Identities Among Certain Triangular Matrices

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ABSTRACT

The object of this paper is to develop the ideas introduced in the author's paper [1] on matrices which generate families of polynomials and associated infinite series. A family of infinite one-subdiagonal non-commuting matrices $\mathbf{Q}_m$ is defined, and a number of identities among its members are given. The matrix $\mathbf{Q}_1$ is applied to solve a problem concerning the derivative of a family of polynomials, and it is shown that the solution is remarkably similar to a conventional solution employing a scalar generating function. Two sets of infinite triangular matrices are then defined. The elements of one set are related to the terms of Laguerre, Hermite, Bernoulli, Euler, and Bessel polynomials, while the elements of the other set consist of Stirling numbers of both kinds, the two-parameter Eulerian numbers, and numbers introduced in a note on inverse scalar relations by Touchard. It is then shown that these matrices are related by a number of identities, several of which are in the form of similarity transformations. Some well-known and less well-known pairs of inverse scalar relations arising in combinatorial analysis are shown to be derivable from simple and obviously inverse pairs of matrix relations. This work is an explicit matrix version of the umbral calculus as presented by Rota et al. [24–26].

1. INTRODUCTION

In an earlier paper [1] the author showed that certain families of polynomials and their associated infinite series can be generated by means of elementary functions of certain simple infinite constant matrices. For example, the matrix which generates the polynomials $(1+x)^n$, $n = 0, 1, 2, \ldots,$ by

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rows and the infinite series $(1-x)^{-n}$, $n=1,2,\ldots$, by columns is

$$\mathbf{M}(x)=\left[\binom{i-1}{j-1}x^{i-j}\right], \qquad i,j=1,2,3,\ldots \tag{1.1}$$

$$=\begin{bmatrix} 1 \\ x & 1 \\ x^2 & 2x & 1 \\ x^3 & 3x^2 & 3x & 1 \\ x^4 & 4x^3 & 6x^2 & 4x & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix},$$

$$=e^{x\mathbf{Q}}, \tag{1.2}$$

where

$$\mathbf{Q}=\begin{bmatrix} 0 \\ 1 & 0 \\ & 2 & 0 \\ & & 3 & 0 \\ & & & 4 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}. \tag{1.3}$$

As no proof of this relation was given in [1], two proofs will be given here.

The first uses the fact that the matrix $\mathbf{M}(x)$ arose as a solution of the Cauchy-Bellman equation

$$\mathbf{M}(x)\mathbf{M}(y)=\mathbf{M}(x+y), \tag{1.4}$$

so that a relation of the form

$$\mathbf{M}(x)=e^{x\mathbf{Q}},$$

where $\mathbf{Q}$ is a constant matrix, is expected. Hence

$$\mathbf{Q}=\mathbf{M}'(0),$$

which gives (1.3). We note in passing that (1.4) has the more general solution

$$\mathbf{M}(x)=\mathbf{K}e^{x\mathbf{Q}}\mathbf{K}^{-1},$$

where $\mathbf{K}$ is an arbitrary nonsingular constant matrix. The second proof is given in the next section. We note also that $\mathbf{M}(x)$ and the Cauchy-Bellman equation appear in a recent paper by Kalman [28].

Note that, if

$$\mathbf{A} = [a_0, a_1, a_2, a_3, \dots]^T,$$

$$\mathbf{X} = [1, x, x^2, x^3, \dots]^T,$$

$$\mathbf{Y} = [1, (1+x), (1+x)^2, (1+x)^3, \dots]^T,$$

$$\mathbf{F} = [f_0(x), f_1(x), f_2(x), f_3(x), \dots]^T,$$

where $f_i(x)$ is the Appell polynomial defined by

$$f_i(x) = \sum_{j=0}^{i} \binom{i}{j} a_j x^{i-j}, \tag{1.5}$$

then

$$\begin{aligned} \mathbf{X}' &= \mathbf{QX}, \\ \mathbf{Y} &= e^{\mathbf{Q}}\mathbf{X}, \\ \mathbf{X} &= e^{-\mathbf{Q}}\mathbf{Y}, \\ \mathbf{Y}' &= e^{\mathbf{Q}}\mathbf{QX}, \\ \mathbf{F} &= e^{x\mathbf{Q}}\mathbf{A}, \\ \Delta^m\mathbf{X} &= (e^{\mathbf{Q}} - I)^m\mathbf{X}, \end{aligned} \tag{1.6}$$

where

$$\Delta x^n = (x+1)^n - x^n.$$

The object of this paper is to develop the ideas introduced in [1] to the stage where they can be applied to the proofs of a number of combinatorial identities and other relations.

A family of simple matrices $\mathbf{Q}_m$ is defined, and a number of relations among its members are given. Two sets of triangular matrices are then defined. The elements of one set are related to the terms of Laguerre, Hermite, Bernoulli, Euler, and Bessel polynomials, while the elements of the other set consist of Stirling numbers of both kinds, the two-parameter Eulerian numbers, and numbers introduced in a note on inverse scalar relations by Touchard. It is then shown that these matrices are related by a number of identities, several of which are in the form of similarity transformations. Some well-known and less well-known pairs of inverse scalar relations arising in combinatorial analysis are shown to be derivable from simple and obviously inverse pairs of matrix relations.

The referee has pointed out that this work is an explicit matrix version of the umbral calculus as presented by Rota et al. [24–26].

2. SOME ONE-SUBDIAGONAL MATRICES

Let

$$\mathbf{A} = \begin{bmatrix} 0 & & & & & \\ a_1 & 0 & & & & \\ & a_2 & 0 & & & \\ & & a_3 & 0 & & \\ & & & a_4 & 0 & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

Then

$$\mathbf{A}^2 = \begin{bmatrix} 0 & & & & & \\ 0 & 0 & & & & \\ a_2a_1 & 0 & 0 & & & \\ & a_3a_2 & 0 & 0 & & \\ & & a_4a_3 & 0 & 0 & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

To economize on space, matrices containing a single nonzero subdiagonal will be expressed in vector form with reference to its precise location. All subdiagonals parallel to the principal diagonal have an "equation" of the form $i - j = n$. Thus

$$\mathbf{A} = \underset{i-j=1}{\operatorname{subdiag}}\left[a_1, a_2, a_3, a_4, \ldots\right].$$

To economize still further, only the element in the jth column will be given:

$$\mathbf{A} = \underset{i-j=1}{\operatorname{subdiag}} [a_j], \qquad j = 1,2,3,\ldots,$$

$$\mathbf{AB} = \underset{i-j=2}{\operatorname{subdiag}} \left[a_{j+1} b_j\right], \qquad j = 1,2,3,\ldots, \tag{2.1}$$

$$\mathbf{ABC} = \underset{i-j=3}{\operatorname{subdiag}} \left[a_{j+2} b_{j+1} c_j\right], \qquad j = 1,2,3,\ldots,$$

etc. Using obvious notation, the only nonzero elements in the product $\mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_n$ lie in the subdiagonal with equation $i - j = n$.

Define two matrices $\mathbf{Q}_m, \mathbf{R}_\alpha$ as follows:

$$\mathbf{Q}_m = \underset{i-j=1}{\operatorname{subdiag}} [j^m],$$

$$\mathbf{Q} = \mathbf{Q}_1, \tag{2.2}$$

$$\mathbf{R}_\alpha = \underset{i-j=1}{\operatorname{subdiag}} [j(\alpha + j)]$$

$$= \alpha\mathbf{Q} + \mathbf{Q}_2. \tag{2.3}$$

Note that

$$D^2(x\mathbf{X}) = (\mathbf{Q} + \mathbf{Q}_2)\mathbf{X},$$

$$D(xD)^m\mathbf{X} = \mathbf{Q}_{m+1}\mathbf{X} \qquad (D = d/dx). \tag{2.4}$$

The definition of $\mathbf{Q}_m$ given in [1] excludes $m = 0$ and has therefore been discarded. A few of the later formulae in that paper require slight modification.

Using the notation of Comtet [5], define the falling and rising factorial functions $(x)_n$, $\langle x \rangle_n$ as follows:

$$(x)_n = x(x-1)(x-2)\cdots(x-n+1), \qquad n = 1,2,3,\ldots,$$

$$(x)_0 = 1,$$

$$\langle x \rangle_n = x(x+1)(x+2)\cdots(x+n-1), \qquad n = 1,2,3,\ldots$$

$$= (-1)^n(-x)_n,$$

$$\langle x \rangle_0 = 1.$$

The notation $x^{(n)}$ is often used for $(x)_n$, especially in the finite-difference calculus.

It may easily be verified that

$$\mathbf{Q}_m^n = \operatorname*{subdiag}_{i-j=n}[\langle j\rangle_n^m], \qquad n = 0,1,2,\ldots,$$

$$\frac{\mathbf{Q}_m^n}{(n!)^m} = \operatorname*{subdiag}_{i-j=n}\left[\binom{n+j-1}{n}^m\right], \qquad n = 0,1,2,\ldots, \tag{2.5}$$

$$\mathbf{R}_\alpha^n = \operatorname*{subdiag}_{i-j=n}[\langle j\rangle_n\langle j+\alpha\rangle_n], \qquad n = 0,1,2,\ldots. \tag{2.6}$$

LEMMA 2.1.

$$[x_{i-j}] = \sum_{n=0}^{\infty} x_n \mathbf{Q}_0^n, \qquad i \geqslant j.$$

The proof consists of the observation that the matrix on the left can be expressed in the form

$$\sum_{n=0}^{\infty} x_n \operatorname*{subdiag}_{i-j=n}[1].$$

For example, when $x_n = x^n$,

$$[x^{i-j}] = \sum_{n=0}^{\infty} (x\mathbf{Q}_0)^n = \frac{\mathbf{I}}{\mathbf{I} - x\mathbf{Q}_0}, \qquad i \geqslant j.$$

LEMMA 2.2.

$$\left[\binom{i-1}{j-1} x_{i-j}\right] = \sum_{n=0}^{\infty} \frac{x_n \mathbf{Q}^n}{n!}.$$

For the matrix on the left can be expressed in the form

$$\sum_{n=0}^{\infty} x_n \operatorname*{subdiag}_{i-j=n}\left[\binom{n+j-1}{n}\right].$$

The result follows from (2.5) with $m = 1$.

ILLUSTRATIONS.

(i)

$$\mathbf{M}(x) = \left[\binom{i-1}{j-1} x^{i-j}\right] = \sum_{n=0}^{\infty} \frac{(x\mathbf{Q})^n}{n!} = e^{x\mathbf{Q}},$$

which is a second proof of the result stated in the Introduction. The author has recently come across yet another proof in a paper by Redheffer [18].

(ii)

$$\left[\binom{i-1}{j-1}(x)_{i-j}\right] = \sum_{n=0}^{\infty} \frac{(x)_n \mathbf{Q}^n}{n!} = (\mathbf{I}+\mathbf{Q})^x$$

(iii)

$$\left[\binom{i-1}{j-1}\langle x\rangle_{i-j}\right] = \sum_{n=0}^{\infty} \frac{\langle x\rangle_n \mathbf{Q}^n}{n!} = (\mathbf{I}-\mathbf{Q})^{-x}.$$

Note that (1.6) can now be extended as follows:

$$\mathbf{F} = e^{x\mathbf{Q}}\mathbf{A} = \left[\sum_{n=0}^{\infty} \frac{a_n \mathbf{Q}^n}{n!}\right]\mathbf{X}$$

LEMMA 2.3.

$$(j+1)^n - j^n = \sum_{s=0}^{n-1} (j+1)^s j^{n-1-s}.$$

The proof follows from the identity

$$\frac{a^n - b^n}{a-b} = \sum_{s=0}^{n-1} a^s b^{n-1-s}.$$

THEOREM 2.1.

$$\mathbf{Q}_m\mathbf{Q}_n - \mathbf{Q}_n\mathbf{Q}_m = \sum_{s=0}^{m-n-1} \mathbf{Q}_{n+s}\mathbf{Q}_{m-s-1}, \qquad m > n.$$

Proof. Referring to (2.2) and the lemma,

$$\mathbf{Q}_m\mathbf{Q}_n - \mathbf{Q}_n\mathbf{Q}_m = \underset{i-j=2}{\operatorname{subdiag}}\left[q(j)\right],$$

where

$$\begin{aligned} q(j) &= (j+1)^m j^n - (j+1)^n j^m \\ &= (j+1)^n j^n\left[(j+1)^{m-n} - j^{m-n}\right] \\ &= (j+1)^n j^n \sum_{s=0}^{m-n-1} (j+1)^s j^{m-n-1-s} \\ &= \sum_{s=0}^{m-n-1} (j+1)^{n+s} j^{m-s-1}. \end{aligned}$$

The result follows. ■

In particular,

$$\mathbf{Q}_{m+1}\mathbf{Q}_m - \mathbf{Q}_m\mathbf{Q}_{m+1} = \mathbf{Q}_m^2, \tag{2.7}$$

from which it follows by induction that

$$\mathbf{Q}_{m+1}\mathbf{Q}_m^n - \mathbf{Q}_m^n\mathbf{Q}_{m+1} = n\mathbf{Q}_m^{n+1}. \tag{2.8}$$

Other identities of a similar nature include

$$\begin{aligned} (\mathbf{Q}_{m+1} - \mathbf{Q}_m)(\mathbf{Q}_{m+1} + \mathbf{Q}_m) &= \mathbf{Q}_{m+1}^2, \\ (\mathbf{Q}_{m+1} + \mathbf{Q}_m)(\mathbf{Q}_{m+1} - \mathbf{Q}_m) &= 2\mathbf{Q}_m^2 - \mathbf{Q}_{m+1}^2, \\ \mathbf{Q}_{m+1}^2 - \mathbf{Q}_m\mathbf{Q}_{m+2} &= \mathbf{Q}_m\mathbf{Q}_{m+1}, \\ \mathbf{Q}_{m+1}^2 - \mathbf{Q}_{m+2}\mathbf{Q}_m &= -\mathbf{Q}_{m+1}\mathbf{Q}_m. \end{aligned} \tag{2.9}$$

Let

$$\mathbf{H}_n = \sum_{r=0}^{n} (-1)^r \binom{n}{r} \mathbf{A}^{n-r}\mathbf{H}_0\mathbf{B}^r, \qquad n = 0,1,2,\ldots.$$

THEOREM 2.2.

