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On sums of Apéry polynomials and related congruences ☆

Zhi-Wei Sun

Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China

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ABSTRACT

The Apéry polynomials are given by

$$A_n(x) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 x^k \quad (n = 0, 1, 2, \dots).$$

(Those $A_n = A_n(1)$ are Apéry numbers.) Let p be an odd prime. We show that

$$\sum_{k=0}^{p-1} (-1)^k A_k(x) \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{16^k} x^k \pmod{p^2},$$

and that

$$\sum_{k=0}^{p-1} A_k(x) \equiv \left(\frac{x}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{4k}{k,k,k,k}}{(256x)^k} \pmod{p}$$

for any p -adic integer $x \not\equiv 0 \pmod{p}$. This enables us to determine explicitly $\sum_{k=0}^{p-1} (\pm 1)^k A_k \pmod{p}$, and $\sum_{k=0}^{p-1} (-1)^k A_k \pmod{p^2}$ in the case $p \equiv 2 \pmod{3}$. Another consequence states that

$$\begin{aligned} & \sum_{k=0}^{p-1} (-1)^k A_k(-2) \\ & \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 4y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \end{aligned}$$

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E-mail address: zwsun@nju.edu.cn.

URL: <http://math.nju.edu.cn/~zwsun>.

We also prove that for any prime $p > 3$ we have

$$\sum_{k=0}^{p-1} (2k+1)A_k \equiv p + \frac{7}{6}p^4 B_{p-3} \pmod{p^5}$$

where B_0, B_1, B_2, \dots are Bernoulli numbers.

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1. Introduction

The well-known Apéry numbers given by

$$A_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^n \binom{n+k}{2k}^2 \binom{2k}{k}^2 \quad (n \in \mathbb{N} = \{0, 1, 2, \dots\})$$

play a central role in Apéry's proof of the irrationality of $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$ (see Apéry [Ap] and van der Poorten [Po]). They also have close connections to modular forms (cf. Ono [O, pp. 198–203]). The Dedekind eta function in the theory of modular forms is defined by

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \quad \text{with } q = e^{2\pi i \tau},$$

where $\tau \in \mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ and hence $|q| < 1$. In 1987 Beukers [B] conjectured that

$$A_{(p-1)/2} \equiv a(p) \pmod{p^2} \quad \text{for any prime } p > 3,$$

where $a(n)$ ($n = 1, 2, 3, \dots$) are given by

$$\eta^4(2\tau)\eta^4(4\tau) = q \prod_{n=1}^{\infty} (1 - q^{2n})^4 (1 - q^{4n})^4 = \sum_{n=1}^{\infty} a(n)q^n.$$

This was finally confirmed by Ahlgren and Ono [AO] in 2000.

We define Apéry polynomials by

$$A_n(x) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 x^k = \sum_{k=0}^n \binom{n+k}{2k}^2 \binom{2k}{k}^2 x^k \quad (n \in \mathbb{N}). \quad (1.1)$$

Clearly $A_n(1) = A_n$. Motivated by the Apéry polynomials, we also introduce a new kind of polynomials:

$$W_n(x) := \sum_{k=0}^n \binom{n}{k}^2 \binom{n-k}{k}^2 x^k = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}^2 \binom{2k}{k}^2 x^k \quad (n \in \mathbb{N}). \quad (1.2)$$

Recall that Bernoulli numbers B_0, B_1, B_2, \dots are rational numbers given by

$$B_0 = 1 \quad \text{and} \quad \sum_{k=0}^n \binom{n+1}{k} B_k = 0 \quad \text{for } n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}.$$

It is well known that $B_{2n+1} = 0$ for all $n \in \mathbb{Z}^+$ and

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \quad (|x| < 2\pi).$$

Also, Euler numbers E_0, E_1, E_2, \dots are integers defined by

$$E_0 = 1 \quad \text{and} \quad \sum_{\substack{k=0 \\ 2|k}}^n \binom{n}{k} E_{n-k} = 0 \quad \text{for } n \in \mathbb{Z}^+.$$

It is well known that $E_{2n+1} = 0$ for all $n \in \mathbb{N}$ and

$$\sec x = \sum_{n=0}^{\infty} (-1)^n E_{2n} \frac{x^{2n}}{(2n)!} \quad \left(|x| < \frac{\pi}{2}\right).$$

Now we state our first theorem.

Theorem 1.1.

(i) Let p be an odd prime. Then

$$\sum_{k=0}^{p-1} (-1)^k A_k(x) \equiv \sum_{k=0}^{p-1} (-1)^k W_k(-x) \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{16^k} x^k \pmod{p^2}. \quad (1.3)$$

Also, for any p -adic integer $x \not\equiv 0 \pmod{p}$, we have

$$\begin{aligned} \sum_{k=0}^{p-1} A_k(x) &\equiv \sum_{k=0}^{p-1} W_k(x) \pmod{p^2} \\ &\equiv \left(\frac{x}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{4k}{k,k,k,k}}{(256x)^k} \pmod{p}, \end{aligned} \quad (1.4)$$

where $(-)$ denotes the Legendre symbol.

(ii) For any positive integer n we have

$$\frac{1}{n} \sum_{k=0}^{n-1} (2k+1) A_k(x) = \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{n+k}{k} \binom{n+k}{2k+1} \binom{2k}{k} x^k. \quad (1.5)$$

If $p > 3$ is a prime, then

$$\sum_{k=0}^{p-1} (2k+1)A_k \equiv p + \frac{7}{6}p^4 B_{p-3} \pmod{p^5} \quad (1.6)$$

and

$$\sum_{k=0}^{p-1} (2k+1)A_k(-1) \equiv \left(\frac{-1}{p}\right)p - p^3 E_{p-3} \pmod{p^4}. \quad (1.7)$$

(iii) Given $\varepsilon \in \{\pm 1\}$ and $m \in \mathbb{Z}^+$, for any prime p we have

$$\sum_{k=0}^{p-1} (2k+1)\varepsilon^k A_k^m \equiv 0 \pmod{p}.$$

Remark 1.1. (i) Let p be an odd prime. The author [Su1, Su2] had conjectures on $\sum_{k=0}^{p-1} \binom{2k}{k}^3 / m^k \pmod{p^2}$ with $m = 1, -8, 16, -64, 256, -512, 4096$. Motivated by the author's conjectures on $\sum_{k=0}^{p-1} A_k(x) \pmod{p^2}$ with $x = 1, -4, 9$ in an initial version of this paper, Guo and Zeng [GZ, Theorem 1.3] recently showed that

$$\sum_{k=0}^{p-1} A_k(x) \equiv \sum_{k=0}^{(p-1)/2} \binom{p+2k}{4k+1} \binom{2k}{k}^2 x^k \pmod{p^2}.$$

(ii) The values of

$$s_n = \frac{1}{n} \sum_{k=0}^{n-1} (2k+1)A_k \in \mathbb{Z}$$

with $n = 1, \dots, 8$ are

$$1, 8, 127, 2624, 61501, 1552760, 41186755, 1131614720$$

respectively. On June 6, 2011 Richard Penner informed the author an interesting application of (1.5): (1.5) with $x = 1$ implies that s_n is the trace of the inverse of nH_n where H_n refers to the Hilbert matrix $(\frac{1}{i+j-1})_{1 \leq i, j \leq n}$.

Can we find integers a_0, a_1, a_2, \dots such that $\sum_{k=0}^{p-1} a_k \equiv 4x^2 - 2p \pmod{p^2}$ if $p = x^2 + y^2$ is a prime with x odd and y even? The following corollary provides an affirmative answer!

