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A generalization of Fibonacci and Lucas matrices

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Abstract

We define the matrix $\mathcal{U}_n^{(a,b,s)}$ of type s, whose elements are defined by the general second-order non-degenerated sequence and introduce the notion of the generalized Fibonacci matrix $\mathcal{F}_n^{(a,b,s)}$, whose nonzero elements are generalized Fibonacci numbers. We observe two regular cases of these matrices ($s = 0$ and $s = 1$). Generalized Fibonacci matrices in certain cases give the usual Fibonacci matrix and the Lucas matrix. Inverse of the matrix $\mathcal{U}_n^{(a,b,s)}$ is derived. In partial case we get the inverse of the generalized Fibonacci matrix $\mathcal{F}_n^{(a,b,0)}$ and later known results from [Gwang-Yeon Lee, Jin-Soo Kim, Sang-Gu Lee, Factorizations and eigenvalues of Fibonaci and symmetric Fibonaci matrices, Fibonacci Quart. 40 (2002) 203–211; P. Stănică, Cholesky factorizations of matrices associated with r-order recurrent sequences, Electron. J. Combin. Number Theory 5 (2) (2005) #A16] and [Z. Zhang, Y. Zhang, The Lucas matrix and some combinatorial identities, Indian J. Pure Appl. Math. (in press)]. Correlations between the matrices $\mathcal{U}_n^{(a,b,s)}$, $\mathcal{F}_n^{(a,b,s)}$ and the generalized Pascal matrices are considered. In the case $a = 0, b = 1$ we get known result for Fibonacci matrices [Gwang-Yeon Lee, Jin-Soo Kim, Seong-Hoon Cho, Some combinatorial identities via Fibonacci numbers, Discrete Appl. Math. 130 (2003) 527–534]. Analogous result for Lucas matrices, originated in [Z. Zhang, Y. Zhang, The Lucas matrix and some combinatorial identities, Indian J. Pure Appl. Math. (in press)], can be derived in the partial case $a = 2, b = 1$. Some combinatorial identities involving generalized Fibonacci numbers are derived.

Keywords: Fibonacci number; Lucas number; Fibonacci matrix; Lucas matrix

1. Introduction

The Fibonacci numbers $\{F_n\}_{n=0}^{\infty}$ are the terms of the sequence $0, 1, 1, 2, 3, 5, \ldots$ where each term is the sum of the two preceding terms, and we get things started with 0 and 1 as F_0 and F_1. You cannot go very far in the lore of Fibonacci numbers without encountering the companion sequence of Lucas numbers $\{L_n\}_{n=0}^{\infty}$, which follows the same recursive pattern as the Fibonacci numbers, but begins with $L_0 = 2$ and $L_1 = 1$. The sequence of Lucas numbers is therefore $2, 1, 3, 4, 7, \ldots$ [13].

We also observe so-called generalized Fibonacci numbers, $\{F_n^{(a,b)}\}_{n=0}^{\infty}$, which satisfy the same recursive formula $F_{n+2}^{(a,b)} = F_{n+1}^{(a,b)} + F_n^{(a,b)}$, $n = 0, 1, \ldots$, but starting with arbitrary initial values $F_0^{(a,b)} = a$ and $F_1^{(a,b)} = b$, (see for example [9,6,12], ([10], Chapter 7)).

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The $n \times n$ Fibonacci matrix $\mathcal{F}_n = [f_{i,j}]$ $(i, j = 1, \ldots, n)$ is defined by [7]:

$$f_{i,j} = \begin{cases} F_{i-j+1}, & i-j+1 \geq 0, \\ 0, & i-j+1 < 0. \end{cases} \tag{1.1}$$

The inverse and Cholesky factorization of the Fibonacci matrix are given in [7]. The relations between the Pascal matrix and the Fibonacci matrix are studied in [8].

As an analogy of the Fibonacci matrix, the $n \times n$ Lucas matrix $\mathcal{L}_n = [l_{i,j}]$ $(i, j = 1, \ldots, n)$ is defined in [16]:

$$l_{i,j} = \begin{cases} L_{i-j+1}, & i-j \geq 0, \\ 0, & i-j < 0. \end{cases} \tag{1.2}$$

In the paper [11] the author investigated the inverse and Cholesky factorization of the matrix $\mathcal{U}_n$ with entries

$$u_{i,j} = \begin{cases} U_{i-j+1}, & i-j+1 \geq 0, \\ 0, & i-j+1 < 0, \end{cases} \tag{1.3}$$

where U_n is the non-degenerated second order sequence $U_{n+1} = AU_n + BU_{n-1}$, $\delta = \sqrt{A^2 + 4B}$ real, and where A, B, U_1 are integers and $U_0 = 0$ (i.e. $A = B$). In [11] the author also generalized these results to r-order recurrent sequence satisfying $U_0 = U_{-1} = \cdots = U_{2-r} = 0$, U_1 arbitrary. Results obtained in [11] include known facts about the Fibonacci matrix [7,8] in the case $U_1 = 1$, $A = B = 1$. But, results about the Lucas matrices from [16] are not included. Lucas sequence is generated by the associated sequence V_n which satisfy $V_0 = 2$, $V_1 = a$. Our goal in this paper is to generalize all results about the Fibonacci and Lucas matrices. The purpose of this paper is to demonstrate that known properties of Fibonacci, Lucas matrices and the matrices defined in [11] are valid for a more general class of matrices, introduced in Section 2.

Throughout the paper we adopt the following two conventions: $0^0 = 1$ and $\binom{n}{k} = 0$ for $k > n$, even in the case $k = 0$. By $\operatorname{rank}(A)$ we denote the rank of matrix A.

The paper is organized as follows. In Section 2 we define the matrix $\mathcal{U}_n^{(a,b,s)}$ of type s, whose entries are numbers $U_n^{(a,b)}$ satisfying the general second order non-degenerated recurrence formula $U_{n+1}^{(a,b)} = AU_n^{(a,b)} + BU_{n-1}^{(a,b)}$, $\delta = \sqrt{A^2 + 4B}$ real, and initial conditions $U_0^{(a,b)} = a$, $U_1^{(a,b)} = b$. In the case $A = B = 1$ we introduce the generalized Fibonacci matrix $\mathcal{F}_n^{(a,b,s)}$ of type s, whose nonzero elements are generalized Fibonacci numbers $F_n^{(a,b)}$. Only two cases generating regular matrices are $s = 0$ and $s = 1$. Generalized Fibonacci matrices reduce to known definition of the usual Fibonacci matrix in the cases $s = 0, a = 0, b = 1$ and $s = 1, a = 0, b = 1$. In the case $a = 2, b = 1, s = 0$ we obtain the matrix whose nonzero entries are Lucas numbers, and arranged as in the Fibonacci matrix. This matrix is called the Lucas matrix [16]. At this moment we consider the matrices $\mathcal{U}_n^{(a,b,0)}$ and $\mathcal{F}_n^{(a,b,0)}$. Inverses of the generalized Fibonacci matrix and for the matrix $\mathcal{U}_n^{(a,b,0)}$ are derived. In the partial case $a = 0, b = 1$ we get known result about the inversion of the usual Fibonacci matrix from [7]. Similarly, in the case $a = 2, b = 1$ we obtain the inverse of the Lucas matrix, originated in [16]. Moreover, in Section 2 we consider the matrix $\mathcal{U}_n^{(a,b,0)}$ defined by means of the general non-degenerated second-order recurrent sequence, and generalize Proposition 2 from [11]. Various correlations between the matrix $\mathcal{U}_n^{(a,b,s)}$ and the Pascal matrix of the first and the second kind are considered in Section 3. Corresponding results for the generalized Fibonacci matrix $\mathcal{F}_n^{(a,b,0)}$ are given as corollaries. Partial case $a = 0, b = 1$ produces known result from [8]. In the case $a = 2, b = 1$ we derive analogous results for Lucas matrices, investigated in [16]. In Section 4 we get some combinatorial identities involving generalized Fibonacci numbers and binomial coefficients.

