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# Matrices Which Generate Families of Polynomials and Associated Infinite Series

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In 1928, Polya [1] gave a solution of Cauchy's functional equation for matrices (Aczel [2] calls it Bellman's functional equation)

$$\mathbf{M}(x)\,\mathbf{M}(y) = \mathbf{M}(x+y) \tag{1}$$

in the form

$$\mathbf{M}(x) = \begin{bmatrix} 1 & & & & \\ x & 1 & & & \\ x^2 & 2x & 1 & & \\ x^3 & 3x^2 & 3x & 1 & \\ x^4 & 4x^3 & 6x^2 & 4x & 1 \\ \multicolumn{5}{c}{\cdots\cdots\cdots\cdots} \end{bmatrix}. \tag{2}$$

The elements in row $(n+1)$ are the terms in the polynomial expansion of $(1+x)^n$ and the elements in column $(n+1)$ are the terms in the infinite series expansion of $(1-x)^{-n-1}$.

In view of the form of Eq. (1) we expect a relation of the form

$$\mathbf{M}(x) = e^{x\mathbf{Q}}, \tag{3}$$

where $\mathbf{Q}$ is a constant matrix, and it is found that

$$\mathbf{Q} = [q_{ij}], \tag{4}$$

where

$$\begin{aligned} q_{ij} &= j, && \text{if} \quad i = j+1, \\ &= 0, && \text{otherwise}, \end{aligned}$$

i.e.,

$$\mathbf{Q} = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 2 & 0 & & \\ & & 3 & 0 & \\ & & & 4 & 0 \\ \multicolumn{5}{c}{\cdots\cdots\cdots} \end{bmatrix}. \tag{5}$$





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The matrix $e^{x\mathbf{Q}}$ may be said to generate the family of polynomials $(1 + x)^n$, $n = 0, 1, 2,\ldots$, by rows and the associated family of infinite series $(1 - x)^{-n-1}$, $n = 0, 1, 2,\ldots$, by columns.

There is another relationship between the exponential and binomial functions, namely the well-known limit

$$e^x = \lim_{n\to\infty}(1 + (x/n))^n. \tag{6}$$

These observations suggest an investigation into the families of polynomials and their associated infinite series which are generated by other functions of the constant matrix $\mathbf{Q}$ and other constant matrices and, conversely, the function of some constant matrix which will generate a given family of polynomials or infinite series.

The Pochhammer notation [3] is defined as

$$(z)_0 = 1,$$
$$(z)_r = z(z + 1)(z + 2) \cdots (z + r - 1).$$

This notation may be extended to matrices. If $\mathbf{Z}$ is a square matrix

$$(\mathbf{Z})_0 = \mathbf{I},$$
$$(\mathbf{Z})_r = \mathbf{Z}(\mathbf{Z} + \mathbf{I})(\mathbf{Z} + 2\mathbf{I}) \cdots (\mathbf{Z} + \overline{r - 1}\mathbf{I}).$$

If $\mathbf{B}$ is a nonsingular square matrix and the product is commutative,

$$(\mathbf{A} + a\mathbf{B})(\mathbf{A} + \overline{a + 1}\mathbf{B})(\mathbf{A} + \overline{a + 2}\mathbf{B}) \cdots (\mathbf{A} + \overline{a + r - 1}\mathbf{B})$$
$$= (\mathbf{AB}^{-1} + a\mathbf{I})(\mathbf{AB}^{-1} + \overline{a + 1}\mathbf{I}) \cdots (\mathbf{AB}^{-1} + \overline{a + r - 1}\mathbf{I})\,\mathbf{B}^r,$$

that is,

$$\prod_{s=0}^{r-1} (\mathbf{A} + \overline{a + s}\mathbf{B}) = (\mathbf{AB}^{-1} + a\mathbf{I})_r\,\mathbf{B}^r. \tag{7}$$

If $\mathbf{B}$ is singular and/or the product is not commutative, this relation is no longer valid but it will be convenient in the analysis which follows to continue to use it in a purely symbolic fashion. The validity of Theorem 2 below does not depend on acceptance of this convention which affects only the notation in which the theorem is presented.