(a) $\mathbf{H}_n = \mathbf{A}\mathbf{H}_{n-1} - \mathbf{H}_{n-1}\mathbf{B}, \quad n = 1,2,3,\ldots.$

(b) $\mathbf{H}_n = \sum_{r=0}^{p} (-1)^r \binom{p}{r} \mathbf{A}^{p-r}\mathbf{H}_{n-p}\mathbf{B}^r, \quad p = 0,1,2,\ldots,n.$

The proof of (a) is elementary and applies the identity

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}.$$

The particular case of (b) in which $p = 1$ is identical with (a). The general form of (b) is obtained by applying (a) repeatedly.

Let

$$\mathbf{K}_{ns}(m) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} \mathbf{Q}_{m+1}^{n-r}\mathbf{Q}_m^s\mathbf{Q}_{m+1}^r, \qquad n,s = 0,1,2,\ldots,$$

$$\mathbf{K}_{0s}(m) = \mathbf{Q}_m^s.$$

THEOREM 2.3.

(a) $\mathbf{K}_{ns}(m) = s\mathbf{K}_{n-1,s+1}(m)$,

(b) $\mathbf{K}_{ns}(m) = \langle s \rangle_n \mathbf{Q}_m^{n+s}$,

(c) $\mathbf{K}_{n1}(m) = n!\mathbf{Q}_m^{n+1}$.

Proof. Put

$$\mathbf{A} = \mathbf{B} = \mathbf{Q}_{m+1},$$

$$\mathbf{H}_n = \mathbf{K}_{ns}(m)$$

$$\mathbf{H}_0 = \mathbf{Q}_m^s$$

in the previous theorem, and refer to (2.8):

$$\begin{aligned}\mathbf{K}_{ns}(m) &= \mathbf{Q}_{m+1}\mathbf{K}_{n-1,s}(m) - \mathbf{K}_{n-1,s}(m)\mathbf{Q}_{m+1}\\ &= \sum_{r=0}^{n-1}(-1)^r\binom{n-1}{r}\mathbf{Q}_{m+1}^{n-1-r}(\mathbf{Q}_{m+1}\mathbf{Q}_m^s - \mathbf{Q}_m^s\mathbf{Q}_{m+1})\mathbf{Q}_{m+1}^r\\ &= s\sum_{r=0}^{n-1}(-1)^r\binom{n-1}{r}\mathbf{Q}_{m+1}^{n-1-r}\mathbf{Q}_m^{s+1}\mathbf{Q}_{m+1}^r,\end{aligned}$$

which proves (a). Applying (a) repeatedly yields (b), and then (c) follows as the special case in which $s = 1$. ∎

It may also be verified that

$$e^{-x\mathbf{Q}}(x\mathbf{Q}_2)e^{x\mathbf{Q}} = x\mathbf{Q}_2 + x^2\mathbf{Q}^2,$$

$$e^{-x\mathbf{Q}}(x\mathbf{Q}_3)e^{x\mathbf{Q}} = x\mathbf{Q}_3 + x^2(\mathbf{Q}^2 + 2\mathbf{Q}\mathbf{Q}_2) + x^3\mathbf{Q}^3.$$

Let

$$\begin{aligned}\mathbf{U}_m &= \tfrac{1}{2}\operatorname*{subdiag}_{i-j=1}\left[(m-1+j)(m+j)\right], \qquad m = 1,2,3,\ldots\\ &= \tfrac{1}{2}\left[m(m-1)\mathbf{Q}_0 + (2m-1)\mathbf{Q}_1 + \mathbf{Q}_2\right],\\ \mathbf{V}_m &= \mathbf{U}_m\mathbf{U}_{m-1}\cdots\mathbf{U}_2\mathbf{U}_1,\\ \mathbf{V}_0 &= \mathbf{I}.\end{aligned}$$

Note that the suffixes in the definition of $\mathbf{V}_m$ are in descending order of magnitude. The $\mathbf{U}_m$ do not commute. $\mathbf{V}_m$ arises in connection with Bessel polynomials.

The pair of identities (2.9) suggests a direction in which they can be generalized.

Let $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ be square matrices of the same order where $\mathbf{DA} = \mathbf{AD}$, $\mathbf{CB} = \mathbf{BC}$. Then the symbol

$$\begin{Vmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{Vmatrix}$$

can represent the matrix $\mathbf{AD} - \mathbf{BC}$ and should not be confused with the determinant of a partitioned matrix. It is a matrix, not a determinant. However, if the matrices do not commute in pairs as stipulated, then the symbol becomes ambiguous.

Thus, the persymmetric matrix symbol

$$\begin{Vmatrix} \mathbf{Q}_m & \mathbf{Q}_{m+1} \\ \mathbf{Q}_{m+1} & \mathbf{Q}_{m+2} \end{Vmatrix}$$

could represent $\mathbf{Q}_m\mathbf{Q}_{m+2} - \mathbf{Q}_{m+1}^2$ or $\mathbf{Q}_{m+2}\mathbf{Q}_m - \mathbf{Q}_{m+1}^2$ which, from (2.9), are not equal. The ambiguity can be removed by defining two interpretations of the matrix symbol.

Let

$$\Uparrow\mathbf{M}_2\Uparrow = \Uparrow\begin{Vmatrix} \mathbf{Q}_m & \mathbf{Q}_{m+1} \\ \mathbf{Q}_{m+1} & \mathbf{Q}_{m+2} \end{Vmatrix}\Uparrow = \mathbf{Q}_m\mathbf{Q}_{m+2} - \mathbf{Q}_{m+1}^2,$$

where the upward-pointing arrows denote that the suffices in each term should be arranged in ascending or constant order of magnitude. Similarly, let

$$\Downarrow\mathbf{M}_2\Downarrow = \Downarrow\begin{Vmatrix} \mathbf{Q}_m & \mathbf{Q}_{m+1} \\ \mathbf{Q}_{m+1} & \mathbf{Q}_{m+2} \end{Vmatrix}\Downarrow = \mathbf{Q}_{m+2}\mathbf{Q}_m - \mathbf{Q}_{m+1}^2,$$

where the downward-pointing arrows denote that the suffices in each term should be arranged in descending or constant order of magnitude.

These definitions can be extended to third- and higher-order persymmetric matrix symbols. The symbol

$$\begin{Vmatrix} \mathbf{Q}_m & \mathbf{Q}_{m+1} & \mathbf{Q}_{m+2} \\ \mathbf{Q}_{m+1} & \mathbf{Q}_{m+2} & \mathbf{Q}_{m+3} \\ \mathbf{Q}_{m+2} & \mathbf{Q}_{m+3} & \mathbf{Q}_{m+4} \end{Vmatrix}$$

does not represent a unique matrix. There are six distinct products of $\mathbf{Q}_m, \mathbf{Q}_{m+2}, \mathbf{Q}_{m+4}$, six distinct products of $\mathbf{Q}_{m+1}, \mathbf{Q}_{m+2}, \mathbf{Q}_{m+3}$ (which occurs twice), three distinct products of $\mathbf{Q}_m, \mathbf{Q}_{m+3}, \mathbf{Q}_{m+3}$, and three distinct products of $\mathbf{Q}_{m+1}, \mathbf{Q}_{m+1}, \mathbf{Q}_{m+4}$. Hence the total number of possible interpretations of the symbol is $6^3 3^2 = 1944$; but it is possible to define two principal interpretations, one in which the suffixes in each term are arranged in ascending order of magnitude and one in which they are arranged in descending order of magnitude. (Ascending should be interpreted as nondescending and descending should be interpreted as nonascending.)

Let

$$\Uparrow\!\|\mathbf{M}_3\|\!\Uparrow = \begin{Vmatrix} \mathbf{Q}_m & \mathbf{Q}_{m+1} & \mathbf{Q}_{m+2} \\ \mathbf{Q}_{m+1} & \mathbf{Q}_{m+2} & \mathbf{Q}_{m+3} \\ \mathbf{Q}_{m+2} & \mathbf{Q}_{m+3} & \mathbf{Q}_{m+4} \end{Vmatrix}^{\Uparrow\;\Uparrow}$$

$$= \mathbf{Q}_m\mathbf{Q}_{m+2}\mathbf{Q}_{m+4} + 2\mathbf{Q}_{m+1}\mathbf{Q}_{m+2}\mathbf{Q}_{m+3} - \mathbf{Q}_{m+2}^3 - \mathbf{Q}_m\mathbf{Q}_{m+3}^2 - \mathbf{Q}_{m+1}^2\mathbf{Q}_{m+4}$$

$$= -\operatorname*{subdiag}_{i-j=3}\left[(j+2)^m(j+1)^{m+1}j^{m+2}(2j^2+5j+4)\right].$$

Similarly,

$$\Downarrow\!\|\mathbf{M}_3\|\!\Downarrow = \mathbf{Q}_{m+4}\mathbf{Q}_{m+2}\mathbf{Q}_m + 2\mathbf{Q}_{m+3}\mathbf{Q}_{m+2}\mathbf{Q}_{m+1} - \mathbf{Q}_{m+2}^3 - \mathbf{Q}_{m+3}^2\mathbf{Q}_m - \mathbf{Q}_{m+4}\mathbf{Q}_{m+1}^2$$

$$= \operatorname*{subdiag}_{i-j=3}\left[(j+2)^{m+2}(j+1)^{m+1}j^m(2j^2+3j+2)\right].$$

Hence

$$\Uparrow\!\|\mathbf{M}_3\|\!\Uparrow + \Downarrow\!\|\mathbf{M}_3\|\!\Downarrow = 2\operatorname*{subdiag}_{i-j=3}\left[(j+2)^m(j+1)^{m+2}j^m(3j^2+6j+4)\right]$$

$$= 2\operatorname*{subdiag}_{i-j=3}\left[3(j+2)^{m+1}(j+1)^{m+2}j^{m+1} + 4(j+2)^m(j+1)^{m+2}j^m\right]$$

$$= 6\mathbf{Q}_{m+1}\mathbf{Q}_{m+2}\mathbf{Q}_{m+1} + 8\mathbf{Q}_m\mathbf{Q}_{m+2}\mathbf{Q}_m.$$

The three suffixes in each of the terms on the right are symmetric about their centers, which is not unexpected.

The name suggested for these determinantal matrix symbols is *materminants*. It is possible to define two principal interpretations of materminants only if the elements can be arranged in a linear sequence, as they can, for example, when the arrays are persymmetric or circulant.

3. AN APPLICATION OF THE MATRIX Q

The finite-difference analogue of the Appell relation

$$D\phi_m(x) = m\phi_{m-1}(x) \qquad (D = d/dx) \tag{3.1}$$

is

$$\Delta\psi_m(x) = m\psi_{m-1}(x), \tag{3.2}$$

where

$$\Delta\psi_m(x) = \psi_m(x+1) - \psi_m(x).$$

The polynomial solution of (3.2) is

$$\psi_m(x) = \sum_{r=0}^{m} \binom{m}{r} \alpha_{m-r}(x)_r, \qquad m = 0,1,2,\ldots.$$

PROBLEM. *Prove that*

$$\psi'_m(x) = \sum_{r=0}^{m-1} (-1)^{m+r+1} \binom{m}{r} (m-r-1)!\psi_r(x).$$

Two proofs are given. The first is conventional and employs a scalar generating function $G(x,t)$. The second employs the constant matrix **Q**, whose role is remarkably similar to that of t.

