# Riordan arrays and the Abel-Gould identity 

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#### Abstract

We gencralize the well-known identities of Abel and Gould in the context of Riordan arrays. This allows us to prove analogous formulas for Stirling numbers of both kinds and also for other quantities.


## 1. Introduction

Recently, Shapiro et al. [8] have formally introduced the concept of a Riordan group; it corresponds to a set of infinite, low-triangular arrays characterized by two analytic functions: the first is invertible and the second has a compositional inverse. Even though the concept can be traced back to a paper by Rogers [6] on renewal arrays, the authors give a clear formulation of the theory of Riordan arrays and relate it to the 1 -umbral calculus, as described, for instance, by Roman [7].

We believe that Riordan arrays are particularly important not only theoretically but also because they constitute a practical device for solving combinatorial sums by means of generating functions. These arrays are precisely the class of objects that allow us to translate a sum $\sum_{k=0}^{n} d_{n, k} f_{k}$ into a suitable transformation of the generating function $f(t)=\mathscr{G}_{t}\left\{f_{k}\right\}_{k \in \mathbb{N}}=\mathscr{G}\left\{f_{k}\right\}$ of the sequence $\left\{f_{k}\right\}_{k \in \mathbb{N}}$. In [9] we tried to give an accurate description of this fact.
Moreover, the concept of Riordan group is strictly related to the Lagrange inversion formula (LIF) which, in turn, is the natural device for inverting the elements in the group. In particular, many traditional applications of the LIF can be approached from a Riordan array point of view. In this paper, we focus our attention on the following two identities of Abel and Gould, respectively:

$$
\begin{equation*}
\sum_{k=0}^{n}\binom{n}{k} a(a+k)^{k-1}(b+n-k)^{n-k}=(a+b+n)^{n} \tag{1.1}
\end{equation*}
$$

$$
\begin{equation*}
\sum_{k=0}^{n} \frac{r}{r-q k}\binom{r-q k}{k}\binom{p+q k}{n-k}=\binom{p+r}{n} . \tag{1.2}
\end{equation*}
$$

We show that they are special cases of a general theorem within the theory of Riordan arrays and this theorem allows us to prove similar formulas for other quantities, such as Stirling numbers of both kinds.

In Section 2 we introduce the concept of the Lagrange group, whose properties are pre-requisite to the introduction of the Riordan group. In Section 3 we give our formulation of the Riordan array concept (which is slightly different from Shapiro's) and prove our main theorem on sums. The rest of the paper is devoted to the applications of this theorem.

As far as our notations are concerned, we write $[f(y) \mid y=g(t)]$ instead of the more traditional $\left.f(y)\right|_{y=g(t)}$, because the latter can become rather cumbersome when $g(t)$ is not simple. For example, we give the associative law explicitly:

$$
[f(y) \mid y=[g(z) \mid z=h(t)]]=[[f(y) \mid y=g(z)] \mid z=h(t)]
$$

in which both expression equal $f(g(h(t)))$.

## 2. The Lagrange group

Let $\mathscr{F}$ be the set of the formal power series on some indeterminate $t$, i.e. $\mathscr{F}=\mathbb{R} \llbracket t \rrbracket$. Even though the set $\mathbb{R}$ of real numbers can be substituted by any field $F$ with 0 characteristics, here we are mainly interested in formal power series with real coefficients. The order $\omega(f(t))$ of a formal power series $f(t) \in \mathscr{F}$ is the smallest integer $k$ for which the coefficient $f_{k}$ of $t^{k}$ is different from 0 . So we have $\omega(0)=\infty$. If $\mathscr{F}_{k}$ denotes the set of formal power series of order $k$, it is well known that a series $f(t)$ is invertible if and only if $f(t) \in \mathscr{F}_{0}$.

Two operations are defined in $\mathscr{F}$ : the sum, denoted by + , and the convolution or Cauchy product, denoted by - or the simple juxtaposition. With these operations, $\mathscr{F}$ is an integrity domain $(\mathscr{F},+, \cdot)$. If $f(t) \in \mathscr{F}_{0}$, then the convolution $f(t) g(t)=h(t)$ is invertible in the sense that knowing $h(t)$, we can go back to $g(t)$. In fact, by multiplying both sides by $f(t)^{-1}$, we obtain $g(t)=f(t)^{-1} h(t)$. In particular, $\left(\mathscr{F}_{0}, \cdot\right)$ is a group.

Let us now introduce the following operation, which we call Lagrange product:

$$
\begin{equation*}
f(t) \otimes g(t)=f(t) g(t f(t)) \tag{2.1}
\end{equation*}
$$

This product is associative:

$$
\begin{aligned}
f(t) \otimes(g(t) \otimes h(t)) & =f(t) \otimes g(t) h(t g(t))=f(t) g(t f(t)) h(t f(t) g(t f(t))) \\
& =(f(t) g(t f(t))) \otimes h(t)=(f(t) \otimes g(t)) \otimes h(t) ;
\end{aligned}
$$

it has an identity:

$$
f(t) \otimes 1=f(t)=1 \otimes f(t)
$$

and can be distributed to the terms of a sum:

$$
\begin{aligned}
f(t) \otimes(g(t)+h(t)) & =f(t)(g(t f(t))+h(t f(t))) \\
& =f(t) g(t f(t))+f(t) h(t f(t))=f(t) \otimes g(t)+f(t) \otimes h(t) .
\end{aligned}
$$

Again, $(\mathscr{F},+, \otimes)$ is an integrity domain and if $f(t) \in \mathscr{F}_{0}$, the Lagrange product can be inverted and we obtain

$$
\begin{aligned}
& f(t) \otimes g(t)=h(t)=f(t) g(t f(t)), \\
& g(t f(t))=f(t)^{-1} h(t) .
\end{aligned}
$$

By now setting $y=t f(t)$, we find

$$
g(y)=\left[f(t)^{-1} h(t) \mid t=y f(t)^{-1}\right] .
$$

When $f(t) \in \mathscr{F}_{0}$, the hypotheses of the LIF are satisfied, so $g(t)$ exists and is uniquely determined. In particular, $\left(\mathscr{F}_{0}, \otimes\right)$ is a group called the Lagrange group. The inverse of a series $f(t) \in \mathscr{F}_{0}$ is denoted by

$$
\begin{equation*}
\bar{f}(y)=\left[f(t)^{-1} \mid t=y f(t)^{-1}\right] . \tag{2.2}
\end{equation*}
$$

Some examples are now in order. First, let us consider $f(t)=(1-t)^{-1}$, and determine $\bar{f}(y)$ by setting $y=t /(1-t)$, i.e. $t=y /(1+y)$. Hence we have $\bar{f}(y)=[1-t \mid t=$ $y /(1+y)]=1 /(1+\mathrm{y})$. A more complex example is $f(t)=1 / \sqrt{1-4 t}$; in this case, too, we can find an explicit form for $t$, i.e., $t=y \sqrt{1+4 y^{2}}-2 y^{2}$, from which we have $\bar{f}(t)=\sqrt{1+4 y^{2}}-2 y$. Finally, let us consider $f(t)=\mathrm{e}^{-t}$; by (2.2),

$$
\bar{f}(y)=\left[\mathrm{e}^{t} \mid t=y \mathrm{e}^{t}\right]
$$

and by a simple application of the LIF,

$$
\begin{aligned}
\bar{f}_{n} & =\left[y^{n}\right]\left[\mathrm{e}^{t} \mid t=y \mathrm{e}^{t}\right]=\frac{1}{n}\left[t^{n-1}\right]\left(D \mathrm{e}^{t}\right) \mathrm{e}^{n t} \\
& =\frac{1}{n}\left[t^{n-1}\right] \mathrm{e}^{(n+1) t}=\frac{1}{n} \frac{(n+1)^{n-1}}{(n-1)!}=\frac{(n+1)^{n-1}}{n!} .
\end{aligned}
$$

Since $\bar{f}_{0}=f_{0}^{-1}=1$, this formula is valid for every $n \in \mathbb{N}$, and hence:

$$
\bar{f}(t)=\sum_{n=0}^{\infty} \frac{(n+1)^{n-1}}{n!} t^{n}=\mathscr{E}(t) .
$$

This is a rather complicated expression and, as far as I know, it cannot be expressed in terms of elementary functions.