Corollary 1.1. Let p be any odd prime. Then

$$\begin{aligned} \sum_{k=0}^{p-1} (-1)^k A_k(-2) &\equiv \sum_{k=0}^{p-1} (-1)^k A_k\left(\frac{1}{4}\right) \\ &\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{4} \text{ \& } p = x^2 + y^2 \text{ (} 2 \nmid x \text{),} \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \end{aligned} \quad (1.8)$$

Proof. It is known (cf. Ishikawa [I]) that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{64^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{4} \text{ \& } p = x^2 + y^2 \ (2 \nmid x), \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

The author conjectured that we can replace 64^k by $(-8)^k$ in the congruence, and this was recently confirmed by Z.H. Sun [S3]. So, applying (1.3) with $x = -2, 1/4$ we obtain (1.8). \square

Corollary 1.2. Let p be an odd prime. Then

$$\sum_{k=0}^{p-1} A_k \equiv c(p) \pmod{p} \quad (1.9)$$

where

$$c(p) := \begin{cases} 4x^2 - 2p & \text{if } p \equiv 1, 3 \pmod{8} \text{ \& } p = x^2 + 2y^2 \ (x, y \in \mathbb{Z}), \\ 0 & \text{if } \left(\frac{-2}{p}\right) = -1, \text{ i.e., } p \equiv 5, 7 \pmod{8}. \end{cases}$$

Also,

$$\begin{aligned} \sum_{k=0}^{p-1} (-1)^k A_k &\equiv \left(\frac{-1}{p}\right) \sum_{k=0}^{p-1} (-1)^k A_k \left(\frac{1}{16}\right) \\ &\equiv \begin{cases} 4x^2 - 2p \pmod{p} & \text{if } p \equiv 1 \pmod{3} \text{ and } p = x^2 + 3y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases} \end{aligned} \quad (1.10)$$

Proof. By [M05] and [Su4], we have

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{k,k,k,k}}{256^k} \equiv c(p) \pmod{p^2}$$

as conjectured in [RV]. (Here we only need the mod p version which was proved in [M05].) So (1.9) follows from (1.4). The author [Su2] conjectured that

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{16^k} &\equiv \left(\frac{-1}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{256^k} \\ &\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{3} \text{ and } p = x^2 + 3y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases} \end{aligned}$$

This was confirmed by Z.H. Sun [S3] in the case $p \equiv 2 \pmod{3}$, and the mod p version in the case $p \equiv 1 \pmod{3}$ follows from (4)–(5) in Ahlgren [A, Theorem 5]. So we get (1.10) by applying (1.3) with $x = 1, 1/16$. \square

Remark 1.2. The author conjectured that (1.9) also holds modulo p^2 , and that (1.10) is also valid modulo p^2 in the case $p \equiv 1 \pmod{3}$.

Corollary 1.3. For any odd prime p and integer x , we have

$$\sum_{k=0}^{p-1} (2k+1)A_k(x) \equiv p \left(\frac{x}{p} \right) \pmod{p^2}. \quad (1.11)$$

Proof. This follows from (1.5) in the case $n = p$, for, $p \mid \binom{p+k}{2k+1}$ for every $k = 0, \dots, (p-3)/2$, and $p \mid \binom{2k}{k}$ for all $k = (p+1)/2, \dots, p-1$. \square

We deduce Theorem 1.1(i) from our following result which has its own interest.

Theorem 1.2. Let p be an odd prime and let x be any p -adic integer.

(i) If $x \equiv 2k \pmod{p}$ with $k \in \{0, \dots, (p-1)/2\}$, then we have

$$\sum_{r=0}^{p-1} (-1)^r \binom{x}{r}^2 \equiv (-1)^k \binom{x}{k} \pmod{p^2}. \quad (1.12)$$

(ii) If $x \equiv k \pmod{p}$ with $k \in \{0, \dots, p-1\}$, then

$$\sum_{r=0}^{p-1} \binom{x}{r}^2 \equiv \binom{2x}{k} \pmod{p^2}. \quad (1.13)$$

Remark 1.3. In contrast with (1.12) and (1.13), we recall the following identities (cf. [G, (3.32) and (3.66)]):

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^2 = (-1)^n \binom{2n}{n} \quad \text{and} \quad \sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}.$$

Corollary 1.4. Let p be an odd prime.

(i) (Conjectured in [RV] and proved in [M03].) We have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p} \right) \pmod{p^2}.$$

(ii) (Conjectured by the author [Su1] and confirmed in [S2].) If $p \equiv 1 \pmod{4}$ and $p = x^2 + y^2$ with $x \equiv 1 \pmod{4}$ and $y \equiv 0 \pmod{2}$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(-16)^k} \equiv (-1)^{(p-1)/4} \left(2x - \frac{p}{2x} \right) \pmod{p^2}. \quad (1.14)$$

Proof. Since $\binom{-1/2}{r} = \binom{2r}{r}/(-4)^r$ for all $r = 0, 1, \dots$, applying (1.13) with $x = -1/2$ and $k = (p-1)/2$ we immediately get the congruence in part (i).

When $p = x^2 + y^2$ with $x \equiv 1 \pmod{4}$ and $y \equiv 0 \pmod{2}$, by (1.12) with $x = -1/2$ and $k = (p-1)/4$ we have

$$\begin{aligned}
\sum_{r=0}^{p-1} \frac{\binom{2r}{r}^2}{(-16)^r} &\equiv (-1)^{(p-1)/4} \binom{-1/2}{(p-1)/4} = \frac{\binom{(p-1)/2}{(p-1)/4}}{4^{(p-1)/4}} = \frac{\binom{(p-1)/2}{(p-1)/4}}{2^{(p-1)/2}} \\
&\equiv \frac{2^{p-1} + 1}{2 \times 2^{(p-1)/2}} \left(2x - \frac{p}{2x} \right) \pmod{p^2} \quad (\text{by [CDE] or [BEW, (9.0.2)]}) \\
&\equiv (-1)^{(p-1)/4} \left(2x - \frac{p}{2x} \right) \pmod{p^2}
\end{aligned}$$

since $((-1)^{(p-1)/4} 2^{(p-1)/2} - 1)^2 \equiv 0 \pmod{p^2}$. This proves (1.14). \square

Corollary 1.5. Let $a_n := \sum_{k=0}^n \binom{n}{k}^2 C_k$ for $n = 0, 1, 2, \dots$, where C_k denotes the Catalan number $\binom{2k}{k}/(k+1) = \binom{2k}{k} - \binom{2k}{k+1}$. Then, for any odd prime p we have

$$a_1 + \dots + a_{p-1} \equiv 0 \pmod{p^2}. \quad (1.15)$$

Remark 1.4. We find no prime $p \leq 5000$ with $\sum_{k=1}^{p-1} a_k \equiv 0 \pmod{p^3}$ and no composite number $n \leq 70\,000$ satisfying $\sum_{k=1}^{n-1} a_k \equiv 0 \pmod{n^2}$. We conjecture that (1.15) holds for no composite $p > 1$.

The author [Su1, Remark 1.2] conjectured that for any prime $p > 5$ with $p \equiv 1 \pmod{4}$ we have

$$\sum_{k=0}^{p^a-1} \frac{k^3 \binom{2k}{k}^3}{64^k} \equiv 0 \pmod{p^{2a}} \quad \text{for } a = 1, 2, 3, \dots$$

This was recently confirmed by Z.H. Sun [S3] in the case $a = 1$. Note that

$$\frac{k^3 \binom{2k}{k}^3}{64^k} = (-1)^k k^3 \binom{-1/2}{k}^3 = \frac{(-1)^{k-1}}{8} \binom{-3/2}{k-1}^3 \quad \text{for all } k = 1, 2, 3, \dots$$

So, for any prime $p > 5$ with $p \equiv 1 \pmod{4}$ we have

$$\sum_{r=0}^{p-1} (-1)^r \binom{-3/2}{r}^3 \equiv 0 \pmod{p^2}.$$

Since $-3/2 \equiv -2(p+3)/4 \pmod{p}$, the result just corresponds to the case $x = -3/2$ of our following general theorem.

Theorem 1.3. Let $p > 3$ be a prime and let x be a p -adic integer with $x \equiv -2k \pmod{p}$ for some $k \in \{1, \dots, \lfloor (p-1)/3 \rfloor\}$. Then we have

$$\sum_{r=0}^{p-1} (-1)^r \binom{x}{r}^3 \equiv 0 \pmod{p^2}. \quad (1.16)$$

Similar to Apéry numbers, the central Delannoy numbers (see [CHV]) are defined by

$$D_n = \sum_{k=0}^n \binom{n+k}{2k} \binom{2k}{k} = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \quad (n \in \mathbb{N}).$$

Such numbers arise naturally in many enumeration problems in combinatorics (cf. Sloane [S]); for example, D_n is the number of lattice paths from $(0, 0)$ to (n, n) with steps $(1, 0)$, $(0, 1)$ and $(1, 1)$.

Now we give our result on central Delannoy numbers.