First proof. Let

$$\begin{aligned} G(x,t) &= \sum_{m=0}^{\infty} \frac{\psi_m(x)t^m}{m!} \\ &= \sum_{r=0}^{\infty} (x)_r \sum_{m=r}^{\infty} \binom{m}{r} \alpha_{m-r} \frac{t^m}{m!} \\ &= \sum_{r=0}^{\infty} \frac{(x)_r t^r}{r!} \sum_{n=0}^{\infty} \frac{\alpha_n t^n}{n!} \\ &= (1+t)^x G(0,t). \end{aligned} \tag{3.3}$$

Hence

$$\frac{\partial G}{\partial x} = \log(1+t)\,G,$$

$$\sum_{m=0}^{\infty} \frac{\psi'_m(x)t^m}{m!} = \sum_{r=1}^{\infty} \frac{(-1)^{r+1}t^r}{r} \sum_{s=0}^{\infty} \frac{\psi_s(x)t^s}{s!}.$$

The proof is completed by equating coefficients of t^m. ∎

Second proof. Let

$$\mathbf{G}(x) = [\psi_0(x), \psi_1(x), \psi_2(x), \dots]^T.$$

Then, referring to Lemma 2.2,

$$\begin{aligned}\mathbf{G}(x) &= \left[\binom{i-1}{j-1}(x)_{i-j}\right]\mathbf{G}(0)\\ &= (\mathbf{I}+\mathbf{Q})^x\mathbf{G}(0)\end{aligned}$$

[Note the similarity between this equation and (3.3).] Thus

$$\begin{aligned}\mathbf{G}'(x) &= \log(\mathbf{I}+\mathbf{Q})\,\mathbf{G}(x)\\ &= \left[\sum_{r=1}^{\infty} \frac{(-1)^{r+1}\mathbf{Q}^r}{r}\right]\mathbf{G}(x)\\ &= \left[(-1)^{i+j+1}\binom{i-1}{j-1}(i-j-1)!\right]\mathbf{G}(x)\end{aligned}$$

The proof is completed by carrying out the matrix multiplication. ∎

4. SOME SPECIAL TRIANGULAR MATRICES

Laguerre polynomials $L_n^{(\alpha)}(x)$, Hermite polynomials $H_n(x)$, Bernoulli polynomials $B_n(x)$, and Euler polynomials $E_n(x)$ are defined by Abramowitz and Stegun [2], Erdelyi et al. [3], and Rainville [4]. The lesser-known Bessel polynomials $y_n(x)$ are defined by Erdelyi, Rainville, and Riordan [6]. They differ from Laguerre, Hermite, Legendre, and Chebyshev polynomials in that they are orthogonal not along the real axis, but round a contour embracing the origin in the complex plane.

The Stirling numbers s_{ij}, S_{ij} of the first and second kinds are defined by Comtet [5], Riordan [6], Jordan [8], Cohen [9], and Abramowitz and Stegun. Apart from displaying both parameters as suffices as is customary in matrix analysis, the notation s_{ij}, S_{ij} is that of Comtet and Riordan. Cohen uses the notation

$$\begin{bmatrix} i \\ j \end{bmatrix} = |s_{ij}|, \qquad \begin{Bmatrix} i \\ j \end{Bmatrix} = S_{ij}$$

in order to emphasize the similarity between the Stirling numbers and the binomial coefficients $\binom{i}{j}$. Abramowitz and Stegun use $\mathfrak{S}$ in place of S. Jordan remarks that the utility of Stirling numbers has not been fully recognized; they have been neglected and are seldom used. Their importance in certain similarity and related transformations will be demonstrated below.

The little-known two-parameter Eulerian numbers A_{ij}, not to be confused with the one-parameter Euler numbers E_r, are defined by Comtet, who gives a long list of references, to which may be added Stalley, Lawden, Klamkin, Zeitlin, Takacs, and Lehmer [10–15] and Mortini [27]. Lawden evaluates the persymmetric (Hankel) determinant whose elements are Eulerian polynomials. Given a family of polynomials

$$P_n(x) = \sum_{r=0}^{n} c_{nr} x^r, \qquad n = 0, 1, 2, \ldots,$$

define a triangular matrix $\mathbf{P}(x)$ with elements

$$P_{ij} = c_{i-1,i-j} x^{i-j}, \qquad i \geqslant j.$$

The polynomials are said to be generated by the rows of $\mathbf{P}(x)$.

The Laguerre polynomials are defined by the generating-function relation

$$(1-t)^{-\alpha-1} \exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n. \tag{4.1}$$

Explicitly,

$$L_n^{(\alpha)}(x) = \sum_{r=0}^{n} \frac{(-1)^r}{r!}\binom{n+\alpha}{n-r}x^r.$$

The polynomials $\binom{n+\alpha}{n}^{-1}L_n^{(\alpha)}(x)$, which have a unit constant term, are generated by the rows of the matrix

$$\mathbf{L}^{(\alpha)}(x) = \alpha!\left[\binom{i-1}{j-1}\frac{(-x)^{i-j}}{(i-j+\alpha)!}\right], \qquad i,j=1,2,3,\ldots. \tag{4.2}$$

Let

$$\begin{aligned}\bar{L}_n^{(\alpha)}(x) &= (-x)^n n! L_n^{(\alpha)}\left(\frac{1}{x}\right)\\ &= \sum_{r=0}^{n} \frac{\langle r+1\rangle_{n-r}\langle r+1+\alpha\rangle_{n-r}(-x)^{n-r}}{(n-r)!}\\ &= \sum_{r=0}^{n} r!\binom{n}{r}\binom{n+\alpha}{r}(-x)^r. \end{aligned} \tag{4.3}$$

The $\bar{L}_n^{(\alpha)}(x)$ are the reversed Laguerre polynomials with unit constant term. The matrix which generates these polynomials by rows is

$$\bar{\mathbf{L}}^{(\alpha)}(x) = \left[\frac{\langle j\rangle_{i-j}\langle j+\alpha\rangle_{i-j}(-x)^{i-j}}{(i-j)!}\right], \qquad i,j=1,2,3,\ldots \tag{4.4}$$

$$= \left[\frac{(i-1)!}{(j-1)!}\binom{i-1+\alpha}{j-1+\alpha}(-x)^{i-j}\right]. \tag{4.5}$$

The numerical coefficients in the elements of the matrix satisfy the recurrence relation

$$a_{ij} = a_{i-1,j-1} - (i+j-2+\alpha)a_{i-1,j}.$$

The reversed simple Laguerre polynomials are obtained by putting $\alpha = 0$ and are generated by the rows of the matrix

$$\bar{\mathbf{L}}(x)=\left[\frac{(i-1)!}{(j-1)!}\binom{i-1}{j-1}(-x)^{i-j}\right] \tag{4.6}$$

$$=\begin{bmatrix} 1 &&&& \\ -x & 1 &&& \\ 2x^2 & -4x & 1 && \\ -6x^3 & 18x^2 & -9x & 1 & \\ 24x^4 & -96x^3 & 72x^2 & -16x & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix},$$

where the numerical coefficients satisfy the recurrence relation

$$a_{ij}=a_{i-1,j-1}-(i+j-2)a_{i-1,j}.$$

Also,

$$\bar{\mathbf{L}}^{(1)}(x)=\begin{bmatrix} 1 &&&& \\ -2x & 1 &&& \\ 6x^2 & -6x & 1 && \\ -24x^3 & 36x^2 & -12x & 1 & \\ 120x^4 & -240x^3 & 120x^2 & -20x & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}. \tag{4.7}$$

The numerical coefficients satisfy the recurrence relation

$$a_{ij}=a_{i-1,j-1}-(i+j-1)a_{i-1,j}$$

and may be identified as $(-1)^j L_{ij}$, where the L_{ij} are the Lah numbers [5, 6] defined by

$$L_{ij}=(-1)^i\frac{(i-1)!}{(j-1)!}\binom{i}{j}. \tag{4.8}$$

The Hermite polynomials are defined by the generating-function relation

$$e^{2xt-t^2}=\sum_{n=0}^{\infty}\frac{H_n(x)t^n}{n!}. \tag{4.9}$$

Explicitly,

$$H_n(x) = n! \sum_{r=0}^{N} \frac{(-1)^r (2x)^{n-2r}}{r!(n-2r)!}, \qquad N = \left[\tfrac{1}{2}n\right].$$

The even-order polynomials

$$\frac{(-1)^n n!}{(2n)!} H_{2n}(x)$$

are generated by the rows of the matrix

$$\mathbf{H}_e(x) = \left[\binom{i-1}{j-1} \frac{(i-j)!(-4x^2)^{i-j}}{(2i-2j)!}\right], \qquad i, j = 1,2,3,\dots. \tag{4.10}$$

The odd-order polynomials

$$\frac{(-1)^n n!}{2(2n+1)!x} H_{2n+1}(x)$$

are generated by the rows of the matrix

$$\mathbf{H}_o(x) = \left[\binom{i-1}{j-1} \frac{(i-j)!\{-4x^2\}^{i-j}}{(2i-2j+1)!}\right], \qquad i, j = 1,2,3,\dots. \tag{4.11}$$

The reversed polynomials

$$\left(\tfrac{1}{2}x\right)^n H_n\left(\frac{1}{x}\right)$$

are generated by the rows of the matrix

$$\overline{\mathbf{H}}(x) = \left[\binom{i-1}{j-1} \frac{(i-j)!\frac{1}{2}\{1+(-1)^{i-j}\}\left(-\frac{1}{2}x\right)^{i-j}}{\left\{\frac{1}{2}(i-j)\right\}!}\right], \qquad i, j = 1,2,3,\dots. \tag{4.12}$$

The Bernoulli polynomials are defined by the generating-function relation

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x) t^n}{n!}. \tag{4.13}$$

The reversed Bernoulli polynomials are generated by the rows of the matrix

$$\bar{\mathbf{B}}(x) = \left[\binom{i-1}{j-1} B_{i-j}(0) x^{i-j}\right], \qquad i, j = 1,2,3,\ldots. \tag{4.14}$$

The numerical coefficients in the first column are the Bernoulli numbers $B_r = B_r(0)$.

The Euler polynomials are defined by the generating-function relation

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} \frac{E_n(x)t^n}{n!}. \tag{4.15}$$

The reversed Euler polynomials are generated by the rows of the matrix

$$\bar{\mathbf{E}}(x) = \left[\binom{i-1}{j-1} E_{i-j}(0) x^{i-j}\right], \qquad i, j = 1,2,3,\ldots. \tag{4.16}$$

The numerical coefficients in the first column are the numbers $E_r(0)$, not the Euler numbers $E_r = 2^r E_r(\frac{1}{2})$.

The Bessel polynomials are defined by the generating-function relation

$$\exp\left[\frac{1-(1-2xt)^{1/2}}{x}\right] = \sum_{n=0}^{\infty} \frac{y_{n-1}(x)t^n}{n!}. \tag{4.17}$$

Explicitly,

$$y_n(x) = \sum_{r=0}^{n} \frac{(n+r)!}{(n-r)!r!}\left(\tfrac{1}{2}x\right)^r.$$

These polynomials are generated by the rows of the matrix

$$\mathbf{Y}(x) = \left[a_{ij} x^{i-j}\right], \qquad i, j = 1,2,3,\ldots,$$

where

$$a_{ij} = \frac{(2i-j-1)!}{(j-1)!(i-j)!2^{i-j}}. \tag{4.18}$$

We have

$$\mathbf{Y}(x)=\begin{bmatrix} 1 &&&& \\ x & 1 &&& \\ 3x^2 & 3x & 1 && \\ 15x^3 & 15x^2 & 6x & 1 & \\ 105x^4 & 105x^3 & 45x^2 & 10x & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}, \tag{4.19}$$

where

$$a_{ij}=a_{i-1,j-1}+(2i-j-2)a_{i-1,j}. \tag{4.20}$$

Stirling numbers of the first and second kinds are defined by the relations

$$(x)_r=\sum_{k=0}^{r} s_{rk}x^k, \qquad s_{r0}=\delta_{r0}, \tag{4.21}$$

$$x^r=\sum_{k=0}^{r} \mathrm{S}_{rk}(x)_k, \quad \mathrm{S}_{r0}=\delta_{r0}. \tag{4.22}$$

Define Stirling matrices $\mathrm{s}(x), \mathrm{S}(x)$ of the first and second kinds as follows:

$$\mathrm{s}(x)=\left[s_{ij}x^{i-j}\right] = \begin{bmatrix} 1 &&&& \\ -x & 1 &&& \\ 2x^2 & -3x & 1 && \\ -6x^3 & 11x^2 & -6x & 1 & \\ 24x^4 & -50x^3 & 35x^2 & -10x & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}, \tag{4.23}$$

where

$$s_{ij}=s_{i-1,j-1}-(i-1)s_{i-1,j}; \tag{4.24}$$

and

$$\mathrm{S}(x)=\left[\mathrm{S}_{ij}x^{i-j}\right] = \begin{bmatrix} 1 &&&& \\ x & 1 &&& \\ x^2 & 3x & 1 && \\ x^3 & 7x^2 & 6x & 1 & \\ x^4 & 15x^3 & 25x^2 & 10x & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}, \tag{4.25}$$

where

$$S_{ij} = S_{i-1,j-1} + jS_{i-1,j}. \tag{4.26}$$

The matrices s(1), S(1) serve as tables of Stirling numbers.