It is not difficult to establish a number of elementary properties of the Lagrange product, and, therefore, of the Lagrange group, too. For example ( $\alpha \in \mathbb{R}$ ):

$$
\begin{aligned}
& \alpha \otimes f(t)=\alpha f(\alpha t), \quad f(t) \otimes t^{k}=(t f(t))^{k} f(t) \\
& f(t) \otimes(g(t) h(t))=f(t)^{-1}(f(t) \otimes g(t))(f(t) \otimes h(t))
\end{aligned}
$$

and so on. However, it seems to us that the most important properties of the Lagrange product are its two identities, which we are now going to prove. For $f(t) \in \mathscr{F}_{0}$, let us introduce the following notation:

$$
\hat{g}_{(f)}(t)=\left[g(y) \mid y=t f(y)^{-1}\right] .
$$

Note that $\hat{f}_{(f)}(t)=\bar{f}(t)^{-1}$. When $f(t)$ is fixed, the subscript $(f)$ is understood. We now have

$$
\begin{aligned}
\hat{g}_{(f)}(t f(t)) & =\left[\left[g(y) \mid y=z f(y)^{-1}\right] \mid z=t f(t)\right] \\
& =\left[g(y) \mid y=t f(t) f(y)^{-1}\right]=[g(y) \mid y f(y)=t f(t)]=g(t)
\end{aligned}
$$

since when $f(t)$ is invertible, i.e. $f(t) \in \mathscr{F}_{0}, y f(y)=t f(t)$ implies $y=t$. We also find

$$
\begin{aligned}
g(t \bar{f}(t)) & =\left[g(z) \mid z=t\left[f(y)^{-1} \mid y=t f(y)^{-1}\right]\right] \\
& =\left[\left[g(z) \mid z=y f(y) f(y)^{-1}\right] \mid y=t f(y)^{-1}\right]=\left[g(y) \mid y=t f(y)^{-1}\right]=\hat{g}_{(f)}(t) .
\end{aligned}
$$

Consequently, we can state formally the following result.

Theorem 2.1. If $f(t)$ is invertible, in particular if $f(t)$ and $g(t)$ belong to the Lagrange group, then the two identities hold:

$$
\begin{equation*}
\hat{g}_{(f)}(t f(t))=g(t) \quad g(t \bar{f}(t))=\hat{g}_{(f)}(t) . \tag{2.3}
\end{equation*}
$$

As a simple example, let us consider $g(t)=\mathrm{e}^{t}$ and $f(t)=\mathrm{e}^{-t}$; as we have already seen, $\bar{f}(t)=\mathscr{E}(t)=\hat{g}_{(f)}(t)=\left[\mathrm{e}^{y} \mid y=t \mathrm{e}^{t}\right]$. Hence, we have

$$
\mathscr{E}\left(t \mathrm{e}^{-t}\right)=\mathrm{e}^{t} \quad \exp (t \mathscr{E}(t))=\mathscr{E}(t) .
$$

## 3. The Riordan group

The set $\mathscr{F}$ of formal power series can be identified with the set of sequences $\mathscr{S}=\left\{\left\{f_{n}\right\}_{n \in N}\right\}$ of real numbers. If $\left\{f_{n}\right\}_{n \in \mathbb{N}} \in \mathscr{P}$, the corresponding formal power series $f(t) \in \mathscr{F}$ is $f(t)=\sum_{n=0}^{\infty} f_{n} t^{n}$, and $f(t)$ is called the generating function of the sequence. We write $f(t)=\mathscr{G}_{t}\left\{f_{n}\right\}_{n \in \mathbb{N}}=\mathscr{G}\left\{f_{n}\right\}$, when there can be no confusion on the binding of $n$ and $t$.

Let $\mathscr{L}$ be the set of linear operators on $\mathscr{S}$. If $A \in \mathscr{L}, A$ can be represented by an infinite, bidimensional array $A=\left\{a_{n, k} \mid n, k \in \mathbb{N}\right\}$, such that every row only contains a finite number of elements different from 0 . If $f \in \mathscr{P}$, i.e., $f=\left\{f_{n}\right\}_{n \in \mathbb{N}}, f$ is represented as a column vector, and if $A f=g$, then we have $g_{n}=\sum_{k=0}^{\infty} a_{n, k} f_{k}$, but the sum is actually finite. The product of two linear operators $A$ and $B$ is denoted by $A B$ or $A * B$ and is defined by $(A B) f=A(B f)$, for every $f \in \mathscr{F}$. In the array representation, $A B$ corresponds to the usual row-by-column product of the two arrays $A$ and $B$. Hence, the product is
associative and has the identity operator $I$ as the only identity; moreover, the product can be distributed to the terms of a sum $A+B$, and some operators $A \in \mathscr{L}$ have an inverse $A^{-1}$ such that $A A^{-1}=A^{-1} A=I$. Since for some $A, B \in \mathscr{L}$ and different from 0 , we can have $A B=0$, we conclude that ( $\mathscr{L},+, *$ ) is a ring with zero divisors.

An operator $A=\left\{a_{n, k} \mid n, k \in \mathbb{N}\right\}$ can also be represented by its bivariate generating function $A(t, w)=\sum_{n, k} a_{n, k} t^{n} w^{k}$, and we often refer to its column generating functions $\left\{\sum_{n=0}^{\infty} a_{n, k} t^{n}\right\}_{k \in \mathbb{N}}$. An important class of linear operators is defined in terms of column generating functions. Let $d(t), h(t)$ be two invertible formal power series, i.e., $d(t), h(t) \in \mathscr{F}_{0}$. A Riordan array is an array $D=(d(t), h(t))=\left\{d_{n, k} \mid n, k \in \mathbb{N}\right\}$, whose $k$ th column generating function is defined as $d(t)(t h(t))^{k}$. By this definition, it can be easily seen that $D$ is a low-triangular bidimensional array and, therefore, it is a linear operator in $\mathscr{L}$. The following is a simple example, based on the Riordan array $D=\left((1-t)^{-1},(1-t)^{-1}\right)$; the element $d_{n, k}$ is defined as the $n$th coefficient in the $k t h$ column generating function for $D$, i.e.

$$
d_{n, k}=\left[t^{n}\right] \frac{1}{1-t}\left(\frac{t}{1-t}\right)^{k}=\left[t^{n-k}\right] \frac{1}{(1-t)^{k+1}}=\binom{-k-1}{n-k}(-1)^{n-k}=\binom{n}{k} .
$$

Hence $D$ is the Pascal triangle.
A basic property of Riordan arrays is the following one: let $D=(d(t), h(t))$ be a Riordan array and let $f(t)=\mathscr{G}\left\{f_{n}\right\}$ be the generating function of any sequence $\left\{f_{n}\right\}_{n \in \mathbb{N}} \in \mathscr{P}$. We then have

$$
\begin{align*}
\sum_{k} d_{n, k} f_{k} & =\sum_{k}\left[t^{n}\right] d(t)(t h(t))^{k}\left[y^{k}\right] f(y) \\
& =\left[t^{n}\right] d(t) \sum_{k}\left[y^{k}\right] f(y)(t h(t))^{k}=\left[t^{n}\right] d(t) f(t h(t)) . \tag{3.1}
\end{align*}
$$

This means that the sum $\sum_{k} d_{n, k} f_{k}$ can be reduced to the extraction of the coefficient of $t^{n}$ from the function $d(t) f(t h(t)$ ), which is a simple transformation of the generating function $f(t)$ and the two functions defining the Riordan array. For example

$$
\sum_{k}\binom{n}{k} f_{k}=\left[t^{n}\right] \frac{1}{1-t} f\left(\frac{t}{1-t}\right)
$$

is the well-known Euler transformation. In [9] we illustrated many applications of this result about Riordan arrays.