Theorem 1.4. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} D_k \equiv \left(\frac{-1}{p}\right) - p^2 E_{p-3} \pmod{p^3}. \quad (1.17)$$

We also have

$$\sum_{k=0}^{p-1} (2k+1)(-1)^k D_k \equiv p - \frac{7}{12} p^4 B_{p-3} \pmod{p^5} \quad (1.18)$$

and

$$\sum_{k=0}^{p-1} (2k+1) D_k \equiv p + 2p^2 q_p(2) - p^3 q_p(2)^2 \pmod{p^4}, \quad (1.19)$$

where $q_p(2)$ denotes the Fermat quotient $(2^{p-1} - 1)/p$.

Remark 1.5. In [Su3] the author determined $\sum_{k=1}^{p-1} D_k/k$ and $\sum_{k=1}^{p-1} D_k/k^2$ modulo an odd prime p .

In the next section we will show Theorems 1.1–1.2 and Corollary 1.5. Section 3 is devoted to our proofs of Theorems 1.3 and 1.4. In Section 4 we are going to raise some related conjectures for further research.

2. Proofs of Theorems 1.1–1.2 and Corollary 1.5

We first prove Theorem 1.2.

Proof of Theorem 1.2. (i) We now consider the first part of Theorem 1.2. Set

$$f_k(y) := \sum_{r=0}^{p-1} (-1)^r \binom{2k+py}{r}^2 \quad \text{for } k \in \mathbb{N}. \quad (2.1)$$

We want to prove that

$$f_k(y) \equiv (-1)^k \binom{2k+py}{k} \pmod{p^2} \quad (2.2)$$

for any p -adic integer y and $k \in \{0, 1, \dots, (p-1)/2\}$.

Applying the Zeilberger algorithm (cf. [PWZ]) via Mathematica 7, we find that

$$\begin{aligned} & (py + 2k + 2)f_{k+1}(y) + 4(py + 2k + 1)f_k(y) \\ &= \frac{(p(y-1) + 2k + 3)^2 F_k(y)}{(py + 2k + 1)(py + 2k + 2)^2} \binom{py + 2k + 2}{p-1}^2, \end{aligned} \quad (2.3)$$

where

$$F_k(y) = 14 + 34k + 20k^2 - 10p - 12kp + 2p^2 + 17py + 20kpy - 6p^2y + 5p^2y^2.$$

Now fix a p -adic integer y . Observe that

$$\begin{aligned} f_{(p-1)/2}(y) &= \sum_{r=0}^{p-1} (-1)^r \binom{p-1+py}{r}^2 = \sum_{r=0}^{p-1} (-1)^r \prod_{0 < s \leq r} \left(1 - \frac{p(y+1)}{s}\right)^2 \\ &\equiv \sum_{r=0}^{p-1} (-1)^r \left(1 - \sum_{0 < s \leq r} \frac{2p(y+1)}{s}\right) = 1 - \sum_{r=1}^{p-1} (-1)^r \sum_{s=1}^r \frac{2p(y+1)}{s} \\ &= 1 - 2p(y+1) \sum_{s=1}^{p-1} \frac{1}{s} \sum_{r=s}^{p-1} (-1)^r = 1 - p(y+1) \sum_{j=1}^{(p-1)/2} \frac{1}{j} \\ &\equiv (-1)^{(p-1)/2} \binom{p-1+py}{(p-1)/2} \pmod{p^2}. \end{aligned}$$

For each $k \in \{0, \dots, (p-3)/2\}$, clearly $py + 2k + 1, py + 2k + 2 \not\equiv 0 \pmod{p}$, and also

$$(p(y-1) + 2k + 3) \binom{py + 2k + 2}{p-1}^2 \equiv 0 \pmod{p^2}$$

since $\binom{py+2k+2}{p-1} = \frac{p}{py+2k+3} \binom{py+2k+3}{p} \equiv 0 \pmod{p}$ if $0 \leq k < (p-3)/2$. Thus, by (2.3) we have

$$f_k(y) \equiv -\frac{py + 2k + 2}{4(py + 2k + 1)} f_{k+1}(y) \pmod{p^2} \quad \text{for } k = 0, \dots, \frac{p-3}{2}.$$

If $0 \leq k < (p-1)/2$ and

$$f_{k+1}(y) \equiv (-1)^{k+1} \binom{2(k+1)+py}{k+1} \pmod{p^2},$$

then

$$\begin{aligned} f_k(y) &\equiv -\frac{py + 2k + 2}{4(py + 2k + 1)} (-1)^{k+1} \binom{2(k+1)+py}{k+1} \\ &= \frac{(-1)^k (py + 2k + 2)^2}{4(k+1)(py + k + 1)} \binom{2k+py}{k} \equiv (-1)^k \binom{2k+py}{k} \pmod{p^2}. \end{aligned}$$

Therefore (2.2) holds for all $k = 0, 1, \dots, (p-1)/2$. This proves Theorem 1.2(i).

(ii) The second part of Theorem 1.2 can be proved in a similar way. Here we mention that if we define

$$g_k(y) := \sum_{r=0}^{p-1} \binom{k+py}{r}^2 \quad \text{for } k \in \mathbb{N}, \quad (2.4)$$

then by the Zeilberger algorithm (cf. [PWZ]) we have the recursion

$$\begin{aligned} & (py + k + 1)g_{k+1}(y) - 2(2py + 2k + 1)g_k(y) \\ &= -\frac{(p(y-1) + k + 2)^2(3py - 2p + 3k + 3)}{(py + k + 1)^2} \binom{py + k + 1}{p-1}^2. \end{aligned}$$

It follows that if $k \in \{0, \dots, p-2\}$ and y is a p -adic integer then

$$\begin{aligned} g_{k+1}(y) &\equiv \binom{2(k+1) + 2py}{k+1} \pmod{p^2} \\ \implies g_k(y) &\equiv \binom{2k + 2py}{k} \pmod{p^2}. \end{aligned} \quad (2.5)$$

In view of this, we have the second part of Theorem 1.2 by induction.

The proof of Theorem 1.2 is now complete. \square

Proof of Corollary 1.5. Observe that

$$\sum_{n=0}^{p-1} a_n = \sum_{k=0}^{p-1} C_k \sum_{n=k}^{p-1} \binom{n}{k}^2 = \sum_{k=0}^{p-1} C_k \sum_{j=0}^{p-1-k} \binom{k+j}{k}^2.$$

If $0 \leq k \leq p-1$ and $p-k \leq j \leq p-1$, then

$$\binom{k+j}{k} = \frac{(k+j)!}{k!j!} \equiv 0 \pmod{p}.$$

Therefore

$$\sum_{n=0}^{p-1} a_n \equiv \sum_{k=0}^{p-1} C_k \sum_{j=0}^{p-1} \binom{k+j}{k}^2 = \sum_{k=0}^{p-1} C_k \sum_{j=0}^{p-1} \binom{x_k}{j}^2,$$

where $x_k = -k-1 \equiv p-1-k \pmod{p}$. Applying Theorem 1.2(ii) we get

$$\sum_{n=0}^{p-1} a_n \equiv \sum_{k=0}^{p-1} C_k \binom{2x_k}{p-1-k} = \sum_{k=0}^{p-1} (-1)^k \binom{p+k}{2k+1} C_k \pmod{p^2}.$$

So it suffices to show that for any $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1} (-1)^k \binom{n+k}{2k+1} C_k = 1. \quad (2.6)$$

We prove (2.6) by induction. Clearly, (2.6) holds for $n=1$. Let n be any positive integer. By the Chu–Vandermonde identity

$$\sum_{k=0}^n \binom{x}{k} \binom{y}{n-k} = \binom{x+y}{n}$$

(see, e.g., [GKP, p. 169]), we have

$$\sum_{k=0}^{n-1} \binom{n+1}{k+1} \binom{n+k}{k} (-1)^k = \sum_{k=0}^{n-1} \binom{n+1}{n-k} \binom{-n-1}{k} = -\binom{-n-1}{n}.$$

Thus

$$\begin{aligned} \sum_{k=0}^n (-1)^k \binom{n+1+k}{2k+1} C_k - \sum_{k=0}^{n-1} (-1)^k \binom{n+k}{2k+1} C_k &= (-1)^n C_n + \sum_{k=0}^{n-1} (-1)^k \binom{n+k}{2k} C_k \\ &= (-1)^n C_n + \frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k+1} \binom{n+k}{k} (-1)^k \\ &= (-1)^n C_n - \frac{1}{n+1} \binom{-n-1}{n} = 0. \end{aligned}$$

This concludes the induction step. We are done. \square

Now we can apply Theorem 1.2 to deduce the first part of Theorem 1.1.