Define a column vector **C**, an augmented column vector $\hat{\mathbf{C}}$, a matrix **M**, and an augmented matrix $\hat{\mathbf{M}}$ as follows:

$$\begin{aligned}
\mathbf{C} &= [a_i]^T, \qquad i = 1,2,3,\ldots \\
&= [a_1, a_2, a_3, \ldots]^T, \\
\hat{\mathbf{C}} &= [a_{i-1}]^T, \qquad i = 1,2,3,\ldots, \quad a_0 = 1 \\
&= [1, a_1, a_2, a_3, \ldots]^T, \\
\mathbf{M} &= [m_{ij}], \qquad i, j = 1,2,3,\ldots, \\
\tilde{\mathbf{M}} &= [m_{i-1,j-1}], \qquad i, j = 1,2,3,\ldots, \qquad m_{ii} = 1, \quad m_{io} = m_{oi} = \delta_{io} \\
&= \begin{bmatrix} 1 & & & \\ \cdot & 1 & & \\ \cdot & m_{21} & 1 & \\ \cdot & m_{31} & m_{32} & 1 \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}.
\end{aligned}$$

Let

$$\mathbf{X} = [x, x^2, x^3, \ldots]^T,$$

$$\mathbf{X}_1 = [(x)_1, (x)_2, (x)_3, \ldots]^T, \tag{4.27a}$$

$$\mathbf{X}_2 = [\langle x\rangle_1, \langle x\rangle_2, \langle x\rangle_3, \ldots]^T, \tag{4.27b}$$

$$\hat{\mathbf{s}}(x) = \left[s_{i-1,j-1}x^{i-j}\right], \tag{4.28a}$$

$$\hat{\mathbf{S}}(x) = \left[S_{i-1,j-1}x^{i-j}\right]. \tag{4.28b}$$

Define another matrix $\tilde{\mathbf{S}}$ as follows:

$$\tilde{\mathbf{S}}(x) = \left[S_{i,i-j+1}(i-j+1)!x^{i-j}\right]. \tag{4.29}$$

The numerical coefficients in the elements satisfy

$$a_{ij} = (i-j+1)(a_{i-1,j-1} + a_{i-1,j}).$$

The relations (4.21), (4.22) can now be expressed in the form

$$\mathbf{X}_1 = \mathrm{s}(1)\mathbf{X},$$

$$\mathbf{X} = \mathrm{S}(1)\mathbf{X}_1.$$

Hence

$$\mathrm{s}(1)\mathrm{S}(1) = \mathbf{I} = \mathrm{S}(1)\mathrm{s}(1),$$

$$\sum_{k=j}^{i} s_{ik}S_{kj} = \delta_{ij} = \sum_{k=j}^{i} S_{ik}s_{kj}. \tag{4.30}$$

These relations are well known. It follows that

$$\mathrm{s}(x)\mathrm{S}(x) = \mathbf{I} = \mathrm{S}(x)\mathrm{s}(x)$$

for all values of x. Also

$$\begin{aligned} \mathbf{X}_2 &= \mathrm{s}(-1)\mathbf{X} \\ &= \mathrm{s}(-1)\mathrm{S}(1)\mathbf{X}_1. \end{aligned} \tag{4.31}$$

Similarly,

$$\hat{\mathbf{X}}_1 = \hat{\mathrm{s}}(1)\hat{\mathbf{X}},$$

$$\hat{\mathbf{X}} = \hat{\mathrm{S}}(1)\hat{\mathbf{X}}_1,$$

$$\hat{\mathrm{s}}(x)\hat{\mathrm{S}}(x) = \mathbf{I} = \hat{\mathrm{S}}(x)\hat{\mathrm{s}}(x)$$

$$\hat{\mathbf{X}}_2 = \hat{\mathrm{s}}(-1)\hat{\mathbf{X}}$$

$$= \hat{\mathrm{s}}(-1)\hat{\mathrm{S}}(1)\hat{\mathbf{X}}_1.$$

Having recorded these identities, it is convenient to discard the notation $\hat{\mathbf{X}}$ and revert to the notation $\mathbf{X}$ as in Sections 1,2.

The proofs of the following relations are left as exercises for the reader:

$$D^i(x^{i-1}\mathbf{X}) = \left[\sum_{j=1}^{i} (-1)^{i-j} s_{ij}\mathbf{Q}_j\right]\mathbf{X} \qquad (D = d/dx),$$

$$\begin{aligned}(xD)^i\mathbf{X} &= x\mathbf{Q}(1+xD)^{i-1}\mathbf{X}\\ &= \left[\sum_{j=1}^{i} S_{ij}(x\mathbf{Q})^j\right]\mathbf{X}.\end{aligned} \tag{4.32}$$

Let

$$W_n = [(x)_n, (x)_{n+1}, (x)_{n+2}, \ldots]^T.$$

Then

$$(\mathbf{I}+\mathbf{Q})^r\mathbf{W}_n = (x-n+r)_r\mathbf{W}_{n-r}.$$

In particular,

$$(\mathbf{I}+\mathbf{Q})^n\mathbf{W}_n = (x)_n\mathbf{W}_0.$$

The Eulerian polynomials $A_i(x)$ and the Eulerian numbers A_{ij} are defined by the generating-function relation

$$\begin{aligned}(1-x)^{i+1}\sum_{k=0}^{\infty} k^i x^k &= A_i(x)\\ &= \sum_{j=1}^{i} A_{ij}x^j.\end{aligned}$$

Explicitly,

$$A_{ij} = \sum_{k=0}^{j} (-1)^k \binom{i+1}{k}(j-k)^i. \tag{4.33}$$

The polynomials $A_i(x)/x$ are generated by the rows of the matrix

$$\mathbf{A}(x) = \left[A_{ij}x^{i-j}\right], \tag{4.34}$$

where

$$A_{ij} = (i-j+1)A_{i-1,j-1} + jA_{i-1,j}.$$

5. IDENTITIES

It will be convenient to list a number of identities in this section and to prove some of them in the next.

If $\mathbf{AB}^{-1} = \mathbf{B}^{-1}\mathbf{A}$, then both products will be written $\mathbf{A}/\mathbf{B}$ without ambiguity. If $\mathbf{B}$ is singular, then $\mathbf{B}^{-1}$ does not exist, but it will be used in a symbolic manner. For example,

$$\sum_{r=0}^{\infty} \frac{(-\mathbf{B})^{2r}}{(2r+1)!} = \frac{\sin \mathbf{B}}{\mathbf{B}},$$

where the infinite sum is well defined for all square matrices $\mathbf{B}$.

We have

$$\mathbf{L}^{(\alpha)}(x) = \alpha!(x\mathbf{Q})^{-\frac{1}{2}\alpha} J_\alpha\left\{2(x\mathbf{Q})^{\frac{1}{2}\alpha}\right\}, \tag{5.1}$$

$$\overline{\mathbf{L}}^{(\alpha)}(x) = e^{-x(\alpha\mathbf{Q}+\mathbf{Q}_2)}, \tag{5.2}$$

$$\overline{\mathbf{L}}(x) = e^{-x\mathbf{Q}_2}, \tag{5.3}$$

$$\mathbf{H}_e(x) = \cos(2x\mathbf{Q}^{\frac{1}{2}}), \tag{5.4}$$

$$\mathbf{H}_o(x) = \frac{\sin(2x\mathbf{Q}^{\frac{1}{2}})}{2x\mathbf{Q}^{\frac{1}{2}}}, \tag{5.5}$$

$$\overline{\mathbf{H}}(x) = e^{-(\frac{1}{2}x\mathbf{Q})^2}, \tag{5.6}$$

$$\overline{\mathbf{B}}(x) = \frac{x\mathbf{Q}}{e^{x\mathbf{Q}} - \mathbf{I}}, \tag{5.7}$$

$$\overline{\mathbf{E}}(x) = \frac{2\mathbf{I}}{e^{x\mathbf{Q}} + \mathbf{I}}, \tag{5.8}$$

$$\overline{\mathbf{B}}(x)\overline{\mathbf{E}}(x) = \overline{\mathbf{E}}(x)\overline{\mathbf{B}}(x) = \overline{\mathbf{B}}(2x), \tag{5.9}$$

$$\hat{\mathbf{s}}(x)\mathbf{S}(x) = \mathbf{I} + x\mathbf{Q}, \tag{5.10}$$

$$\mathbf{S}(x)\hat{\mathbf{s}}(x) = e^{x\mathbf{Q}} = \hat{\mathbf{S}}(-x)\mathbf{s}(-x), \tag{5.11}$$

$$\mathbf{s}(x)e^{x\mathbf{Q}}\mathbf{S}(x) = \mathbf{I} + x\mathbf{Q}, \tag{5.12}$$

$$\hat{\mathbf{s}}(x)e^{x\mathbf{Q}}\hat{\mathbf{S}}(x) = \mathbf{I} + x\mathbf{Q}, \tag{5.13}$$

$$\mathbf{s}(x)(e^{x\mathbf{Q}} - \mathbf{I})\mathbf{S}(x) = x\mathbf{Q}, \tag{5.14}$$

$$\hat{\mathbf{s}}(x)(e^{x\mathbf{Q}} - \mathbf{I})\hat{\mathbf{S}}(x) = x\mathbf{Q}, \tag{5.15}$$

$$\mathbf{S}(x)\log(\mathbf{I} + x\mathbf{Q})\mathbf{s}(x) = x\mathbf{Q}, \tag{5.16}$$

$$\hat{\mathbf{S}}(x)\log(\mathbf{I} + x\mathbf{Q})\hat{\mathbf{s}}(x) = x\mathbf{Q}, \tag{5.17}$$

$$\hat{\mathbf{s}}(-x)\mathbf{S}(x) = e^{x\mathbf{Q}_2} = \mathbf{s}(-x)\hat{\mathbf{S}}(x), \tag{5.18}$$

$$\mathbf{s}(-x)\mathbf{S}(x) = e^{x(\mathbf{Q}+\mathbf{Q}_2)}, \tag{5.19}$$

$$\hat{\mathbf{s}}(-x)\hat{\mathbf{S}}(x) = (\mathbf{I} - x\mathbf{Q})e^{x\mathbf{Q}_2} = e^{x\mathbf{Q}_2}(\mathbf{I} + x\mathbf{Q})^{-1}, \tag{5.20}$$

$$\mathbf{S}(x)e^{-x\mathbf{Q}_2}\mathbf{s}(-x) = e^{x\mathbf{Q}}, \tag{5.21}$$

$$\hat{\mathbf{S}}(x)e^{-x\mathbf{Q}_2}\hat{\mathbf{s}}(-x) = e^{-x\mathbf{Q}}, \tag{5.22}$$

$$e^{x\mathbf{Q}_{m+1}}\left(\frac{x\mathbf{Q}_m}{\mathbf{I} + x\mathbf{Q}_m}\right)e^{-x\mathbf{Q}_{m+1}} = x\mathbf{Q}_m, \tag{5.23}$$

$$e^{-x\mathbf{Q}_{m+1}}(\mathbf{I} - x\mathbf{Q}_m)^{-1}e^{x\mathbf{Q}_{m+1}} = \mathbf{I} + x\mathbf{Q}_m, \tag{5.24}$$

$$e^{-x(\alpha\mathbf{Q}+\mathbf{Q}_2)}e^{x(\overline{\alpha+1}\,\mathbf{Q}+\mathbf{Q}_2)} = \mathbf{I} + x\mathbf{Q}, \tag{5.25}$$