2. Generalized Fibonacci matrix and its inverse

By $F_n^{(a,b)}$ we denote the n-th generalized Fibonacci number, generated by the Fibonacci recursive formula and by the initial values $F_0^{(a,b)} = a$, $F_1^{(a,b)} = b$. Notions of Fibonacci and Lucas matrix are generalized in the following definition.

Definition 2.1. Let $F_n^{(a,b)}$ be the n-th generalized Fibonacci number, where the starting members of the Fibonacci array are $F_0^{(a,b)} = a$ and $F_1^{(a,b)} = b$, and where $a, b \in \mathbf{C}$. The generalized Fibonacci matrix of type s and of the order

n, denoted by $\mathcal{F}_n^{(a,b,s)} = [f_{i,j}^{(a,b,s)}]$, is defined by

$$f_{i,j}^{(a,b,s)} = \begin{cases} F_{i-j+1}^{(a,b)}, & i-j+s \geq 0 \\ 0, & i-j+s < 0 \end{cases}, \ i, j = 1, \ldots, n. \tag{2.1}$$

It is clear that the integer s means the shift of non-zero elements with respect to main diagonal. We also define a generalization $\mathcal{U}_n^{(a,b,s)} = [u_{i,j}^{(a,b,s)}]$ of the matrix $\mathcal{F}_n^{(a,b,s)}$ and the matrix $\mathcal{U}_n$ from (1.3).

Definition 2.2. The matrix $\mathcal{U}_n^{(a,b,s)} = [u_{i,j}^{(a,b,s)}]$ is defined by

$$u_{i,j}^{(a,b,s)} = \begin{cases} U_{i-j+1}^{(a,b)}, & i-j+s \geq 0, \\ 0, & i-j+s < 0, \end{cases} \tag{2.2}$$

where the second order recurrent sequence $U_n^{(a,b)}$ satisfies the following conditions:

$$U_n^{(a,b)} = AU_{n-1}^{(a,b)} + BU_{n-2}^{(a,b)}, \qquad U_0^{(a,b)} = a, \qquad U_1^{(a,b)} = b, \qquad A^2 + 4B > 0. \tag{2.3}$$

Remark 2.1. **(a)** Generalized Fibonacci matrices $\mathcal{F}_n^{(0,1,1)}$ and $\mathcal{F}_n^{(0,1,0)}$ are both identical to the usual Fibonacci matrix defined in (1.1).

(b) The generalized Fibonacci matrix $\mathcal{F}_n^{(2,1,0)}$ corresponds to Lucas matrix, defined in (1.2).

(c) The matrix $\mathcal{U}_n^{(0,b,1)}$ reduces to the matrix $\mathcal{U}_n$ defined in (1.3).

Example 2.1. The 6×6 generalized Fibonacci matrix of type 0 is equal to

$$\mathcal{F}_6^{(a,b,0)} = \begin{pmatrix} b & 0 & 0 & 0 & 0 & 0 \\ a+b & b & 0 & 0 & 0 & 0 \\ a+2b & a+b & b & 0 & 0 & 0 \\ 2a+3b & a+2b & a+b & b & 0 & 0 \\ 3a+5b & 2a+3b & a+2b & a+b & b & 0 \\ 5a+8b & 3a+5b & 2a+3b & a+2b & a+b & b \end{pmatrix}.$$

The 6×6 generalized Fibonacci matrix of type 1 is defined by

$$\mathcal{F}_6^{(a,b,1)} = \begin{pmatrix} b & a & 0 & 0 & 0 & 0 \\ a+b & b & a & 0 & 0 & 0 \\ a+2b & a+b & b & a & 0 & 0 \\ 2a+3b & a+2b & a+b & b & a & 0 \\ 3a+5b & 2a+3b & a+2b & a+b & b & a \\ 5a+8b & 3a+5b & 2a+3b & a+2b & a+b & b \end{pmatrix}.$$

The matrix $\mathcal{U}_4^{(a,b,0)}$ is equal to

$$\mathcal{U}_4^{(a,b,0)} = \begin{pmatrix} b & 0 & 0 & 0 \\ Ab+aB & b & 0 & 0 \\ bB+A(Ab+aB) & Ab+aB & b & 0 \\ B(Ab+aB)+A(bB+A(Ab+aB)) & bB+A(Ab+aB) & Ab+aB & b \end{pmatrix}.$$

Proposition 2.1. *The nondegenerated second-order recurrent sequence $U_n^{(a,b)}$, defined in (2.3), satisfies the following generalization of the Binet's Fibonacci number formula*

$$U_n^{(a,b)} = c_1\alpha^n + c_2\beta^n, \tag{2.4}$$

where

$$c_1 = \frac{a(A^2+4B)+(2b-aA)\sqrt{A^2+4B}}{2(A^2+4B)}, \tag{2.5}$$

$$c_2 = \frac{a(A^2+4B)-(2b-aA)\sqrt{A^2+4B}}{2(A^2+4B)}, \tag{2.6}$$

$$\alpha = \frac{A+\sqrt{A^2+4B}}{2}, \qquad \beta = \frac{A-\sqrt{A^2+4B}}{2}. \tag{2.7}$$

In the case $s > 1$ it is necessary to use generalized Fibonacci numbers $F_n^{(a,b)}$ and the numbers $U_n^{(a,b)}$ with negative indices n. Recurrent definition of the generalized Fibonacci numbers can be expanded for negative indices n using (2.4)–(2.7), similarly as for the Fibonacci numbers in [10].