Formula (3.1) can also be written as

$$
(d(t), h(t)) * f(t)=d(t) f(t h(t))
$$

and the same argument can be extended to every array $F$ whose column generating functions $f_{k}(t)=\sum_{n} f_{n, k} t^{n}$ are known. In fact, we obtain

$$
(d(t), h(t)) * f(t, w)=\sum_{k} d(t) f_{k}(t h(t)) w^{k}
$$

and this corresponds to the row-by-column product $D * F=D F$. When $F$ is a Riordan array, say $F=(a(t), b(t))$, we have $f_{k}(t)=a(t)(t b(t))^{k}$ and therefore

$$
\begin{aligned}
& (d(t), h(t)) *(a(t), b(t))=\sum_{k=0}^{\infty} d(t)\left[a(y)(y b(y))^{k} \mid y=t h(t)\right] w^{k} \\
& \quad=\sum_{k=0}^{\infty} d(t) a(t h(t))(\operatorname{th}(t) b(t h(t)))^{k} w^{k}=(d(t) a(t h(t)), h(t) b(t h(t)))
\end{aligned}
$$

This is the proof that the product of two Riordan arrays is a Riordan array. Furthermore, since $I=(1,1)$ is a Riordan array, we obtain that the identity $I$ is in the set $R$ of Riordan arrays, and from:

$$
(d(t), h(t)) *(a(t), b(t))=(d(t) a(t h(t)), h(t) b(t h(t)))=(1,1),
$$

we find that

$$
\begin{aligned}
(a(y), b(y)) & =\left(\left[d(t)^{-1} \mid t=y h(t)^{-1}\right],\left[h(t)^{-1} \mid t=y h(t)^{-1}\right]\right) \\
& =\left(\hat{d}_{(k)}(t)^{-1}, \hat{h}_{(t)}(t)^{-1}\right)=\left(\hat{d}_{(h)}(t)^{-1}, \bar{h}(t)\right),
\end{aligned}
$$

in the notations of the Lagrange group. Therefore, $(d(t), h(t))$ has an inverse and $(\mathscr{R}, *)$ is a group, called the Riordan group. Many properties of the Riordan group are described in [8,9]. In the present paper, we are primarily interested in the following result, which we call the Abel-Gould identity.

Theorem 3.1. Let $D=(d(t), h(t))$ be a Riordan array and let $f(t)$ be the generating function of a sequence $\left\{f_{n}\right\}_{n \in N} \in \mathscr{S}$. If $\left\{\hat{f}_{n}\right\}_{n \in \mathbb{N}}$ is the sequence, whose generating function is $\hat{f}_{(h)}(t)=\left[f(t) \mid t=y h(t)^{-1}\right]$, then:

$$
\begin{equation*}
\sum_{k=0}^{\infty} d_{n, k} \hat{f}_{k}=\left[t^{n}\right] d(t) f(t) \tag{3.2}
\end{equation*}
$$

Proof. By means of the elementary properties of the Riordan and Lagrange groups (see formulas (3.1) and (2.3)) we immediately find

$$
\sum_{k=0}^{\infty} d_{n, k} \hat{f}_{k}=\left[t^{n}\right] d(t) \dot{f}_{(k)}(t h(t))=\left[t^{n}\right] d(t) f(t)
$$

The most common example of this is the Abel identity. Let $D$ be the Riordan array $\left(\mathrm{e}^{p r}, \mathrm{e}^{q t}\right.$ ) and let $f(t)=\mathrm{e}^{r t}$, with $p, q, r \in \mathbb{R}$. Then we have

$$
\begin{aligned}
& d_{n, k}=\left[t^{n}\right] \mathrm{e}^{p t}\left(t \mathrm{e}^{q}\right)^{k}=\left[t^{n-k}\right] \mathrm{e}^{(p+q k) t}=\frac{(p+q k)^{n-k}}{(n-k)!}, \\
& \hat{f}_{\mathrm{k}}=\left[t^{k}\right]\left[\mathrm{e}^{r y} \mid y=t \mathrm{c}^{-q \nu}\right]=\frac{1}{k}\left[y^{k-1}\right]\left(D \mathrm{c}^{r y}\right) \mathrm{e}^{-q k y}=\frac{r}{k}\left[y^{k-1}\right] \mathrm{c}^{(r-q k) y} \\
& \\
& \quad=\frac{r(r-q k)^{k-1}}{k}=r \frac{(r-q k)^{k-1}}{k!} .
\end{aligned}
$$

The theorem now gives

$$
\sum_{k=0}^{\infty} \frac{(p+q k)^{n-k}}{(n-k)!} r \frac{(r-q k)^{k-1}}{k!}=\left[t^{n}\right] \mathrm{e}^{p t} \mathrm{e}^{r}=\frac{(p+r)^{n}}{n!},
$$

which is usually written as

$$
\begin{equation*}
\sum_{k=0}^{\infty}\binom{n}{k} r(r-q k)^{k-1}(p+q k)^{n-k}=(p+r)^{n} \tag{3.3}
\end{equation*}
$$

It is now sufficient to set $r=a, q=-1, p=b+n$ to obtain (1.1).

## 4. The Abel identity

Identities (2.1) and (3.3) are special cases of a more general identity. Let us re-examine the Riordan array $D=\left(\mathrm{e}^{p t}, \mathrm{e}^{q t}\right)$, with $p, q \in \mathbb{R}$, and let $f(t)=t^{s} \mathrm{e}^{r t}$. For $k \neq 0$, $k \geqslant s$, we have

$$
\begin{aligned}
& \hat{f}_{k}=\left[t^{k}\right]\left[y^{s} \mathrm{e}^{r y} \mid y=t \mathrm{e}^{-q y}\right]=\frac{1}{k}\left[y^{k-1}\right]\left(D y^{s} \mathrm{e}^{r y}\right) \mathrm{e}^{-q k y} \\
&=\frac{1}{k}\left[y^{k-1}\right]\left(s y^{s-1}+r y^{s}\right) \mathrm{e}^{(r-q k) y}=\frac{s}{k} \frac{(r-q k)^{k-s}}{(k-s)!}+\frac{r}{k} \frac{(r-q k)^{k-s-1}}{(k-s-1)!} \\
&=\frac{s(r-q k)^{k-s}}{k} \frac{r(k-s)}{(k-s)!}+\frac{(r-q k)^{k-s}}{k(r-q k)} \frac{r-q s}{(k-s)!}=\frac{(r-q k)^{k-s}}{r-q k} \\
&(k-s)!
\end{aligned}
$$

For $k=0$ we use the formula (see [4, p. 17]):

$$
\begin{equation*}
\hat{f}_{0}=\left[t^{0}\right] f(t)-\left[t^{-1}\right] f(t) \frac{\phi^{\prime}(t)}{\phi(t)} \tag{4.1}
\end{equation*}
$$

When $s<0$ we find

$$
\hat{f}_{0}=\left[t^{0}\right] t^{s} \mathrm{e}^{r t}-\left[t^{-1}\right] t^{s} \mathrm{e}^{r t} \frac{-q \mathrm{e}^{-q t}}{\mathrm{e}^{-q t}}=\frac{r^{-s}}{(-s)!}+q \frac{r^{-s-1}}{(-s-1)!}=\frac{r-q s}{r} \frac{r^{-s}}{(-s)!},
$$

which coincides with the previous expression for $\hat{f}_{k}$ having $k=0$. Theorem 3.1 now gives

$$
\sum_{k=s}^{\infty} \frac{(p+q k)^{n-k}}{(n-k)!} \frac{r-q s}{r-q k} \frac{(r-q k)^{k-s}}{(k-s)!}=\frac{(p+r)^{n-s}}{(n-s)!}
$$

and this can be written as

$$
\begin{equation*}
\sum_{k=s}^{\infty}\binom{n-s}{n-k}(r-q s)(r-q k)^{k-s-1}(p+q k)^{n-k}=(p+r)^{n-s} . \tag{4.2}
\end{equation*}
$$

Obviously, formula (3.3) is obtained by setting $s=0$, and the constant $r-q s$ can also be moved to the right-hand side. Form (4.2) is usually preferred, because it emphasizes the fact that the right-hand member does not depend on $q$, and this may seem rather surprising.

