum_{i=0}^{\infty} a_i t^i.$$

Differentiating the above series and identifying the coefficients of t^{i-1} yield

$$a_i = \frac{k-n}{k-n+i}[t^i]\left(\frac{g(t)}{t}\right)^{k-n-1} g'(t),$$

where $g'(t)$ is the formal derivative of $g(t)$. Hence

$$\hat{B}_n^{(k)} = n! f_1^k [t^n]\left(\frac{g(t)}{t}\right)^{k-n-1} g'(t), \tag{6.4}$$

from which we can obtain (6.2). □

In fact, Eqs. (6.3) and (6.4) were proposed by Adelberg [2, Proposition 2.1], who also pointed out the duality between the generalized higher order Bernoulli numbers of both kinds. However, the relations between the generalized higher order Bernoulli numbers and the generalized Stirling numbers have not been given.

Theorem 6.2. *For integers $n \geqslant 1$ and $k \geqslant 1$, the following relations hold*:

$$\hat{B}_n^{(-k)} = \frac{\hat{S}(n+k,k)}{f_1^k\binom{n+k}{k}} = g_1^{-n}\frac{k}{n+k}\hat{b}_n^{(n+k)} \tag{6.5}$$

$$= \frac{g_1^k}{\binom{n+k}{k}}\sum_{i=0}^{n+k-1}\binom{n+k-1}{i}f_{n+k-i}\hat{S}(i,k-1) = \frac{kf_1^{-1}}{n+k}\sum_{i=0}^{n}\binom{n}{i}f_{n-i+1}\hat{B}_i^{(1-k)} \tag{6.6}$$

$$= g_1^{-n}\sum_{j=0}^{n}(-1)^j\binom{2n+k}{n-j}\binom{n+k+j}{j}\frac{k}{n+k+j}\hat{b}_n^{(-j)}. \tag{6.7}$$

Proof. Of course, following a similar way to the proof of Theorem 6.1 gives us Eqs. (6.5) and (6.6). However, there exists a more straightforward proof. Replacing k by $n+k$ in Theorem 6.1, multiplying each term by a factor $g_1^n k/(n+k)$, and applying the substitutions $\hat{B}_n^{(z)} \rightleftharpoons \hat{b}_n^{(z)}$, $\hat{S}(n,k) \rightleftharpoons \hat{s}(n,k)$ and $f_n \rightleftharpoons g_n$, we can also obtain the final results. From the proof, it can be found that Theorems 6.1 and 6.2 are actually dual.

Next, let us verify Eq. (6.7). In [15, p. 142, Theorem C], it is shown that for any complex number z, the potential polynomials satisfy

$$P_n^{(-z)} = \sum_{j=0}^{n}(-1)^j\binom{n+z}{n-j}\binom{z+j-1}{j}P_n^{(j)}, \tag{6.8}$$

thus we have

$$\hat{B}_n^{(-k)} = \left[\frac{t^n}{n!}\right]\left(\frac{f_1 t}{f(t)}\right)^{-k} = \frac{n!}{f_1^k}\left[t^{n+k}\right]\left(f(t)\right)^k = \frac{n!}{f_1^k}\frac{k}{n+k}\left[t^n\right]\left(\frac{g(t)}{t}\right)^{-n-k}$$
$$= g_1^{-n}\frac{k}{n+k}\left[\frac{t^n}{n!}\right]\left(\frac{g(t)}{g_1 t}\right)^{-n-k}$$
$$= g_1^{-n}\frac{k}{n+k}\sum_{j=0}^{n}(-1)^j\binom{2n+k}{n-j}\binom{n+k+j-1}{j}\hat{b}_n^{(-j)},$$

from which Eq. (6.7) can be obtained. □

Combining (6.1) with (6.5) and taking into account the duality, we can see for $n \geqslant 0$ that

$$\hat{B}_n^{(k)} = g_1^{-n}\frac{k}{k-n}\hat{b}_n^{(n-k)}, \qquad \hat{b}_n^{(k)} = f_1^{-n}\frac{k}{k-n}\hat{B}_n^{(n-k)} \quad (k > n \text{ or } k \leqslant -1).$$

The following four relations also deserve emphasis:

$$\hat{B}_n^{(k)} = f_1^k\frac{\hat{s}(k,k-n)}{\binom{k-1}{n}}, \qquad \hat{b}_n^{(k)} = g_1^k\frac{\hat{S}(k,k-n)}{\binom{k-1}{n}} \quad (0 \leqslant n < k), \tag{6.9}$$

$$\hat{B}_n^{(-k)} = \frac{\hat{S}(n+k,k)}{f_1^k\binom{n+k}{k}}, \qquad \hat{b}_n^{(-k)} = \frac{\hat{s}(n+k,k)}{g_1^k\binom{n+k}{k}} \quad (n,k \geqslant 0). \tag{6.10}$$

