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On Appell sequences of polynomials of Bernoulli and Euler type

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Abstract

A construction of new sequences of generalized Bernoulli polynomials of first and second kind is proposed. These sequences share with the classical Bernoulli polynomials many algebraic and number theoretical properties. A class of Euler-type polynomials is also presented.

Keywords: Generalized Bernoulli polynomials; Finite operator theory; Appell sequences

1. Introduction

In this paper several classes of polynomials of Appell type, which generalize the classical Bernoulli and Euler polynomials, will be introduced and discussed.

We recall that the Bernoulli polynomials [6] are defined by the generating function

$$\frac{t}{e^t-1}e^{xt}=\sum_{k=0}^{\infty}\frac{B_k(x)}{k!}t^k. \tag{1.1}$$

For $x=0$, formula (1.1) reduces to the generating function of the Bernoulli numbers

$$\frac{t}{e^t-1}=\sum_{k=0}^{\infty}\frac{B_k}{k!}t^k. \tag{1.2}$$

The Bernoulli polynomials are also determined by the two properties

$$DB_n(x)=nB_{n-1}(x), \tag{1.3}$$

$$\Delta B_n(x)=nx^{n-1}, \tag{1.4}$$

with the condition $B_0(x)=1$. Here D is the continuous derivative, $\Delta:=(T-1)$ is the discrete one, and T is the shift operator, defined by $Tf(x)=f(x+1)$. Standard references are [14] and [37] and the books [17] and [23]. By virtue of relation (1.3), the Bernoulli polynomials belong to the class of Appell polynomials.

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The Bernoulli polynomials play an important rôle in several branches of mathematical analysis, such as the theory of distributions in p-adic analysis [25], the theory of modular forms [26], the study of polynomial expansions of analytic functions [7], etc. Formula (1.4) is the basis for the application of these polynomials in interpolation theory [32]. Several generalizations of Bernoulli polynomials have also been proposed. Particularly important is that of Leopold and Iwasawa, motivated by a connection with the theory of p-adic L-functions [24]. Another example is provided by the work of Carlitz [9]. Recently, new applications of the Bernoulli polynomials have also been found in mathematical physics, in connection with the theory of the Korteweg–de Vries equation [18] and Lamé equation [20], and in the study of vertex algebras [13].

The Bernoulli numbers are relevant in several branches of number theory, in particular in the computation of special values of zeta functions (see, e.g., [16,44], in the theory of cyclotomic fields [42] and, since Kummer's work, in connection with Fermat's last theorem [23]. They are also useful in singularity theory [2] and in the study of Coxeter groups [3]. Standard applications in algebraic topology are found in the computation of Todd characteristic classes and in the Hirzebruch signature theorem, as well as, more recently, in complex homology theory [5,33]. In the last years Bernoulli number identities have found applications in Quantum Field Theory [15] and in the computation of Gromov–Witten invariants [19].

In the papers [39,40] the following generalization of the Bernoulli polynomials was introduced.

Definition 1. Let us consider the polynomial ring $\mathbb{Q}[c_1, c_2, \dots]$ and the formal power series

$$F(s) = s + c_1 \frac{s^2}{2!} + c_2 \frac{s^3}{3!} + \cdots. \tag{1.5}$$

Let $G(t)$ be the compositional inverse series

$$G(t) = t - c_1 \frac{t^2}{2} + \left(3c_1^2 - 2c_2\right)\frac{t^3}{6} + \cdots \tag{1.6}$$

so that $F(G(t)) = t$. The *universal higher-order Bernoulli polynomials* $B_{k,a}^{G}(x, c_1, \dots, c_n) \equiv B_{k,a}^{G}(x)$ are defined by

$$\left(\frac{t}{G(t)}\right)^a e^{xt} = \sum_{k\geqslant 0} B_{k,a}^{G}(x)\frac{t^k}{k!}, \tag{1.7}$$

where $a \neq 0$.

This definition is clearly motivated by the works by Clarke [12], Ray [33] and Adelberg [1] on *universal Bernoulli numbers*. We observe that many known polynomial sequences can be obtained by suitable choices of the coefficients appearing in the generating function (1.7). For instance, the case of the standard Bernoulli polynomials corresponds to the choice $a = 1$, $c_i = (-1)^i$, since now $F(s) = \log(1+s)$ and $G(t) = e^t - 1$. When a is a rational integer, $c_i = (-1)^i$ we obtain the higher-order Bernoulli polynomials, which have also been extensively studied (e.g. in [10,11,22,35]). When $x = 0$, and a is a variable, formula (1.7) can be reduced to the generating function of the Nörlund polynomials and their generalizations.

By construction, $B_{k,1}^{U}(0) \equiv \widehat{B_k} \in \mathbb{Q}[c_1, \dots, c_n]$, where $\widehat{B_k}$ are the universal Bernoulli numbers introduced by Clarke in [12]. The connection with algebraic topology relies on the fact that $G(F(s_1) + F(s_2))$ represents the *universal formal group* [21]; the series F is called the formal group logarithm and G the formal group exponential. The universal formal group is defined over the Lazard ring L, which is the subring of $\mathbb{Q}[c_1, \dots, c_n]$ generated by the coefficients of the power series $G(F(s_1) + F(s_2))$. In algebraic topology, in particular in complex cobordism theory, the coefficients c_n are identified with the cobordism classes of $\mathbb{C}P^n$ (see [4,31] and [33]). Generalizations of famous congruences valid for the classical Bernoulli numbers, like the celebrated Kummer and Clausen–von Staudt congruences [1,23], are satisfied by these universal numbers.

The universal polynomials (1.7) possess as well several interesting general properties studied in [39,40].