If we start with (4.2) and specialize the values of $p, q, r$ and $s$, a great number of familiar identities can be obtained. For example, by setting $p=0, q=-1, r=x, s=0$, we have

$$
\sum_{k=0}^{\infty}\binom{n}{k}(-1)^{n+k} k^{n-k} x(x+k)^{k-1}=x^{n}
$$

which is in [5, p. 97]. Analogously, for $p=n+1, q=-1, r=0, s=1$, we obtain

$$
\sum_{k=1}^{\infty}\binom{n-1}{n-k} k^{k-2}(n-k+1)^{n-k}=(n+1)^{n-1}
$$

However, since $\binom{n-1}{n-k}=\binom{n}{k} k / n$, by multiplying both members by $n$ we find

$$
\sum_{k=1}^{\infty}\binom{n}{k} k^{k-1}(n-k+1)^{n-k}=n(n+1)^{n-1}
$$

and this is Problem 1 in [5, p. 116]. A third example, also taken from [5, p. 117] is obtained for $p=n, q=0, r=1, s=1$ :

$$
\sum_{k=1}^{\infty}\binom{n-1}{n-k} n^{n-k}=\sum_{k=1}^{\infty}\binom{n}{k} k n^{n-k-1}=(n+1)^{n-1} .
$$

It is worth noting that $r=2$ gives the second identity in the same problem. The inverse relations are obtained for $p=0, q=-1, r=1, s=0$ :

$$
\sum_{k=0}^{\infty}\binom{n}{k}(-1)^{n+k} k^{n-k}(k+1)^{k-1}=1,
$$

while for $p=0, q=-1, r=2, s=0$ :

$$
\sum_{k=0}^{\infty}\binom{n}{k}(-1)^{n+k} k^{n-k} 2(k+2)^{k-2}=2^{n}
$$

Formula (4.2) generalizes these identities to every value of $r$.
Another couple of inverse relations [5, p. 119] are obtained for $p=n, q=0, r=-1$, $s=1$ and $p=0, q=1, r=1, s=0$ :

$$
\begin{aligned}
& \sum_{k=1}^{\infty}\binom{n-1}{n-k}(-1)^{k-1} n^{n-k}(-1)^{k-2}=\sum_{k=1}^{\infty}\binom{n}{k}(-1)^{k-1} k n^{n-k-1}=(n-1)^{n-1}, \\
& \sum_{k=0}^{\infty}\binom{n}{k}(-1)^{k-1} k^{n-k}(k-1)^{k-1}=1^{n}=1
\end{aligned}
$$

The following two examples are taken from [2, Exercises 2.4.2.c and 2.4.2.a]. By setting $p=x+n, q=-1, r=\alpha, s=0$ and performing the transformation $k \rightarrow n-k$, we have

$$
\sum_{k=0}^{\infty}\binom{n}{k} \alpha(\alpha+n-k)^{n-k-1}(x+k)^{k}=(x+n+\alpha)^{n} .
$$

By isolating the term for $k=n$, which evaluates at $(x+n)^{n}$, and by moving it to the right-hand member, we eventually find

$$
\sum_{k=0}^{n-1}\binom{n}{k}(\alpha+n-k)^{n-k-1}(x+k)^{k}=\frac{(x+n+\alpha)^{n}-(x+n)^{n}}{\alpha} .
$$

Analogously, for $p=n, q=-1, r=0, s=1$, we obtain

$$
\sum_{k=1}^{n}\binom{n-1}{n-k} k^{k-2}(n-k)^{n-k}=n^{n-1}
$$

We can now observe that $\binom{n-1}{n-k}=\binom{n-1}{k} k /(n-k)$ except when $k=n$. By isolating the corresponding term and moving it to the right-hand member, we eventually obtain the identity desired:

$$
\sum_{k=1}^{n-1}\binom{n-1}{k} k^{k-1}(n-k)^{n-k-1}=n^{n-1}-n^{n-2}=(n-1) n^{n-2} .
$$

The applications of (4.2) are not always so direct. For instance, let us consider the Csorgo-Bhaskaranada identity $(\beta \neq n)$ :

$$
\begin{aligned}
& (\alpha-1)(\beta-n) \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}(\alpha+k)^{k}(\beta-k)^{n-k-1} \\
& \quad=\frac{1}{n+1}\left((\alpha+\beta)^{n}(\alpha+\beta-n-1)-(\beta+1)^{n}(\beta-n)\right)
\end{aligned}
$$

which is Exercise 2.4.2.b in [2]. By setting $s=0$ and $n=n+1$, formula (4.2) becomes

$$
\sum_{k=0}^{n+1}\binom{n+1}{n+1-k} r(r-q k)^{k-1}(p+q k)^{n+1-k}=(p+r)^{n+1}
$$

By isolating the term for $k=n+1$ and by moving it to the right-hand member, we observe that $\binom{n+1}{n+1-k}=\binom{n}{k}(n+1) /(n-k+1)$ except for $k=n+1$, and the identity becomes

$$
\sum_{k=0}^{n}\binom{n}{k} \frac{r}{n-k+1}(r-q k)^{k-1}(p+q k)^{n+1-k}=\frac{(p+r)^{n+1}-r(r-q-q n)^{n}}{n+1}
$$

We can now perform the substitution $k \rightarrow n-k$ and set $q=-1$ :

$$
\sum_{k=0}^{n}\binom{n}{k} \frac{r}{k+1}(r+n-k)^{n-k-1}(p-n+k)^{k+1}=\frac{(p+r)^{n+1}-r(r+1+n)^{n}}{n+1} .
$$

Finally, we set $\alpha=p-n, \beta=r+n$, and obtain

$$
\begin{equation*}
(\beta-n) \sum_{k=0}^{n}\binom{n}{k} \frac{1}{k+1}(\alpha+k)^{k+1}(\beta-k)^{n-k-1}=\frac{(\alpha+\beta)^{n+1}-(\beta+1)^{n}(\beta-n)}{n+1} . \tag{4.3}
\end{equation*}
$$

By using formula (4.2) again with $s=0, q=-1$, we find

$$
\sum_{k=0}^{n}\binom{n}{k} r(r+k)^{k-1}(p-k)^{n-k}=(p+r)^{n} .
$$

The substitution $k \rightarrow n-k$ now gives

$$
\sum_{k=0}^{n}\binom{n}{k} r(r+n-k)^{n-k-1}(p-n+k)^{k}=(p+r)^{n}
$$

and by setting $\alpha=p-n, \beta=r+n$ again, we obtain

$$
(\beta-n) \sum_{k=0}^{n}\binom{n}{k}(\alpha+k)^{k}(\beta-k)^{n-k-1}=(\alpha+\beta)^{n} .
$$

At this point, we substract this formula from (4.3), perform all possible simplifications and eventually obtain the Csorgo-Bhaskaranada identity.