Now, replacing the numbers $\hat{B}_n^{(-k)}$ and $\hat{b}_n^{(-j)}$ in (6.7) by (6.10) and doing some transformations, we can finally obtain the Schlömilch formula (see [15, Section 5.7, Theorem A] and [22, Section 2, Eq. (14)]):

$$\hat{S}(n,k) = \sum_{j=0}^{n-k}(-1)^j\binom{n+j-1}{n+j-k}\binom{2n-k}{n-k-j}g_1^{-n-j}\hat{s}(n-k+j,j), \tag{6.11}$$

where $n \geqslant k \geqslant 1$. In [22, Section 2], Hsu proved that the Schlömilch formula (6.11) is equivalent to the Lagrange inversion formula (3.2), i.e., they are deducible from each other. Since (6.11) is equivalent to (6.7), then (6.7) is also equivalent to the Lagrange inversion formula.

Additionally, define $F_n(k) = \hat{S}(n+k,k)/f_1^k$ and $G_n(k) = (-1)^n\hat{s}(k,k-n)/g_1^k$, then $F_n(k)$ and $G_n(k)$ are polynomials in k. By the fundamental theorem of algebra, relations (6.1) and (6.5) hold for all $k \in \mathbb{Z}$. Thus, replacing k by $-k$ in (6.1) and combining with (6.5), we obtain

$$F_n(k) = G_n(-k), \quad \text{for all } k \in \mathbb{Z},$$

which coincides with the classical result (e.g., see [1, pp. 12–13] and [16, Proposition 1.2]).

Theorem 6.3. *For any complex number z and integers $n \geqslant 1$ and $k \geqslant 0$, we have*

$$\hat{B}_n^{(kz)} = \sum_{j=0}^{n} (-1)^j \binom{n+z}{n-j}\binom{z+j-1}{j} \hat{B}_n^{(-kj)} \tag{6.12}$$

$$= \sum_{j=0}^{n} (-1)^j f_1^{-kj} \binom{n+z}{n-j}\binom{z+j-1}{j}\binom{n+kj}{n}^{-1} \hat{S}(n+kj, kj) \tag{6.13}$$

$$= g_1^{-n} \sum_{j=0}^{n} (-1)^j \binom{n+z}{n-j}\binom{z+j-1}{j} \frac{kj}{n+kj} \hat{b}_n^{(n+kj)}. \tag{6.14}$$

Proof. According to the property (6.8) of the potential polynomials, we have

$$\hat{B}_n^{(kz)} = \left[\frac{t^n}{n!}\right]\left(\frac{f_1 t}{f(t)}\right)^{kz} = \left[\frac{t^n}{n!}\right]\left(\frac{f(t)^k}{f_1^k t^k}\right)^{-z}$$
$$= \sum_{j=0}^{n} (-1)^j \binom{n+z}{n-j}\binom{z+j-1}{j} \left[\frac{t^n}{n!}\right]\left(\frac{f(t)^k}{f_1^k t^k}\right)^{j},$$

which yields identity (6.12). Next, by means of (6.5), identities (6.13) and (6.14) can also be established. Moreover, replacing k by $k-n$ in Eq. (6.7), applying the substitutions $\hat{b}_n^{(z)} \rightleftharpoons \hat{B}_n^{(z)}$ and $g_1 \rightleftharpoons f_1$, and then using Eq. (6.1), we can obtain the $k=1$ case of (6.12). □

The substitution $z=1$ in Theorem 6.3 leads us to the corollary below.

Corollary 6.4. *For integers $n \geqslant 1$ and $k \geqslant 0$, we have*

$$\hat{B}_n^{(k)} = \sum_{j=0}^{n} (-1)^j \binom{n+1}{j+1} \hat{B}_n^{(-kj)} = \sum_{j=0}^{n} (-1)^j f_1^{-kj} \binom{n+1}{j+1}\binom{n+kj}{n}^{-1} \hat{S}(n+kj, kj)$$
$$= g_1^{-n} \sum_{j=0}^{n} (-1)^j \binom{n+1}{j+1} \frac{kj}{n+kj} \hat{b}_n^{(n+kj)}. \tag{6.15}$$