$$e^{-x\mathbf{Q}_2}e^{x(\alpha\mathbf{Q}+\mathbf{Q}_2)} = (\mathbf{I} + x\mathbf{Q})^{\alpha}, \tag{5.26}$$

$$\mathbf{S}(-x)e^{x(\alpha\mathbf{Q}+\mathbf{Q}_2)}\mathbf{s}(x) = e^{(\alpha-1)x\mathbf{Q}}, \tag{5.27}$$

$$\hat{\mathbf{S}}(-x)e^{x(\alpha\mathbf{Q}+\mathbf{Q}_2)}\hat{\mathbf{s}}(x) = e^{(\alpha+1)x\mathbf{Q}}, \tag{5.28}$$

$$e^{-x(\overline{\alpha+1}\,\mathbf{Q}+\mathbf{Q}_2)}e^{x(\alpha\mathbf{Q}+\mathbf{Q}_2)} = \mathbf{I} + x(\alpha\mathbf{Q}_0 + \mathbf{Q}), \tag{5.29}$$

$$\mathbf{s}(x)\left[\frac{x\mathbf{Q}}{e^{x\mathbf{Q}} - \mathbf{I}}\right]\mathbf{S}(x) = \frac{\log(\mathbf{I} + x\mathbf{Q})}{x\mathbf{Q}}, \tag{5.30}$$

$$\mathbf{s}(x)\left[\frac{2\mathbf{I}}{e^{x\mathbf{Q}}+\mathbf{I}}\right]\mathbf{S}(x)=\frac{\mathbf{I}}{\mathbf{I}+\frac{1}{2}x\mathbf{Q}},\tag{5.31}$$

$$\mathbf{A}(x)=\tilde{\mathbf{S}}(x)e^{-x\mathbf{Q}},\tag{5.32}$$

$$\mathbf{Y}^{-1}(x)\hat{\mathbf{Y}}(x)=\mathbf{I}-x\mathbf{Q},\tag{5.33}$$

$$\hat{\mathbf{Y}}(x)\mathbf{Y}^{-1}(x)=(\mathbf{I}-2x\mathbf{Q})^{\frac{1}{2}},\tag{5.34}$$

$$\mathbf{Y}^{-1}(x)(\mathbf{I}-2x\mathbf{Q})^{\frac{1}{2}}\mathbf{Y}(x)=\mathbf{I}-x\mathbf{Q},\tag{5.35}$$

$$\mathbf{Y}(x)=\sum_{r=0}^{\infty}\frac{\mathbf{V}_r x^r}{r!},\tag{5.36}$$

$$\mathbf{S}(1)e^{x\mathbf{Q}}\mathbf{s}(1)=e^{-x}e^{x\mathbf{Q}D}e^{x}\qquad (D=d/dx).\tag{5.37}$$

Note that corresponding to each identity of the form

$$\mathbf{A}^{-1}\mathbf{B}\mathbf{A}=\mathbf{C},$$

there are identities of the form

$$\mathbf{A}^{-1}\mathbf{B}^r\mathbf{A}=\mathbf{C}^r,$$

$$\mathbf{A}^{-1}e^{x\mathbf{B}}\mathbf{A}=e^{x\mathbf{C}},$$

etc.

6. PROOFS

The identities (5.1), (5.4), (5.5) were proved in [1] but are reproduced here for comparison with (5.2), (5.6).

To prove (5.2), note that the elements in the subdiagonal $i-j=n$ of the matrix in (4.4) are

$$\langle j\rangle_n\langle j+\alpha\rangle_n\frac{(-x)^n}{n!},\qquad j=1,2,3,\ldots.$$

Hence

$$\begin{aligned}\bar{\mathbf{L}}^{(\alpha)}(x) &= \mathbf{I} + \sum_{n=1}^{\infty} \frac{(-x)^n}{n!} \operatorname*{subdiag}_{i-j=n} [\langle j \rangle_n \langle j+\alpha \rangle_n] \\ &= \mathbf{I} + \sum_{n=1}^{\infty} \frac{(-x)^n}{n!} \mathbf{R}_\alpha^n \\ &= e^{-x\mathbf{R}_\alpha}.\end{aligned}$$

The result follows from (2.3). It follows that

$$e^{-x(\alpha\mathbf{Q}+\mathbf{Q}_2)} = \left[\frac{(i-1)!}{(j-1)!} \binom{i-1+\alpha}{j-1+\alpha} (-x)^{i-j} \right]. \tag{6.1}$$

The identities (5.6) to (5.8) are illustrations of a more general theorem.

THEOREM 6.1. *If the Appell polynomials $\alpha_n(x)$ are generated by the relation*

$$e^{xt}A(t) = \sum_{n=0}^{\infty} \frac{\alpha_n(x)t^n}{n!},$$

where

$$A(t) = \sum_{r=0}^{\infty} \frac{a_r t^r}{r!}, \qquad a_0 = 1,$$

then the reversed polynomials $x^n\alpha_n(1/x)$ are generated by the rows of the matrix $A(x\mathbf{Q})$.

Proof. Equating coefficients of $t^n/n!$ in the generating relation, we have

$$\alpha_n(x) = \sum_{r=0}^{n} \binom{n}{r} a_r x^{n-r}, \qquad n = 0, 1, 2, \ldots .$$

Hence

$$x^n \alpha_n\left(\frac{1}{x}\right) = \sum_{r=0}^{n} \binom{n}{r} a_r x^r, \qquad n = 0, 1, 2, \ldots .$$

The matrix which generates these polynomials by rows with the unit elements in the principal diagonal is

$$\left[\binom{i-1}{j-1}a_{i-j}x^{i-j}\right], \qquad i,j=1,2,3,\ldots.$$

It follows from Lemma 2.2 that this matrix is equal to

$$\sum_{r=0}^{\infty}\frac{a_r(x\mathbf{Q})^r}{r!}=A(x\mathbf{Q}).$$

■

Equation (5.6) now follows by putting $A(t)=e^{-t^2}$ and $\alpha_n(x)=H_n(\frac{1}{2}x)$. Equations (5.7), (5.8) are proved in a similar manner, and (5.9) follows easily.

The identities (5.10) to (5.17) relate the Stirling matrices to $\mathbf{Q}$ and depend on two lemmas on Stirling numbers.

LEMMA 6.1.

$$\sum_{k=j}^{i}s_{i-1,k-1}S_{kj}=\begin{cases}1, & i=j,\\ j, & i=j+1,\\ 0 & \textit{otherwise}.\end{cases}$$

Proof. Referring to (4.24), (4.30),

$$\begin{aligned}\sum_{k=j}^{i}s_{i-1,k-1}S_{kj}&=\sum_{k=j}^{i}\left[s_{ik}+(i-1)s_{i-1,k}\right]S_{kj}\\&=\sum_{k=j}^{i}s_{ik}S_{kj}+(i-1)\sum_{k=j}^{i-1}s_{i-1,k}S_{kj}\\&=\delta_{ij}+(i-1)\delta_{i-1,j}.\end{aligned}$$

The result follows. ■

(5.10) now follows immediately but (5.11) requires another lemma.

LEMMA 6.2 (Cohen [8, p. 137]).

$$\sum_{k=j}^{i}S_{ik}s_{k-1,j-1}=\binom{i-1}{j-1}.$$

The left side of (5.11) now follows with the aid of (1.2). The right side is obtained by changing the sign of x and inverting. (5.12), (5.13) are obtained by eliminating first $\hat{\mathrm{s}}(x)$ and then $\mathrm{S}(x)$ from (5.10), (5.11). Equations (5.14), (5.15) then follow easily. (5.16) is obtained by raising (5.12) to the power r [see the note which follows (5.37)], and then differentiating with respect to r at $r = 0$. (5.17) is obtained in a similar manner. Note that the functions which appear in (5.14) to (5.17), namely $e^t - 1, \log(1+t)$, are mutually inverse in the scalar sense.

One more lemma on Stirling numbers is required to prove the identities which contain the matrix $\mathbf{Q}_2$.

LEMMA 6.3 (Riordan [7, p. 44]).

$$\sum_{k=j}^{i} (-1)^k s_{ik} S_{kj} = \frac{(-1)^i i!}{j!} \binom{i-1}{j-1}.$$

The left side of (5.18) can now be proved as follows:

$$\begin{aligned} \hat{\mathrm{s}}(-x)\mathrm{S}(x) &= \left[s_{i-1,j-1}(-x)^{i-j}\right]\left[S_{ij}x^{i-j}\right] \\ &= \left[\beta_{ij}x^{i-j}\right], \end{aligned}$$

where

$$\begin{aligned} \beta_{ij} &= (-1)^i \sum_{k=j}^{i} (-1)^k s_{i-1,k-1} S_{kj} \\ &= (-1)^i \sum_{k=j}^{i} (-1)^k \left[s_{ik} + (i-1)s_{i-1,j}\right] S_{kj} \\ &= (-1)^i \sum_{k=j}^{i} (-1)^k s_{ik} S_{kj} + (-1)^i (i-1) \sum_{k=j}^{i} (-1)^k s_{i-1,j} S_{kj} \\ &= \frac{i!}{j!}\binom{i-1}{j-1} - \frac{(i-1)(i-1)!}{(j-1)!}\binom{i-2}{j-1} \\ &= \frac{(i-1)!}{(j-1)!}\binom{i-1}{j-1}. \end{aligned}$$

Hence, referring to (6.1),

$$\hat{\mathrm{s}}(-x)\mathbf{S}(x) = \left[\frac{(i-1)!}{(j-1)!}\binom{i-1}{j-1}x^{i-j}\right]$$

$$= e^{x\mathbf{Q}_2}.$$

The right side of (5.18) now follows by changing the sign of x and inverting. The same lemma can be applied to prove (5.19):

$$\mathrm{s}(-x)\mathbf{S}(x) = \left[s_{ij}(-x)^{i-j}\right]\left[S_{ij}x^{i-j}\right]$$

$$= \left[\varepsilon_{ij}x^{i-j}\right],$$

where

$$\varepsilon_{ij} = (-1)^i \sum_{k=j}^{i} (-1)^k s_{ik}S_{kj}$$

$$= \frac{i!}{j!}\binom{i-1}{j-1}.$$

The result follows. Note that $\mathbf{Q}, \mathbf{Q}_2$ do not commute and hence

$$e^{x(\mathbf{Q}+\mathbf{Q}_2)} \neq e^{x\mathbf{Q}}e^{x\mathbf{Q}_2} \text{ or } e^{x\mathbf{Q}_2}e^{x\mathbf{Q}}. \tag{6.2}$$

This point is emphasized by (5.25), (5.26). Note also that no new identity is obtained from (5.19) by changing the sign of x and inverting. The resulting equation is identical with the original.

The left side of (5.20) is obtained by changing the sign of x in (5.10) and using the right side of (5.18). The right side follows easily. Both (5.21), (5.22) can be obtained from (5.11), (5.18), changing the sign of x where necessary.

The identities (5.23) to (5.26) are independent of Stirling matrices and may be proved with the aid of further lemmas.

LEMMA 6.4.

$$e^{\mathbf{A}}\mathbf{B}e^{-\mathbf{A}} = \sum_{n=0}^{\infty} \frac{\mathbf{M}_n}{n!},$$

where

$$\mathbf{M}_n = \sum_{r=0}^{n} (-1)^r \binom{n}{r} \mathbf{A}^{n-r}\mathbf{B}\mathbf{A}^r.$$

The proof is elementary.

Put

$$\mathbf{A} = x\mathbf{Q}_{m+1},$$

$$\mathbf{B} = x\mathbf{Q}_m.$$

Then, referring to Theorem 2.3,

$$\begin{aligned}\mathbf{M}_n &= x^{n+1} \sum_{r=0}^{n} (-1)^r \binom{n}{r} \mathbf{Q}_{m+1}^{n-r}\mathbf{Q}_m\mathbf{Q}_{m+1}^r \\ &= x^{n+1}\mathbf{K}_{n1}(m) \\ &= n!(x\mathbf{Q}_m)^{n+1}.\end{aligned}$$

Hence

$$\begin{aligned}e^{x\mathbf{Q}_{m+1}}(x\mathbf{Q}_m)e^{-x\mathbf{Q}_{m+1}} &= \sum_{n=0}^{\infty} (x\mathbf{Q}_m)^{n+1} \\ &= \frac{x\mathbf{Q}_m}{\mathbf{I} - x\mathbf{Q}_m},\end{aligned}$$

which is equivalent to (5.23). Equation (5.24) now follows easily.