From (4) it is found that

$$\mathbf{Q}^r = [q_{ij}^{(r)}]$$

where

$$q_{ij}^{(r)} = (j)_r, \qquad \text{if} \qquad i = j + r,$$
$$= 0, \qquad \text{otherwise.}$$

Define a matrix $\mathbf{M}(x)$ as the following infinite series:

$$\mathbf{M}(x) = \sum_0^{\infty} \lambda_r (x\mathbf{Q})^r, \qquad \lambda_0 = 1, \qquad \mathbf{Q}^0 = \mathbf{I},$$

$$= \begin{bmatrix} 1 & & & \\ (1)_1\lambda_1 x & 1 & & \\ (1)_2\lambda_2 x^2 & (2)_1\lambda_1 x & 1 & \\ (1)_3\lambda_3 x^3 & (2)_2\lambda_2 x^2 & (3)_1\lambda_1 x & 1 \\ (1)_4\lambda_4 x^4 & (2)_3\lambda_3 x^3 & (3)_2\lambda_2 x^2 & (4)_1\lambda_1 x \\ \hdashline \end{bmatrix}.$$

The elements in row $(n+1)$ are the terms in the polynomial

$$p_n(x) = \sum_0^{n} \lambda_r (n+1-r)_r\, x^r, \qquad \lambda_0 = 1.$$

The elements in column $(n+1)$ are the terms in the infinite series

$$f_n(x) = \sum_0^{\infty} \lambda_r (n+1)_r\, x^r, \qquad \lambda_0 = 1.$$

Using the relations

$$\begin{aligned} (n+1-r)_r &= (-1)^r (-n)_r\,, \qquad && \text{when} \quad r \leqslant n, \\ &= 0, && \text{when} \quad r > n, \end{aligned} \tag{8}$$

the above observations may be summarized in the following theorem:

THEOREM 1. *The matrix*

$$\mathbf{M}(x) = \sum_0^{\infty} \lambda_r (x\mathbf{Q})^r, \qquad \lambda_0 = 1, \qquad \mathbf{Q}^0 = \mathbf{I},$$

*generates the family of polynomials*

$$p_n(x) = \sum_0^{\infty} \lambda_r (-n)_r\, (-x)^r, \qquad n = 0, 1, 2, \ldots,$$

*by rows and the family of associated infinite series*

$$p_{-n-1}(-x) = \sum_0^{\infty} \lambda_r (n+1)_r\, x^r, \qquad n = 0, 1, 2, \ldots,$$

*by columns.*

For example, when $\lambda_r = (r!)^{-1}$, we get Polya's matrix (2) and the binomial series.

When

$$\lambda_r = (-1)^r/(r!)^2$$

we get

$$\mathbf{M}(x) = \sum_0^\infty (-x\mathbf{Q})^r/(r!)^2) = J_0(2(x\mathbf{Q})^{1/2}, \tag{9}$$

$$p_n(x) = n! \sum_0^\infty (-x)^r/(n-r)!\,(r!)^2 = L_n(x), \tag{10}$$

giving a new relation between the Bessel function of order zero and the Laguerre polynomial. There is a limiting relationship between these two functions analogous to (6), viz.

$$J_0(2x^{1/2}) = \lim_{n\to\infty} L_n(x/n).$$

This and other relationships between these two functions can be found in [3, 4] and in tables of Laplace transforms [5].