7. Applications of the results in Section 6

Example 7.1. Applying Theorem 6.1 to entries (A1) and (A2), we have

$$B_n^{(k)} = \frac{s(k, k-n)}{\binom{k-1}{n}} = \frac{k}{k-n} b_n^{(n-k)}$$
$$= \frac{1}{\binom{k-1}{n}} \sum_{i=0}^{n} (-1)^i i! \binom{k-1}{i} s(k-1-i, k-1-n) = \sum_{i=0}^{n} (-1)^i i! \binom{n}{i} b_{n-i}^{(n+1-k)},$$
$$b_n^{(k)} = \frac{S(k, k-n)}{\binom{k-1}{n}} = \frac{k}{k-n} B_n^{(n-k)}$$
$$= \frac{1}{\binom{k-1}{n}} \sum_{i=0}^{n} \binom{k-1}{i} S(k-1-i, k-1-n) = \sum_{i=0}^{n} \binom{n}{i} B_{n-i}^{(n+1-k)},$$

where $0 \leqslant n < k$. Next, using Theorem 6.2, we have

$$B_n^{(-k)} = \frac{S(n+k, k)}{\binom{n+k}{k}} = \frac{k}{n+k} b_n^{(n+k)}$$
$$= \frac{1}{\binom{n+k}{k}} \sum_{i=0}^{n+k-1} \binom{n+k-1}{i} S(i, k-1) = \frac{k}{n+k} \sum_{i=0}^{n} \binom{n}{i} B_i^{(1-k)}$$

$$= \sum_{j=0}^{n} (-1)^j \binom{2n+k}{n-j}\binom{n+k+j}{j}\frac{k}{n+k+j} b_n^{(-j)},$$

$$b_n^{(-k)} = \frac{s(n+k,k)}{\binom{n+k}{k}} = \frac{k}{n+k} B_n^{(n+k)}$$

$$= \frac{1}{\binom{n+k}{k}} \sum_{i=0}^{n+k-1} (-1)^{n+k-1-i} \frac{(n+k-1)!}{i!} s(i,k-1) = \frac{k}{n+k}\sum_{i=0}^{n}(-1)^{n-i}\frac{n!}{i!} b_i^{(1-k)}$$

$$= \sum_{j=0}^{n} (-1)^j \binom{2n+k}{n-j}\binom{n+k+j}{j}\frac{k}{n+k+j} B_n^{(-j)},$$

where $n, k \geqslant 1$. Finally, using Theorem 6.3, we have

$$B_n^{(kz)} = \sum_{j=0}^{n}(-1)^j \binom{n+z}{n-j}\binom{z+j-1}{j} B_n^{(-kj)}$$

$$= \sum_{j=0}^{n}(-1)^j \binom{n+z}{n-j}\binom{z+j-1}{j}\binom{n+kj}{n}^{-1} S(n+kj,kj)$$

$$= \sum_{j=0}^{n}(-1)^j \binom{n+z}{n-j}\binom{z+j-1}{j}\frac{kj}{n+kj} b_n^{(n+kj)} \tag{7.1}$$

and

$$b_n^{(kz)} = \sum_{j=0}^{n}(-1)^j \binom{n+z}{n-j}\binom{z+j-1}{j} b_n^{(-kj)}$$

$$= \sum_{j=0}^{n}(-1)^j \binom{n+z}{n-j}\binom{z+j-1}{j}\binom{n+kj}{n}^{-1} s(n+kj,kj)$$

$$= \sum_{j=0}^{n}(-1)^j \binom{n+z}{n-j}\binom{z+j-1}{j}\frac{kj}{n+kj} B_n^{(n+kj)},$$

where $n \geqslant 1$ and $k \geqslant 0$. When $z = 1$, identity (7.1) reduces to

$$B_n^{(k)} = \sum_{j=0}^{n}(-1)^j \binom{n+1}{j+1}\binom{n+kj}{n}^{-1} S(n+kj,kj),$$

which can be found in [23, p. 60]. Setting further $k = 1$ in the above identity yields

$$B_n = \sum_{j=0}^{n}(-1)^j \binom{n+1}{j+1}\binom{n+j}{n}^{-1} S(n+j,j),$$

which is a known result also (see [17, p. 49, Eq. (17)], [18, p. 317, Exercise 78] and [24, p. 219]). Additionally, setting $k = 1$ in (7.1) gives us a result due to Todorov [36, p. 665, Eq. (3)]. As Example 5.1, by applying the theorems to entries (B1) and (B2), we can obtain similar results involving the higher order degenerate Bernoulli numbers and the degenerate Stirling numbers. The readers may consult the paper of Cenkci and Howard [13].