The proof of (5.25) can be made to depend on Theorem 2.2. Put

$$\mathbf{A} = \alpha\mathbf{Q} + \mathbf{Q}_2,$$

$$\mathbf{B} = \overline{\alpha+1}\,\mathbf{Q} + \mathbf{Q}_2,$$

$$\mathbf{H}_0 = \mathbf{I}.$$

Then, referring to (2.7),

$$\begin{aligned}\mathbf{H}_1 &= \mathbf{A}-\mathbf{B}\\ &= -\mathbf{Q}\\ \mathbf{H}_2 &= \mathbf{A}^2-2\mathbf{AB}+\mathbf{B}^2\\ &= (\mathbf{A}^2-\mathbf{AB}-\mathbf{BA}+\mathbf{B}^2)-(\mathbf{AB}-\mathbf{BA})\\ &= (\mathbf{A}-\mathbf{B})^2-(\mathbf{Q}_2\mathbf{Q}-\mathbf{Q}\mathbf{Q}_2)\\ &= \mathbf{Q}^2-\mathbf{Q}^2\\ &= \mathbf{0}.\end{aligned}$$

Hence, from the recurrence relation of Theorem 2.2(a),

$$\mathbf{H}_n = \mathbf{0}, \qquad n \geqslant 2.$$

(5.25) can now be proved as follows:

$$\begin{aligned}e^{-x\mathbf{A}}e^{x\mathbf{B}} &= \mathbf{I}-x\mathbf{H}_1+\sum_{n=2}^{\infty}\frac{(-1)^n\mathbf{H}_n x^n}{n!}\\ &= \mathbf{I}+x\mathbf{Q},\end{aligned}$$

which is the stated result.

Applying (5.25) repeatedly,

$$\begin{aligned}e^{x(\alpha\mathbf{Q}+\mathbf{Q}_2)} &= e^{x(\overline{\alpha-1}\,\mathbf{Q}+\mathbf{Q}_2)}(\mathbf{I}+x\mathbf{Q})\\ &= e^{x(\overline{\alpha-2}\,\mathbf{Q}+\mathbf{Q}_2)}(\mathbf{I}+x\mathbf{Q})^2\\ &= e^{x(\overline{\alpha-s}\,\mathbf{Q}+\mathbf{Q}_2)}(\mathbf{I}+x\mathbf{Q})^s, \qquad s=1,2,\ldots,\alpha,\\ &= e^{x\mathbf{Q}_2}(\mathbf{I}+x\mathbf{Q})^\alpha,\end{aligned}$$

which is equivalent to (5.26).

From (5.26), (5.12)

$$e^{-x\mathbf{Q}_2}e^{x(\alpha\mathbf{Q}+\mathbf{Q}_2)} = \mathbf{s}(x)e^{\alpha x\mathbf{Q}}\mathbf{S}(x),$$

$$e^{\alpha x\mathbf{Q}} = \mathbf{S}(x)e^{-x\mathbf{Q}_2}e^{x(\alpha\mathbf{Q}+\mathbf{Q}_2)}\mathbf{s}(x).$$

Premultiply both sides by $e^{-x\mathbf{Q}}$ and use (5.21):

$$\begin{aligned} e^{(\alpha-1)x\mathbf{Q}} &= \left[e^{-x\mathbf{Q}}\mathbf{S}(x)e^{-x\mathbf{Q}_2}\right]e^{x(\alpha\mathbf{Q}+\mathbf{Q}_2)}\mathbf{s}(x) \\ &= \mathbf{S}(-x)e^{x(\alpha\mathbf{Q}+\mathbf{Q}_2)}\mathbf{s}(x), \end{aligned}$$

which proves (5.27). Equation (5.28) is proved in a similar manner.

The proof of (5.29) may be made to depend on a binomial identity. Let

$$\omega_{ij} = \sum_{r=0}^{i-j}(-1)^r\binom{i-j}{r}(r+j-1).$$

Lemma.

$$\omega_{ij} = \begin{cases} i-1, & j=i, \\ -1, & j=i-1, \\ 0 & \textit{otherwise}. \end{cases}$$

The proof is elementary.

Referring to (6.1) and the $\mathbf{A},\mathbf{B}$ defined in the proof of (5.25),

$$\begin{aligned} \widehat{e^{-x\mathbf{B}}e^{x\mathbf{A}}} &= \left[\frac{(i-2)!(i-1+\alpha)!(-x)^{i-j}}{(j-2)!(j-1+\alpha)!(i-j)!}\right]\left[\frac{(i-1)!(i-1+\alpha)!x^{i-j}}{(j-1)!(j-1+\alpha)!(i-j)!}\right] \\ &= \left[a_{ij}x^{i-j}\right], \end{aligned}$$

where

$$a_{ij} = \frac{(-1)^{i-j}(i-2)!(i-1+\alpha)!}{(j-1)!(j-1+\alpha)!(i-j)!}\omega_{ij}.$$

Hence, applying the lemma,

$$a_{ij} = \begin{cases} 1, & j=i, \\ i-1+\alpha, & j=i-1, \\ 0 & \text{otherwise}, \end{cases}$$

$$\widehat{e^{-x\mathbf{B}}e^{x\mathbf{A}}} = \mathbf{I} + x(\alpha\mathbf{Q}_0 + \mathbf{Q}),$$

which proves (5.29).

The generating-function relation (4.13) with $x=0$ will be used twice, once in each direction, to prove (5.30):

$$\begin{aligned}
\mathrm{s}(x)\left[\frac{x\mathbf{Q}}{e^{x\mathbf{Q}}-\mathbf{I}}\right]\mathrm{S}(x) &= \mathrm{s}(x)\left[\sum_{n=0}^{\infty}\frac{B_n(0)(x\mathbf{Q})^n}{n!}\right]\mathrm{S}(x)\\
&= \sum_{n=0}^{\infty}\frac{B_n(0)}{n!}\mathrm{s}(x)(x\mathbf{Q})^n\mathrm{S}(x)\\
&= \sum_{n=0}^{\infty}\frac{B_n(0)\log^n(\mathbf{I}+x\mathbf{Q})}{n!}\\
&= \frac{\log(\mathbf{I}+x\mathbf{Q})}{(\mathbf{I}+x\mathbf{Q})-\mathbf{I}}.
\end{aligned}$$

The result follows. It is interesting to note that a shorter proof can be found by raising (5.14) to the power -1 despite the fact that $(e^{x\mathbf{Q}}-\mathbf{I})^{-1}$, $(x\mathbf{Q})^{-1}$ in isolation are meaningless. Referring to (5.16),

$$\begin{aligned}
\mathrm{s}(x)(e^{x\mathbf{Q}}-\mathbf{I})^{-1}(x\mathbf{Q})\mathrm{S}(x) &= \left[\mathrm{s}(x)(e^{x\mathbf{Q}}-\mathbf{I})^{-1}\mathrm{S}(x)\right]\left[\mathrm{s}(x)(x\mathbf{Q})\mathrm{S}(x)\right]\\
&= (x\mathbf{Q})^{-1}\log(\mathbf{I}+x\mathbf{Q}).
\end{aligned}$$

This function illustrates the note at the beginning of Section 5. It can be expanded as a power series in $x\mathbf{Q}$ and is therefore well defined. It appears that certain dubious intermediate steps are justified provided that there are no objections to the final result.

The proof of (5.31) is simpler. Expand the function $(e^{x\mathbf{Q}}+\mathbf{I})^{-1}$ as a power series in $e^{x\mathbf{Q}}$ and apply (5.12).

Some properties of the finite forms of $\mathrm{s}(x), \mathrm{S}(x)$ are given in [16].

The proof of (5.32) depends on two lemmas.

LEMMA 6.4 (Comtet [5, p. 244]).

$$\sum_{k=1}^{i} A_{ik}u^k = A_i(u) = u\sum_{k=1}^{i} k!\mathrm{S}_{ik}(u-1)^{i-k}.$$

LEMMA 6.5.

$$\sum_{k=j}^{i} A_{ik}\binom{k-1}{j-1} = (i-j+1)!\mathrm{S}_{i,i-j+1}.$$

Proof. Divide through the previous lemma by u, put $u = 1 + v$, and expand $(1+v)^{k-1}$ in a binomial series:

$$\sum_{k=1}^{i} A_{ik} \sum_{r=0}^{k-1} \binom{k-1}{r} v^r = \sum_{k=1}^{i} k!\mathrm{S}_{ik} v^{i-k}.$$

The result follows by equating coefficients of v^{j-1}. ■

It follows that

$$\left[A_{ij}x^{i-j}\right]\left[\binom{i-1}{j-1}x^{i-j}\right] = \left[(i-j+1)!\mathrm{S}_{i,i-j+1}x^{i-j}\right],$$

that is,

$$\mathbf{A}(x)e^{x\mathbf{Q}} = \tilde{\mathbf{S}}(x),$$

which is equivalent to (5.32), which then yields

$$A_{ij} = (-1)^{i-j-1} \sum_{r=1}^{i-j+1} (-1)^r r! \binom{i-r}{j-1} \mathrm{S}_{ir}.$$

This identity may not be well known and is an example of a combinatorial relation which can be obtained by the matrix method.

The proofs of the remaining identities are left as exercises for the reader. Numerical calculations suggest that

$$\mathbf{Y}^{-1}(x) = \left[b_{ij}x^{i-j}\right],$$

where

$$b_{ij} = (-1)^{i-j} a_{j+1,2j+1-i}$$

and where a_{ij} is defined in (4.17); and also that

$$b_{ij} = \frac{(-1)^{i-j} i!}{(2j-i)!(i-j)!2^{i-j}},$$

$$b_{ij} = b_{i-1,j-1} - (i-1)b_{i-2,j-1}.$$

Apart from the alternating signs, the nonzero elements in the jth column of $\mathbf{Y}^{-1}(x)$, read from top to bottom, are identical with the elements in the $(j+1)$th row of $\mathbf{Y}(x)$ read from right to left. For example, the 4th column of $\mathbf{Y}^{-1}(x)$ is

$$[0,0,0,1,-10x,45x^2,-105x^3,105x^4,0,0,0,\ldots]^T.$$

Milne [17] has investigated the relationship between the recurrence relations satisfied by the elements of mutually inverse pairs of triangular matrices.

The proof of (5.37), and also that of (4.32), can be made to depend on the exponential polynomial identity

$$\sum_{j=0}^{i} S_{ij} x^j = e^{-x}(xD)^n e^x \qquad (D = d/dx),$$

which is proved by Roman and Rota [25, p. 134].

7. TWO GENERAL SIMILARITY TRANSFORMATIONS

The identities in Section 5 can be applied to prove two general similarity transformations.

Let

$$A(x\mathbf{Q}) = \sum_{r=0}^{\infty} \frac{a_r (x\mathbf{Q})^r}{r!},$$

$$B(x\mathbf{Q}) = \sum_{r=0}^{\infty} \frac{b_r (x\mathbf{Q})^r}{r!},$$

and let a matrix with a suffix 1 denote the first column of that matrix:

$$A_1(x\mathbf{Q}) = [a_0, a_1 x, a_2 x^2, \ldots]^T,$$

$$B_1(x\mathbf{Q}) = [b_0, b_1 x, b_2 x^2, \ldots]^T,$$

$$\mathbf{S}_1(x) = [1, x, x^2, \ldots]^T,$$

where $\mathbf{S}(x)$ is a Stirling matrix.

LEMMA 7.1.

$$e^{-x\mathbf{Q}} A(x\mathbf{Q}) \mathbf{S}_1(x) = A_1(x\mathbf{Q}).$$

Proof:

$$\begin{aligned} A(x\mathbf{Q})\mathbf{S}_1(x) &= \left[\binom{i-1}{j-1} a_{i-j} x^{i-j}\right][x^{i-1}] \\ &= \left[x^{i-1} \sum_{k=1}^{i} \binom{i-1}{i-k} a_{i-k}\right] \\ &= \left[\sum_{r=1}^{i} \binom{i-1}{r-1} x^{i-r} a_{r-1} x^{r-1}\right] \\ &= \left[\binom{i-1}{j-1} x^{i-j}\right][a_{i-1} x^{i-1}] \\ &= e^{x\mathbf{Q}} A_1(x\mathbf{Q}), \end{aligned}$$

which is equivalent to the stated result.