Lemma 2.1. *The following identity is valid for the second order non-degenerated recurrent sequence $U_n^{(a,b)}$ satisfying $b \neq 0$ and for two arbitrary integers i, j satisfying $i \geq j+2$:*

$$(a^2B+abA-b^2)\sum_{k=j+2}^{i}(-1)^{k-j}\frac{a^{k-j-2}B^{k-j-1}}{b^{k-j+1}}U_{i-k+1}^{(a,b)} = \frac{aB}{b^2}U_{i-j}^{(a,b)} - \frac{B}{b}U_{i-j-1}^{(a,b)}. \tag{2.8}$$

Proof. By using (2.7) we obtain

$$\alpha\beta = -B, \qquad \alpha+\beta = A, \qquad \alpha-\beta = \sqrt{A^2+4B}. \tag{2.9}$$

By applying (2.4) and simple transformations, we obtain the following:

$$\begin{aligned}
&(a^2B+abA-b^2)\sum_{k=j+2}^{i}(-1)^{k-j}\frac{a^{k-j-2}B^{k-j-1}}{b^{k-j+1}}U_{i-k+1}^{(a,b)} \\
&= (a^2B+abA-b^2)\sum_{k=j+2}^{i}(-1)^{k-j}\cdot\frac{a^{k-j-2}B^{k-j-1}}{b^{k-j+1}}\left(c_1\alpha^{i-k+1}+c_2\beta^{i-k+1}\right) \\
&= \frac{a^2B+abA-b^2}{b^3}\sum_{k=j+2}^{i}\left(\left(-\frac{aB}{b\alpha}\right)^{k-j-2}B\alpha^{i-j-1}c_1+\left(-\frac{aB}{b\beta}\right)^{k-j-2}B\beta^{i-j-1}c_2\right).
\end{aligned}$$

Using

$$\sum_{k=j+2}^{i}\left(-\frac{aB}{b\alpha}\right)^{k-j-2} = \frac{1-\left(-\frac{aB}{b\alpha}\right)^{i-j-1}}{1+\frac{aB}{b\alpha}}, \qquad \sum_{k=j+2}^{i}\left(-\frac{aB}{b\beta}\right)^{k-j-2} = \frac{1-\left(-\frac{aB}{b\beta}\right)^{i-j-1}}{1+\frac{aB}{b\beta}}$$

we get

$$\begin{aligned}
&(a^2B+abA-b^2)\sum_{k=j+2}^{i}(-1)^{k-j}\frac{a^{k-j-2}B^{k-j-1}}{b^{k-j+1}}U_{i-k+1}^{(a,b)} \\
&= \frac{a^2B+abA-b^2}{b^3}\left(c_1\frac{1-(-\frac{aB}{b\alpha})^{i-j-1}}{1+\frac{aB}{b\alpha}}\alpha^{i-j-1}+c_2\frac{1-(-\frac{aB}{b\beta})^{i-j-1}}{1+\frac{aB}{b\beta}}\beta^{i-j-1}\right)B.
\end{aligned}$$

With consideration of (2.9), we have

$$(a^2B+abA-b^2)\sum_{k=j+2}^{i}(-1)^{k-j}\frac{a^{k-j-2}B^{k-j-1}}{b^{k-j+1}}U_{i-k+1}^{(a,b)}$$

$$= \frac{a^2B + abA - b^2}{b^3} \cdot \frac{(\frac{aB}{b} + \beta)\left(\alpha^{i-j} - \alpha(-\frac{aB}{b})^{i-j-1}\right)c_1 + (\frac{aB}{b} + \alpha)\left(\beta^{i-j} - \beta(-\frac{aB}{b})^{i-j-1}\right)c_2}{-1 + \frac{aA}{b} + \frac{a^2B}{b^2}}$$

$$= \frac{1}{b}\left[c_1\left(\frac{aB}{b}\alpha^{i-j} + \alpha\left(-\frac{aB}{b}\right)^{i-j} - B\alpha^{i-j-1} + B\left(-\frac{aB}{b}\right)^{i-j-1}\right)\right.$$

$$\left. + c_2\left(\frac{aB}{b}\beta^{i-j} + \beta\left(-\frac{aB}{b}\right)^{i-j} - B\beta^{i-j-1} + B\left(-\frac{aB}{b}\right)^{i-j-1}\right)\right].$$

By grouping similar members, using $c_1 + c_2 = a$, $c_1\alpha + c_2\beta = U_1^{(a,b)}$ and using (2.4) and (2.9), one can verify the following:

$$(a^2B + abA - b^2)\sum_{k=j+2}^{i}(-1)^{k-j}\frac{a^{k-j-2}B^{k-j-1}}{b^{k-j+1}}U_{i-k+1}^{(a,b)}$$

$$= \frac{1}{b}\left[\frac{aB}{b}\left(c_1\alpha^{i-j} + c_2\beta^{i-j}\right) + (c_1\alpha + c_2\beta)\left(-\frac{aB}{b}\right)^{i-j}\right.$$

$$\left. - B\left(c_1\alpha^{i-j-1} + c_2\beta^{i-j-1}\right) + B(c_1 + c_2)\left(-\frac{aB}{b}\right)^{i-j-1}\right]$$

$$= \frac{aB}{b^2}U_{i-j}^{(a,b)} + \frac{1}{b}U_1^{(a,b)}\left(-\frac{aB}{b}\right)^{i-j} - \frac{B}{b}U_{i-j-1}^{(a,b)} + \frac{aB}{b}\left(-\frac{aB}{b}\right)^{i-j-1}.$$

The proof can be completed by using $U_1^{(a,b)} = b$. □

In the partial case $A = B = 1$ we obtain the following result for the generalized Fibonacci numbers.

Corollary 2.1. *For the generalized Fibonacci numbers $F_n^{(a,b)}$, $b \neq 0$ and for two arbitrary integers i, j satisfying $i \geq j + 2$ the following is valid:*

$$(a^2 + ab - b^2)\sum_{k=j+2}^{i}(-1)^{k-j}\frac{a^{k-j-2}}{b^{k-j+1}}F_{i-k+1}^{(a,b)} = \frac{a}{b^2}F_{i-j}^{(a,b)} - \frac{1}{b}F_{i-j-1}^{(a,b)}. \tag{2.10}$$

In the case $a = 2, b = 1$, from the previous corollary we get known result from [16].

Corollary 2.2. *For the Lucas numbers and each $i \geq j + 2$ the following is valid:*

$$5\sum_{k=j+2}^{i}(-1)^{k-j}2^{k-j-2}L_{i-k+1} = 2L_{i-j} - L_{i-j-1}.$$

Theorem 2.1. *The inverse ${\mathcal{U}^{-1}}_n^{(a,b,0)} = [{u'}_{i,j}^{(a,b,0)}]$ of the matrix $\mathcal{U}_n^{(a,b,0)} = [u_{i,j}^{(a,b,0)}]$ $(b \neq 0)$ is equal to*

$${u'}_{i,j}^{(a,b,0)} = \begin{cases} (-1)^{i-j}\cdot\dfrac{a^2B + abA - b^2}{b^{i-j+1}}a^{i-j-2}B^{i-j-1}, & i \geq j+2, \\ -\dfrac{aB + bA}{b^2}, & i = j+1, \\ \dfrac{1}{b}, & i = j, \\ 0, & i < j. \end{cases} \tag{2.11}$$