Example 7.2. Let us consider entries (C1) and (C2). Now, Theorem 6.1 gives

$$k^n = (k-n-1)\sum_{i=0}^{n}\binom{n}{i}(k-i-1)^{n-i-1}(i+1)^i, \tag{7.2}$$

$$k(k-n)^{n-1} = \sum_{i=0}^{n}\binom{n}{i}(i+1)(k-n-1)^{n-i}, \tag{7.3}$$

where $0 \leqslant n < k$. Theorem 6.2 gives

$$
\begin{aligned}
k^n &= \frac{k}{n+k}\sum_{i=0}^{n}\binom{n}{i}(n-i+1)(k-1)^i \\
&= \sum_{j=0}^{n}(-1)^{n-j}\binom{2n+k}{n-j}\binom{n+k+j}{j}\frac{kj}{n+k+j}(n+j)^{n-1}
\end{aligned}
\tag{7.4}
$$

and

$$
\begin{aligned}
(k+n)^{n-1} &= \frac{k-1}{k+n}\sum_{i=0}^{n}\binom{n}{i}(n-i+1)^{n-i}(k+i-1)^{i-1} \\
&= \sum_{j=0}^{n}(-1)^{n-j}\binom{2n+k}{n-j}\binom{n+k+j}{j}\frac{j^n}{n+k+j},
\end{aligned}
\tag{7.5}
$$

where $n, k \geqslant 1$. Moreover, for integers $n \geqslant 1$ and $k \geqslant 0$, Theorem 6.3 yields

$$
\begin{aligned}
z^n &= \sum_{j=0}^{n}(-1)^{n-j}\binom{n+z}{n-j}\binom{z+j-1}{j}j^n, \\
z(kz-n)^{n-1} &= \sum_{j=0}^{n}(-1)^{n-j}\binom{n+z}{n-j}\binom{z+j-1}{j}j(kj+n)^{n-1}.
\end{aligned}
$$

It is not difficult to verify that (7.2) and (7.5) can be derived from the Abel identity [28, p. 18, Eq. (13)], while (7.3) and (7.4) can be derived from the binomial identity.

Example 7.3. We present the results related to entry (D1). Theorem 6.1 gives

$$
B_n^{(k)}\left(\frac{k}{2}\right) = \sum_{\substack{i=0\\ i\ \text{even}}}^{n}(-1)^{\frac{i}{2}}\binom{n}{i}\binom{i}{\frac{i}{2}}\frac{i!(k-n-1)}{4^i(k-i-1)}B_{n-i}^{(k-i-1)}\left(\frac{k-i-1}{2}\right),
\tag{7.6}
$$

where $0 \leqslant n < k-1$. Theorems 6.2 and 6.3 give

$$
\begin{aligned}
B_n^{(-k)}\left(-\frac{k}{2}\right) &= \frac{k}{n+k}\sum_{\substack{i=0\\ i\ \text{even}}}^{n}\binom{n}{i}\left(\frac{1}{2}\right)^i B_{n-i}^{(1-k)}\left(\frac{1-k}{2}\right) \\
&= \sum_{j=0}^{n}(-1)^j\binom{2n+k}{n-j}\binom{n+k+j}{j}\frac{kj}{(n+k+j)(n+j)}B_n^{(n+j)}\left(\frac{n+j}{2}\right)
\end{aligned}
\tag{7.7}
$$

and

$$
B_n^{(kz)}\left(\frac{kz}{2}\right) = \sum_{j=0}^{n}(-1)^j\binom{n+z}{n-j}\binom{z+j-1}{j}B_n^{(-kj)}\left(-\frac{kj}{2}\right),
$$

respectively, where $n \geqslant 1$ and $k \geqslant 0$. Note that Eqs. (7.6) and (7.7) coincide with the first two identities of Example 5.3.