From the basic relation (4.2), we can obtain some other interesting identities. For example, let us call $S_{n}$ the sum in (4.2); then we have

$$
\begin{aligned}
(n-s) S_{n-1} & =\sum_{k=s}^{n-1}(n-s)\binom{n-s-1}{n-k-1}(r-q s)(r-q k)^{k-s-1}(p+q k)^{n-k-1} \\
& =(n-s)(p+r)^{n-s-1}
\end{aligned}
$$

We can now observe that

$$
(n-s)\binom{n-s-1}{n-k-1}=\frac{(n-s)!}{(n-k)!(k-s)!}=\binom{n-s}{n-k}(n-k)
$$

which is also valid for $k=n$. So we can extend the sum in $(n-s) S_{n-1}$ to $k=n$ and sum that quantity to, or substract it from $S_{n}$ :

$$
\begin{align*}
S_{n} \pm(n-s) S_{n-1} & =\sum_{k=s}^{n}\binom{n-s}{n-k}(r-q s)(r-q k)^{k-s-1}(p+q k)^{n-k-1}(p+q k \pm(n-k)) \\
& =(p+r)^{n-s-1}(p+r \pm(n-s)) \tag{4.4}
\end{align*}
$$

Here we give two applications of formula (4.4). We first examine the case having $p=0, q=1, r=n, s=0$ and the sign + :

$$
\sum_{k=0}^{n}\binom{n}{k} n^{2}(n-k)^{k-1} k^{n-k-1}=n^{n-1}(2 n)
$$

which is the identity in [2, Exercise 2.4.2.a ${ }_{1}$ ]:

$$
\sum_{k=0}^{n}\binom{n}{k}(n-k)^{k-1} k^{n-k-1}=2 n^{n-2} .
$$

Again, by setting $p=n+1, q=-1, r=0, s=1$ and by considering the $-\operatorname{sign}$ in (4.4), we find

$$
\sum_{k=1}^{n}\binom{n-1}{n-k}(n-k+1)^{n-k-1} k^{k-2}(n+1-k-n+k)=(n+1)^{n-2}(n+1-n+1)
$$

However, since $\binom{n-1}{n-k}=\binom{n}{k} k / n$, we immediately have

$$
\sum_{k=1}^{n}\binom{n}{k}(n-k+1)^{n-k-1} k^{k-1}=2 n(n+1)^{n-2}
$$

which is an identity that Egorychev [2] ascribes to Renyi.
Another result strictly connected to the Abel identity can be obtained from Theorem 3.1 by using the Riordan array $D=\left(\mathrm{e}^{p t}, \mathrm{e}^{q t}\right)$ again, but in connection with the function $f(t)=\mathrm{e}^{r t} /(1+q t)$. In fact:

$$
\begin{aligned}
\hat{f}_{k} & =\left[t^{k}\right]\left[\left.\frac{\mathrm{e}^{r y}}{1+q y} \right\rvert\, y=t \mathrm{e}^{-q y}\right]=\frac{1}{k}\left[y^{k-1}\right]\left(D \frac{\mathrm{e}^{r y}}{1+q y}\right) \mathrm{e}^{-q k y} \\
& =\frac{1}{k}\left[y^{k-1}\right]\left(\frac{r \mathrm{e}^{(r-q k) t}}{1+q y}-\frac{q \mathrm{e}^{(r-q k) t}}{(1+q y)^{2}}\right) \\
& =\frac{1}{k}\left(\sum_{j=0}^{k-1}(-q)^{k-j-1} r \frac{(r-q k)^{j}}{j!}-\sum_{j=0}^{k-1}(-q)^{k-j-1}(k-j) q \frac{(r-q k)^{j}}{j!}\right) \\
& =\frac{1}{k} \sum_{j=0}^{k-1}(-q)^{k-j-1}(r-q k+q j) \frac{(r-q k)^{j}}{j!} \\
& =\frac{1}{k}\left(\sum_{j=0}^{k-1}(-q)^{k-j-1} \frac{(r-q k)^{j+1}}{j!}-\sum_{j=0}^{k-1}(-q)^{k-j} \frac{(r-q k)^{j}}{(j-1)!}\right) \\
& =\frac{1}{k}\left(\sum_{j=0}^{k-1}(-q)^{k-j-1} \frac{(r-q k)^{j+1}}{j!}-\sum_{j=0}^{k-2}(-q)^{k-j-1} \frac{(r-q k)^{j+1}}{j!}\right) \\
& =\frac{1}{k}(-q)^{0} \frac{(r-q k)^{k}}{(k-1)!}=\frac{(r-q k)^{k}}{k!} .
\end{aligned}
$$

Theorem 3.1 now gives

$$
\begin{aligned}
& \sum_{k=0}^{n} \frac{(p+q k)^{n-k}}{(n-k)!} \frac{(r-q k)^{k}}{k!}=\left[t^{n}\right] \frac{\mathrm{e}^{(p+r) t}}{1+q t}=\sum_{k=0}^{n} \frac{(p+r)^{k}}{k!}(-q)^{n-k} \\
& \quad=\frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}(p+r)^{k}(-q)^{n-k}(n-k)!=\frac{(p+r+\alpha(-q))^{n}}{n!},
\end{aligned}
$$

in which we use the notation of umbral calculus:

$$
(\alpha(-q))^{k} \equiv \alpha_{k}(-q) \equiv(-q)^{k} k!.
$$

Finally, by taking $n!$ to the right-hand member, we obtain the identity

$$
\left.\sum_{k=0}^{\infty}\binom{n}{k}(r-q k)^{k}(p+q k)^{n-k}=(p+r+\alpha(-q))\right)^{n} .
$$

By setting $p=y+n, q=-1, r=x, \alpha=\alpha(1)$, we find

$$
\sum_{k=0}^{\infty}\binom{n}{k}(x+k)^{k}(y+n-k)^{n-k}=(x+y+n+\alpha)^{n}
$$

which is the case $(0,0)$ in Table 1.2 of [5].

## 5. Gould identities

From the mid-fifties to the mid-seventies, Gould found a considerable amount of combinatorial identities, among which we believe (1.2) to be particularly significant (also see [5]). If we examine the Riordan array $D=\left((1+\alpha t)^{p},(1+\alpha t)^{q}\right)$, the generic element $d_{n, k}$ is clearly:

$$
d_{n, k}=(1+\alpha t)^{p}\left(t(1+\alpha t)^{q}\right)^{k}=\left[t^{n-k}\right](1+\alpha t)^{p+q k}=\binom{p+q k}{n-k} \alpha^{n-k},
$$

in which $p, q \in \mathbb{R}$. Let $f(t)=t^{s}(1+\alpha t)^{r}$, then for $k \neq 0, k \geqslant s$ we have

$$
\begin{aligned}
\hat{f}_{k} & =\left[t^{k}\right]\left[y^{s}(1+\alpha y)^{r} \mid y=t(1+\alpha y)^{-q}\right]=\frac{1}{k}\left[y^{k-1}\right]\left(D y^{s}(1+\alpha y)^{r}\right)(1+\alpha y)^{-q k} \\
& =\frac{1}{k}\left[y^{k-1}\right]\left(s y^{s-1}(1+\alpha y)^{r}+\alpha r y^{s}(1+\alpha y)^{r-1}\right)(1+\alpha y)^{-q k} \\
& =\frac{s}{k}\left[y^{k-s}\right](1+\alpha y)^{r-q k}+\frac{\alpha r}{k}\left[y^{k-s-1}\right](1+\alpha y)^{r-1-q k} \\
& =\frac{s}{k}\binom{r-q k}{k-s} \alpha^{k-s}+\frac{\alpha r}{k}\binom{r-q k-1}{k-s-1} \alpha^{k-s-1} \\
& =\frac{s}{k}\binom{r-q k}{k-s} \alpha^{k-s}+\frac{r}{k} \frac{k-s}{r-q k}\binom{r-q k}{k-s} \alpha^{k-s}=\frac{r-q s}{r-q k}\binom{r-q k}{k-s} \alpha^{k-s} .
\end{aligned}
$$

When $k=0$, we use formula (4.1), which gives, for $s<0$,

$$
\begin{aligned}
\hat{f}_{0} & =\left[t^{0}\right] t^{s}(1+\alpha t)^{r}-\left[t^{-1}\right] t^{s}(1+\alpha t)^{r} \frac{-q \alpha(1+\alpha t)^{-q-1}}{(1+\alpha t)^{-q}} \\
& =\binom{r}{-s} \alpha^{-s}+q \alpha^{-s}\binom{r-1}{-s-1}=\alpha^{-s} \frac{r-q s}{r}\binom{r}{-s} .
\end{aligned}
$$