Proof. Let $\sum_{k=1}^{n} u_{i,k}^{(a,b,0)}{u'}_{k,j}^{(a,b,0)} = c_{i,j}$. Obviously $c_{i,j} = 0$ for $i < j$. In the case $i = j$ one can verify the following:

$$c_{i,i} = u_{i,i}^{(a,b,0)}{u'}_{i,i}^{(a,b,0)} = b\frac{1}{b} = 1.$$

In the case $i = j + 1$ we obtain

$$c_{j+1,j} = u^{(a,b,0)}_{j+1,j} u'^{(a,b,0)}_{j,j} + u^{(a,b,0)}_{j+1,j+1} u'^{(a,b,0)}_{j+1,j} = (Ab + Ba)\frac{1}{b} + b\left(-\frac{aB + bA}{b^2}\right) = 0.$$

For $i \geq j + 2$, by applying results of Lemma 2.1 and (2.3) and (2.11) we obtain

$$\begin{aligned}
c_{i,j} &= \sum_{k=1}^{n} u^{(a,b,0)}_{i,k} u'^{(a,b,0)}_{k,j} \\
&= u^{(a,b,0)}_{i,j} u'^{(a,b,0)}_{j,j} + u^{(a,b,0)}_{i,j+1} u'^{(a,b,0)}_{j+1,j} + \sum_{k=j+2}^{i} u^{(a,b,0)}_{i,k} u'^{(a,b,0)}_{k,j} \\
&= \frac{1}{b} U^{(a,b)}_{i-j+1} - \frac{aB + bA}{b^2} U^{(a,b)}_{i-j} + (a^2 B + abA - b^2) \sum_{k=j+2}^{i} (-1)^{k-j} \frac{a^{k-j-2} B^{k-j-1}}{b^{k-j+1}} U^{(a,b)}_{i-k+1} \\
&= \frac{1}{b} U^{(a,b)}_{i-j+1} - \frac{aB + bA}{b^2} U^{(a,b)}_{i-j} + \frac{aB}{b^2} U^{(a,b)}_{i-j} - \frac{B}{b} U^{(a,b)}_{i-j-1} \\
&= \frac{1}{b}(U^{(a,b)}_{i-j+1} - AU^{(a,b)}_{i-j} - BU^{(a,b)}_{i-j-1}) \\
&= 0.
\end{aligned}$$

Therefore, we verify $\mathcal{U}^{(a,b,0)}_n \mathcal{U}^{-1}{}^{(a,b,0)}_n = I_n$, where I_n is $n \times n$ identity matrix. In a similar way one can verify $\mathcal{U}^{-1}{}^{(a,b,0)}_n \mathcal{U}^{(a,b,0)}_n = I_n$. □

Example 2.2. The inverse of the matrix $\mathcal{U}^{(a,b,0)}_4$ is equal to

$$\begin{pmatrix}
\frac{1}{b} & 0 & 0 & 0 \\
-\frac{Ab + aB}{b^2} & \frac{1}{b} & 0 & 0 \\
\frac{B\left(Ba^2 + Aba - b^2\right)}{b^3} & -\frac{Ab + aB}{b^2} & \frac{1}{b} & 0 \\
\frac{aB^2\left(-Ba^2 - Aba + b^2\right)}{b^4} & \frac{B\left(Ba^2 + Aba - b^2\right)}{b^3} & -\frac{Ab + aB}{b^2} & \frac{1}{b}
\end{pmatrix}.$$

Remark 2.2. Proposition 2 from [11] can be derived by placing $a = 0$ in (2.11).

In the partial case $A = B = 1$ from Theorem 2.1 we obtain the inverse of the generalized Fibonacci matrix of type 0.

Corollary 2.3. *Let $\mathcal{F}^{(a,b,0)}_n = [f^{(a,b,0)}_{i,j}]$, $b \neq 0$, be $n \times n$ generalized Fibonacci matrix of type 0. The inverse of $\mathcal{F}^{(a,b,0)}_n$, denoted by $\mathcal{F}^{-1}{}^{(a,b,0)}_n = [f'^{(a,b,0)}_{i,j}]$, is equal to*

$$f'^{(a,b,0)}_{i,j} = \begin{cases}
(-1)^{i-j} \cdot \dfrac{a^2 + ab - b^2}{b^{i-j+1}} a^{i-j-2}, & i \geq j + 2, \\
-\dfrac{a + b}{b^2}, & i = j + 1, \\
\dfrac{1}{b}, & i = j, \\
0, & i < j.
\end{cases} \tag{2.12}$$

Example 2.3. The inverse of the generalized Fibonacci matrix $\mathcal{F}_6^{(a,b,0)}$ is equal to

$$\begin{pmatrix} \frac{1}{b} & 0 & 0 & 0 & 0 & 0 \\ -\frac{a+b}{b^2} & \frac{1}{b} & 0 & 0 & 0 & 0 \\ \frac{a^2+ab-b^2}{b^3} & -\frac{a+b}{b^2} & \frac{1}{b} & 0 & 0 & 0 \\ -\frac{a(a^2+ab-b^2)}{b^4} & \frac{a^2+ab-b^2}{b^3} & -\frac{a+b}{b^2} & \frac{1}{b} & 0 & 0 \\ \frac{a^2(a^2+ab-b^2)}{b^5} & -\frac{a(a^2+ab-b^2)}{b^4} & \frac{a^2+ab-b^2}{b^3} & -\frac{a+b}{b^2} & \frac{1}{b} & 0 \\ -\frac{a^3(a^2+ab-b^2)}{b^6} & \frac{a^2(a^2+ab-b^2)}{b^5} & -\frac{a(a^2+ab-b^2)}{b^4} & \frac{a^2+ab-b^2}{b^3} & -\frac{a+b}{b^2} & \frac{1}{b} \end{pmatrix}.$$

In the case $a = 2, b = 1$ we get the inverse Lucas matrix, derived in [16].

Corollary 2.4. *The inverse of the Lucas matrix* $\mathcal{L}_n^{-1} = [l'_{i,j}]\,(i, j = 1, \ldots, n)$ *is equal to*

$$l'_{i,j} = \begin{cases} 5(-1)^{i-j}2^{i-j-2}, & i \geq j+2, \\ -3, & i = j+1, \\ 1, & i = j, \\ 0, & \textit{otherwise}. \end{cases}$$

In the case $a = 0, b = 1$ we get the inverse Fibonacci matrix, which is the known result from [7].