Example 7.4. Apply Theorems 6.1–6.3 to entries (E1) and (E2). For $0 \leqslant n < k$, we have

$$
\binom{k}{n} = \sum_{i=0}^{n}(-1)^{n-i}\binom{n+1-k}{n-i}(i+1).
$$

For $n \geqslant 1$ and $k \geqslant 0$, we have

$$
\begin{aligned}
\binom{n+k-1}{n} &= \frac{k}{n+k}\sum_{i=0}^{n}(-1)^i\binom{1-k}{i}(n-i+1) \\
&= \sum_{j=0}^{n}(-1)^{n-j}\binom{2n+k}{n-j}\binom{n+k+j}{j}\binom{n+j-1}{n}\frac{k}{n+k+j}
\end{aligned}
$$

and

$$\binom{kz}{n}=\sum_{j=0}^{n}(-1)^{n-j}\binom{n+z}{n-j}\binom{z+j-1}{j}\binom{n+kj-1}{n}.$$

Example 7.5. Let us consider entries (F1) and (F2) (entries (G1) and (G2) give the same results). Combining Theorem 6.1 with entry (F1), we obtain

$$\binom{k+n-1}{n}=\sum_{i=0}^{n}\binom{2i}{i}\binom{k+n-1-2i}{n-i}\frac{k-1-n}{k+n-1-2i},$$

where $0\leqslant n<k$. Since now $f_i=0$ for $i\geqslant 3$, then combining Theorem 6.1 with entry (F2) gives us trivial results. Applying Eq. (6.7) of Theorem 6.2 to (F1) yields

$$\binom{k-1}{n-1}=\sum_{j=0}^{n}(-1)^{n-j}\binom{2n+k}{n-j}\binom{n+k+j}{j}\binom{2n+j-1}{n-1}\frac{j}{n+k+j},$$

and applying Eq. (6.6) of Theorem 6.2 to (F2) yields

$$\binom{2n+k-1}{n}=\sum_{i=0}^{n}\binom{2i+k-2}{i}\binom{2n-2i}{n-i}\frac{k-1}{i+k-1},$$

where $n,k\geqslant 1$. Finally, from Theorem 6.3, we obtain

$$\binom{kz+n-1}{n}=\sum_{j=0}^{n}(-1)^{n-j}\binom{n+z}{n-j}\binom{z+j-1}{j}\binom{kj}{n},$$

$$\binom{2n-kz-1}{n-1}z=\sum_{j=0}^{n}(-1)^{j-1}\binom{n+z}{n-j}\binom{z+j-1}{j}\binom{2n+kj-1}{n-1}j,$$

where $n\geqslant 1$ and $k\geqslant 0$.

Because $\hat{b}_n^{(-1)}/n!$ are just the Catalan numbers $C_n=\frac{1}{n+1}\binom{2n}{n}$, by setting $z=-1/k$ in Theorem 6.3, we can establish an identity involving the Catalan numbers:

$$C_n=\frac{1}{n+1}\binom{2n}{n}=(n+1)\binom{n-\frac{1}{k}}{n+1}\frac{k}{n}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{2n+kj-1}{n-1}\frac{kj}{kj-1},$$

where $n\geqslant 1$ and $k\geqslant 2$.

8. Relations between any two Bernoulli numbers

In this section, we study the relations between any two generalized higher order Bernoulli numbers. Given three delta series $f(t)$, $h(t)$ and $l(t)$ with $h(l(t))=l(h(t))=t$, define $\hat{B}_n^{(z)}$ and $\hat{\beta}_n^{(z)}$ by

$$\left(\frac{f_1t}{f(t)}\right)^z=\sum_{n=0}^{\infty}\hat{B}_n^{(z)}\frac{t^n}{n!}\quad\text{and}\quad\left(\frac{h_1t}{h(t)}\right)^z=\sum_{n=0}^{\infty}\hat{\beta}_n^{(z)}\frac{t^n}{n!}.$$

Define further $\hat{\gamma}_n^{(z)}$ and $\hat{\mathscr{T}}(n,k)$ by

$$\left(\frac{l_1f_1t}{l(f(t))}\right)^z=\sum_{n=0}^{\infty}\hat{\gamma}_n^{(z)}\frac{t^n}{n!}\quad\text{and}\quad\frac{1}{k!}\big(l(f(t))\big)^k=\sum_{n=k}^{\infty}\hat{\mathscr{T}}(n,k)\frac{t^n}{n!}.$$

Then the next theorem holds.

Theorem 8.1. *For nonnegative integers n and k, if $0\leqslant k\leqslant n$, then*

$$\hat{B}_n^{(k)}=\sum_{j=0}^{k-1}\binom{n}{j}\left(\frac{f_1}{h_1}\right)^j\hat{\gamma}_{n-j}^{(k-j)}\hat{\beta}_j^{(k)}+\binom{n}{k}\left(\frac{f_1}{h_1}\right)^k\sum_{j=k}^{n}\binom{j}{k}^{-1}\hat{\mathscr{T}}(n-k,j-k)\hat{\beta}_j^{(k)};\tag{8.1}$$

if $0 \leqslant n < k$, *then*

$$\hat{B}_n^{(k)} = \sum_{j=0}^{n} \binom{n}{j} \left(\frac{f_1}{h_1}\right)^j \hat{\gamma}_{n-j}^{(k-j)} \hat{\beta}_j^{(k)}. \tag{8.2}$$