Proof of Theorem 1.1(i). Let $\varepsilon \in \{\pm 1\}$. Then

$$\begin{aligned} \sum_{m=0}^{p-1} \varepsilon^m A_m(x) &= \sum_{m=0}^{p-1} \varepsilon^m \sum_{k=0}^m \binom{m+k}{2k}^2 \binom{2k}{k}^2 x^k \\ &= \sum_{k=0}^{p-1} \binom{2k}{k}^2 x^k \sum_{m=k}^{p-1} \varepsilon^m \binom{m+k}{2k}^2 \\ &= \sum_{k=0}^{p-1} \binom{2k}{k}^2 x^k \sum_{r=0}^{p-1-k} \varepsilon^{k+r} \binom{2k+r}{r}^2 \\ &= \sum_{k=0}^{p-1} \binom{2k}{k}^2 \varepsilon^k x^k \sum_{r=0}^{p-1-k} \varepsilon^r \binom{p-1-2k-p}{r}^2. \end{aligned}$$

Set $n = (p-1)/2$. Clearly $\binom{2k}{k} \equiv 0 \pmod{p}$ for $k = n+1, \dots, p-1$, and

$$\binom{p-1-2k-p}{r} \equiv \binom{p-1-2k}{r} = 0 \pmod{p}$$

if $0 \leq k \leq n$ and $p-1-2k < r \leq p-1$. Therefore

$$\sum_{m=0}^{p-1} \varepsilon^m A_m(x) \equiv \sum_{k=0}^n \binom{2k}{k}^2 \varepsilon^k x^k \sum_{r=0}^{p-1-k} \varepsilon^r \binom{2(n-k)-p}{r}^2 \pmod{p^2}.$$

Similarly,

$$\begin{aligned}
\sum_{m=0}^{p-1} \varepsilon^m W_m(\varepsilon x) &= \sum_{m=0}^{p-1} \varepsilon^m \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m}{2k}^2 \binom{2k}{k}^2 (\varepsilon x)^k \\
&= \sum_{k=0}^n \binom{2k}{k}^2 \varepsilon^k x^k \sum_{m=2k}^{p-1} \varepsilon^m \binom{m}{2k}^2 \\
&= \sum_{k=0}^n \binom{2k}{k}^2 \varepsilon^k x^k \sum_{r=0}^{p-1-2k} \varepsilon^{2k+r} \binom{2k+r}{r}^2 \\
&\equiv \sum_{k=0}^n \binom{2k}{k}^2 \varepsilon^k x^k \sum_{r=0}^{p-1} \varepsilon^r \binom{2(n-k)-p}{r}^2 \pmod{p^2}.
\end{aligned}$$

So we have

$$\sum_{m=0}^{p-1} \varepsilon^m A_m(x) \equiv \sum_{m=0}^{p-1} \varepsilon^m W_m(\varepsilon x) \equiv \sum_{k=0}^n \binom{2k}{k}^2 \varepsilon^k x^k S_k(\varepsilon) \pmod{p^2}, \quad (2.7)$$

where

$$S_k(\varepsilon) := \sum_{r=0}^{p-1} \varepsilon^r \binom{2(n-k)-p}{r}^2.$$

Applying Theorem 1.2(i) we get

$$\begin{aligned}
S_k(-1) &\equiv (-1)^{n-k} \binom{2(n-k)-p}{n-k} = (-1)^{n-k} \binom{-2k-1}{n-k} \\
&= \binom{n+k}{n-k} = \binom{n+k}{2k} \equiv \frac{\binom{2k}{k}}{(-16)^k} \pmod{p^2}.
\end{aligned}$$

(The last congruence can be easily deduced, see, e.g., [S2, Lemma 2.2].) Combining this with (2.7) in the case $\varepsilon = -1$ we immediately obtain (1.3).

In view of Theorem 1.2(ii),

$$S_k(1) \equiv \binom{4(n-k)-2p}{2(n-k)} \pmod{p^2}.$$

Recall that $\binom{n+k}{n-k}(-16)^k \equiv \binom{2k}{k} \pmod{p^2}$. So, in view of (2.7) with $\varepsilon = 1$, we have

$$\begin{aligned}
\sum_{m=0}^{p-1} A_m(x) &\equiv \sum_{m=0}^{p-1} W_m(x) \equiv \sum_{k=0}^n \binom{n+k}{n-k}^2 (-16)^{2k} x^k \binom{4(n-k)-2p}{2(n-k)} \\
&= \sum_{j=0}^n \binom{n+(n-j)}{j}^2 256^{n-j} x^{n-j} \binom{4j-2p}{2j} \\
&= 16^{p-1} \sum_{k=0}^n \frac{\binom{4k-2p}{2k} \binom{2k-p}{k}^2}{256^k} x^{n-k} \pmod{p^2}.
\end{aligned}$$

If x is a p -adic integer with $x \not\equiv 0 \pmod{p}$, then

$$16^{p-1} \sum_{k=0}^n \frac{\binom{4k-2p}{2k} \binom{2k-p}{k}^2}{256^k} x^{n-k} \equiv \left(\frac{x}{p}\right) \sum_{k=0}^n \frac{\binom{4k}{2k} \binom{2k}{k}^2}{(256x)^k} \equiv \left(\frac{x}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{4k}{k,k,k,k}}{(256x)^k} \pmod{p},$$

and therefore (1.4) holds. \square

Lemma 2.1. Let $k \in \mathbb{N}$. Then, for any $n \in \mathbb{Z}^+$ we have the identity

$$\sum_{m=0}^{n-1} (2m+1) \binom{m+k}{2k}^2 = \frac{(n-k)^2}{2k+1} \binom{n+k}{2k}^2. \quad (2.8)$$

Proof. Obviously (2.8) holds when $n = 1$.

Now assume that $n > 1$ and (2.8) holds. Then

$$\begin{aligned} \sum_{m=0}^n (2m+1) \binom{m+k}{2k}^2 &= \frac{(n-k)^2}{2k+1} \binom{n+k}{2k}^2 + (2n+1) \binom{n+k}{2k}^2 \\ &= \frac{(n+k+1)^2}{2k+1} \binom{n+k}{2k}^2 = \frac{(n+1-k)^2}{2k+1} \binom{(n+1)+k}{2k}^2. \end{aligned}$$

Combining the above, we have proved the desired result by induction. \square

Lemma 2.2. Let $p > 3$ be a prime. Then

$$\sum_{\substack{k=0 \\ k \neq (p-1)/2}}^{p-1} \frac{(-1)^k}{2k+1} \equiv -pE_{p-3} \pmod{p^2}. \quad (2.9)$$

Proof. Observe that

$$\begin{aligned} \sum_{\substack{k=0 \\ k \neq (p-1)/2}}^{p-1} \frac{(-1)^k}{2k+1} &= \frac{1}{2} \sum_{\substack{k=0 \\ k \neq (p-1)/2}}^{p-1} \left(\frac{(-1)^k}{2k+1} + \frac{(-1)^{p-1-k}}{(2(p-1-k)+1)} \right) \\ &= -p \sum_{\substack{k=0 \\ k \neq (p-1)/2}}^{p-1} \frac{(-1)^k}{(2k+1)(2k+1-2p)} \\ &\equiv -\frac{p}{4} \sum_{k=0}^{p-1} (-1)^k \left(k + \frac{1}{2}\right)^{p-3} \pmod{p^2}. \end{aligned}$$

So we have reduced (2.9) to the following congruence

$$\sum_{k=0}^{p-1} (-1)^k \left(k + \frac{1}{2}\right)^{p-3} \equiv 4E_{p-3} \pmod{p}. \quad (2.10)$$

Recall that the Euler polynomial of degree n is defined by

$$E_n(x) = \sum_{k=0}^n \binom{n}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{n-k}.$$

It is well known that

$$E_n(x) + E_n(x+1) = 2x^n.$$

Thus

$$\begin{aligned} 2 \sum_{k=0}^{p-1} (-1)^k \left(k + \frac{1}{2}\right)^{p-3} &= \sum_{k=0}^{p-1} \left((-1)^k E_{p-3} \left(k + \frac{1}{2}\right) - (-1)^{k+1} E_{p-3} \left(k + 1 + \frac{1}{2}\right) \right) \\ &= E_{p-3} \left(\frac{1}{2}\right) - (-1)^p E_{p-3} \left(p + \frac{1}{2}\right) \\ &\equiv 2E_{p-3} \left(\frac{1}{2}\right) = 2 \frac{E_{p-3}}{2^{p-3}} \equiv 8E_{p-3} \pmod{p} \end{aligned}$$

and hence (2.10) follows. We are done. \square

For each $m = 1, 2, 3, \dots$ those rational numbers

$$H_n^{(m)} := \sum_{0 < k \leq n} \frac{1}{k^m} \quad (n = 0, 1, 2, \dots)$$

are called harmonic numbers of order m . We simply write H_n for $H_n^{(1)}$. A well-known theorem of Wolstenholme asserts that $H_{p-1} \equiv 0 \pmod{p^2}$ and $H_{p-1}^{(2)} \equiv 0 \pmod{p}$ for any prime $p > 3$.