The first aim of this paper is to provide nontrivial realizations of polynomial sequences of the type (1.7). In particular, we will focus on a specific class of Bernoulli-type polynomials, constructed using the finite operator calculus, as formulated by G.C. Rota and S. Roman [35,36]. For a recent review of the vast literature existing on this approach (also known, in its earlier formulations, as the Umbral Calculus) see, e.g., [8]. In other words, the *Bernoulli-type* *polynomials of order p* considered here correspond to a class of formal power series $G(t)$ representing suitable *difference delta operators of order p* (denoted by Δ_p). These delta operators have been introduced in [28,29], where a version of Rota's operator approach, based on the theory of representations of the Heisenberg–Weyl algebra, has been outlined. In this paper we wish to further illustrate the connection between Rota's approach and the theory of Appell and Sheffer polynomials. The basic sequences associated with the delta operators Δ_p are here studied in details and some of their combinatorial properties derived. In addition, their connection constants with the basic sequence $\{x^n\}$ (and the associated inverse relations) define generalized Stirling numbers of the first and second kind. The problem of classifying all *Sheffer sequences* associated with the delta operators considered in this paper is essentially open.

By analogy with the classical case, the Bernoulli-type polynomials, parametrized by a real variable a, are uniquely determined by the relations

$$DB_{n,a}^{p}(x) = nB_{n-1,a}^{p}(x), \tag{1.8}$$

$$\Delta_p B_{n,a}^{p}(x) = nx^{n-1}, \tag{1.9}$$

and by the condition $B_{0,a}^{p}(x) = 1$.

The use of the Roman–Rota formalism has the advantage of providing a natural language which allows the mathematical treatment of many polynomial sequences to be unified.

The second aim of this paper is to introduce the notions of *Bernoulli polynomials of the second kind* and of *Euler-type polynomials*.

All these polynomials can also play a role in discrete mathematics. Indeed, they satisfy certain interesting linear difference equations of order p in one variable, defined in a two-dimensional space of parameters.

The future research plans on generalized Bernoulli polynomials include applications in analytic number theory, in connection with the theory of Dirichlet L-series and Riemann–Hurwitz zeta function. Results in this direction have been obtained in [39] and [40], where a new construction relating generalized Bernoulli polynomials, formal groups and L-series is proposed. An application of these polynomial sequences to the determination of hyperfunctions and generalized Lipschitz summation formulae is given in [30].

The paper is organized as follows. In Section 2, a brief introduction to finite operator calculus is presented, with a discussion in Section 3 of difference delta operators. In Section 4, generalized Stirling numbers are considered. In Section 5, new Bernoulli-type polynomials of the first kind are introduced. The Bernoulli-type polynomials of the second kind are introduced and discussed in Section 6. A generalization of Euler polynomials is discussed in the final Section 7.

2. Finite operator calculus

In this section, some basic results concerning the theory of finite difference operators and their relation with polynomial sequences of Appell and Sheffer type are reviewed. For further details and proofs, see the monographs [35,36], where an extensive and modern treatment of this topic is proposed.

Let $\mathcal{F}$ denote the algebra of formal power series in one variable t, endowed with the operations of sum and multiplication of series. Let $\mathcal{P}$ be the algebra of polynomials in one variable x and $\mathcal{P}^*$ the vector space of all linear functionals on $\mathcal{P}$. If $L \in \mathcal{P}^*$, following Dirac we will denote the action of L on $p(x) \in \mathcal{P}$ by $\langle L|p(x)\rangle$. A remarkable fact of the finite operator calculus is that any element of $\mathcal{F}$ can play a threefold rôle: It can be regarded as a formal power series, as a linear functional on $\mathcal{P}$ and also as a linear operator on $\mathcal{P}$. To prove this, let us first notice that, given a formal power series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k, \tag{2.1}$$

we can associate it with a linear functional via the correspondence

$$\langle f(t) | x^n \rangle = a_n. \tag{2.2}$$

Therefore

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t)|x^k\rangle}{k!} t^k. \tag{2.3}$$

In particular, since

$$\langle t^k | x^n \rangle = n!\delta_{n,k}, \tag{2.4}$$

it follows that, for any polynomial $p(x)$,

$$\langle t^k | p(x) \rangle = p^{(k)}(0), \tag{2.5}$$

where $p^{(k)}(0)$ denotes the k-derivative of $p(x)$ evaluated at $x = 0$. It is easily shown that any linear functional $L \in \mathcal{P}^*$ is of the form (2.3).

If we interpret t^k as the kth-order derivative operator on $\mathcal{P}$, given a polynomial $p(x)$ we have that $t^k p(x) = p^{(k)}(x)$. Hence the formal series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k t^k}{k!} \tag{2.6}$$

is also regarded as a linear operator acting on $\mathcal{P}$. Depending on the context, t will play the rôle of a formal variable or that of a derivative operator.

Now, some basic definitions and theorems of finite operator theory, necessary in the subsequent considerations, are in order.

Definition 2. An operator S commuting with the shift-operator T, i.e. $[S, T] = 0$, is said to be shift-invariant.

Relevant examples of operators belonging to this class are provided by the delta operators.

Definition 3. A delta operator Q is a shift-invariant operator such that $Qx = \text{const} \neq 0$.

As has been proved in [36], there is an isomorphism between the ring of formal power series in a variable t and the ring of shift-invariant operators, carrying

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k t^k}{k!} \quad \text{into} \quad \sum_{k=0}^{\infty} \frac{a_k Q^k}{k!}. \tag{2.7}$$

In the following, let us denote by $p_n(x)$, $n = 0, 1, 2, \ldots,$ a sequence of polynomials in x, where $p_i(x)$ is of order i for all i.