Proof. Let $l(f(t)) = x$, then $f(t) = h(x)$. From the generating function of $\hat{B}_n^{(z)}$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \hat{B}_n^{(k)} \frac{t^n}{n!} &= \left(\frac{f_1 t}{h_1 x}\right)^k \left(\frac{h_1 x}{h(x)}\right)^k = \left(\frac{f_1 t}{h_1 x}\right)^k \sum_{j=0}^{\infty} \hat{\beta}_j^{(k)} \frac{x^j}{j!} = \left(\frac{f_1 t}{h_1}\right)^k \sum_{j=0}^{\infty} \hat{\beta}_j^{(k)} \frac{x^{j-k}}{j!} \\
&= \left(\frac{f_1 t}{h_1}\right)^k \sum_{j=0}^{k-1} \hat{\beta}_j^{(k)} \frac{x^{j-k}}{j!} + \left(\frac{f_1 t}{h_1}\right)^k \sum_{j=k}^{\infty} \hat{\beta}_j^{(k)} \frac{x^{j-k}}{j!} \\
&= \sum_{j=0}^{k-1} \left(\frac{f_1}{h_1}\right)^j \hat{\beta}_j^{(k)} \frac{t^j}{j!} \left(\frac{l_1 f_1 t}{x}\right)^{k-j} + \left(\frac{f_1}{h_1}\right)^k t^k \sum_{j=0}^{\infty} \frac{\hat{\beta}_{j+k}^{(k)}}{(j+k)_k} \frac{x^j}{j!} \\
&= \sum_{j=0}^{k-1} \left(\frac{f_1}{h_1}\right)^j \hat{\beta}_j^{(k)} \frac{t^j}{j!} \sum_{n=0}^{\infty} \hat{\gamma}_n^{(k-j)} \frac{t^n}{n!} + \left(\frac{f_1}{h_1}\right)^k t^k \sum_{j=0}^{\infty} \frac{\hat{\beta}_{j+k}^{(k)}}{(j+k)_k} \sum_{n=j}^{\infty} \hat{\mathscr{T}}(n,j) \frac{t^n}{n!} \\
&= \sum_{j=0}^{k-1} \sum_{n=0}^{\infty} \left(\frac{f_1}{h_1}\right)^j \frac{\hat{\beta}_j^{(k)} \hat{\gamma}_n^{(k-j)}}{j!n!} t^{n+j} + \left(\frac{f_1}{h_1}\right)^k \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{\hat{\beta}_{j+k}^{(k)} \hat{\mathscr{T}}(n,j)}{(j+k)_k n!} t^{n+k}.
\end{aligned}$$

Equating the coefficients of t^n in the first and last members gives us the desired result. $\square$

Remark 8.2. Using the conventions $\hat{\gamma}_n^{(z)} = 0$ for $n < 0$ and $\sum_{j=k}^{n} a(j) = 0$ for $0 \leqslant n < k$, we can see that relation (8.1) holds for all nonnegative integers n and k.

When $l(t) = \log(1+t)$, Theorem 8.1 reduces to a result due to Cenkci and Howard [13, Theorem 3.1]. Moreover, setting $k = 1$ in (8.1) yields a corollary between any two generalized Bernoulli numbers.

Corollary 8.3. *For positive integer n, we have*

$$\hat{B}_n = \hat{\gamma}_n + \frac{f_1}{h_1} \sum_{j=1}^{n} \frac{n}{j} \hat{\mathscr{T}}(n-1, j-1) \hat{\beta}_j. \tag{8.3}$$

Example 8.1. We now expand the classical higher order Bernoulli numbers $B_n^{(k)}$ in terms of the higher order degenerate Bernoulli numbers $\beta_n^{(k)}(\lambda)$.