Corollary 2.5. *The inverse Fibonacci matrix* $\mathcal{F}_n^{-1} = [f'_{i,j}]\,(i, j = 1, \ldots, n)$ *is equal to*

$$f'_{i,j} = \begin{cases} -1, & j+1 \leq i \leq j+2, \\ 1, & i = j, \\ 0, & \textit{otherwise}. \end{cases}$$

In the following theorem we study rank of the matrix $\mathcal{U}_n^{(a,b,s)}$ $(b \neq 0)$:

Theorem 2.2. *Matrices* $\mathcal{U}_n^{(a,b,s)}$ *of the order* $n > 2$ *of an arbitrary type* $s > 1$ *or* $s < 0$ *are singular. The generalized Fibonacci matrices* $\mathcal{U}_2^{(a,b,s)}$ *are always regular. In the case* $b \neq 0$ *matrices* $\mathcal{U}_n^{(a,b,0)}$ *and* $\mathcal{U}_n^{(a,b,1)}$ *are regular.*

Proof. In the case $s < 0$ the proof is trivial, since $|s|$ diagonal parallels below the main diagonal in $\mathcal{U}_n^{(a,b,s)}$ are filled by zeros (i.e. the last $|s|$ columns are zero columns), and therefore $\operatorname{rank}(\mathcal{U}_n^{(a,b,s)}) = n - |s| < n$. Denote by R_i the i-th row of the matrix $\mathcal{U}_n^{(a,b,s)}$. In the case $s \geq 0$ the last $s+1$ rows (i.e. the rows $R_{n-s}, \ldots, R_n$) in $\mathcal{U}_n^{(a,b,s)}$ are completely filled by the elements $U_{i-j+1}^{(a,b)}$. For these rows it is not difficult to verify from (2.3)

$$R_i = AR_{i+1} + BR_{i+2}, \quad i = n-s+2, \ldots, n.$$

Therefore, between the rows $R_{n-s}, \ldots, R_n$ there is only one linearly independent row in the case $s = 0$, and only two linearly independent in the case $s > 0$. On the other hand, it is clear that rows $R_1, \ldots, R_{n-s-1}$ are linearly independent. Hence, in the case $s > 1$

$$\operatorname{rank}(\mathcal{U}_n^{(a,b,s)}) = \begin{cases} n-s+1 < n, & s \leq n-1, \\ 2, & n \geq s > n-1 \end{cases}$$

so the matrix $\mathcal{U}_n^{(a,b,s)}$ is singular.

From the previous argumentation, it is not difficult to verify that both $\mathcal{U}_n^{(a,b,0)}$ and $\mathcal{U}_n^{(a,b,1)}$ are regular matrices. □

Example 2.4. The last two columns of the matrix

$$\mathcal{U}_4^{(a,b,-2)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A^2b + aAB + bB & 0 & 0 & 0 \\ A^3b + aA^2B + 2Ab\ B + aB^2 & A^2b + aAB + bB & 0 & 0 \end{bmatrix}$$

are the zero columns, and $\operatorname{rank}(\mathcal{U}_4^{(a,b,-2)}) = 2$.

On the other hand, the rank of the matrix

$$\mathcal{U}_4^{(a,b,2)} = \begin{bmatrix} b & a & \dfrac{b-aA}{B} & 0 \\ Ab + aB & b & a & \dfrac{b-aA}{B} \\ bA^2 + aBA + bB & Ab + aB & b & a \\ bA^3 + aBA^2 + 2bBA + aB^2 & bA^2 + aBA + bB & Ab + aB & b \end{bmatrix}$$

is 3, because of $R_4 = AR_3 + BR_2$.

3. Generalized Fibonacci matrix and Pascal matrices

Various types of Pascal matrices are investigated in [1,2,4,5,14,15]. The generalized Pascal matrix of the first kind $\mathcal{P}_n[x] = [p_n(x; i, j)]$, $i, j = 1, \ldots, n$ is defined in [4]:

$$p_n(x; i, j) = \begin{cases} x^{i-j}\binom{i-1}{j-1}, & i \geq j, \\ 0, & i < j. \end{cases} \tag{3.1}$$

In the case $x = 1$, the generalized Pascal matrix of the first kind reduces to ther well-known Pascal matrix $\mathcal{P}_n = [p_n(i, j)]$, $i, j = 1, \ldots, n$, which is defined in [3,4]:

$$p_n(i, j) = \begin{cases} \binom{i-1}{j-1}, & i \geq j, \\ 0, & i < j. \end{cases} \tag{3.2}$$

In the following theorem we define the matrix $\mathcal{G}_n[x; a, b] = [g_{i,j}(x; a, b)]$, $i, j = 1, \ldots, n$ which gives a correlation between the matrix $\mathcal{U}_n^{(a,b,0)}$ and the generalized Pascal matrix of the first kind:

Theorem 3.1. *The matrix $\mathcal{G}_n[x; a, b]$ ($x \neq 0, b \neq 0$), whose entries are defined by*

$$g_{i,j}(x; a, b) = x^{-j}\left[\frac{1}{b}x^i\binom{i-1}{j-1} - \frac{aB + bA}{b^2}x^{i-1}\binom{i-2}{j-1} + \sum_{k=j}^{i-2}(-1)^{i-k}\frac{a^2B + abA - b^2}{b^{i-k+1}}a^{i-k-2}B^{i-k-1}x^k\binom{k-1}{j-1}\right], \tag{3.3}$$

satisfies

$$\mathcal{P}_n[x] = \mathcal{U}_n^{(a,b,0)}\mathcal{G}_n[x; a, b]. \tag{3.4}$$

Proof. It is sufficient to verify

$$\mathcal{U}^{-1}{}_n^{(a,b,0)}\mathcal{P}_n[x] = \mathcal{G}_n[x; a, b].$$

It is evident that $g_{i,j}(x; a, b) = 0$ for $i < j$, which is of the form (3.3). So, it remains to verify all the other cases. The cases $i = j$ and $i = j + 1$ can be simply verified:

$$g_{j,j}(x;a,b) = u'^{(a,b,0)}_{j,j}\, p_n(x;j,j) = \frac{1}{b} = x^{-j}\frac{1}{b}x^j\binom{j-1}{j-1};$$

$$\begin{aligned} g_{j+1,j}(x;a,b) &= u'^{(a,b,0)}_{j+1,j}\, p_n(x;j,j) + u'^{(a,b,0)}_{j+1,j+1} p_n(x;j+1,j) \\ &= -\frac{aB+bA}{b^2} + \frac{jx}{b} \\ &= x^{-j}\left[\frac{1}{b}x^{j+1}\binom{j+1-1}{j-1} - \frac{aB+bA}{b^2}x^j\binom{j+1-2}{j-1}\right]. \end{aligned}$$

In the last case, $i \geq j+2$, by applying results of Theorem 2.1 we get

$$\begin{aligned} g_{i,j}(x;a,b) &= u'^{(a,b,0)}_{i,i}\, p_n(x;i,j) + u'^{(a,b,0)}_{i,i-1} p_n(x;i-1,j) + \sum_{k=j}^{i-2} u'^{(a,b,0)}_{i,k}\, p_n(x;k,j) \\ &= \frac{1}{b}x^{i-j}\binom{i-1}{j-1} - \frac{aB+bA}{b^2}x^{i-1-j}\binom{i-2}{j-1} \\ &\quad + \sum_{k=j}^{i-2}(-1)^{i-k}\frac{a^2B+abA-b^2}{b^{i-k+1}}a^{i-k-2}B^{i-k-1}x^{k-j}\binom{k-1}{j-1}, \end{aligned}$$

which is also of the form (3.3). □

In the case $A = B = 1$ we get analogous result for the generalized Fibonacci matrix.