Lemma 2.3. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{(p-3)/2} \frac{H_k^{(2)}}{2k+1} \equiv -\frac{7}{4} B_{p-3} \pmod{p}. \quad (2.11)$$

Proof. Clearly,

$$\sum_{k=1}^{p-1} \frac{1}{k^3} = \sum_{k=1}^{(p-1)/2} \left(\frac{1}{k^3} + \frac{1}{(p-k)^3} \right) \equiv 0 \pmod{p}.$$

By [ST, (5.4)], $\sum_{k=1}^{p-1} H_k/k^2 \equiv B_{p-3} \pmod{p}$. Therefore

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{H_k^{(2)}}{k} &= \sum_{k=1}^{p-1} \frac{1}{k} \sum_{j=1}^k \frac{1}{j^2} = \sum_{j=1}^{p-1} \frac{H_{p-1} - H_{j-1}}{j^2} \\ &\equiv -\sum_{k=1}^{p-1} \frac{H_k}{k^2} + \sum_{k=1}^{p-1} \frac{1}{k^3} \equiv -B_{p-3} \pmod{p}. \end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{k=1}^{p-1} \frac{H_k^{(2)}}{k} &= \sum_{k=1}^{(p-1)/2} \left(\frac{H_k^{(2)}}{k} + \frac{H_{p-k}^{(2)}}{p-k} \right) \\ &\equiv \sum_{k=1}^{(p-1)/2} \left(\frac{H_k^{(2)}}{k} + \frac{H_{p-1}^{(2)} - H_{k-1}^{(2)}}{-k} \right) \equiv 2 \sum_{k=1}^{(p-1)/2} \frac{H_k^{(2)}}{k} - H_{(p-1)/2}^{(3)} \pmod{p}.\end{aligned}$$

It is known (see, e.g., [S1, Corollary 5.2]) that

$$H_{(p-1)/2}^{(3)} = \sum_{k=1}^{(p-1)/2} \frac{1}{k^3} \equiv -2B_{p-3} \pmod{p}.$$

So we have

$$\sum_{k=1}^{(p-1)/2} \frac{H_k^{(2)}}{k} \equiv \frac{1}{2} \left(\sum_{k=1}^{p-1} \frac{H_k^{(2)}}{k} + H_{(p-1)/2}^{(3)} \right) \equiv \frac{-B_{p-3} - 2B_{p-3}}{2} = -\frac{3}{2}B_{p-3} \pmod{p}.$$

Clearly

$$H_{(p-1)/2}^{(2)} \equiv \frac{1}{2} \sum_{k=1}^{(p-1)/2} \left(\frac{1}{k^2} + \frac{1}{(p-k)^2} \right) = \frac{1}{2} H_{p-1}^{(2)} \equiv 0 \pmod{p}.$$

Observe that

$$\begin{aligned}\sum_{k=0}^{(p-3)/2} \frac{H_k^{(2)}}{2k+1} &\equiv - \sum_{k=0}^{(p-3)/2} \frac{H_k^{(2)}}{p-1-2k} = - \sum_{k=1}^{(p-1)/2} \frac{H_{(p-1)/2-k}^{(2)}}{2k} \\ &\equiv -\frac{1}{2} \sum_{k=1}^{(p-1)/2} \frac{1}{k} \left(H_{(p-1)/2}^{(2)} - \sum_{j=0}^{k-1} \frac{1}{((p-1)/2-j)^2} \right) \\ &\equiv 2 \sum_{k=1}^{(p-1)/2} \frac{1}{k} \sum_{j=0}^{k-1} \frac{1}{(2j+1)^2} \equiv 2 \sum_{k=1}^{(p-1)/2} \frac{1}{k} \left(H_{2k}^{(2)} - \sum_{j=1}^k \frac{1}{(2j)^2} \right) \\ &= 4 \sum_{k=1}^{(p-1)/2} \frac{H_{2k}^{(2)}}{2k} - \frac{1}{2} \sum_{k=1}^{(p-1)/2} \frac{H_k^{(2)}}{k} \pmod{p}\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^{(p-1)/2} \frac{H_{2k}^{(2)}}{2k} &= \sum_{\substack{k=1 \\ 2|k}}^{p-1} \frac{1}{k} \sum_{j=1}^k \frac{1}{j^2} = \sum_{\substack{k=1 \\ 2|k}}^{p-1} \frac{1}{k^3} + \sum_{\substack{1 \leq j < k \leq p-1 \\ 2|k}} \frac{1}{j^2 k} \\ &\equiv \frac{1}{8} H_{(p-1)/2}^{(3)} - \frac{3}{8} B_{p-3} \quad (\text{by Pan [P, (2.4)]})\end{aligned}$$

$$\equiv \frac{1}{8}(-2B_{p-3}) - \frac{3}{8}B_{p-3} = -\frac{5}{8}B_{p-3} \pmod{p}.$$

So we finally get

$$\sum_{k=0}^{(p-3)/2} \frac{H_k^{(2)}}{2k+1} \equiv 4\left(-\frac{5}{8}B_{p-3}\right) - \frac{1}{2}\left(-\frac{3}{2}B_{p-3}\right) = -\frac{7}{4}B_{p-3} \pmod{p}.$$

This concludes the proof of (2.11). \square

Proof of Theorem 1.1(ii). (i) Let n be any positive integer. Then

$$\begin{aligned} \sum_{m=0}^{n-1} (2m+1)A_m(x) &= \sum_{m=0}^{n-1} (2m+1) \sum_{k=0}^m \binom{m+k}{2k}^2 \binom{2k}{k}^2 x^k \\ &= \sum_{k=0}^{n-1} \binom{2k}{k}^2 x^k \sum_{m=0}^{n-1} (2m+1) \binom{m+k}{2k}^2 \\ &= \sum_{k=0}^{n-1} \binom{2k}{k}^2 x^k \frac{(n-k)^2}{2k+1} \binom{n+k}{2k}^2 \quad (\text{by (2.8)}) \\ &= \sum_{k=0}^{n-1} \frac{(n-k)^2}{2k+1} \binom{n}{k}^2 \binom{n+k}{k}^2 x^k. \end{aligned}$$

Since

$$(n-k) \binom{n}{k} = n \binom{n-1}{k} \quad \text{for all } k = 0, \dots, n-1,$$

we have

$$\begin{aligned} \frac{1}{n} \sum_{m=0}^{n-1} (2m+1)A_m(x) &= \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{n-k}{2k+1} \binom{n}{k} \binom{n+k}{k}^2 x^k \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{n-k}{2k+1} \binom{n+k}{2k} \binom{2k}{k} \binom{n+k}{k} x^k \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{n+k}{k} \binom{n+k}{2k+1} \binom{2k}{k} x^k. \end{aligned}$$

This proves (1.5).

Now we fix a prime $p > 3$. By the above,

$$\sum_{m=0}^{p-1} (2m+1)A_m(x) = \sum_{k=0}^{p-1} \frac{p^2}{2k+1} \binom{p-1}{k}^2 \binom{p+k}{k}^2 x^k. \quad (2.12)$$

For $k \in \{0, \dots, p-1\}$, clearly

$$\begin{aligned} \binom{p-1}{k}^2 \binom{p+k}{k}^2 &= \prod_{0 < j \leq k} \left(\frac{p-j}{j} \cdot \frac{p+j}{j} \right)^2 = \prod_{0 < j \leq k} \left(1 - \frac{p^2}{j^2} \right)^2 \\ &\equiv \prod_{0 < j \leq k} \left(1 - \frac{2p^2}{j^2} \right) \equiv 1 - 2p^2 H_k^{(2)} \pmod{p^4}. \end{aligned}$$

Thus (2.12) implies that

$$\sum_{m=0}^{p-1} (2m+1) A_m(x) = \sum_{k=0}^{p-1} \frac{p^2}{2k+1} (1 - 2p^2 H_k^{(2)}) x^k \pmod{p^5}. \quad (2.13)$$

Since $H_{(p-1)/2}^{(2)} \equiv 0 \pmod{p}$, taking $x = -1$ in (2.13) and applying (2.9) we obtain

$$\sum_{m=0}^{p-1} (2m+1) A_m(-1) \equiv \sum_{k=0}^{p-1} \frac{p^2 (-1)^k}{2k+1} \equiv \frac{p^2 (-1)^{(p-1)/2}}{2(p-1)/2+1} - p^3 E_{p-3} \pmod{p^4}$$

and hence (1.7) holds.