More generally, when

$$\lambda_r = \frac{(\alpha_1)_r\,(\alpha_2)_r \cdots (\alpha_p)_r}{(\beta_1)_r\,(\beta_2)_r \cdots (\beta_q)_r}\,\frac{1}{r!},$$

where $\alpha_i$, $\beta_i$ are independent of $n$, then

$$\mathbf{M}(x) = {}_pF_q\left[\begin{matrix}\alpha_1, \alpha_2, \ldots, \alpha_p\\ \beta_1, \beta_2, \ldots, \beta_q\end{matrix}\,\middle|\, x\mathbf{Q}\right], \tag{11}$$

$$p_n(x) = {}_{p+1}F_q\left[\begin{matrix}-n, \alpha_1, \alpha_2, \ldots, \alpha_p\\ \beta_1, \beta_2, \ldots, \beta_q\end{matrix}\,\middle|\, -x\right]. \tag{12}$$

Special cases of this result are given in Table I. It will be noticed that there is a relationship between the Hermite polynomials and the circular functions. Once again there is a limiting relationship between these two functions (see [3, Sect. 22.15]).

The converse of Theorem I is straightforward:

The family of polynomials

$$p_n(x) = \sum_0^n b_{nr}x^r, \qquad b_{n0} = 1,$$

is generated by the matrix

$$\mathbf{M}(x) = \sum_0^\infty b_{nr}(-x\mathbf{Q})^r/(-n)_r,$$

provided that $b_{nr}/(-n)_r$ is independent of $n$.

TABLE I

Illustrations of Theorem 1

| Family of polynomials | | | Generating matrix | |
|---|---|---|---|---|
| $p_n(x)$ | Name | Hypergeometric form | Hypergeometric form | Special function |
| $(1+x)^n$ | Binomial | ${}_1F_0\left[\begin{matrix}-n\\ -\end{matrix}\middle\vert -x\right]$ | ${}_0F_0\left[\begin{matrix}-\\ -\end{matrix}\middle\vert x\mathbf{Q}\right]$ | $e^{x\mathbf{Q}}$ |
| $L_n(x)$ | Laguerre | ${}_1F_1\left[\begin{matrix}-n\\ 1\end{matrix}\middle\vert x\right]$ | ${}_0F_1\left[\begin{matrix}-\\ 1\end{matrix}\middle\vert -x\mathbf{Q}\right]$ | $J_0(2(x\mathbf{Q})^{1/2})$ |
| $\dfrac{n!\,\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)}L_n^{(\alpha)}(x)$ | Generalized Laguerre | ${}_1F_1\left[\begin{matrix}-n\\ \alpha+1\end{matrix}\middle\vert x\right]$ | ${}_0F_1\left[\begin{matrix}-\\ \alpha+1\end{matrix}\middle\vert -x\mathbf{Q}\right]$ | $\Gamma(\alpha+1)(x\mathbf{Q})^{-\alpha/2}J_\alpha(2(x\mathbf{Q})^{1/2})$* |
| $\dfrac{(-1)^n\,n!}{(2n)!}H_{2n}(x)$ | Hermite | ${}_1F_1\left[\begin{matrix}-n\\ \frac{1}{2}\end{matrix}\middle\vert x^2\right]$ | ${}_0F_1\left[\begin{matrix}-\\ \frac{1}{2}\end{matrix}\middle\vert -x^2\mathbf{Q}\right]$ | $\cos(2x\mathbf{Q}^{1/2})$ |
| $\dfrac{(-1)^n\,n!}{2(2n+1)!\,x}H_{2n+1}(x)$ | Hermite | ${}_1F_1\left[\begin{matrix}-n\\ \frac{3}{2}\end{matrix}\middle\vert x^2\right]$ | ${}_0F_1\left[\begin{matrix}-\\ \frac{3}{2}\end{matrix}\middle\vert -x^2\mathbf{Q}\right]$ | $(2x\mathbf{Q}^{1/2})^{-1}\sin(2x\mathbf{Q}^{1/2})$* |

* Symbolically.