Definition 4. A polynomial sequence $p_n(x)$ is called a sequence of basic polynomials for a delta operator Q if it satisfies the following conditions.

(1) $p_0(x) = 1$;
(2) $p_n(0) = 0$ for all $n > 0$;
(3) $Qp_n(x) = np_{n-1}(x)$.

For each delta operator there exists a unique sequence of associated basic polynomials.

Definition 5. A polynomial sequence $s_n(x)$ is called a set of Sheffer polynomials for the delta operator Q if

(1) $s_0(x) = c \neq 0$;
(2) $Qs_n(x) = ns_{n-1}(x)$.

Definition 6. An Appell sequence of polynomials is a Sheffer sequence for the delta operator D.

Any shift invariant operator S can be expanded into a formal power series in terms of a delta operator Q

$$S = \sum_{k \geqslant 0} \frac{a_k}{k!} Q^k,$$

with $a_k = [Sp_k(x)]|_{x=0}$, where p_k is the basic polynomial of order k associated with Q. By using the isomorphism (2.7), a formal power series $s(t)$ is defined, which is called the *indicator of* S.

Remark 1. A shift invariant operator is a delta operator if and only if it corresponds, under the isomorphism (2.7), to a formal power series $G(t)$ such that $G(0) = 0$ and $G'(0) \neq 0$. This series admits a unique compositional inverse.

The umbral formalism also allows us to characterize the generating functions of polynomial sequences of Sheffer and Appell type.

If $s_n(x)$ is a Sheffer sequence for the operator Q, then there exists an invertible shift-invariant operator S such that

$$Ss_n(x) = q_n(x),$$

where $q_n(x)$ is a basic set for Q.

Let us denote by $s(t)$ and $q(t)$ the indicators of the operators S and Q. We will say that $s_n(x)$ is the Sheffer sequence associated with $(s(t), q(t))$. The following result holds:

$$\frac{1}{s(q^{-1}(t))} e^{xq^{-1}(t)} = \sum_{n \geqslant 0} \frac{s_n(x)}{n!} t^n, \tag{2.8}$$

where $q^{-1}(t)$ denotes the compositional inverse of $q(t)$.

Another characterization of a Sheffer sequence is provided by the identity

$$s_n(x+y) = \sum_{k \geqslant 0} \binom{n}{k} q_k(y) s_{n-k}(x).$$

In the specific case of the Appell sequences, which obey the equation

$$Ds_n(x) = ns_{n-1}(x), \tag{2.9}$$

there exists an invertible operator $g(t)$ such that

$$g(t)s_n(x) = x^n. \tag{2.10}$$

Therefore, we will say that $s_n(x)$ is the Appell sequence associated with $g(t)$. Its generating function is defined by

$$\sum_{k=0}^{\infty} \frac{s_k(x)}{k!} t^k = \frac{1}{g(t)} e^{xt}. \tag{2.11}$$

The Appell identity is

$$s_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} s_k(y) x^{n-k}. \tag{2.12}$$

We also recall the polynomial expansion theorem. If $s_n(x)$ is a Sheffer set for the operators $(s(t), q(t))$, then for any polynomial $p(x)$ we have

$$p(x) = \sum_{k \geqslant 0} \frac{\langle s(t) q(t)^k | p(x) \rangle}{k!} s_k(x). \tag{2.13}$$

In particular, if $q_n(x)$ is a basic sequence for $q(t)$, this formula reduces to

$$p(x) = \sum_{k \geqslant 0} \frac{\langle q(t)^k | p(x) \rangle}{k!} q_k(x). \tag{2.14}$$

3. Difference delta operators

In this paper, we will consider a specific class of difference operators, introduced in [28], having the general form

$$Q \equiv \Delta_p = \frac{1}{\sigma} \sum_{k=l}^{m} a_k T^k, \quad l, m \in \mathbb{Z},\ l < m,\ m - l = p, \tag{3.1}$$

where a_k and σ are constants. In order to satisfy the definition of delta operator, we must assume that

$$\sum_{k=l}^{m} a_k = 0, \qquad \sum_{k=l}^{m} k a_k = c. \tag{3.2}$$

We also require that in the continuous limit analogue, Δ_p reproduces the standard derivative D; this implies $c = 1$, i.e.

$$\sum_{k=l}^{m} k a_k = 1. \tag{3.3}$$

Definition 7. A difference operator of the form (3.1) which satisfies Eqs. (3.2) and (3.3) is called a delta operator of order $p = m - l$.

We observe that Eq. (3.1) involves $m - l + 1$ constants a_k, subject to two conditions (3.2) and (3.3). To fix all constants a_k we have to impose $m - l - 1$ further conditions. A possible choice is, for instance,

$$\gamma_p \equiv \sum_{k=l}^{m} k^p a_k = 0, \quad p = 2, 3, \ldots, m - l. \tag{3.4}$$

When conditions (3.2), (3.3) and (3.4) are satisfied, the operator (3.1) provides an approximation of order p of the continuous derivative D, since

$$\Delta_p f \underset{\sigma \to 0}{\sim} f'(x) + \frac{\sigma^{m-l}}{(m-l+1)!} f^{(m-l-1)}(x) \sum_{k=l}^{m} a_k k^{m-l-1}.$$