This expression coincides with the previous one for $\hat{f}_{k}$ having $k=0$. Finally, we find

$$
\left[t^{n}\right] d(t) f(t)=\left[t^{n}\right](1+\alpha t)^{p} t^{s}(1+\alpha t)^{r}=\left[t^{n-s}\right](1+\alpha t)^{p+r}=\binom{p+r}{n-s} \alpha^{n-s}
$$

Consequently, Theorem 3.1 gives the identity

$$
\begin{equation*}
\sum_{k=s}^{n} \frac{r-q s}{r-q k}\binom{r-q k}{k-s}\binom{p+q k}{n-k}=\binom{p+r}{n-s}, \tag{5.1}
\end{equation*}
$$

which reduces to (1.2) when we set $s=0$.
We could now specialize the values of $p, q, r, s$ and obtain a number of well-known identities, as we previously did for Abel's identity. At this point, however, this would not be very meaningful, so we just observe that when $q$ divides $r$, a term in the sum (5.1) becomes $((r-q s) / 0)\binom{0}{k-s)}\binom{p+q k}{n-k}$. This term can be changed into a manageable form by using the rule $(1 / m)\binom{m}{h}=(1 / h)\binom{m-1}{h-1}$, except when $h=0$. We can also consider the expression $(1 / 0)\binom{0}{k}$ as the limit of $\frac{1}{x}\binom{x}{k}$ when $x \rightarrow 0$. When $k \neq 0$, we have:

$$
\begin{aligned}
\frac{1}{0}\binom{0}{k} & =\lim _{x \rightarrow 0} \frac{1}{x}\binom{x}{k}=\lim _{x \rightarrow 0} \frac{1}{x} \frac{x(x-1) \cdots(x-k+1)}{k!} \\
& =\lim _{x \rightarrow 0} \frac{(x-1) \cdots(x-k+1)}{k!}=\frac{(-1)^{k-1}}{k}
\end{aligned}
$$

The reader can easily verify that this position is correct; for example, by setting $p=n$, $q=1, r=n$ and $s=0$ in (5.1), we obtain the identity

$$
\sum_{k=0}^{n} \frac{n}{n-k}\binom{n-k}{k}\binom{n+k}{n-k}=\binom{2 n}{n}
$$

and this is true only if we set $(n / 0)\binom{0}{n}\binom{n}{n}=(-1)^{n-1}$ for $k=n$. In other words, we have

$$
\sum_{k=0}^{n-1} \frac{n}{n-k}\binom{n-k}{k}\binom{n+k}{n-k}=\sum_{k=0}^{n-1} \frac{n}{n-k}\binom{n}{n-2 k}\binom{n+k}{k}=\binom{2 n}{n}+(-1)^{n} .
$$

Another identity found by Gould is obtained by using the Riordan array $D=\left((1+t)^{p},(1+t)^{-q}\right)$ and the function $f(t)=(1+t)^{r+1} /(1-(q-1) t)$. The array's generic element is $d_{n, k}=\binom{p-q k}{n-k}$ and we can find $\hat{f}_{k}$ in the following way:

$$
\begin{aligned}
\hat{f}_{k} & =\left[t^{k}\right]\left[\left.\frac{(1+y)^{r+1}}{1-(q-1) y} \right\rvert\, y=t(1+y)^{q}\right] \\
& =\frac{1}{k}\left[y^{k-1}\right]\left((r+1) \frac{(1+y)^{r+q k}}{1-(q-1) y}+(q-1) \frac{(1+y)^{r+q k+1}}{(1-(q-1) y)^{2}}\right) \\
& =\frac{r+1}{k} \sum_{j=0}^{k-1}\binom{r+q k}{j}(q-1)^{k-j-1}+\frac{q-1}{k} \sum_{j=0}^{k-1}\binom{r+q k+1}{j}(k-j)(q-1)^{k-j-1} .
\end{aligned}
$$

We now write $r+1$ as $r+q k+1-q k$ and obtain

$$
\begin{aligned}
\hat{f}_{k}= & \frac{1}{k} \sum_{j=0}^{k-1}(r+q k+1)\binom{r+q k}{j}(q-1)^{k-j-1}-\frac{q}{k} \sum_{j=0}^{k-1}\binom{r+q k}{j} k(q-1)^{k-j-1} \\
& +\frac{q-1}{k} \sum_{j=0}^{k-1}\binom{r+q k+1}{j} k(q-1)^{k-j-1}-\frac{1}{k} \sum_{j=0}^{k-1}\binom{r+q k+1}{j} j(q-1)^{k-j}
\end{aligned}
$$

When we simplify, the terms of the first and fourth sums are annulled and only leave the terms for $j=k$ and $j=0$ and the latter is zero. We can now use the recurrence relation for binomial coefficients:

$$
\begin{aligned}
\hat{f}_{k}= & \binom{r+q k+1}{k}-q \sum_{j=0}^{k-1}\binom{r+q k}{j}(q-1)^{k-j-1} \\
& +(q-1) \sum_{j=0}^{k-1}\binom{r+q k}{j}(q-1)^{k-j-1}+(q-1) \sum_{j=0}^{k-1}\binom{r+q k}{j-1}(q-1)^{k-j-1}
\end{aligned}
$$

The second and third sums are simplified and by performing the transformation $j \rightarrow j+1$, we get

$$
\begin{aligned}
\hat{f}_{k}= & \binom{r+q k+1}{k}-\sum_{j=0}^{k-1}\binom{r+q k}{j}(q-1)^{k-j-1}+\sum_{j=0}^{k-2}\binom{r+q k}{j}(q-1)^{k-j-1} \\
& =\binom{r+q k+1}{k}-\binom{r+q k}{k-1}=\binom{r+q k}{k} .
\end{aligned}
$$

Hence, Theorem 3.1 gives the identity

$$
\begin{equation*}
\sum_{k=0}^{n}\binom{r+q k}{k}\binom{p-q k}{n-k}=\left[t^{n}\right] \frac{(1+t)^{p+r+1}}{1-(q-1) t}: \tag{5.2}
\end{equation*}
$$

The last coefficient is

$$
\sum_{k=0}^{n}\binom{p+r+1}{n-k}(q-1)^{k}=\sum_{k=0}^{n}\binom{p+r-k}{n-k} q^{k}
$$

and the Gould identity is usually given with one of these two sums as the right-hand member. When the parameters $p, q, r$ are appropriately chosen, we can obtain a closed form as in the following case:

$$
\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\binom{x-m k}{n}=\sum_{k=0}^{n}\binom{-x-1+(m+1) k}{k}\binom{x-(m+1) k}{n-k} .
$$

This is the left-hand member of (5.2) if we set $p=x, q=m+1, r=-x-1$. We therefore have

$$
\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\binom{x-m k}{n}=\left[t^{n}\right] \frac{(1+t)^{x-x-1+1}}{1-m t}=\left[t^{n}\right] \frac{1}{1-m t}=m^{n}
$$

We conclude this section by giving a formula involving the harmonic numbers $H_{n}=\sum_{k=1}^{n} 1 / k$. The following is a well-known generating functioning:

$$
\mathscr{G}\left\{\left(H_{m+n}-H_{m}\right)\binom{m+n}{m}\right\}=\frac{1}{(1-t)^{m+1}} \ln \frac{1}{1-t},
$$

(see [3]). Let us take the Riordan array

$$
D=\left(\frac{1}{(1-t)^{p+1}} \ln \frac{1}{1-t}, \frac{1}{(1-t)^{q}}\right),
$$

whose generic element is

$$
\begin{aligned}
d_{n, k}= & {\left[t^{n}\right]\left(\frac{1}{(1-t)^{p+1}} \ln \frac{1}{1-t}\right)\left(\frac{t}{(1-t)^{q}}\right)^{k}=\left[t^{n-k}\right] \frac{1}{(1-t)^{p+q k+1}} \ln \frac{1}{1-t} } \\
& =\left(H_{p+n+(q-1) k}-H_{p+q k}\right)\binom{p+n+(q-1) k}{n-k} .
\end{aligned}
$$