The family of infinite series

$$f_n(x) = \sum_0^\infty c_{nr} x^r, \qquad c_{n0} = 1,$$

is generated by the matrix

$$\mathbf{M}(x) = \sum_0^\infty c_{nr}(-x\mathbf{Q})^r/(n+1)_r \,,$$

provided that $c_{nr}/(n+1)_r$ is independent of $n$. Should a given family of polynomials or infinite series not satisfy the appropriate condition, it cannot be generated solely by means of the matrix $\mathbf{Q}$ but might possibly be generated by other constant matrices. A variety of polynomials and infinite series not covered by Theorem I can be generated by generalizing $\mathbf{Q}$ as follows:

$$\mathbf{Q}_m = (1/m)\,[q_{ij}^{(m)}], \tag{13}$$

where

$$\begin{aligned} q_{ij}^{(m)} &= j^m, && \text{if} \quad i = j+1, \\ &= 0, && \text{otherwise.} \end{aligned}$$

$$\mathbf{Q}_m{}^r = (1/m^r)\,[q_{ij}^{(m,r)}],$$

where

$$\begin{aligned} q_{ij}^{(m,r)} &= \{(j)_r\}^m, && \text{if} \quad i = j+r, \\ &= 0, && \text{otherwise.} \end{aligned}$$

The polynomials of Legendre and Chebyshev, among others, can be expressed in the hypergeometric form [3]

$$d_n(x)\,{}_2F_1\left[\begin{matrix}-n, n+2a \\ b\end{matrix}\,\middle|\, g(x)\right].$$

These polynomials can be generated by means of $\mathbf{Q}_1$, $\mathbf{Q}_2$, where

$$\mathbf{Q}_1 = \mathbf{Q}$$

$$\mathbf{Q}_2 = \tfrac{1}{2}\begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 4 & 0 & & \\ & & 9 & 0 & \\ & & & 16 & \\ - & - & - & - & - \end{bmatrix}.$$

Let

$$p_n(x) = \sum_0^\infty (-n)_r\,(n+2a)_r\,\lambda_r x^r, \qquad (\lambda_r \text{ independent of } n).$$

The matrix which generates these polynomials by rows is

$$\mathbf{M}(x) = \begin{bmatrix} 1 & & & & \\ (-1)_1(1+2a)_1\lambda_1 x & 1 & & & \\ (-2)_2(2+2a)_2\lambda_2 x^2 & (-2)_1(2+2a)_1\lambda_1 x & 1 & & \\ (-3)_3(3+2a)_3\lambda_3 x^3 & (-3)_2(3+2a)_2\lambda_2 x^2 & (-3)_1(3+2a)_1\lambda_1 x & 1 & \\ (-4)_4(4+2a)_4\lambda_4 x^4 & (-4)_3(4+2a)_3\lambda_3 x^3 & (-4)_2(4+2a)_2\lambda_2 x^2 & (-4)_1(4+2a)_1\lambda_1 x & \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$$= \sum_0^\infty \mathbf{P}_r \lambda_r (-2x)^r,$$

where

$$\mathbf{P}_1 = \tfrac{1}{2} \begin{bmatrix} 0 & & & & \\ (1)_1\,(1+2a)_1 & 0 & & & \\ & (2)_1\,(2+2a)_1 & 0 & & \\ & & (3)_1\,(3+2a)_1 & 0 & \\ & & & (4)_1\,(4+2a)_1 & \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$$= \mathbf{Q}_2 + a\mathbf{Q}_1\,,$$

$$\mathbf{P}_r = (1/2^r)\,[p_{ij}^{(r)}],$$

where

$$\begin{aligned} p_{ij}^{(r)} &= (j)_r\,(j+r-1+2a)_r\,, && \text{if} \quad i = j+r, \\ &= 0, && \text{otherwise.} \end{aligned}$$

It may be verified by elementary processes that

$$\mathbf{P}_r = \prod_{s=0}^{r-1} (\mathbf{Q}_2 + \overline{a+s}\,\mathbf{Q}_1). \tag{14}$$

If we now put

$$\lambda_r = \{(b)_r\, r!\}^{-1}$$

we may summarize the above result in the following theorem:

THEOREM 2. *The family of polynomials*

$$p_n(x) = \sum_0^\infty \frac{(-n)_r\,(n+2a)_r\,x^r}{(b)_r\,r!} = {}_2F_1\left[\begin{matrix}-n,\, n+2a\\ b\end{matrix}\,\middle|\, x\right]$$

*is generated by rows by the matrix*

$$\mathbf{M}(x) = \sum_0^\infty \frac{(-2x)^r}{(b)_r\,r!}\prod_{s=0}^{r-1}(\mathbf{Q}_2 + \overline{a+s}\mathbf{Q}_1).$$

Using the convention defined in (7), the matrix $\mathbf{M}$ can be expressed purely symbolically in the form

$$\mathbf{M}(x) = \sum_0^\infty \frac{(\mathbf{Q}_2\mathbf{Q}_1^{-1} + a\mathbf{I})_r\,(-2x\mathbf{Q}_1)^r}{(b)_r\,r!}$$

$$= {}_1F_1\left[\begin{matrix}\mathbf{Q}_2\mathbf{Q}_1^{-1} + a\mathbf{I}\\ b\end{matrix}\,\middle|\, -2x\mathbf{Q}_1\right].$$

Special cases of Theorem 2 are given in Table II.

TABLE II

Illustrations of Theorem 2

| Family of polynomials | | | |
|---|---|---|---|
| $p_n(x)$ | Name | Hypergeometric form | Generating matrix |
| $\frac{(-1)^n\,2^{2n}(n!)^2}{(2n)!}P_{2n}(x)$ | Legendre | ${}_2F_1\left[\begin{matrix}-n,\, n+\frac{1}{2}\\ \frac{1}{2}\end{matrix}\,\middle|\, x^2\right]$ | ${}_1F_1\left[\begin{matrix}\mathbf{Q}_2\mathbf{Q}_1^{-1}+\frac{1}{4}\mathbf{I}\\ \frac{1}{2}\end{matrix}\,\middle|\, -2x^2\mathbf{Q}_1\right]$ |
| $\frac{(-1)^n\,2^{2n}(n!)^2}{(2n+1)!\,x}P_{2n+1}(x)$ | Legendre | ${}_2F_1\left[\begin{matrix}-n,\, n+\frac{3}{2}\\ \frac{3}{2}\end{matrix}\,\middle|\, x^2\right]$ | ${}_1F_1\left[\begin{matrix}\mathbf{Q}_2\mathbf{Q}_1^{-1}+\frac{3}{4}\mathbf{I}\\ \frac{3}{2}\end{matrix}\,\middle|\, -2x^2\mathbf{Q}_1\right]$ |
| $P_n(1-2x)$ | Legendre | ${}_2F_1\left[\begin{matrix}-n,\, n+1\\ 1\end{matrix}\,\middle|\, x\right]$ | ${}_1F_1\left[\begin{matrix}\mathbf{Q}_2\mathbf{Q}_1^{-1}+\frac{1}{2}\mathbf{I}\\ 1\end{matrix}\,\middle|\, -2x\mathbf{Q}_1\right]$ |
| $T_n(1-2x)$ | Chebyshev | ${}_2F_1\left[\begin{matrix}-n,\, n\\ \frac{1}{2}\end{matrix}\,\middle|\, x\right]$ | ${}_1F_1\left[\begin{matrix}\mathbf{Q}_2\mathbf{Q}_1^{-1}\\ \frac{1}{2}\end{matrix}\,\middle|\, -2x\mathbf{Q}_1\right]$ |
| $\frac{1}{n+1}U_n(1-2x)$ | Chebyshev | ${}_2F_1\left[\begin{matrix}-n,\, n+2\\ \frac{3}{2}\end{matrix}\,\middle|\, x\right]$ | ${}_1F_1\left[\begin{matrix}\mathbf{Q}_2\mathbf{Q}_1^{-1}+\mathbf{I}\\ \frac{3}{2}\end{matrix}\,\middle|\, -2x\mathbf{Q}_1\right]$ |
| | Piessens [6] | ${}_2F_2\left[\begin{matrix}-n,\, n\\ \frac{1}{2},\, b\end{matrix}\,\middle|\, x\right]$ | ${}_1F_2\left[\begin{matrix}\mathbf{Q}_2\mathbf{Q}_1^{-1}\\ \frac{1}{2},\, b\end{matrix}\,\middle|\, -2x\mathbf{Q}_1\right]$ |