Corollary 3.1. *The matrix $\mathcal{G}_n[x;a,b]$ $(x \neq 0, b \neq 0)$, whose entries are defined by*

$$g_{i,j}(x;a,b) = x^{-j}\left[\frac{1}{b}x^i\binom{i-1}{j-1} - \frac{a+b}{b^2}x^{i-1}\binom{i-2}{j-1} + \sum_{k=j}^{i-2}(-1)^{i-k}\frac{a^2+ab-b^2}{b^{i-k+1}}a^{i-k-2}x^k\binom{k-1}{j-1}\right],$$

satisfies

$$\mathcal{P}_n[x] = \mathcal{F}_n^{(a,b,0)}\mathcal{G}_n[x;a,b].$$

Moreover, the last corollary produces a known result from [8] in partial case $a = 0$, $b = 1$ and $x = 1$:

Corollary 3.2. *Let $\mathcal{M}_n$ be the matrix with elements defined by*

$$m_{ij} = \binom{i-1}{j-1} - \binom{i-2}{j-1} - \binom{i-3}{j-1}.$$

The Pascal matrix and the Fibonacci matrix are related with $\mathcal{P}_n = \mathcal{F}_n\mathcal{M}_n$.

Proof. The proof follows from $\mathcal{M}_n = \mathcal{G}_n[1;0,1]$. □

In the case $a = 2$, $b = 1$, from Corollary 3.1 we give a corresponding result for Lucas matrices [16]:

Corollary 3.3. *The generalized Pascal matrix of the first kind and the Lucas matrix satisfy $\mathcal{P}_n[x] = \mathcal{L}_n\mathcal{G}_n[x;2,1]$, where*

$$g_{i,j}(x;2,1) = x^{-j}\left[x^i\binom{i-1}{j-1} - 3x^{i-1}\binom{i-2}{j-1} + 5(-1)^i 2^{i-2}\sum_{k=j}^{i-2}(-1)^k\binom{k-1}{j-1}\left(\frac{x}{2}\right)^k\right].$$

After the substitution $x = 1$ in the previous result, the following result immediately follows:

Corollary 3.4. *The Pascal matrix and the Lucas matrix satisfy $\mathcal{P}_n = \mathcal{L}_n\mathcal{G}_n[1;2,1]$, where*

$$g_{i,j}(1;2,1) = \binom{i-1}{j-1} - 3\binom{i-2}{j-1} + 5(-2)^{i-2}\sum_{k=j}^{i-2}(-2)^{-k}\binom{k-1}{j-1}.$$

In the following theorem we define the matrix $\mathcal{H}_n[x; a, b] = [h_{i,j}(x; a, b)]$, $i, j = 1, \ldots, n$ which gives a similar correlation between the matrix $\mathcal{U}_n^{(a,b,0)}$ and the generalized Pascal matrix of the first kind:

Theorem 3.2. *The matrix $\mathcal{H}_n[x; a, b]$, $(b \neq 0)$, defined by*

$$h_{i,j}(x; a, b) = x^i \left[\frac{1}{b} x^{-j} \binom{i-1}{j-1} - \frac{aB + bA}{b^2} x^{-j-1} + \sum_{k=j+2}^{i} (-1)^{k-j} \frac{a^2 B + abA - b^2}{b^{k-j+1}} a^{k-j-2} B^{k-j-1} x^{-k} \binom{i-1}{k-1} \right] \tag{3.5}$$

satisfies

$$\mathcal{P}_n[x] = \mathcal{H}_n[x; a, b]\mathcal{U}_n^{(a,b,0)}. \tag{3.6}$$

Proof. Similar as the proof of Theorem 3.1. □

An analogous result for the generalized Fibonacci matrix can be derived in the case $A = B = 1$.

Corollary 3.5. *The matrix $\mathcal{H}_n[x; a, b]$, $(b \neq 0)$, defined by*

$$h_{i,j}(x; a, b) = x^i \left[\frac{1}{b} x^{-j} \binom{i-1}{j-1} - \frac{a+b}{b^2} x^{-j-1} \binom{i-1}{j} + \sum_{k=j+2}^{i} (-1)^{k-j} \frac{a^2 + ab - b^2}{b^{k-j+1}} a^{k-j-2} x^{-k} \binom{i-1}{k-1} \right]$$

satisfies

$$\mathcal{P}_n[x] = \mathcal{H}_n[x; a, b]\mathcal{F}_n^{(a,b,0)}.$$

An analogous result for Lucas matrices is [16]:

Corollary 3.6. *The Lucas matrix satisfies $\mathcal{P}_n[x] = \mathcal{H}_n[x; 2, 1]\mathcal{L}_n$, where*

$$h_{i,j}(x; 2, 1) = x^{i-j-1} \left[x \binom{i-1}{j-1} - 3 \binom{i-1}{j} + (-1)^j \frac{5x^{j+1}}{2^{j+2}} \sum_{k=j+2}^{i} (-1)^k \binom{i-1}{k-1} 2^k x^{-k} \right].$$

The generalized Pascal matrix of the second kind $\mathcal{Q}_n[x] = [q_n(x; i, j)]$, $i, j = 1, \ldots, n$ is defined by [4]:

$$q_n(x; i, j) = \begin{cases} x^{i+j-2} \binom{i-1}{j-1}, & i \geq j, \\ 0, & i < j. \end{cases} \tag{3.7}$$

Theorem 3.3. *The matrices $\mathcal{S}_n[x; a, b] = [s_{i,j}(x; a, b)]$ and $\mathcal{T}_n[x; a, b] = [t_{i,j}(x; a, b)]$, $i, j = 1, \ldots, n$, $(b \neq 0)$ whose entries are defined by*

$$s_{i,j}(x; a, b) = x^j \left[\frac{1}{b} x^{i-2} \binom{i-1}{j-1} - \frac{aB + bA}{b^2} x^{i-3} \binom{i-2}{j-1} + \sum_{k=j}^{i-2} (-1)^{i-k} \frac{a^2 B + abA - b^2}{b^{i-k+1}} a^{i-k-2} B^{i-k-1} x^{k-2} \binom{k-1}{j-1} \right], \tag{3.8}$$

$$t_{i,j}(x;a,b) = x^i\left[\frac{1}{b}x^{j-2}\binom{i-1}{j-1} - \frac{aB+bA}{b^2}x^{j-1}\binom{i-1}{j} + \sum_{k=j+2}^{i}(-1)^{k-j}\frac{a^2B+abA-b^2}{b^{k-j+1}}a^{k-j-2}B^{k-j-1}x^{k-2}\binom{i-1}{k-1}\right] \tag{3.9}$$