From now on, we will put $\sigma = 1$. The Pincherle derivative of a delta operator is defined by the relation

$$Q' = [Q, x].$$

Now, let us introduce a shift-invariant operator β such that

$$[\Delta_n, x\beta] = 1. \tag{3.5}$$

It follows that $\beta = (\Delta_n')^{-1}$ (see [27–29]). Such an operator is invertible [36] and finite. Indeed, using the identity $\beta\beta^{-1} = 1$, the action of β on a monomial of order n is easily seen to be

$$\beta x^n = x^n - \sum_{j=0}^{n-1} \alpha_j^n x^{n-1-j}, \tag{3.6}$$

where α_j^n are defined via the recursion relation

$$\alpha_j^n = \binom{n}{j+1} \gamma_{j+2} - \sum_{l=0}^{j-1} \binom{n}{l+1} \gamma_{l+2} \alpha_{j-l-1}^{n-l-1},$$

with $\gamma_j = \sum_{k=l}^{m} k^j a_k$. We see that β preserves polynomial structures. When $\Delta = D$, $\beta = 1$. Other specific cases are listed below: for

$$\Delta = \Delta^+ = T - 1, \quad \beta = T^{-1}; \tag{3.7}$$

for

$$\Delta = \Delta^- = 1 - T^{-1}, \quad \beta = T, \tag{3.8}$$

when

$$\Delta_2 = \Delta^s = \frac{T - T^{-1}}{2}, \quad \beta = \left(\frac{T + T^{-1}}{2}\right)^{-1}. \tag{3.9}$$

Other nontrivial examples of higher-order operators are provided by, for instance,

$$\Delta_3 = -\left(T^2 - 2T + T^{-1}\right), \tag{3.10}$$

$$\Delta_4 = T^2 - \frac{3}{2}T + \frac{3}{2}T^{-1} - T^{-2}, \tag{3.11}$$

$$\Delta_5 = T^3 - 2T^2 + 2T - 2T^{-1} + T^{-2}, \tag{3.12}$$

$$\Delta_7 = T^4 - T^3 + T^2 - 2T + T^{-1} - T^{-2} + T^{-3}. \tag{3.13}$$

Finally, we recall that Q is a delta operator if and only if it is of the form $Q = DP$, where P is an invertible shift-invariant operator [36]. It follows that the sequence of basic polynomials for Q is expressible in the form

$$p_n(x) = xP^{-n}x^{n-1}. \tag{3.14}$$

4. A generalization of the Stirling numbers

According to Rota's approach, with any delta operator of the form (3.1)–(3.3) it is possible to associate a sequence of basic polynomials. In perfect analogy with the classical theory of Stirling numbers, this allows us to study generalized Stirling numbers. In this section, several concrete examples of basic polynomials as well as generalized Stirling numbers are obtained. This construction represents a natural realization of the very general scheme proposed by Ray in [33], who first studied the basic sequences and defined the universal Stirling numbers for any delta operator constructed via the complex homology theory. Here we will follow a different but equivalent philosophy: the basic sequences for the operators (3.1)–(3.3) are simply derived from the "discrete" representations of the Heisenberg–Weyl algebra; the associated Stirling numbers furnish the connection coefficients between the basic sequences and the standard power sequence $\{x^n\}_{n\in\mathbb{N}}$.

We recall that the classical Stirling numbers of the first and second kind $s(n,k)$ and $S(n,k)$ are defined by the relations (see, e.g., [34])

$$(x)_n = \sum_{k=0}^{n} s(n,k)x^k \tag{4.1}$$

and

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k, \tag{4.2}$$

respectively. Here $(x)_n = x(x-1)\dots(x-n+1)$ denotes the lower factorial polynomial of order n. The numbers $S(n,k)$ also admit a representation in terms of a generating function

$$\sum_{n=0}^{\infty} S(n,k)\frac{t^n}{n!} = \frac{(e^x-1)^k}{k!}.$$

As is well known, the lower factorial polynomials are the basic sequence associated with the forward derivative: $\Delta(x)_n = n(x)_{n-1}$. More generally, the basic sequence associated with the operator Δ_p, denoted by $(x)_n^p$, is [28]

$$(x)_n^p = (x\beta)^n 1. \tag{4.3}$$

Therefore,

$$\Delta_p (x)_n^p = n(x)_{n-1}^p. \tag{4.4}$$

Explicitly, the first basic polynomials are expressed by

$$\begin{aligned}
&(x)_0^p = (x\beta)^0 1 = 1,\\
&(x)_1^p = (x\beta)^1 1 = x,\\
&(x)_2^p = (x\beta)^2 1 = x^2 - \gamma_2 x,\\
&(x)_3^p = (x\beta)^3 1 = x^3 - 3\gamma_2 x^2 - \left(\gamma_3 - 3\gamma_2^2\right)x,\\
&(x)_4^p = (x\beta)^4 1 = x^4 - 6\gamma_2 x^3 + \left(-4\gamma_3 + 15\gamma_2^2\right)x^2 + \left(-\gamma_4 + 10\gamma_2\gamma_3 - 15\gamma_2^3\right)x,
\end{aligned} \tag{4.5}$$

and so on, with γ_j given by

$$\gamma_j = \sum_{k=l}^{m} a_k k^j, \quad \gamma_0 = 0, \quad \gamma_1 = 1, \quad j = 0, 1, 2, \ldots. \tag{4.6}$$

Definition 8. The *generalized Stirling numbers of the first kind* and order p associated with the operators (3.1)–(3.3), denoted by $s^p(n,k)$, are defined by

$$(x)_n^p = \sum_{k=0}^{n} s^p(n,k) x^k. \tag{4.7}$$

Remark 2. In the subsequent considerations, with an abuse of notation, we will use Roman numerals for denoting the values of p, when p appears as an index in sequences of polynomials or numbers.