Now we prove (1.6). In view of (2.13) with $x = 1$, we have

$$\begin{aligned} \sum_{m=0}^{p-1} (2m+1) A_m &\equiv \frac{p^2}{2(p-1)/2+1} (1 - 2p^2 H_{(p-1)/2}^{(2)}) \\ &\quad + p^2 \sum_{k=0}^{(p-3)/2} \left(\frac{1 - 2p^2 H_k^{(2)}}{2k+1} + \frac{1 - 2p^2 H_{p-1-k}^{(2)}}{2(p-1-k)+1} \right) \\ &= p - 2p^3 H_{(p-1)/2}^{(2)} + 2p^3 \sum_{k=0}^{(p-3)/2} \frac{2p+2k+1}{(2k+1)(4p^2 - (2k+1)^2)} \\ &\quad - 2p^4 \sum_{k=0}^{(p-3)/2} \left(\frac{H_k^{(2)}}{2k+1} + \frac{H_{p-1}^{(2)} - \sum_{0 < j \leq k} (p-j)^{-2}}{2p - (2k+1)} \right) \\ &\equiv p - 2p^3 H_{(p-1)/2}^{(2)} - 4p^4 \sum_{k=0}^{(p-3)/2} \frac{1}{(2k+1)^3} \\ &\quad - 2p^3 \sum_{k=0}^{(p-3)/2} \frac{1}{(2k+1)^2} - 4p^4 \sum_{k=0}^{(p-3)/2} \frac{H_k^{(2)}}{2k+1} \pmod{p^5}. \end{aligned}$$

By [S1, Corollaries 5.1 and 5.2],

$$\begin{aligned} H_{p-1}^{(2)} &\equiv \frac{2}{3} p B_{p-3} \pmod{p^2}, \quad H_{(p-1)/2}^{(2)} \equiv \frac{7}{3} p B_{p-3} \pmod{p^2}, \\ \sum_{k=0}^{(p-3)/2} \frac{1}{(2k+1)^2} &= H_{p-1}^{(2)} - \frac{H_{(p-1)/2}^{(2)}}{4} \equiv \frac{p}{12} B_{p-3} \pmod{p^2}, \end{aligned}$$

and

$$\sum_{k=0}^{(p-3)/2} \frac{1}{(2k+1)^3} = H_{p-1}^{(3)} - \frac{H_{(p-1)/2}^{(3)}}{8} \equiv 0 - \frac{-2B_{p-3}}{8} = \frac{B_{p-3}}{4} \pmod{p}.$$

Combining these with Lemma 2.3, we finally obtain

$$\begin{aligned} \sum_{k=0}^{p-1} (2k+1)A_k &\equiv p - 2p^3 \frac{7}{3} p B_{p-3} - 4p^4 \frac{B_{p-3}}{4} - 2p^3 \frac{p}{12} B_{p-3} - 4p^4 \left(-\frac{7}{4} B_{p-3}\right) \\ &= p + \frac{7}{6} p^4 B_{p-3} \pmod{p^5}. \end{aligned}$$

So far we have proved the second part of Theorem 1.1. \square

Part (iii) of Theorem 1.1 is easy.

Proof of Theorem 1.1(iii). As $A_0 = 1$ and $A_1 = 3$, the desired congruence with $p = 2$ holds trivially. Below we assume that p is an odd prime. If $k \in \{0, 1, \dots, p-1\}$, then

$$\begin{aligned} A_{p-1-k} &= \sum_{j=0}^{p-1} \binom{(p-1-k)+j}{2j}^2 \binom{2j}{j}^2 \\ &\equiv \sum_{j=0}^{p-1} \binom{j-k-1}{2j}^2 \binom{2j}{j}^2 = \sum_{j=0}^k \binom{j+k}{2j}^2 \binom{2j}{j}^2 = A_k \pmod{p}. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{k=0}^{p-1} (2k+1) \varepsilon^k A_k^m &= \sum_{k=0}^{p-1} (2(p-1-k)+1) \varepsilon^{p-1-k} A_{p-1-k}^m \\ &\equiv - \sum_{k=0}^{p-1} (2k+1) \varepsilon^k A_k^m \pmod{p} \end{aligned}$$

and hence we have the desired congruence. \square

3. Proofs of Theorems 1.3 and 1.4

Proof of Theorem 1.3. Define

$$w_k(y) := \sum_{r=0}^{p-1} (-1)^r \binom{py-2k}{r}^3 \quad \text{for } k \in \mathbb{N}. \quad (3.1)$$

We want to show that $w_k(y) \equiv 0 \pmod{p^2}$ for any p -adic integer y and $k \in \{1, \dots, \lfloor (p-1)/3 \rfloor\}$.

By the Zeilberger algorithm (cf. [PWZ]), for $k = 0, 1, 2, \dots$ we have

$$\begin{aligned} &(py-2k)^2 w_k(y) + 3(3py-2(3k+1))(3py-2(3k+2)) w_{k+1}(y) \\ &= \frac{P(k, p, y)(p(1-y)+2k-1)^3}{(py-2k)^3(py-2k-1)^3} \binom{py-2k}{p-1}^3 \end{aligned} \quad (3.2)$$

where $P(k, p, y)$ is a suitable polynomial in k, p, y with integer coefficients such that $P(0, p, y) \equiv 0 \pmod{p^2}$. (Here we omit the explicit expression of $P(k, p, y)$ since it is complicated.) Note also that

$$w_1(0) = \sum_{r=0}^{p-1} (-1)^r \binom{-2}{r}^3 = \sum_{r=0}^{p-1} (r+1)^3 = \frac{p^2(p+1)^2}{4} \equiv 0 \pmod{p^2}.$$

Fix a p -adic integer y . If $y \neq 0$, then (3.2) with $k = 0$ yields

$$\begin{aligned} & 3(3py - 2)(3py - 4)w_1(y) \\ & \equiv \frac{P(0, p, y)(p(1-y) - 1)^3}{(py)^3(py - 1)^3} \left(\frac{py}{p-1} \binom{p(y-1) + p - 1}{p-2} \right)^3 \equiv 0 \pmod{p^2} \end{aligned}$$

and hence $w_1(y) \equiv 0 \pmod{p^2}$. If $1 < k+1 \leq \lfloor (p-1)/3 \rfloor$, then by (3.2) we have

$$(py - 2k)^2 w_k(y) + 3(3py - 2(3k+1))(3py - 2(3k+2))w_{k+1}(y) \equiv 0 \pmod{p^3}$$

since

$$\binom{py - 2k}{p-1} = \frac{p}{py - 2k + 1} \binom{py - 2k + 1}{p} \equiv 0 \pmod{p}.$$

Thus, when $1 < k+1 \leq \lfloor (p-1)/3 \rfloor$ we have

$$w_k(y) \equiv 0 \pmod{p^2} \implies w_{k+1}(y) \equiv 0 \pmod{p^2}.$$

So, by induction, $w_k(y) \equiv 0 \pmod{p^2}$ for all $k = 1, \dots, \lfloor (p-1)/3 \rfloor$.