Let $f(t)=t^{s}(1-t)^{-r}$; we therefore have

$$
\begin{aligned}
\hat{f}_{k} & =\left[t^{k}\right]\left[\left.\frac{y^{s}}{(1-y)^{r}} \right\rvert\, y=t(1-y)^{q}\right]=\frac{1}{k}\left[y^{k-1}\right] \frac{s y^{s-1}(1-y)+r y^{s}}{(1-y)^{r+1}}(1-y)^{q k} \\
& =\frac{s}{k}\left[y^{k-s}\right](1-y)^{-r+q k}+\frac{r}{k}\left[y^{k-s-1}\right](1-y)^{-r+q k-1} \\
& =\frac{s}{k}\binom{r-s-1-(q-1) k}{k-s}+\frac{r}{k}\binom{r-s-1-(q-1) k}{k-s-1}=\frac{r-q s}{r-q k}\binom{r-s-1-(q-1) k}{k-s} .
\end{aligned}
$$

When $k=0$, we use formula (4.1), which gives for $s<0$ :

$$
\begin{aligned}
\hat{f}_{0} & =\left[t^{0}\right] \frac{t^{s}}{(1-t)^{r}}-\left[t^{-1}\right] \frac{t^{s}}{(1-t)^{r}} \frac{-q(1-t)^{q-1}}{(1-t)^{q}} \\
& =\binom{-r}{-s}(-1)^{-s}-q\binom{-r-1}{-s-1}(-1)^{-s-1} \\
& =\binom{r-s-1}{-s}-q\binom{r-s-1}{-s-1}=\frac{r-q s}{r}\binom{r-s-1}{-s} .
\end{aligned}
$$

This expression coincides with the previous one for $\hat{f}_{k}$ having $k=0$. Therem 3.1 now gives

$$
\left.\begin{array}{rl}
\sum_{k=s}^{n} & \left(H_{p+n+(q-1) k}-H_{p+q k}\right)
\end{array} \begin{array}{c}
p+n+(q-1) k \\
n-k
\end{array}\right) \frac{r-q s}{r-q k}\binom{r-s-1-(q-1) k}{k-s} .
$$

It is worth noting case $s=0$ :

$$
\begin{aligned}
& \sum_{k=0}^{n}\left(H_{p+n+(q-1) k}-H_{p+q k}\right)\binom{p+n+(q-1) k}{n-k} \frac{r}{r-(q-1) k}\binom{r-(q-1) k}{k} \\
& \quad=\left(H_{p+r+n}-H_{p+r}\right)\binom{p+r+n}{n},
\end{aligned}
$$

and the identity obtained when $p=n, q=2, r=n$ :

$$
\sum_{k=0}^{n}\left(H_{2 n+k}-H_{n+2 k}\right)\binom{2 n+k}{n-k} \frac{n}{n-k}\binom{n-k}{k}=\left(H_{3 n}-H_{2 n}\right)\binom{3 n}{2 n} .
$$

## 6. Stirling numbers

Since

$$
\begin{aligned}
& {\left[t^{k}\right]\left(\ln \frac{1}{1-t}\right)^{p}=\frac{p!}{k!}\left[\begin{array}{l}
k \\
p
\end{array}\right] \quad \text { (Stirling numbers of the 1st kind), }} \\
& {\left[t^{k}\right]\left(\mathrm{e}^{t}-1\right)^{p}=\frac{p!}{k!}\left\{\begin{array}{l}
k \\
p
\end{array}\right\} \text { (Stirling numbers of the 2nd kind), }}
\end{aligned}
$$

it is possible to consider the two Riordan arrays:

$$
\begin{aligned}
& D^{\prime}=\left(\left(\frac{1}{t} \ln \frac{1}{1-t}\right)^{p},\left(\frac{1}{t} \ln \frac{1}{1-t}\right)^{q}\right) \text { and } \\
& D^{\prime \prime}=\left(\left(\frac{\mathrm{e}^{t}-1}{t}\right)^{p},\left(\frac{\mathrm{e}^{t}-1}{t}\right)^{q}\right)
\end{aligned}
$$

whose generic elements can be easily expressed in terms of the Stirling numbers as follows:

$$
\begin{aligned}
d_{n, k}^{\prime} & =\left[t^{n}\right]\left(\frac{1}{t} \ln \frac{1}{1-t}\right)^{p}\left(t\left(\frac{1}{t} \ln \frac{1}{1-t}\right)^{q}\right)^{k}=\left[t^{n+p+(q-1) k}\right]\left(\ln \frac{1}{1-t}\right)^{p+q k} \\
& =\frac{(p+q k)!}{(n+p+(q-1) k)!}\left[\begin{array}{c}
n+p+(q-1) k \\
p+q k
\end{array}\right], \\
d_{n, k}^{\prime \prime} & =\left[t^{n}\right]\left(\frac{\mathrm{e}^{t}-1}{t}\right)^{p}\left(t\left(\frac{\mathrm{e}^{t}-1}{t}\right)^{q}\right)^{k}=\left[t^{n+p+(q-1) k}\right]\left(\mathrm{e}^{t}-1\right)^{p+q k} \\
& =\frac{(p+q k)!}{(n+p+(q-1) k)!}\left\{\begin{array}{c}
n+p+(q-1) k \\
p+q k
\end{array}\right\} .
\end{aligned}
$$

Let us now consider the function $f(t)=(1 / t) \ln (1 /(1-t)))^{r}$ and the Riordan array $D^{\prime}$ :

$$
\begin{aligned}
\hat{f}_{k} & =\left[t^{k}\right]\left[\left.\left(\frac{1}{y} \ln \frac{1}{1-y}\right)^{r} \right\rvert\, y=t y^{q}\left(\ln \frac{1}{1-y}\right)^{-q}\right] \\
& =\frac{1}{k}\left[y^{k-1}\right]\left(-\frac{r}{y^{r+1}}\left(\ln \frac{1}{1-y}\right)^{r}+\frac{r}{y^{r}} \frac{1}{1-y}\left(\ln \frac{1}{1-y}\right)^{r-1}\right) y^{q k}\left(\ln \frac{1}{1-y}\right)^{-q k} \\
& =\frac{r}{k}\left(\left[y^{k-1+r-q k}\right] \frac{1}{1-y}\left(\ln \frac{1}{1-y}\right)^{r-1-q k}-\left[y^{k+r-q k}\right]\left(\ln \frac{1}{1-y}\right)^{r-q k}\right) .
\end{aligned}
$$

Here we observe that by differentiating:

$$
D\left(\ln \frac{1}{1-t}\right)^{m+1}=\frac{m+1}{1-t}\left(\ln \frac{1}{1-t}\right)^{m}
$$

and since $\left[t^{k}\right] D f(t)=(k+1)\left[t^{k+1}\right] f(t)$ we immediately find

$$
\left[t^{k}\right] \frac{1}{1-t}\left(\ln \frac{1}{1-t}\right)^{m}=\sum_{j=0}^{k} \frac{m!}{j!}\left[\begin{array}{c}
j \\
m
\end{array}\right]=\frac{m!}{k!}\left[\begin{array}{c}
k+1 \\
m+1
\end{array}\right] .
$$

We can now go on with our computations by using the recurrence for the Stirling numbers of the first kind:

$$
\begin{aligned}
\hat{f}_{k} & =\frac{r}{k}\left(\frac{(r-1-q k)!}{(r-1-(q-1) k)!}\left[\begin{array}{c}
k+r-q k \\
r-q k
\end{array}\right]-\frac{(r-q k)!}{(k+r-q k)!}\left[\begin{array}{c}
k+r-q k \\
r-q k
\end{array}\right]\right) \\
& =\frac{r}{k} \frac{(r-q k)!}{(r-(q-1) k)!}\left[\begin{array}{c}
r-(q-1) k \\
r-q k
\end{array}\right]\left(\frac{k+r-q k}{r-q k}-1\right) \\
& =\frac{r}{r-q k} \frac{(r-q k)!}{(r-(q-1) k)!}\left[\begin{array}{c}
r-(q-1) k \\
r-q k
\end{array}\right] .
\end{aligned}
$$