Observe that $(x)_n^I = (x)_n$ and $s^I(n,k) = s(n,k)$. Since for any polynomial $p(x)$ the following expansion holds:

$$p(x) = \sum_{k\geqslant 0} \frac{\langle t^k | p(x)\rangle}{k!} x^k, \tag{4.8}$$

it emerges that

$$s^p(n,k) = \frac{1}{k!}\left\langle t^k \middle| (x)_n^p \right\rangle. \tag{4.9}$$

Definition 9. The *generalized Stirling numbers of the second kind and order* p, denoted by $S^p(n,k)$, are defined by

$$x^n = \sum_{k=0}^{n} S^p(n,k)(x)_k^p. \tag{4.10}$$

They admit the generating function

$$\sum_{n=k}^{\infty} S^p(n,k)\frac{t^n}{n!} = \frac{[\Delta_p(t)]^k}{k!}, \tag{4.11}$$

where $\Delta_p(t) = \sum_j a_j e^{jt}$ is the indicator of the operator Δ_p. As a consequence of formula (2.14), one immediately obtains that

$$S^p(n,k) = \frac{1}{k!}\left\langle \left(\sum_{j=l}^{m} a_j e^{jt}\right)^k \middle| x^n \right\rangle, \quad m - l = p. \tag{4.12}$$

The polynomials $(x)_n^p$ satisfy the binomial identity

$$(x+y)_n^p = \sum_{k=0}^{n} \binom{n}{k} (x)_k^p (y)_{n-k}^p. \tag{4.13}$$

A relevant feature of relations (4.7) and (4.10) is that they are inverse to each other. This immediately implies that

$$\delta_{mn} = \sum_k s^p(n,k) S^p(k,m) = \sum_k S^p(n,k) s^p(k,m). \tag{4.14}$$

Remark 3. Due to formulae (3.5) and (4.4), it is natural to interpret the operators Δ and $x\beta$ as quantum mechanical annihilation and creation operators a and $a^\dagger$ acting on a finitely generated space, as noticed in [38,41]. For instance,

$$(x\beta)^n \cdot 1 = \left(a^\dagger\right)^n |0\rangle = |n\rangle.$$
$$\Delta_p (x\beta)^n \cdot 1 = n|n-1\rangle.$$

5. Bernoulli-type polynomials and numbers of the first kind

In this section, some realizations of the polynomial class (1.7) are presented. We will restrict to sequences of polynomials obtainable in the umbral context described above. We propose the following definition.

Definition 10. The higher-order *Bernoulli-type polynomials of the first kind* are the polynomials generated by the relation

$$\sum_{k=0}^{\infty} \frac{B_{k,a}^p(x)}{k!} t^k = J_p(t)^a e^{xt}, \tag{5.1}$$

with $a \neq 0$ and

$$J_p(t) = \frac{t}{\sum_{j=l}^{m} a_j e^{tj}}, \tag{5.2}$$

where $\sum_{j=l}^{m} a_j e^{tj}$ is the indicator of a delta operator of order p.

Remark 4. We will assume that $a_j \in \mathbb{Q}$, as turns out to be in many applications. However, more general operator structures with real coefficients a_j could be considered within the same framework. In the following, when $a = 1$, the index a will be omitted for simplicity.

Definition 11. The Bernoulli-type numbers $B_{k,a}^p$ are defined, for every p and a, by the relation

$$\sum_{k=0}^{\infty} \frac{B_{k,a}^p}{k!} t^k = J_p(t)^a. \tag{5.3}$$

Remark 5. The operators $J_p(t)$ and the corresponding numbers for $a = 1$ have been studied in [33] in the context of complex oriented homology theory. Here again the construction of these operators is realized using a difference operator approach.

We can now prove a result stated in the introduction.

Lemma 1. *The polynomials $B_{m,a}^p(x)$ are uniquely determined by the two properties*

$$D B_{m,a}^p(x) = m B_{m-1,a}^p, \tag{5.4}$$
$$\Delta_p B_{m,a}^p(x) = n x^{m-1}, \tag{5.5}$$

with the condition $B_{0,a}^p(x) = 1$.

Proof. Identity (5.4) follows immediately from (5.1). To obtain (5.5), we note that

$$B_{m,a}^p(x+h) = \sum_{\nu=0}^{m} \frac{h^\nu}{\nu!} D^\nu B_{m,a}^p(x) = \sum_{\nu=0}^{m} \binom{m}{\nu} h^\nu B_{m-\nu,a}^p(x) = \sum_{\nu=0}^{m} \binom{m}{\nu} h^{m-\nu} B_{\nu,a}^p(x), \tag{5.6}$$

after having replaced ν with $m-\nu$. Equation (5.5) involves Bernoulli-type polynomials evaluated at shifted points. Substituting Eq. (5.6) into Eq. (5.5) and using the initial condition, one can get recursively all polynomials of degree $\leqslant m$. $\square$

Examples. The case $p=2$. We get a sequence of polynomials which we will call the *central Bernoulli polynomials* (of the first kind). The generating function reads

$$\sum_{k=0}^{\infty} \frac{B_{k,a}^{II}(x)}{k!} t^k = \left(\frac{t}{\sinh t}\right)^a e^{xt}. \tag{5.7}$$

For $x=0$ and $a=1$, one obtains the numbers B_n^{II}. From the generating function it emerges that $B_k^{II}=0$ for k odd. The first central Bernoulli numbers B_{2k}^{II} are: $1, -\frac{1}{3}, \frac{7}{15}, -\frac{31}{21}, \frac{127}{15}, -\frac{2555}{33}$, etc.