In view of the above, we have completed the proof of Theorem 1.3. \square

Lemma 3.1. Let $n \in \mathbb{N}$. Then we have

$$\sum_{k=0}^n \binom{x+k-1}{k} = \binom{x+n}{n}. \quad (3.3)$$

Proof. By the Chu–Vandermonde identity (see, e.g., [GKP, p. 169]),

$$\sum_{k=0}^n \binom{-x}{k} \binom{-1}{n-k} = \binom{-x-1}{n}$$

which is equivalent to (3.3). Of course, it is easy to prove (3.3) by induction. \square

Proof of Theorem 1.4. (i) Observe that

$$\begin{aligned} \sum_{n=0}^{p-1} D_n &= \sum_{n=0}^{p-1} \sum_{k=0}^n \binom{n+k}{2k} \binom{2k}{k} = \sum_{k=0}^{p-1} \binom{2k}{k} \sum_{n=k}^{p-1} \binom{n+k}{2k} \\ &= \sum_{k=0}^{p-1} \binom{2k}{k} \sum_{j=0}^{p-1-k} \binom{j+2k}{j} \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{p-1} \binom{2k}{k} \binom{2k+1+p-1-k}{p-1-k} \quad (\text{by Lemma 3.1}) \\
&= \sum_{k=0}^{p-1} \binom{2k}{k} \binom{p+k}{2k+1} = \sum_{k=0}^{p-1} \frac{k+1}{2k+1} \binom{2k+1}{k} \binom{p+k}{2k+1}
\end{aligned}$$

and thus

$$\sum_{n=0}^{p-1} D_n = \sum_{k=0}^{p-1} \frac{k+1}{2k+1} \binom{p+k}{k} \binom{p}{k+1} = p + \sum_{k=1}^{p-1} \frac{p}{2k+1} \binom{p-1}{k} \binom{p+k}{k}.$$

For $k = 1, \dots, p-1$ we clearly have

$$\binom{p-1}{k} \binom{p+k}{k} = (-1)^k \prod_{j=1}^k \left(1 - \frac{p^2}{j^2}\right) \equiv (-1)^k (1 - p^2 H_k^{(2)}) \pmod{p^4}; \quad (3.4)$$

in particular,

$$\binom{p-1}{(p-1)/2} \binom{p+(p-1)/2}{(p-1)/2} \equiv (-1)^{(p-1)/2} = \left(\frac{-1}{p}\right) \pmod{p^3}$$

since $H_{(p-1)/2}^{(2)} \equiv 0 \pmod{p}$. Therefore

$$\begin{aligned}
\sum_{n=0}^{p-1} D_n &\equiv \sum_{\substack{k=0 \\ k \neq (p-1)/2}}^{p-1} \frac{p}{2k+1} (-1)^k + \left(\frac{-1}{p}\right) \\
&\equiv \left(\frac{-1}{p}\right) - p^2 E_{p-3} \pmod{p^3} \quad (\text{by (2.9)}).
\end{aligned}$$

This proves (1.17).

(ii) Now we prove (1.18) and (1.19).

Let n be any positive integer. Then

$$\begin{aligned}
\sum_{m=0}^{n-1} (2m+1)(-1)^m D_m &= \sum_{m=0}^{n-1} (2m+1)(-1)^m \sum_{k=0}^m \binom{m+k}{2k} \binom{2k}{k} \\
&= \sum_{k=0}^{n-1} \binom{2k}{k} \sum_{m=0}^{n-1} (2m+1)(-1)^m \binom{m+k}{2k}.
\end{aligned}$$

By induction, we have the identity

$$\sum_{m=0}^{n-1} (2m+1)(-1)^m \binom{m+k}{2k} = (-1)^n (k-n) \binom{n+k}{2k}. \quad (3.5)$$

Thus

$$\begin{aligned}\sum_{m=0}^{n-1} (2m+1)(-1)^m D_m &= (-1)^{n-1} \sum_{k=0}^{n-1} \binom{2k}{k} (n-k) \binom{n+k}{2k} \\ &= (-1)^{n-1} \sum_{k=0}^{n-1} (n-k) \binom{n}{k} \binom{n+k}{k} \\ &= (-1)^{n-1} n \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{n+k}{k}.\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{m=0}^{n-1} (2m+1) D_m &= \sum_{m=0}^{n-1} (2m+1) \sum_{k=0}^m \binom{m+k}{2k} \binom{2k}{k} \\ &= \sum_{k=0}^{n-1} \binom{2k}{k} \sum_{m=0}^{n-1} (2m+1) \binom{m+k}{2k} \\ &= n \sum_{k=0}^{n-1} C_k (n-k) \binom{n+k}{2k} = \sum_{k=0}^{n-1} \frac{n^2}{k+1} \binom{n-1}{k} \binom{n+k}{k}.\end{aligned}$$

In view of (3.4) and the above,

$$\begin{aligned}\frac{1}{p} \sum_{m=0}^{p-1} (2m+1)(-1)^m D_m &= \sum_{k=0}^{p-1} \binom{p-1}{k} \binom{p+k}{k} \\ &\equiv \sum_{k=0}^{p-1} (-1)^k - p^2 \sum_{k=1}^{p-1} \sum_{0 < j \leq k} \frac{(-1)^k}{j^2} = 1 - p^2 \sum_{j=1}^{p-1} \frac{1}{j^2} \sum_{k=j}^{p-1} (-1)^k \\ &\equiv 1 - p^2 \sum_{i=1}^{(p-1)/2} \frac{1}{(2i)^2} = 1 - \frac{p^2}{4} H_{(p-1)/2}^{(2)} \equiv 1 - \frac{7}{12} p^3 B_{p-3} \pmod{p^4}\end{aligned}$$

and hence (1.18) holds. Similarly,

$$\begin{aligned}\frac{1}{p} \sum_{m=0}^{p-1} (2m+1) D_m &= \sum_{k=0}^{p-1} \frac{p}{k+1} \binom{p-1}{k} \binom{p+k}{k} \\ &\equiv \binom{p+(p-1)}{p-1} + p \sum_{k=0}^{p-2} \frac{(-1)^k}{k+1} (1 - p^2 H_k^{(2)}) \pmod{p^5} \\ &\equiv \binom{2p-1}{p-1} - p \sum_{k=1}^{p-1} \frac{1+(-1)^k}{k} \equiv 1 - p H_{(p-1)/2} \pmod{p^3}.\end{aligned}$$

(We have employed Wolstenholme's congruences $\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}$ and $H_{p-1} \equiv 0 \pmod{p^2}$.) To obtain (1.19) it suffices to apply Lehmer's congruence (cf. [L])

$$H_{(p-1)/2} \equiv -2q_p(2) + pq_p^2(2) \pmod{p^2}.$$

The proof of Theorem 1.4 is now complete. \square

4. Some related conjectures

Our following conjecture was motivated by Theorem 1.1(i).

Conjecture 4.1. *Let $p > 3$ be a prime.*

(i) *If $p \equiv 1 \pmod{3}$, then*

$$\sum_{k=0}^{p-1} (-1)^k A_k \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{16^k} \pmod{p^3}. \quad (4.1)$$

If $p \equiv 1, 3 \pmod{8}$, then

$$\sum_{k=0}^{p-1} A_k \equiv \sum_{k=0}^{p-1} \frac{\binom{4k}{k,k,k,k}}{256^k} \pmod{p^3}. \quad (4.2)$$

(ii) *If x belongs to the set*

$$\{1, -4, 9, -48, 81, -324, 2401, 9801, -25920, -777924, 96059601\} \\ \cup \left\{ \frac{81}{256}, -\frac{9}{16}, \frac{81}{32}, -\frac{3969}{256} \right\}$$

and $x \not\equiv 0 \pmod{p}$, then we must have

$$\sum_{k=0}^{p-1} A_k(x) \equiv \left(\frac{x}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{4k}{k,k,k,k}}{(256x)^k} \pmod{p^2}.$$

Remark 4.1. For those

$$x = -4, 9, -48, 81, -324, 2401, 9801, -25920, -777924, 96059601, \frac{81}{256},$$

the author (cf. [Su2]) had conjectures on $\sum_{k=0}^{p-1} \binom{4k}{k,k,k,k} / (256x)^k \pmod{p^2}$. Motivated by this, Z.H. Sun [S2] guessed $\sum_{k=0}^{p-1} \binom{4k}{k,k,k,k} / (256x)^k \pmod{p^2}$ for $x = -9/16, 81/32, -3969/256$ in a similar way.

We have checked Conjecture 4.1 as well as all the other conjectures in this paper via Mathematica 7. Below we provide numerical evidences for (4.1) and (4.2).