satisfy

$$\mathcal{Q}_n[x] = \mathcal{U}_n^{(a,b,0)}\mathcal{S}_n[x;a,b], \tag{3.10}$$

$$\mathcal{Q}_n[x] = \mathcal{T}_n[x;a,b]\mathcal{U}_n^{(a,b,0)}. \tag{3.11}$$

Proof. Similar as the proof of Theorem 3.1. □

Corollary 3.7. *The matrices* $\mathcal{S}_n[x;a,b] = [s_{i,j}(x;a,b)]$ *and* $\mathcal{T}_n[x;a,b] = [t_{i,j}(x;a,b)]$, $i,j = 1,\dots,n$, $(b \neq 0)$ *whose entries are defined by*

$$s_{i,j}(x;a,b) = x^j\left[\frac{1}{b}x^{i-2}\binom{i-1}{j-1} - \frac{a+b}{b^2}x^{i-3}\binom{i-2}{j-1} + \sum_{k=j}^{i-2}(-1)^{i-k}\frac{a^2+ab-b^2}{b^{i-k+1}}a^{i-k-2}x^{k-2}\binom{k-1}{j-1}\right],$$

$$t_{i,j}(x;a,b) = x^i\left[\frac{1}{b}x^{j-2}\binom{i-1}{j-1} - \frac{a+b}{b^2}x^{j-1}\binom{i-1}{j} + \sum_{k=j+2}^{i}(-1)^{k-j}\frac{a^2+ab-b^2}{b^{k-j+1}}a^{k-j-2}x^{k-2}\binom{i-1}{k-1}\right]$$

satisfy

$$\mathcal{Q}_n[x] = \mathcal{F}_n^{(a,b,0)}\mathcal{S}_n[x;a,b],$$

$$\mathcal{Q}_n[x] = \mathcal{T}_n[x;a,b]\mathcal{F}_n^{(a,b,0)}.$$

Theorem 3.4. *In the case* $b \neq 0$ *the matrix* $\mathcal{G}_n\left[-\frac{a}{b};a,b\right]$ *is defined by*

$$g_{i,j}\left(-\frac{a}{b};a,b\right) = \frac{(-a)^{i-j-2}}{b^{i-j+1}}\left[a^2\binom{i-1}{j-1} + (a+b)a\binom{i-2}{j-1} + (a^2+ab-b^2)\binom{i-2}{j}\right] \tag{3.12}$$

and satisfies

$$\mathcal{P}_n\left[-\frac{a}{b}\right] = \mathcal{F}_n^{(a,b,0)}\mathcal{G}_n\left[-\frac{a}{b};a,b\right]. \tag{3.13}$$

Proof. Follows from Corollary 3.1 and the following simple combinatorial identity:

$$\sum_{k=j}^{i-2}\binom{k-1}{j-1} = \binom{i-2}{j}. \quad \square$$

In a similar way as Theorem 3.4, the following result can be proved:

Theorem 3.5. *The matrix* $\mathcal{S}_n\left[-\frac{a}{b};a,b\right]$ $(b \neq 0)$ *is defined by*

$$s_{i,j}\left(-\frac{a}{b};a,b\right) = \frac{(-a)^{i+j-4}}{b^{i+j-1}}\left[a^2\binom{i-1}{j-1} + (a^2+ab)\binom{i-1}{j} - b^2\binom{i-2}{j}\right] \tag{3.14}$$

and satisfies

$$\mathcal{Q}_n\left[-\frac{a}{b}\right] = \mathcal{F}_n^{(a,b,0)}\mathcal{S}_n\left[-\frac{a}{b}; a, b\right]. \tag{3.15}$$

In the partial case $a = 2, b = 1$ Theorems 3.4, 3.5 and Corollary 3.7 yield the following results:

Corollary 3.8. *The Lucas matrix satisfies:*

$$P_n[-2] = \mathcal{L}_n\mathcal{G}_n[-2; 2, 1],$$
$$Q_n[-2] = \mathcal{L}_n\mathcal{S}_n[-2; 2, 1],$$
$$P_n[-2] = \mathcal{H}_n[-2; 2, 1]\mathcal{L}_n,$$
$$Q_n[-2] = \mathcal{T}_n[-2; 2, 1]\mathcal{L}_n,$$

where

$$g_{i,j}(-2; 2, 1) = (-2)^{i-j-2}\left[4\binom{i-1}{j-1} + 6\binom{i-2}{j-1} + 5\binom{i-2}{j}\right],$$
$$s_{i,j}(-2; 2, 1) = (-2)^{i+j-4}\left[4\binom{i-1}{j} + 6\binom{i-2}{j-1} - \binom{i-2}{j}\right],$$
$$h_{i,j}(-2; 2, 1) = (-2)^{i-j-2}\left[4\binom{i-1}{j-1} + 6\binom{i-1}{j} + 5\sum_{k=j+2}^{i}\binom{i-1}{k-1}\right],$$
$$t_{i,j}(-2; 2, 1) = (-2)^{i+j-4}\left[4\binom{i-1}{j-1} + 6\binom{i-1}{j} + 5\binom{i-1}{j+1} + 5\sum_{k=j+3}^{i}2^{2j+2k}\binom{i-1}{k-1}\right].$$

4. Some combinatorial identities

In this section we investigate some combinatorial identities involving the generalized Fibonacci numbers.

Theorem 4.1. *If i, j are positive integers satisfying $i \geq j + 2$, and $b \neq 0$, we have*

$$\left(-\frac{a}{b}\right)^{i-j}\binom{i-1}{j-1} = \frac{F_{i-j+1}^{(a,b)}}{b} - F_{i-j}^{(a,b)}\frac{b + (j+1)a}{b^2} + \sum_{k=j+2}^{i}F_{i-k+1}^{(a,b)}\frac{(-a)^{k-j-2}}{b^{k-j-1}} \times\left[\frac{a}{b^2}\binom{k-1}{j-1} + \frac{(a+b)a}{b^2}\binom{k-2}{j-1} + \frac{a^2+ab-b^2}{b^2}\binom{k-2}{j}\right]. \tag{4.1}$$

Proof. From (3.12) we derive the following identities:

$$g_{j,j}\left(-\frac{a}{b}; a, b\right) = \frac{1}{b},$$
$$g_{j+1,j}\left(-\frac{a}{b}; a, b\right) = \frac{(-a)^{-1}}{b^2}\left(a^2 j + (a+b)a\right) = -\frac{a+b}{b^2} + \frac{j}{b}\left(-\frac{a}{b}\right) = -\frac{b + (j+1)a}{b^2}. \tag{4.2}$$

Now, the proof can be derived by applying identities (4.2) and the next identity

$$p_n\left(-\frac{a}{b}; i, j\right) = \begin{cases}\left(-\frac{a}{b}\right)^{i-j}\binom{i-1}{j-1}, & i \geq j,\\ 0, & i < j\end{cases}$$

together with (3.12), (3.13) and (2.1). □

Theorem 4.2. *If i, j are positive integers satisfying $i \geq j+2$ and $b \neq 0$, we have*

$$\left(-\frac{a}{b}\right)^{i+j-2}\binom{i-1}{j-1} = F^{(a,b)}_{i-j+1}\frac{a^{2j-2}}{b^{2j-1}} - F^{(a,b)}_{i-j}\frac{a^{2j-2}}{b^{2j}}\left[(j+1)a+b\right] + \sum_{k=j+2}^{i} F^{(a,b)}_{i-k+1}\frac{(-a)^{k+j-4}}{b^{k+j-1}} \times\left[a^2\binom{k-1}{j-1} + (a^2+ab)\binom{k-1}{j} - b^2\binom{k-2}{j}\right]. \tag{4.3}$$