The polynomials (5.7) can be directly expressed in terms of the classical Bernoulli polynomials

$$B_n^{II}(x) = 2^n B_n^I\left(\frac{x+1}{2}\right), \qquad B_n^I(x) = \frac{B_n^{II}(2x-1)}{2^n}.$$

Other properties are now briefly discussed. From the polynomial expansion theorem we get

$$B_n^{II}(x) = \sum_{k=0}^{n} \frac{\langle (\frac{e^t-e^{-t}}{2})^k | B_n^{II}(x)\rangle}{k!} (x)_k^{II}. \tag{5.8}$$

Since

$$\begin{aligned}\left\langle \frac{(e^t-e^{-t})^k}{2} \middle| B_{n,1}^{II}(x)\right\rangle &= \left\langle \left(\frac{e^t-e^{-t}}{2}\right)^k \middle| \frac{2t}{e^t-e^{-t}} x^n\right\rangle = \left\langle \left(\frac{e^t-e^{-t}}{2}\right)^{k-1} \middle| n x^{n-1}\right\rangle \\ &= n(k-1)!S^{II}(n-1,k-1),\end{aligned} \tag{5.9}$$

we have

$$B_n^{II}(x) = B_n^{II}(0) + \sum_{k=1}^{n} \frac{n}{k} S^{II}(n-1,k-1)(x)_k^{II}. \tag{5.10}$$

For $p=3$, the corresponding delta operator is given by formula (3.10), with the generating function

$$\sum_{k=0}^{\infty} \frac{B_k^{III}(x)}{k!} t^k = -\frac{t e^{xt}}{e^{2t}-2e^t+e^{-t}}. \tag{5.11}$$

For the other cases, we get similar results. For instance,

$$\sum_{k=0}^{\infty} \frac{B_k^{V}(x)}{k!} t^k = \frac{t e^{xt}}{e^{3t}-2e^{2t}+2e^t-2e^{-t}+e^{-2t}}, \tag{5.12}$$

$$\sum_{k=0}^{\infty} \frac{B_k^{VII}(x)}{k!} t^k = \frac{-t e^{xt}}{e^{4t}-e^{3t}+e^{2t}-2e^t+e^{-t}-e^{-2t}+e^{-3t}}, \tag{5.13}$$

and so on. As an example, the first polynomials $B_n^{III}(x)$ are given by

$$B_0^{III}(x)=1, \qquad B_1^{III}(x) = x+\frac{3}{2}, \qquad B_2^{III}(x) = x^2+3x+\frac{37}{6},$$

$$B_3^{III}(x) = x^3+\frac{9}{2}x^2+\frac{37}{2}x+39,$$

$$B_4^{III}(x) = x^4+6x^3+37x^2+156x+\frac{9719}{30},$$

$$B_5^{III}(x) = x^5+\frac{15}{2}x^4+\frac{185}{3}x^3+390x^2+\frac{9719}{6}x+3365,$$

$$B_6^{III}(x) = x^6+9x^5+\frac{185}{2}x^4+780x^3+\frac{9719}{2}x^2+20190x+\frac{1762237}{42}, \ldots.$$

The relation (5.10) between higher-order Stirling-type numbers, Bernoulli-type numbers and Bernoulli-type polynomials can be generalized as follows:

$$B_n^p(x) = B_n^p(0) + \sum_{k=1}^{n} \frac{n}{k} S^p(n-1, k-1)(x)_k^p. \tag{5.14}$$

The recurrence relation for the polynomials $B_{n,a}^p(x)$ is formally derived from the defining relation (5.1)

$$\sum_{j=l}^{m} a_j e^{jt} B_{n,a}^p(x) = \left(\frac{t}{\sum_{j=l}^{m} a_j e^{tj}}\right)^{a-1} t x^n = n B_{n-1,a-1}^p(x). \tag{5.15}$$

Therefore, the difference equation solved by the Bernoulli-type polynomials is, for every p,

$$\sum_{j=l}^{m} a_j B_{n,a}^p(x+j) = n B_{n-1,a-1}^p(x), \tag{5.16}$$

where the a_k satisfy the constraints (3.2)–(3.3). When $p=2$, we obtain

$$B_{n,a}^{II}(x+1) - B_{n,a}^{II}(x-1) - n B_{n-1,a-1}^{II}(x) = 0. \tag{5.17}$$

Also we have the recurrence

$$B_{n+1,a}^p(x) = \left(x - \frac{g'(t)}{g(t)}\right) B_{n,a}^p(x), \tag{5.18}$$

where $g(t)$ is the operator

$$g(t) = \left(\frac{\sum_{k=l}^{m} a_k e^{kt}}{t}\right)^a.$$

Using the relations $[t, x] = 1$ and $B_{n,a}^p = g(t)^{-1} x^n$, which is a consequence of Eq. (2.10), we get

$$(n+1) B_{n,a}^p(x) = \left(xt + 1 - a \frac{\sum_{k=l}^{m} a_k e^{kt}(tk-1)}{\sum_{k=l}^{m} a_k e^{kt}}\right) B_{n,a}^p(x). \tag{5.19}$$

From the previous equation we derive the formula expressing the Bernoulli-type polynomials of order $a+1$ in terms of those of order a,

$$\sum_k k a_k B_{n,a+1}^p(x+k) = \left(1 - \frac{n}{a}\right) B_{n,a}^p(x) + \frac{nx}{a} B_{n-1,a}^p(x). \tag{5.20}$$

Observe that, in accordance with Eq. (3.14), the polynomials

$$p_n(x) = x \left(\frac{t}{\sum_k a_k e^{kt}}\right)^{na} x^{n-1} = x B_{n-1,na}^p(x) \tag{5.21}$$

represent the basic sequence associated with the operator $Q = (\frac{\Delta_p}{D})^a D$. Consequently, for $a = 1$, we deduce the relation expressing the basic sequence $(x)_n^p$ in terms of Bernoulli-type polynomials

$$(x)_n^p = x B_{n-1,n}^p(x). \tag{5.22}$$

An interesting connection between the Bernoulli numbers and the Stirling numbers of the same order holds. Indeed, using (5.22) and its differential consequences, after some algebraic manipulations we get

$$s^p(n, r) = \frac{1}{r!} \langle t^r \,|\, (x)_n^p \rangle = \frac{1}{r!} \langle t^0 \,|\, t^r (x)_n^p \rangle = \binom{n}{r} B_{n-r,n+1}^p(1). \tag{5.23}$$