Example 4.1. The values of A_0, A_1, \dots, A_{10} are given by

$$1, 5, 73, 1445, 33\,001, 819\,005, 21\,460\,825, \\ 584\,307\,365, 16\,367\,912\,425, 468\,690\,849\,005, 13\,657\,436\,403\,073$$

respectively. Via computation we find that

$$\sum_{k=0}^6 (-1)^k A_k = 20\,673\,445, \quad \sum_{k=0}^6 \frac{\binom{2k}{k}^3}{16^k} = \frac{18\,825\,543}{262\,144},$$

and also

$$20\,673\,445 \times 262\,144 - 18\,825\,543 = 7^3 \times 15\,800\,002\,159.$$

This verifies (4.1) for $p = 7$. By computation we have

$$\sum_{k=0}^{10} A_k = 14\,143\,101\,786\,223, \quad \sum_{k=0}^{10} \frac{\binom{4k}{k,k,k,k}}{256^k} = \frac{22\,821\,835\,381\,970\,859\,184\,405}{18\,889\,465\,931\,478\,580\,854\,784},$$

and also

$$14\,143\,101\,786\,223 \times 18\,889\,465\,931\,478\,580\,854\,784 - 22\,821\,835\,381\,970\,859\,184\,405 \\ = 11^3 \times 200\,717\,985\,992\,690\,007\,194\,123\,778\,899\,817.$$

This verifies (4.2) for $p = 11$.

Inspired by parts (ii) and (iii) of Theorem 1.1, we raise the following conjecture.

Conjecture 4.2. For any $\varepsilon \in \{\pm 1\}$, $m, n \in \mathbb{Z}^+$ and $x \in \mathbb{Z}$, we have

$$\sum_{k=0}^{n-1} (2k+1) \varepsilon^k A_k(x)^m \equiv 0 \pmod{n}. \quad (4.3)$$

If $p > 5$ is a prime, then

$$\sum_{k=0}^{p-1} (2k+1) A_k \equiv p - \frac{7}{2} p^2 H_{p-1} \pmod{p^6}. \quad (4.4)$$

Remark 4.2. After reading an initial version of this paper, Guo and Zeng [GZ] proved the author's following conjectural results:

(a) For any $n \in \mathbb{Z}^+$ and $x \in \mathbb{Z}$ we have

$$\sum_{k=0}^{n-1} (2k+1) (-1)^k A_k(x) \equiv 0 \pmod{n}.$$

If p is an odd prime and x is an integer, then

$$\sum_{k=0}^{p-1} (2k+1)(-1)^k A_k(x) \equiv p \left(\frac{1-4x}{p} \right) \pmod{p^2}.$$

(b) For any prime $p > 3$ we have

$$\sum_{k=0}^{p-1} (2k+1)(-1)^k A_k \equiv p \left(\frac{p}{3} \right) \pmod{p^3}$$

and

$$\sum_{k=0}^{p-1} (2k+1)(-1)^k A_k(-2) \equiv p - \frac{4}{3} p^2 q_p(2) \pmod{p^3}.$$

Example 4.2. It is easy to check that

$$\sum_{k=0}^9 (2k+1) A_k^2 = 4\,178\,310\,699\,572\,329\,761\,604\,780 \equiv 0 \pmod{10}$$

and

$$\sum_{k=0}^9 (2k+1)(-1)^k A_k^2 = -4\,169\,201\,796\,654\,383\,947\,725\,970 \equiv 0 \pmod{10}.$$

So (4.3) holds when $m=2$, $n=10$, $x=1$ and $\varepsilon \in \{\pm 1\}$. Via computation we find that

$$\sum_{k=0}^{10} (2k+1) A_k = 295\,998\,598\,024\,613, \quad H_{10} = \frac{7381}{2520},$$

and

$$295\,998\,598\,024\,613 - 11 + \frac{7}{2} \times 11^2 \times \frac{7381}{2520} = 11^6 \times \frac{120\,300\,114\,181}{720}.$$

This verifies (4.4) for $p=11$.

Recall that for a prime p and a rational number x , the p -adic valuation of x is given by

$$v_p(x) = \sup\{a \in \mathbb{Z}: \text{the denominator of } p^{-a}x \text{ is not divisible by } p\}.$$

Just like the Apéry polynomial $A_n(x) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k} x^k$ we define

$$D_n(x) = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} x^k.$$

Actually $D_n((x-1)/2)$ coincides with the Legendre polynomial $P_n(x)$ of degree n .

Our following conjecture involves p -adic valuations.

Conjecture 4.3.

(i) For any $n \in \mathbb{Z}^+$ the numbers

$$s(n) = \frac{1}{n^2} \sum_{k=0}^{n-1} (2k+1)(-1)^k A_k \left(\frac{1}{4} \right)$$

and

$$t(n) = \frac{1}{n^2} \sum_{k=0}^{n-1} (2k+1)(-1)^k D_k \left(-\frac{1}{4} \right)^3$$

are rational numbers with denominators $2^{2\nu_2(n!)}$ and $2^{3(n-1+\nu_2(n!))-\nu_2(n)}$ respectively. Moreover, the numerators of $s(1), s(3), s(5), \dots$ are congruent to 1 modulo 12 and the numerators of $s(2), s(4), s(6), \dots$ are congruent to 7 modulo 12. If p is an odd prime and $a \in \mathbb{Z}^+$, then

$$s(p^a) \equiv t(p^a) \equiv 1 \pmod{p}.$$

For $p = 3$ and $a \in \mathbb{Z}^+$ we have

$$s(3^a) \equiv 4 \pmod{3^2} \quad \text{and} \quad t(3^a) \equiv -8 \pmod{3^5}.$$

(ii) Let p be a prime. For any positive integer n and p -adic integer x , we have

$$\nu_p \left(\frac{1}{n} \sum_{k=0}^{n-1} (2k+1)(-1)^k A_k(x) \right) \geq \min\{\nu_p(n), \nu_p(4x-1)\} \quad (4.5)$$

and

$$\nu_p \left(\frac{1}{n} \sum_{k=0}^{n-1} (2k+1)(-1)^k D_k(x)^3 \right) \geq \min\{\nu_p(n), \nu_p(4x+1)\}. \quad (4.6)$$

Example 4.3. We check Conjecture 4.3 with $n = p = 5$. For $n = 5$ we have

$$2^{2\nu_2(n!)} = 2^{2\nu_2(120)} = 2^6 = 64$$

and

$$2^{3(n-1+\nu_2(n!))-\nu_2(n)} = 2^{3(5-1+3)-0} = 2^{21} = 2097152.$$

Via computation we find that

$$s(5) = \frac{19849}{64} \quad \text{and} \quad t(5) = \frac{82547}{2097152}.$$

Note that $19849 \equiv 1 \pmod{12}$ and $s(5) \equiv t(5) \equiv 1 \pmod{5}$. Also,

$$\nu_5 \left(\frac{1}{5} \sum_{k=0}^4 (2k+1)(-1)^k A_k(-1) \right) = \nu_5(-13\,095) = 1 = \nu_5(\pm 5)$$

and

$$\nu_5 \left(\frac{1}{5} \sum_{k=0}^4 (2k+1)(-1)^k D_k(1)^3 \right) = \nu_5(59\,189\,205) = 1 = \nu_5(5).$$

Motivated by Theorem 1.3, we pose the following conjecture.

Conjecture 4.4. Let p be an odd prime and let $n \geq 2$ be an integer. Suppose that x is a p -adic integer with $x \equiv -2k \pmod{p}$ for some $k \in \{1, \dots, \lfloor (p+1)/(2n+1) \rfloor\}$. Then we have

$$\sum_{r=0}^{p-1} (-1)^r \binom{x}{r}^{2n+1} \equiv 0 \pmod{p^2}. \quad (4.7)$$

Example 4.4. Clearly $1/3 \equiv -2 \pmod{7}$ and $\lfloor (7+1)/5 \rfloor = 1$. Via computation we find that

$$\sum_{r=0}^6 (-1)^r \binom{1/3}{r}^5 = \frac{12\,107\,415\,300\,799\,972\,328}{12\,157\,665\,459\,056\,928\,801} = 7^2 \times \frac{247\,090\,108\,179\,591\,272}{12\,157\,665\,459\,056\,928\,801}.$$

So (4.7) holds when $p = 7$, $n = 2$ and $x = 1/3$.

Conjecture 4.5. Let $p \equiv 3 \pmod{4}$ be a prime, and let $m \in \mathbb{Z}$ with $p \nmid m(4m+1)$. Then

$$\sum_{k=0}^{p-1} \frac{W_k(-m^2)}{(4m+1)^k} \equiv \sum_{k=0}^{p-1} \frac{A_k(-m^2/(4m+1))}{(4m+1)^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{2p}{k}}{(-16)^k} D_{2k} \left(\frac{1}{4m} \right) \pmod{p^2}.$$

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