Proof. From (3.14) we derive the following identities:

$$s_{j,j}\left(-\frac{a}{b};a,b\right) = \frac{a^{2j-2}}{b^{2j-1}},$$
$$s_{j+1,j}\left(-\frac{a}{b};a,b\right) = -\frac{a^{2j-2}}{b^{2j}}\left[(j+1)a+b\right]. \tag{4.4}$$

Now, the proof can be derived by using (4.4), the next identity

$$q_n\left(-\frac{a}{b};i,j\right) = \begin{cases}\left(-\frac{a}{b}\right)^{i+j-2}\binom{i-1}{j-1}, & i \geq j\\ 0, & i<j\end{cases}$$

and (3.14), (3.15) and (2.1) □

Theorem 4.3. *For $1 \leq r \leq n$ and $b \neq 0$ we have*

$$\binom{n-1}{r-1} = \sum_{l=r}^{n} F^{(a,b)}_{n-l+1}\left[\frac{1}{b}\binom{l-1}{r-1} - \frac{a+b}{b^2}\binom{l-2}{r-1} + \sum_{k=r}^{l-2}(-1)^{l-k}\frac{a^2+ab-b^2}{b^{l-k+1}}a^{l-k-2}\binom{k-1}{r-1}\right]. \tag{4.5}$$

Proof. In the partial case $x=1$ from Corollary 3.1 we get

$$g_{i,j}(1;a,b) = \frac{1}{b}\binom{i-1}{j-1} - \frac{a+b}{b^2}\binom{i-2}{j-1} + \sum_{k=j}^{i-2}(-1)^{i-k}\frac{a^2+ab-b^2}{b^{i-k+1}}a^{i-k-2}\binom{k-1}{j-1}.$$

Now, the proof follows from

$$\binom{n-1}{r-1} = p_n(n,r) = \sum_{l=r}^{n} F^{(a,b)}_{n-l+1}g_{l,r}(1;a,b). \quad \square$$

In the partial case $a=0, b=1$ Theorem 4.3 reduces to Corollary 2.2 from [8].

Corollary 4.1. *For $1 \leq r \leq n$*

$$\binom{n-1}{r-1} = \sum_{k=r}^{n} F_{n-k+1}\frac{(k-3)!\left(r(k-1)-2(r-1)-(k-r)^2\right)}{(r-1)!(k-r)!}.$$

Proof. The proof can be completed using Theorem 4.3 and Corollary 3.2, in the same way as in [8]. □

5. Conclusion

In the present paper we introduce the matrix $\mathcal{U}^{(a,b,s)}_n$ of type s, whose entries are numbers $U^{(a,b)}_n$ satisfying the general second order non-degenerated recurrence formula $U_{n+1} = AU_n + BU_{n-1}$, $\delta = \sqrt{A^2+4B}$ real, and initial conditions $U^{(a,b)}_0 = a$, $U^{(a,b)}_1 = b$. In the case $A = B = 1$ we define the generalized Fibonacci matrix $\mathcal{F}^{(a,b,s)}_n$ of type s, whose entries are generalized Fibonacci numbers satisfying known recursive formula and initial conditions $F^{(a,b)}_0 = a$, $F^{(a,b)}_1 = b$. We observe two regular cases ($s=0$ and $s=1$) of these matrices. Generalized Fibonacci

matrices of type $s = 0$ or $s = 1$ correspond to known definition of the usual Fibonacci matrix, in the case $a = 0, b = 1$. In the case $a = 2, b = 1, s = 0$ we obtain definition of the Lucas matrix from [16]. Inversion of the matrix $\mathcal{U}_n^{(a,b,s)}$ and generalized Fibonacci matrix is considered. In certain cases we get known results from [7,11,16]. A correlation between the generalized Fibonacci matrix and the Pascal matrix of the first and the second kind is considered. In two partial cases ($a = 0, b = 1, s = 0$ and $a = 0, b = 1, s = 0$) we get known result from [8]. We get some combinatorial identities involving generalized Fibonacci numbers. In the partial case $a = 2, b = 1, s = 0$ we derive analogous result for Lucas matrices, introduced in [16].

References

[1] R. Aggarwala, M.P. Lamoureux, Inverting the Pascal matrix plus one, Amer. Math. Monthly 109 (2002) 371–377.
[2] A. Ashrafi, P.M. Gibson, An involutory Pascal matrix, Linear Algebra Appl. 387 (2004) 277–286.
[3] R. Brawer, M. Pirovino, The linear algebra of Pascal matrix, Linear Algebra Appl. 174 (1992) 13–23.
[4] G.S. Call, D.J. Vellman, Pascal matrices, Amer. Math. Monthly 100 (1993) 372–376.
[5] Gi-Cheon, Jin-Soo Kim, Stirling matrix via Pascal matrix, Linear Algebra Appl. 329 (2001) 49–59.
[6] A.F. Horadam, A generalized Fibonacci sequence, Amer. Math. Monthly 68 (1961) 455–459.
[7] Gwang-Yeon Lee, Jin-Soo Kim, Sang-Gu Lee, Factorizations and eigenvalues of Fibonaci and symmetric Fibonaci matrices, Fibonacci Quart. 40 (2002) 203–211.
[8] Gwang-Yeon Lee, Jin-Soo Kim, Seong-Hoon Cho, Some combinatorial identities via Fibonacci numbers, Discrete Appl. Math. 130 (2003) 527–534.
[9] D. Kalman, R. Mena, The Fibonacci numbers—exposed, Math. Magazine 76 (2003) 167–181.
[10] T. Koshy, Fibonacci and Lucas Numbers with Applications, Wiley, New York, 2001.
[11] P. Stănică, Cholesky factorizations of matrices associated with r-order recurrent sequences, Electron. J. Combin. Number Theory 5 (2) (2005) #A16.
[12] J.E. Walton, A.F. Horadam, Some further identities for the generalized Fibonacci sequence $\{H_n\}$, Fibonacci Quart. 12 (1974) 272–280.
[13] Eric W. Weisstein, Lucas Number, in: MathWorld—A Wolfram Web Resource. http://mathworld.wolfram.com/LucasNumber.html.
[14] Z. Zhang, The linear algebra of generalized Pascal matrix, Linear Algebra Appl. 250 (1997) 51–60.
[15] Z. Zhang, J. Wang, Bernoulli matrix and its algebraic properties, Discrete Appl. Math. 154 (2006) 1622–1632.
[16] Z. Zhang, Y. Zhang, The Lucas matrix and some combinatorial identities, Indian J. Pure Appl. Math. (in press).