Therefore,

$$(x)_n^p = \sum_{r=0}^{n} \binom{n}{r} B_{n-r,n+1}^p(1) x^r.$$

Analogously,

$$S^p(n,r) = \frac{1}{r!}\left\langle \left(\frac{\sum_{k=l}^m a_k e^{kt}}{t}\right)^r \middle| t^r x^n \right\rangle = \binom{n}{r} B^p_{n-r,-r}(0) \tag{5.24}$$

and

$$x^n = \sum_{r=0}^{n} \binom{n}{r} B^p_{n-r,-r}(0)(x)^p_r. \tag{5.25}$$

6. New Bernoulli-type polynomials of the second kind

By analogy with the standard Bernoulli polynomials of the second kind (see, e.g., [35]), we introduce the following sequences of polynomials.

Definition 12. The higher-order Bernoulli-type polynomials of the second kind are the polynomials defined by

$$b^p_n(x) = J_p(t)(x)^p_n, \quad n \in \mathbb{N},\ p \in \mathbb{N}, \tag{6.1}$$

where $J_p(t)$ is the operator (5.2).

These polynomials represent a Sheffer sequence for the operator Δ_p of Eqs. (3.1), (3.2)

$$\Delta_p b^p_n(x) = n b^p_{n-1}(x), \tag{6.2}$$

and therefore they satisfy the identity

$$b^p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} b^p_k(y)(x)^p_{n-k} \tag{6.3}$$

which relates them with the higher factorial polynomials. In particular, for $y = 0$, we get

$$b^p_n(x) = \sum_{k=0}^{n} \binom{n}{k} b^p_k(0)(x)^p_{n-k}. \tag{6.4}$$

From Eq. (6.2) we deduce the difference equation satisfied by the polynomials (6.1)

$$\sum_k a_k b^p_n(x+k) = n b^p_{n-1}(x). \tag{6.5}$$

The case $J_1(t) = e^t - 1$ reproduces the standard Bernoulli polynomials of the second kind. The generating function of the sequences (6.1) can be explicitly obtained in some specific cases. Let us briefly discuss the second-order case. Since the operator $J^{II} = \frac{e^t - e^{-t}}{2t}$ acts as follows:

$$J^{II} p(x) = \frac{1}{2} \int_{x-1}^{x+1} p(u)\, du,$$

we deduce an explicit expression for the central Bernoulli polynomials of the second kind

$$b^{II}_n(x) = J^{II}(x)^{II}_n = \frac{1}{2} \int_{x-1}^{x+1} (u)^{II}_n\, du. \tag{6.6}$$

The generating function is

$$\sum_{k=0}^{\infty} \frac{b^{II}_k(x) t^k}{k!} = \frac{t}{\log(t + \sqrt{1+t^2})} \left(t + \sqrt{1+t^2}\right)^x. \tag{6.7}$$

For $x = 0$ we get

$$\sum_{k=0}^{\infty} \frac{b_k^{II}(0)t^k}{k!} = \frac{t}{\log(t+\sqrt{1+t^2})}. \tag{6.8}$$

Let us derive the recurrence relation satisfied by the polynomials $b_n^{II}(x)$. From Eq. (6.6) we get

$$t b_n^{II}(x) = \left(\frac{e^t - e^{-t}}{2}\right)(x)_n^{II} = n(x)_{n-1}^{II}. \tag{6.9}$$

Integrating we obtain

$$b_n^{II}(x) - b_n^{II}(0) = n \int_0^x (u)_{n-1}^{II}\, du. \tag{6.10}$$

The difference equation (6.5) reduces to

$$b_n^{II}(x+1) = b_n^{II}(x-1) + 2n b_{n-1}^{II}(x). \tag{6.11}$$

Since

$$b_n^{II}(1) = \frac{1}{2}\int_0^2 (u)_n^{II}\, du, \tag{6.12}$$

another representation of $b_n^{II}(0)$ is

$$b_n^{II}(0) = \frac{1}{2}\int_0^2 (u)_n^{II}\, du - n\int_0^1 (u)_{n-1}^{II}\, du. \tag{6.13}$$

Notice the analogy with the Bernoulli numbers of the second kind, defined by (see [35])

$$b_n(0) = \left\langle \frac{e^t - 1}{t} \middle| (x)_n \right\rangle = \int_0^1 (u)_n\, du. \tag{6.14}$$

There is a connection between Bernoulli-type polynomials of the second kind and generalized Stirling numbers of the first kind. Since

$$b_n^p(x) = \sum_{k=0}^{n} \frac{1}{k!} \langle t^k | b_n^p(x) \rangle x^k,$$

we get

$$b_n^p(x) = b_n^p(0) + \sum_{k=1}^{n} \frac{n}{k} s^p(n-1, k-1) x^k. \tag{6.15}$$

Many other properties and identities, which we will not discuss here for the sake of brevity, can be derived using operator techniques.

7. New Euler-type polynomials

In this section, a new class of polynomial sequences of Appell type is discussed. For many aspects, it can be considered to be a natural generalization of the Euler polynomials. The definition proposed in this paper is different from that in [43].