THEOREM 7.1.

$$\mathbf{s}(x) A(x\mathbf{Q}) \mathbf{S}(x) = B(x\mathbf{Q}),$$

where

$$b_i = \sum_{j=0}^{i} s_{ij} a_j, \qquad n = 0, 1, 2, \ldots.$$

Proof. Using (5.11) and the lemma,

$$\begin{aligned} B_1(x\mathbf{Q}) &= \mathrm{s}(x)A(x\mathbf{Q})\mathbf{S}_1(x) \\ &= \hat{\mathrm{s}}(x)e^{-x\mathbf{Q}}A(x\mathbf{Q})\mathbf{S}_1(x) \\ &= \hat{\mathrm{s}}(x)A_1(x\mathbf{Q}). \end{aligned}$$

The formula for b_i is obtained by carrying out the matrix multiplication with $x = 1$. ∎

LEMMA 7.2.

$$A(x\mathbf{Q})(e^{x(\alpha\mathbf{Q}+\mathbf{Q}_2)})_1 = (\mathbf{I} - x\mathbf{Q})^{-\alpha-1}A_1(x\mathbf{Q}).$$

Proof. Referring to (6.1) and Lemma 2.2(iii),

$$\begin{aligned} A(x\mathbf{Q})(e^{x(\alpha\mathbf{Q}+\mathbf{Q}_2)})_1 &= \left[\binom{i-1}{j-1}a_{i-j}x^{i-j}\right]\left[\frac{(i-1+\alpha)!x^{i-1}}{\alpha!}\right] \\ &= \left[x^{i-1}\sum_{k=1}^{i}\binom{i-1}{k-1}a_{i-k}\frac{(k-1+\alpha)!}{\alpha!}\right] \\ &= \left[\sum_{r=1}^{i}\binom{i-1}{i-r}\frac{(i-r+\alpha)!}{\alpha!}x^{i-r}a_{r-1}x^{r-1}\right] \\ &= \left[\binom{i-1}{i-j}\frac{(i-j+\alpha)!}{\alpha!}x^{i-j}\right]\left[a_{i-1}x^{i-1}\right] \\ &= \left[\sum_{n=0}^{\infty}\frac{\langle\alpha+1\rangle_n(x\mathbf{Q})^n}{n!}\right]A_1(x\mathbf{Q}) \\ &= (\mathbf{I} - x\mathbf{Q})^{-\alpha-1}A_1(x\mathbf{Q}). \end{aligned}$$

∎

THEOREM 7.2.

$$e^{-x(\alpha\mathbf{Q}+\mathbf{Q}_2)}A(x\mathbf{Q})e^{x(\alpha\mathbf{Q}+\mathbf{Q}_2)} = B(x\mathbf{Q}),$$

where

$$\frac{b_i}{i!} = \sum_{j=1}^{i} (-1)^{i-j}\binom{i-1}{j-1}\frac{a_j}{j!}.$$

Proof. Using (5.26) with a change in the sign of x, and the lemma,

$$\begin{aligned} B_1(x\mathbf{Q}) &= e^{-x(\alpha\mathbf{Q}+\mathbf{Q}_2)}\mathbf{A}(x\mathbf{Q})\left(e^{x(\alpha\mathbf{Q}+\mathbf{Q}_2)}\right)_1 \\ &= e^{-x(\alpha\mathbf{Q}+\mathbf{Q}_2)}(\mathbf{I}-x\mathbf{Q})^{-\alpha-1}A_1(x\mathbf{Q}) \\ &= e^{-x\mathbf{Q}_2}(\mathbf{I}-x\mathbf{Q})^{-1}A_1(x\mathbf{Q}) \\ &= e^{x(\mathbf{Q}-\mathbf{Q}_2)}A_1(x\mathbf{Q}) \\ &= \left[\frac{(i-1)!}{(j-1)!}\binom{i-2}{j-2}(-x)^{i-j}\right]A_1(x\mathbf{Q}). \end{aligned}$$

The formula for b_i is obtained by carrying out the matrix multiplication. Note that b_i is independent of α. ∎

These matrix identities yield scalar combinatorial identities.

Lemma 7.3.

$$\sum_{p=j}^{i}\sum_{q=j}^{p} F_{pq}a_{p-q} = \sum_{k=0}^{i-j} a_k \sum_{r=j}^{i-k} F_{r+k,r}.$$

Each of these double sums represents the sum of the same triangular array of elements.

Theorem 7.1 can be expressed in the form

$$\left[\binom{i-1}{j-1}\sum_{k=0}^{i-j} s_{i-j,k}a_k\right] = [s_{ij}]\left[\binom{i-1}{j-1}a_{i-j}\right][S_{ij}].$$

Hence, applying the lemma,

$$\binom{i-1}{j-1}\sum_{k=0}^{i-j} s_{i-j,k}a_k = \sum_{p=j}^{i}\sum_{q=j}^{p} s_{ip}\binom{p-1}{q-1}a_{p-q}S_{qj}$$

$$= \sum_{k=0}^{i-j} a_k \sum_{r=j}^{i-k} s_{i,r+k}\binom{k+r-1}{k}S_{rj}.$$

Equating coefficients of a_k we obtain the identity

$$\binom{i-1}{j-1}s_{i-j,k} = \sum_{r=j}^{i-k} s_{i,r+k}\binom{k+r-1}{k}S_{rj},$$

which remains valid when s_{ij}, S_{ij} are interchanged. Two similar identities are proved by Riordan [6, p. 204] using an operational method.

8. INVERSE RELATIONS

Several of the inverse scalar pairs obtained in this section are given by Riordan [6], who devotes two chapters to the subject. However, Riordan's proofs are based on the use of scalar generating functions, whereas the proofs given below are based on the transformation of one column vector into another by means of functions of the matrices $\mathbf{Q}_m$. The close resemblance between the two methods has already been noted in Section 3.

Let

$$\mathbf{A} = [a_0, a_1, a_2, \ldots]^T,$$

$$\mathbf{B} = [b_0, b_1, b_2, \ldots]^T.$$

The notation $\hat{\mathbf{A}}, \hat{\mathbf{B}}$ is not used in this section.

The inverse pair of matrix relations

$$\mathbf{A} = \mathbf{MB},$$

$$\mathbf{B} = \mathbf{M}^{-1}\mathbf{A},$$

where

$$\mathbf{M} = [\lambda_{ij}], \qquad i, j = 1,2,3,\ldots, \qquad i \geqslant j,$$

$$\mathbf{M}^{-1} = [\mu_{ij}], \qquad i, j = 1,2,3,\ldots, \qquad i \geqslant j,$$

are equivalent to the inverse pair of scalar relations

$$a_i = \sum_{j=0}^{i} \lambda_{i+1,\,j+1} b_j,$$

$$b_i = \sum_{j=0}^{i} \mu_{i+1,\,j+1} a_j.$$

There are a number of interesting special cases of this result. When $\lambda_{ij} = \lambda_{i-j}$, then, referring to Lemma 2.1, we have the following theorem on $\mathbf{Q}_0$.

THEOREM 8.1. *The inverse pair*

$$\mathbf{A} = \mathbf{MB},$$

$$\mathbf{B} = \mathbf{M}^{-1}\mathbf{A},$$

where

$$\mathbf{M} = [\lambda_{i-j}] = \sum_{n=0}^{\infty} \lambda_n \mathbf{Q}_0^n,$$

$$\mathbf{M}^{-1} = [\mu_{i-j}] = \sum_{n=0}^{\infty} \mu_n \mathbf{Q}_0^n,$$

is equivalent to the pair

$$a_i = \sum_{j=0}^{i} \lambda_j b_{i-j} = \sum_{j=0}^{i} \lambda_{i-j} b_j,$$

$$b_i = \sum_{j=0}^{i} \mu_j a_{i-j} = \sum_{j=0}^{i} \mu_{i-j} a_j.$$

The proof is elementary.

For example, the inverse pair

$$\mathbf{A} = (\mathbf{I} - 4\mathbf{Q}_0)^{-1/2}B,$$

$$\mathbf{B} = (\mathbf{I} - 4\mathbf{Q}_0)^{1/2}\mathbf{A}$$

is equivalent to the inverse pair

$$a_i = \sum_{j=0}^{i} \binom{2j}{j} b_{i-j},$$

$$b_i = a_i - 2\sum_{j=1}^{i} \frac{1}{j}\binom{2j-2}{j-1} a_{i-j}.$$

Proof:

$$(\mathbf{I} - 4\mathbf{Q}_0)^{-1/2} = \sum_{n=0}^{\infty} \binom{2n}{n} \mathbf{Q}_0^n.$$

Hence

$$\lambda_n = \binom{2n}{n}, \qquad n = 0,1,2,\ldots,$$

$$(\mathbf{I} - 4\mathbf{Q}_0)^{1/2} = \mathbf{I} - 2\sum_{n=1}^{\infty} \frac{1}{n}\binom{2n-2}{n-1} \mathbf{Q}_0^n.$$

Hence

$$\mu_n = -\frac{2}{n}\binom{2n-2}{n-1}, \qquad n = 1,2,3,\ldots,$$

$$\mu_0 = 1.$$

The result follows. Note the similarity between the transformation matrix $(\mathbf{I} - 4\mathbf{Q}_0)^{-1/2}$ and the scalar ordinary generating function $(1-4x)^{-1/2}$ employed by Riordan [6, p. 101] to solve the same problem. Little additional work is required to prove that the inverse pair

$$\mathbf{A} = \tfrac{1}{2}\mathbf{Q}_0^{-1}\left[\mathbf{I} - (\mathbf{I} - 4\mathbf{Q}_0)^{1/2}\right]\mathbf{B},$$

$$\mathbf{B} = \tfrac{1}{2}\left[\mathbf{I} + (\mathbf{I} - 4\mathbf{Q}_0)^{1/2}\right]\mathbf{A}$$

is equivalent to the inverse pair

$$a_i = \sum_{j=0}^{i} \frac{1}{j+1}\binom{2j}{j} b_{i-j},$$

$$b_i = a_i - \sum_{j=1}^{i} \frac{1}{j}\binom{2j-2}{j-1} a_{i-j}.$$

Note that Riordan gives this pair correctly on p. 102, but reproduces the formula for b_i incorrectly in Table 3.2.

Similarly, the inverse pair

$$\mathbf{A} = (\mathbf{I} + \mathbf{Q}_0)^p \mathbf{B},$$

$$\mathbf{B} = (\mathbf{I} + \mathbf{Q}_0)^{-p} \mathbf{A}$$

is equivalent to the pair

$$a_i = \sum_{j=0}^{i} \binom{p}{j} b_{i-j},$$

$$b_i = \sum_{j=0}^{i} \binom{-p}{j} b_{i-j}$$

$$= \sum_{j=0}^{i} (-1)^j \binom{j+p-1}{j} b_{i-j}.$$

When

$$\lambda_{ij} = \binom{i-1}{j-1} \lambda_{i-j},$$

then, referring to Lemma 2.2, we have the following theorem on $\mathbf{Q}$:

THEOREM 8.2. *The inverse pair*

$$\mathbf{A} = \mathbf{M}\mathbf{B},$$

$$\mathbf{B} = \mathbf{M}^{-1}\mathbf{A},$$

where

$$\mathbf{M} = \left[\binom{i-1}{j-1}\lambda_{i-j}\right] = \sum_{n=0}^{\infty} \frac{\lambda_n \mathbf{Q}^n}{n!},$$

$$\mathbf{M}^{-1} = \left[\binom{i-1}{j-1}\mu_{i-j}\right] = \sum_{n=0}^{\infty} \frac{\mu_n \mathbf{Q}^n}{n!},$$

is equivalent to the inverse pair

$$a_i = \sum_{j=0}^{i} \binom{i}{j} \lambda_j b_{i-j} = \sum_{j=0}^{i} \binom{i}{j} \lambda_{i-j} b_j,$$

$$b_i = \sum_{j=0}^{i} \binom{i}{j} \mu_j a_{i-j} = \sum_{j=0}^{i} \binom{i}{j} \mu_{i-j} a_j.$$

For example, the pair

$$\mathbf{A} = e^{\mathbf{Q}}\mathbf{B},$$

$$\mathbf{B} = e^{-\mathbf{Q}}\mathbf{A}$$

is equivalent to the pair

$$a_i = \sum_{j=0}^{i} \binom{i}{j} b_j,$$

$$b_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} a_j.$$

The proof is elementary and consists of expanding $e^{\mathbf{Q}}, e^{-\mathbf{Q}}$ in series and showing that $\lambda_n = 1$, $\mu_n = (-1)^n$.

The pair

$$\mathbf{A} = (\mathbf{I}+\mathbf{Q})^{p}\mathbf{B},$$

$$\mathbf{B} = (\mathbf{I}+\mathbf{Q})^{-p}\mathbf{A}$$

is equivalent to the pair

$$\alpha_i = \sum_{j=0}^{i} \binom{p}{i-j}\beta_j,$$

$$\beta_i = \sum_{j=0}^{i} \binom{-p}{i-j}\alpha_j$$

$$= \sum_{j=0}^{i} (-1)^{i-j}\binom{i-j+p-1}{i-j}\alpha_j,$$

where $\alpha_i = a_i/i!$, $\beta_i = b_i/i!$.