The next theorem can be proved in a similar manner.

THEOREM 3. *The family of polynomials*

$$p_n(x, a) = {}_2F_1\left[\begin{matrix}-n, -n-2a\\ b\end{matrix}\,\middle|\, x\right]$$

*is generated by rows by the matrix*

$$\mathbf{M}(x) = \sum_0^\infty \frac{\{(\mathbf{Q}_2 + a\mathbf{Q}_1)\, 2x\}^r}{(b)_r\, r!},$$

$$= {}_0F_1\left[\begin{matrix}-\\ b\end{matrix}\,\middle|\, (\mathbf{Q}_2 + a\mathbf{Q}_1)\, 2x\right],$$

*which also generates by columns the family of infinite series*

$$p_{-n-1}(x, -a) = \sum_0^\infty (n+1)_r\, (n+1+2a)_r\, \lambda_r x^r.$$

Theorems 2 and 3 cover between them nine of the polynomials given by Abramowitz and Stegun [3, in the section "Orthogonal Polynomials as Hypergeometric Functions"].

The matrix

$$\mathbf{M}(x) = \sum_0^\infty \lambda_r (x\mathbf{Q}_{-1})^r,$$

where $\mathbf{Q}_{-1}$ is defined by (13), generates by columns the family of infinite series

$$f_n(x) = \sum_0^\infty \frac{\lambda_r(-x)^r}{(n+1)_r}, \qquad n = 0, 1, 2, \ldots.$$

If $\lambda_r = (r!)^{-1}$ we get a relation between the exponential function and Bessel's function $J_n(x)$.

$$\mathbf{M}(x) = e^{x\mathbf{Q}_{-1}}$$

$$f_n(x) = n! x^{-n/2} J_n(2x^{1/2}), \qquad n = 0, 1, 2, \ldots.$$

If $\lambda_r = (-1)^r (-m)_r / r!$ we get a relation between the binomial function and the generalized Laguerre polynomials which appear as columns of finite length.

$$\mathbf{M}(x) = (\mathbf{I} + x\mathbf{Q}_{-1})^m,$$

$$f_n(x) = \frac{m!\,n!}{(m+n)!} L_m^{(n)}(x), \qquad n = 0, 1, 2, \ldots.$$

Finally the matrix

$$\mathbf{M}(x) = \sum_0^\infty \lambda_r(\mathbf{Q}_0 + \mathbf{Q}_{-1})^r x^r,$$

where, from (13),

$$\mathbf{Q}_0 + \mathbf{Q}_{-1} = \begin{bmatrix} 0 & & & & \\ 0 & 0 & & & \\ & \frac{1}{2} & 0 & & \\ & & \frac{2}{3} & 0 & \\ & & & \frac{3}{4} & 0 \\ - & - & - & - & - \end{bmatrix}$$

generates by columns the infinite series

$$f_n(x) = \sum_0^\infty \frac{(n)_r \lambda_r x^r}{(n+1)_r}, \qquad n = 0, 1, 2, \ldots .$$

If $\lambda_r = (-1)^r/r!$ we get a relation between the exponential function and the incomplete gamma function.

$$\mathbf{M}(x) = e^{-x(\mathbf{Q}_0+\mathbf{Q}_{-1})},$$

$$f_n(x) = nx^{-n}\gamma(n, x).$$

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