Take $f(t) \to e^t - 1$ and $h(t) \to (1+\lambda t)^{1/\lambda} - 1$, then $\hat{B}_n^{(z)} \to B_n^{(z)}$ are the classical higher order Bernoulli numbers and $\hat{\beta}_n^{(z)} \to \beta_n^{(z)}(\lambda)$ are the higher order degenerate Bernoulli numbers. Additionally, we have $l(t) \to ((1+t)^\lambda - 1)/\lambda$ and $l(f(t)) \to (e^{\lambda t} - 1)/\lambda$. Then

$$\begin{aligned}
\left(\frac{t}{l(f(t))}\right)^z &= \left(\frac{\lambda t}{e^{\lambda t} - 1}\right)^z = \sum_{n=0}^{\infty} \lambda^n B_n^{(z)} \frac{t^n}{n!}, \\
\frac{1}{k!}\big(l(f(t))\big)^k &= \frac{1}{k!}\left(\frac{e^{\lambda t} - 1}{\lambda}\right)^k = \sum_{n=k}^{\infty} \lambda^{n-k} S(n,k) \frac{t^n}{n!}.
\end{aligned}$$

These indicate that $\hat{\gamma}_n^{(z)} \to \lambda^n B_n^{(z)}$ and $\hat{\mathscr{T}}(n,k) \to \lambda^{n-k} S(n,k)$, where $S(n,k)$ are the classical Stirling numbers of the second kind. Hence, for $0 \leqslant k \leqslant n$, Eq. (8.1) gives

$$B_n^{(k)} = \sum_{j=0}^{k-1} \binom{n}{j} \lambda^{n-j} B_{n-j}^{(k-j)} \beta_j^{(k)}(\lambda) + \binom{n}{k} \sum_{j=k}^{n} \binom{j}{k}^{-1} \lambda^{n-j} S(n-k, j-k) \beta_j^{(k)}(\lambda), \tag{8.4}$$

and for $0 \leqslant n < k$, Eq. (8.2) gives

$$B_n^{(k)} = \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j} B_{n-j}^{(k-j)} \beta_j^{(k)}(\lambda). \tag{8.5}$$

Moreover, Corollary 8.3 yields an expression for the classical Bernoulli numbers:

$$\left(1-\lambda^n\right) B_n = \sum_{j=1}^{n} \frac{n}{j} \lambda^{n-j} S(n-1, j-1) \beta_j(\lambda), \tag{8.6}$$

where n is a positive integer and $\beta_j(\lambda)$ are the degenerate Bernoulli numbers.

Cenkci and Howard showed in [13, Corollary 3.2] how to expand $\beta_n^{(k)}(\lambda)$ in terms of $B_n^{(k)}$. In fact, take $f(t) \to (1+\lambda t)^{1/\lambda} - 1$ and $h(t) \to e^t - 1$, then $l(t) \to \log(1+t)$ and $l(f(t)) \to \log(1+\lambda t)/\lambda$. Thus, we have $\hat{B}_n^{(z)} \to \beta_n^{(z)}(\lambda)$, $\hat{\beta}_n^{(z)} \to B_n^{(z)}$, $\hat{\gamma}_n^{(z)} \to \lambda^n b_n^{(z)}$ and $\hat{\mathscr{T}}(n,k) \to \lambda^{n-k} s(n,k)$, where $b_n^{(z)}$ are the higher order Bernoulli numbers of the second kind and $s(n,k)$ are the classical Stirling numbers of the first kind. According to Theorem 8.1, for $0 \leqslant k \leqslant n$ we have

$$\beta_n^{(k)}(\lambda) = \sum_{j=0}^{k-1} \binom{n}{j} \lambda^{n-j} b_{n-j}^{(k-j)} B_j^{(k)} + \binom{n}{k} \sum_{j=k}^{n} \binom{j}{k}^{-1} \lambda^{n-j} s(n-k, j-k) B_j^{(k)}, \tag{8.7}$$

and for $0 \leqslant n < k$ we have

$$\beta_n^{(k)}(\lambda) = \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j} b_{n-j}^{(k-j)} B_j^{(k)}. \tag{8.8}$$

Moreover, Corollary 8.3 gives us an expression for the degenerate Bernoulli numbers:

$$\beta_n(\lambda) = \lambda^n b_n + \sum_{j=1}^{n} \frac{n}{j} \lambda^{n-j} s(n-1, j-1) B_j,$$

which coincides with the one given by Howard [20, Section 7]. □

Corollary 8.4. *For integers $n \geqslant 1$ and $k \geqslant 1$, the following relation holds between the generalized higher order Bernoulli numbers of both kinds*:

$$\sum_{j=0}^{k-1} \binom{n}{j} f_1^{k-j} \hat{b}_{n-j}^{(k-j)} \hat{B}_j^{(k)} + \binom{n}{k} \sum_{j=k}^{n} \binom{j}{k}^{-1} \hat{s}(n-k, j-k) \hat{B}_j^{(k)} = 0. \tag{8.9}$$