The pair

$$\mathbf{A} = \tfrac{1}{2}\mathbf{Q}^{-1}\left[\mathbf{I}-(\mathbf{I}-4\mathbf{Q})^{1/2}\right]\mathbf{B},$$

$$\mathbf{B} = \tfrac{1}{2}\left[\mathbf{I}+(\mathbf{I}-4\mathbf{Q})^{1/2}\right]\mathbf{A}$$

is equivalent to the pair

$$a_i = \sum_{j=0}^{i} \binom{i}{j}\frac{(2j)!}{(j+1)!}b_{i-j},$$

$$b_i = a_i - \sum_{j=1}^{i} \binom{i}{j}\frac{(2j-2)!}{(j-1)!}a_{i-j}.$$

The pair

$$\mathbf{A} = e^{p\mathbf{R}}\mathbf{B},$$

$$\mathbf{B} = e^{-p\mathbf{R}}\mathbf{A},$$

where $\mathbf{R}$ is defined implicitly by the relation

$$\mathbf{R}e^{-\mathbf{R}} = \mathbf{Q} \tag{8.1}$$

and explicitly by the Lagrange series (Riordan [6, pp. 96, 146]),

$$\mathbf{R} = \sum_{n=1}^{\infty} \frac{n^{n-1}\mathbf{Q}^n}{n!} \tag{8.2}$$

is equivalent to the Abel pair

$$a_i = \sum_{j=1}^{i} \binom{i}{j} p(p+i-j)^{i-j-1} b_j,$$

$$b_i = \sum_{j=1}^{i} (-1)^{i-j} \binom{i}{j} p(p-i+j)^{i-j-1} a_j.$$

Proof:

$$e^{p\mathbf{R}} = \sum_{n=0}^{\infty} \frac{p(n+p)^{n-1}\mathbf{Q}^n}{n!}$$

$$= \left[\binom{i-1}{j-1} p(p+i-j)^{i-j-1} \right], \tag{8.3}$$

$$e^{-p\mathbf{R}} = -\sum_{n=0}^{\infty} \frac{p(n-p)^{n-1}\mathbf{Q}^n}{n!}$$

$$= \left[(-1)^{i-j} \binom{i-1}{j-1} p(p - \quad i)^{i-j-1} \right]. \tag{8.4}$$

The result follows. Note that Riordan [6, p. 93] obtains the inverse pair by applying Abel's generalization of the binomial theorem and then states the orthogonal relation, that is, a relation of the form

$$\sum_k u_{ik} v_{kj} = \delta_{ij},$$

which the result implies. Using the matrix method, the orthogonal relation comes first for it is implied by the identity

$$e^{p\mathbf{R}} e^{-p\mathbf{R}} = \mathbf{I}.$$

Referring to (6.1), it can be seen that the pair

$$\mathbf{A} = e^{\alpha\mathbf{Q} + \mathbf{Q}_2}\mathbf{B},$$

$$\mathbf{B} = e^{-\alpha\mathbf{Q} - \mathbf{Q}_2}\mathbf{A}$$

is equivalent to the pair

$$\alpha_i = \sum_{j=0}^{i} \binom{i+\alpha}{j+\alpha}\beta_j,$$

$$\beta_j = \sum_{j=0}^{i} (-1)^{i-j}\binom{i+\alpha}{j+\alpha}\alpha_j.$$

The pair

$$\mathbf{A} = e^{\mathbf{Q}_2}e^{\mathbf{Q}}\mathbf{B},$$

$$\mathbf{B} = e^{-\mathbf{Q}}e^{-\mathbf{Q}_2}\mathbf{A}$$

is equivalent to the pair

$$\alpha_i = \sum_{j=0}^{i} L_{i-j}^{(j)}(-1)\beta_j,$$

$$\beta_i = \sum_{j=0}^{i} L_{i-j}^{(-i-1)}(1)\alpha_j,$$

which may not be well known. The $L_n^{(\alpha)}(x)$ are the Laguerre polynomials.

The pair

$$\mathbf{A} = e^{\mathbf{P}}\mathbf{B},$$

$$\mathbf{B} = e^{-\mathbf{P}}\mathbf{A},$$

where

$$\mathbf{P} = \operatorname*{subdiag}_{i-j=1}\left[\binom{p+1-j}{1}\right],$$

is equivalent to the pair

$$a_i = \sum_{j=0}^{i} \binom{p-j}{i-j} b_j,$$

$$b_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{p-j}{i-j} a_j,$$

which is due to Stanton and Sprott [19].

Inverse pairs of a different type can be found by replacing the column vectors $\mathbf{A}, \mathbf{B}$ by $\mathbf{F}, \mathbf{X}$, where

$$\mathbf{F} = [f_0(x), f_1(x), f_2(x), \ldots]^T,$$

$$\mathbf{X} = [1, x, x^2, \ldots]^T.$$

For example, the pair

$$\mathbf{F} = e^{-\alpha \mathbf{Q} - \mathbf{Q}_2} \mathbf{X},$$

$$\mathbf{X} = e^{\alpha \mathbf{Q} + \mathbf{Q}_2} \mathbf{F},$$

where

$$f_i(x) = (-1)^i i! L_i^{(\alpha)}(x),$$

is equivalent to the pair

$$L_i^{(\alpha)}(x) = \sum_{j=0}^{i} (-1)^j \binom{i+\alpha}{j+\alpha} \frac{x^j}{j!},$$

$$\frac{x^i}{i!} = \sum_{j=0}^{i} (-1)^j \binom{i+\alpha}{j+\alpha} L_j^{(\alpha)}(x).$$

Using (5.19), the Stirling relation (4.31) gives the pair

$$\mathbf{X}_2 = e^{\mathbf{Q} + \mathbf{Q}_2} \mathbf{X}_1,$$

$$\mathbf{X}_1 = e^{-\mathbf{Q} - \mathbf{Q}_2} \mathbf{X}_2,$$

which is equivalent to the pair

$$\langle x\rangle_i = (-1)^i \sum_{j=0}^{i} L_{ij}(x)_j,$$

$$(x)_i = \sum_{j=0}^{i} (-1)^j L_{ij}\langle x\rangle_j,$$

where the L_{ij} are the Lah numbers defined in (4.8).

The formulae which express $L_i^{(\alpha)}(x)$ in terms of the $(x)_i$ and vice versa can be obtained from the pair

$$\mathbf{F} = e^{-\alpha\mathbf{Q}-\mathbf{Q}_2}\mathbf{S}(1)\mathbf{X},$$

$$\mathbf{X}_1 = \mathbf{s}(1)e^{\alpha\mathbf{Q}+\mathbf{Q}_2}\mathbf{F}.$$

Other inverse scalar pairs are given by Carlitz [20–23].

9. TOUCHARD NUMBERS AND MATRICES

Define numbers t_{ij}, T_{ij} and matrices $\mathbf{t}(x), \mathbf{T}(x)$ in a manner similar to the definitions of Stirling numbers s_{ij}, S_{ij} and Stirling matrices $\mathbf{s}(x), \mathbf{S}(x)$.

Let

$$(i+1+x)^i = \sum_{j=0}^{i} t_{ij}x^j, \qquad t_{i0} = \delta_{i0}, \tag{9.1}$$

$$x^i = \sum_{j=0}^{i} T_{ij}(j+1+x)^j, \qquad T_{i0} = \delta_{i0}. \tag{9.2}$$

Then

$$t_{ij} = \binom{i-1}{j-1} i^{i-j},$$

$$T_{ij} = (-1)^{i-j}\binom{i}{j} j^{i-j}.$$

The numbers t_{ij} have a combinatorial interpretation in the theory of acyclic maps—see Comtet [5, p. 70].

Let

$$\mathbf{t}(x) = \left[t_{ij}x^{i-j}\right]$$
$$\mathbf{T}(x) = \left[T_{ij}x^{i-j}\right] \qquad i, j = 1,2,3\ldots.$$

Then

$$\mathbf{t}(x)\mathbf{T}(x) = \mathbf{I} = \mathbf{T}(x)\mathbf{t}(x),$$
$$\hat{\mathbf{t}}(x)\hat{\mathbf{T}}(x) = \mathbf{I} = \hat{\mathbf{T}}(x)\hat{\mathbf{t}}(x).$$

Using the column vectors $\mathbf{F},\mathbf{X}$ defined in Section 8 with

$$f_i(x) = (i+1+x)^i,$$

(9.1), (9.2) can be expressed in the form

$$\mathbf{F} = \mathbf{t}(1)\mathbf{X},$$
$$\mathbf{X} = \mathbf{T}(1)\mathbf{F}.$$

Let

$$\mathbf{E}(x) = \sum_{n=0}^{\infty} \frac{(n+1)^{n-1}(x\mathbf{Q})^n}{n!}$$
$$= \left[\binom{i-1}{j-1}(i-j+1)^{i-j-1}x^{i-j}\right]$$

then

$$\mathbf{E}^{-1}(x) = -\sum_{n=0}^{\infty} \frac{(n-1)^{n-1}(x\mathbf{Q})^n}{n!} \qquad (0^0 = 1)$$
$$= -\left[\binom{i-1}{j-1}(i-j-1)^{i-j-1}x^{i-j}\right].$$

Note that

$$\mathbf{E}(1) = e^{\mathbf{R}},$$

but

$$\mathbf{E}(x) \neq e^{x\mathbf{R}}.$$

Note also that the functions defined by

$$\mathbf{F} = e^{\pm \mathbf{R}}\mathbf{X}$$

are both Appell polynomials of the form (1.5), where a_j is given respectively by $(j+1)^{j-1}$, $-(j-1)^{j-1}$. They satisfy (3.1).

The proofs of the following identities are left as exercises for the reader:

$$\hat{\mathbf{t}}(x)\mathbf{T}(x) = \mathbf{E}^{-1}(x) - x\mathbf{Q},$$

$$\hat{\mathbf{t}}(1)\mathbf{T}(1) = (\mathbf{I}-\mathbf{R})e^{-x\mathbf{R}} = \frac{d\mathbf{Q}}{d\mathbf{R}},$$

$$\mathbf{T}(x)\hat{\mathbf{t}}(x) = (\mathbf{I}-x\mathbf{Q})e^{-x\mathbf{Q}},$$

$$\mathbf{t}(1)\mathbf{Q}\mathbf{T}(1) = \mathbf{R},$$

$$\mathbf{t}(1)\mathbf{Q}e^{-\mathbf{Q}}\mathbf{T}(1) = \mathbf{Q},$$

$$\widehat{e^{-\mathbf{R}}}e^{\mathbf{R}} = \mathbf{I} + \frac{\mathbf{Q}_0 e^{\mathbf{R}}}{\mathbf{I}-\mathbf{R}},$$

$$e^{-\mathbf{R}}\mathbf{Q}_2 e^{\mathbf{R}} = \mathbf{Q}_2 + \frac{\mathbf{Q}\mathbf{R}}{\mathbf{I}-\mathbf{R}}.$$

The inverse pair (9.1), (9.2) is related to an inverse pair due to Touchard and quoted by Riordan [6, p. 96]. A similar pair is quoted by Rota et al. [26, p. 744] and attributed without a reference to Clarke. In view of the similarities in properties between $t_{ij}, T_{ij}, \mathbf{t}(x), \mathbf{T}(x)$ and $s_{ij}, S_{ij}, \mathbf{s}(x), \mathbf{S}(x)$, it is suggested that t_{ij}, T_{ij} be called the *Touchard numbers of the first and second kinds* and that $\mathbf{t}(x), \mathbf{T}(x)$ be called the *Touchard matrices of the first and second kinds.*

10. A FOOTNOTE ON STIRLING MATRICES

The Stirling matrices $\mathbf{s} = \mathbf{s}(1)$, $\mathbf{S} = \mathbf{S}(1)$ can each be expressed in terms of $\mathbf{Q}, \mathbf{Q}_2$ and one new matrix $\mathbf{Z}$:

$$\mathbf{s} = \mathbf{Z}^{-1} e^{-\frac{1}{2}(\mathbf{Q}+\mathbf{Q}_2)} \mathbf{Z},$$

$$\mathbf{S} = \mathbf{Z}^{-1} e^{\frac{1}{2}(\mathbf{Q}+\mathbf{Q}_2)} \mathbf{Z},$$

where

$$\mathbf{Z} = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & \frac{7}{2} & 1 & & \\ 1 & \frac{28}{3} & \frac{23}{3} & 1 & \\ 1 & \frac{4151}{54} & \frac{3565}{54} & \frac{145}{9} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

It may be verified that the sum of the elements in each row of $\mathbf{Z}^{-1}$, except the first, is zero.

The author is indebted to the referee for several constructive comments.

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Received 11 May 1982; revised 1 October 1985