Euler polynomials and numbers (introduced by Euler in 1740) also possess an extensive literature and several interesting applications in Number Theory (see, for instance, [14,35,43]). In many respects, they are closely related to the theory of Bernoulli polynomials and numbers.

Definition 13. The Euler-type polynomials are the Appell sequence generated by

$$\sum_{k=0}^{\infty} \frac{E_k^{p,a,\omega}(x)}{k!} t^k = \left(1 + \frac{\Delta_p}{\omega}\right)^{-a} e^{xt}, \tag{7.1}$$

where $a, \omega \neq 0$.

For $a = 1$, $\Delta = T - 1$, $\omega = 2$, we obtain the classical Euler polynomials. It will be shown that these new polynomials possess many of the properties of their classical analogues.

We recall that the Euler polynomials of order a, which will be denoted by the symbol $E_n^a(x)$, are the Appell sequence associated with the operator

$$g(t) = \left(1 + \frac{e^t - 1}{2}\right)^a.$$

Consequently

$$E_n^a(x) = \left(1 + \frac{e^t - 1}{2}\right)^{-a} x^n. \tag{7.2}$$

Therefore, we have the following

Lemma 2. *The Euler-type polynomial sequences are given by the relation*

$$E_n^{p,a,\omega}(x) = \left(1 + \frac{\Delta_p}{\omega}\right)^{-a} x^n, \tag{7.3}$$

where $a, \omega \in \mathbb{R}$.

For the sake of clarity, we will omit the superscript ω. The Euler-type polynomials satisfy the Appell property

$$DE_n^{p,a}(x) = nE_{n-1}^{p,a}(x)$$

and the Appell binomial identity

$$E_n^{p,a}(x+y) = \sum_{k=0}^{n} \binom{n}{k} E_k^{p,a}(y) x^{n-k}. \tag{7.4}$$

It also emerges that, for every p, the polynomials (7.1) are an Appell cross-sequence

$$E_n^{p,a+b}(x+y) = \sum_{k=0}^{n} \binom{n}{k} E_k^{p,a}(y) E_{n-k}^{p,b}(x). \tag{7.5}$$

We also have a recurrence formula

$$E_{n+1}^{p,a}(x) = \left(x - \frac{g_p'(t)}{g_p(t)}\right) E_n^{p,a}(x), \tag{7.6}$$

where $g_p(t) = (1 + \frac{\Delta_p(t)}{\omega})$.

There exists an interesting analogue of the Boole summation formula. Since for any function $h(t)$,

$$h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t) | E_k^{p,a}(x) \rangle}{k!} \left(1 + \frac{\Delta_p}{\omega}\right)^a t^k,$$

then, choosing $h(t) = \exp(yt)$, we get

$$\exp(yt) = \sum_{k=0}^{\infty} \frac{E_k^{p,a}(y)}{k!} \left(1 + \frac{\Delta_p}{\omega}\right)^a t^k. \tag{7.7}$$

Once applied to a polynomial $p(x)$, we obtain for any p the expansion

$$p(x+y)=\sum_{k=0}^{\infty}\frac{E_k^{p,a}(y)}{k!}\left(1+\frac{\Delta_p}{\omega}\right)^a p^{(k)}(0).$$

Let us consider in more detail the operator (for $p=2$)

$$g^{II}(t)=\left(1+\frac{e^t-e^{-t}}{\omega}\right)^a. \tag{7.8}$$

Correspondingly,

$$E_n^{II,a}(x)=\left(1+\frac{e^t-e^{-t}}{\omega}\right)^{-a}x^n. \tag{7.9}$$

Since

$$g^{II}(t)^{-1}=\left(\frac{1}{1+\frac{e^t-e^{-t}}{\omega}}\right)^a=\sum_{j=0}^{\infty}\binom{-a}{j}\left(\frac{e^t-e^{-t}}{\omega}\right)^j, \tag{7.10}$$

the polynomials (7.9) are of the form

$$E_n^{II,a}(x)=\sum_{j=0}^{\infty}\binom{-a}{j}\left(\frac{e^t-e^{-t}}{\omega}\right)^j x^n=\sum_{k=0}^{n}\sum_{j=0}^{n}\binom{-a}{j}\frac{j!}{k!}\frac{2^j}{\omega^j}S^{II}(k,j)x^{n-k}. \tag{7.11}$$

From this equation and from the definition (7.9) we derive the difference equation

$$E_n^{II,a}(x+1)-E_n^{II,a}(x-1)=\omega\bigl(E_n^{II,a-1}(x)-E_n^{II,a}(x)\bigr). \tag{7.12}$$

The analogue of the Newton expansion is given by the formula

$$E_n^{II,a}(x)=\sum_{k=0}^{n}\sum_{j=k}^{n}\binom{-a}{k}\frac{2^k}{\omega^k}(j)_k S^{II}(n,j)(x)_{j-k}^{II}, \tag{7.13}$$

which is easily derived from Eq. (7.11) and from the formula

$$\left(\frac{e^t-e^{-t}}{2}\right)^k x^n=\sum_{j=k}^{n}S^{II}(n,k)(j)_k(x)_{j-k}^{II}. \tag{7.14}$$

Finally, from the recurrence formula (7.6) (for $p=2$) we have

$$E_{n+1}^{II,a}(x)=\left(x-\frac{e^t+e^{-t}}{\omega+e^t-e^{-t}}\right)E_n^{II,a}(x),$$

providing another difference equation satisfied by $E_n^{II,a}(x)$,

$$E_{n+1}^{II,a}(x)=xE_n^{II,a}(x)-\frac{1}{\omega}\bigl(E_n^{II,a+1}(x+1)+E_n^{II,a+1}(x-1)\bigr). \tag{7.15}$$

Other cases can be studied similarly.

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