For integer $n \geqslant 1$, the following relation holds between the generalized Bernoulli numbers of both kinds:

$$\hat{B}_n = -\frac{n}{g_1} \sum_{j=1}^{n} \frac{1}{j} \hat{S}(n-1, j-1) \hat{b}_j. \tag{8.10}$$

Proof. In Theorem 8.1, take $f(t) \to t$ and $h(t) \to f(t)$, then $l(t) \to g(t)$ and $l(f(t)) \to g(t)$. Now, we have $\hat{B}_n^{(z)} \to \delta_{n,0}$, $\hat{\beta}_n^{(z)} \to \hat{B}_n^{(z)}$, $\hat{\gamma}_n^{(z)} \to \hat{b}_n^{(z)}$, $\hat{\mathscr{T}}(n,k) \to \hat{s}(n,k)$ and $f_1 \to 1$, $h_1 \to f_1$. Thus, Theorem 8.1 gives

$$\begin{cases} \displaystyle\sum_{j=0}^{k-1} \binom{n}{j} f_1^{k-j} \hat{b}_{n-j}^{(k-j)} \hat{B}_j^{(k)} + \binom{n}{k} \sum_{j=k}^{n} \binom{j}{k}^{-1} \hat{s}(n-k, j-k) \hat{B}_j^{(k)} = 0 \quad (1 \leqslant k \leqslant n); \\ \displaystyle\sum_{j=0}^{n} \binom{n}{j} f_1^{k-j} \hat{b}_{n-j}^{(k-j)} \hat{B}_j^{(k)} = 0 \quad (1 \leqslant n < k). \end{cases}$$

According to Remark 8.2, we can obtain (8.9). Next, setting $k = 1$ in (8.9), we have

$$f_1 \hat{b}_n + n \sum_{j=1}^{n} \frac{1}{j} \hat{s}(n-1, j-1) \hat{B}_j = 0,$$

which is the dual case of (8.10). □

Example 8.2. For the classical higher order Bernoulli numbers of both kinds, we can derive from Eqs. (8.9) and (8.10) the following results:

$$\sum_{j=0}^{k-1}\binom{n}{j}b_{n-j}^{(k-j)}B_j^{(k)}+\binom{n}{k}\sum_{j=k}^{n}\binom{j}{k}^{-1}s(n-k,\, j-k)B_j^{(k)}=0,$$
$$\sum_{j=0}^{k-1}\binom{n}{j}B_{n-j}^{(k-j)}b_j^{(k)}+\binom{n}{k}\sum_{j=k}^{n}\binom{j}{k}^{-1}S(n-k,\, j-k)b_j^{(k)}=0,$$

and

$$\sum_{j=1}^{n}\frac{1}{j}S(n-1,\, j-1)b_j=-\frac{1}{n}B_n,\qquad \sum_{j=1}^{n}\frac{1}{j}s(n-1,\, j-1)B_j=-\frac{1}{n}b_n,$$

where n and k are positive integers.

Example 8.3. For entries (C1) and (C2), from Eq. (8.9), we have

$$\sum_{j=0}^{n}\binom{n}{j}(j-k)(n-k)^{n-j-1}k^j=\begin{cases}0 & \text{if } k\neq n,\\ -n^n & \text{if } k=n,\end{cases}$$
$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}(k-j)^{n-1}=0,$$

where n and k are positive integers. The first equation can be verified by the binomial identity. The second equation can be verified by expanding $(k-j)^{n-1}$ by the binomial identity and then using the explicit expression of the Stirling numbers of the second kind.

Example 8.4. Let us consider entries (F1) and (F2) (or (G1) and (G2)). For positive integers n and k, Eq. (8.9) gives

$$\sum_{j=0}^{n}\binom{k+j-1}{j}\binom{2n-k-j-1}{n-j}\frac{j-k}{n-k}=0\quad (n\neq k),$$
$$\sum_{j=1}^{n}\binom{k+n-2j-1}{n-j}\binom{2j-k-1}{j-1}\frac{k}{j}=\binom{k+n-1}{n}.$$

Setting $k=1$, the above identities will reduce to the following ones involving the Catalan numbers:

$$C_n=\sum_{j=2}^{n+1}\binom{2n-j}{n-1}\frac{j-1}{n},\qquad 1=\sum_{j=1}^{n}\binom{n-2j}{n-j}C_{j-1},$$

where $n\geqslant 1$. Combining these two identities will further give us a double summation:

$$1=\sum_{j=1}^{n}\sum_{i=1}^{j}\binom{n-2j-1}{n-j}\binom{2j-i-1}{j-1}\frac{i}{j}.$$

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