At this point, Theorem 3.1, gives the identity

$$
\begin{align*}
& \sum_{k=0}^{n} \frac{(p+q k)!}{(n+p+(q-1) k)!}\left[\begin{array}{c}
n+p+(q-1) k \\
p+q k
\end{array}\right] \frac{r}{r-q k} \frac{(r-q k)!}{(r-(q-1) k)!}\left[\begin{array}{c}
r-(q-1) k \\
r-q k
\end{array}\right] \\
& \quad=\frac{(p+r)!}{(n+p+r)!}\left[\frac{n+p+r}{p+r}\right] . \tag{6.1}
\end{align*}
$$

We can now multiply and divide by $k!(n-k)$ ! so that by using the notation $\left.\llbracket \begin{array}{l}n \\ k\end{array}\right]$ for $\binom{n}{k}^{-1}\left[\begin{array}{l}n \\ k\end{array}\right]$ we obtain the more concise form:

$$
\sum_{k=0}^{n}\binom{n}{k}\left[\begin{array}{c}
n+p+(q-1) k \\
p+q k
\end{array}\right] \frac{r}{r-q k}\left[\frac{r-(q-1) k}{r-q k}\right]=\left[\begin{array}{c}
n+p+r \\
p+r
\end{array}\right] .
$$

An analogous formula can be obtained for the Stirling numbers of the second kind. This time we use the function $f(t)=\left(\left(\mathrm{e}^{t}-1\right) / t\right)^{r}$ and obtain

$$
\begin{aligned}
\hat{f}_{k}= & {\left[t^{k}\right]\left[\left.\left(\frac{\mathrm{e}^{y}-1}{y}\right)^{r} \right\rvert\, y=t y^{q}\left(\mathrm{e}^{y}-1\right)^{-q}\right] } \\
= & \frac{1}{k}\left[y^{k-1}\right]\left(r \frac{\left(\mathrm{e}^{y}-1\right)^{r-1} \mathrm{e}^{y}}{y^{r}}-r \frac{\left(\mathrm{e}^{y}-1\right)^{r}}{y^{r+1}}\right)\left(\mathrm{e}^{y}-1\right)^{-q k} y^{q k} \\
= & \frac{r}{k}\left(\left[y^{k-1+r-q k}\right]\left(\mathrm{e}^{y}-1\right)^{r-q k}+\left[y^{k-1+r-q k}\right]\left(\mathrm{e}^{y}-1\right)^{r-1-q k}-\left[y^{k+r-q k}\right]\left(\mathrm{e}^{y}-1\right)^{r-q k}\right) \\
= & \frac{r}{k}\left(\frac{(r-1-q k)!}{(r-1-(q-1) k)!}\left((r-q k)\left\{\begin{array}{c}
r-1-(q-1) k \\
r-q k
\end{array}\right\}+\left\{\begin{array}{c}
r-1-(q-1) k \\
r-1-q k
\end{array}\right\}\right)\right. \\
& \left.-\frac{(r-q k)!}{(r-(q-1) k)!}\left\{\begin{array}{c}
r-(q-1) k \\
r-q k
\end{array}\right\}\right) \\
= & \frac{r}{k}\left(\frac{(r-1-q k)!}{(r-1-(q-1) k)!}\left\{\begin{array}{c}
r-(q-1) k \\
r-q k
\end{array}\right\}-\frac{(r-q k)!}{(r-(q-1) k)!}\left\{\begin{array}{c}
r-(q-1) k \\
r-q k
\end{array}\right\}\right) \\
= & \frac{r}{r-q k} \frac{(r-q k)!}{(r-(q-1) k)!}\left\{\begin{array}{c}
r-(q-1) k \\
r-q k
\end{array}\right\} .
\end{aligned}
$$

In the next-to-last passage we used the recurrence relation for Stirling numbers of the second kind. By Theorem 3.1, our result is very similar to the one found for Stirling numbers of the first kind:

$$
\begin{align*}
& \sum_{k=0}^{n} \frac{(p+q k)!}{(n+p+(q-1) k)!}\left\{\begin{array}{c}
n+p+(q-1) k \\
p+q k
\end{array}\right\} \frac{r}{r-q k} \frac{(r-q k)!}{(r-(q-1) k)!}\left\{\begin{array}{c}
r-(q-1) k \\
r-q k
\end{array}\right\} \\
& =\frac{(p+r)!}{(n+p+r)!}\left\{\frac{n+p+r}{p+r}\right\} . \tag{6.2}
\end{align*}
$$

By using the notation $\left.\left\{\begin{array}{l}n \\ k\end{array}\right\}\right\}$ for $\binom{n}{k}^{-1}\left\{\begin{array}{l}n \\ k\end{array}\right\}$ we obtain the more concise form

$$
\sum_{k=0}^{n}\binom{n}{k}\left\{\left\{\begin{array}{c}
n+p+(q-1) k \\
p+q k
\end{array}\right\}\right\} \underset{r-q k}{r}\left\{\left\{\begin{array}{c}
r-(q-1) k \\
r-q k
\end{array}\right\}\right\}=\left\{\left\{\begin{array}{c}
n+p+r \\
p+r
\end{array}\right\}\right\} .
$$

As usual, we can specialize the values for $p, q, r$ and obtain a series of identities. For example, by setting $p=0, q=1, r=n+1$, we find

$$
\begin{aligned}
& \sum_{k=0}^{n}\binom{n}{k}^{-1}\left[\begin{array}{l}
n \\
k
\end{array}\right]\left[\begin{array}{c}
n+1 \\
n-k+1
\end{array}\right]=\left[\begin{array}{c}
2 n+1 \\
n+1
\end{array}\right]\binom{2 n+1}{n+1}^{-1}, \\
& \sum_{k=0}^{n}\binom{n}{k}^{-1}\left\{\begin{array}{l}
n \\
k
\end{array}\right\}\left\{\begin{array}{c}
n+1 \\
n-k+1
\end{array}\right\}=\left\{\begin{array}{c}
2 n+1 \\
n+1
\end{array}\right\}\binom{2 n+1}{n+1}^{-1}
\end{aligned}
$$

More in general, for $p=0, q=1$, we find for $r>n$ :

$$
\begin{align*}
& \sum_{k=0}^{n}\binom{r-1}{k}^{-1}\left[\begin{array}{l}
n \\
k
\end{array}\right]\left[\begin{array}{c}
r \\
r-k
\end{array}\right]=\left[\begin{array}{c}
r+n \\
r
\end{array}\right]\binom{r+n}{r}^{-1},  \tag{6.3}\\
& \sum_{k=0}^{n}\binom{r-1}{k}^{-1}\left\{\begin{array}{l}
n \\
k
\end{array}\right\}\left\{\begin{array}{c}
r \\
r-k
\end{array}\right\}=\left\{\begin{array}{c}
r+n \\
r
\end{array}\right\}\binom{r+n}{r}^{-1}, \tag{6.4}
\end{align*}
$$

The value $r=n$ is to be excluded because a problem arises in formulas (6.1) and (6.2) when $q$ divides $r$ and $r \leqslant q n$. In that case, in fact, we have a factor

$$
\frac{r}{0} \frac{0!}{k!}\left[\begin{array}{l}
k \\
0
\end{array}\right] \quad \text { or } \quad \frac{r}{0} \frac{0!}{k!}\left\{\begin{array}{l}
k \\
0
\end{array}\right\}
$$

which cannot be computed directly. In order to evaluate this factor, we can proceed in the following way. We take the Stirling polynomials $\sigma_{n}(x)$ (see [3]):

$$
\sigma_{n}(x)=\left[\begin{array}{c}
x \\
x-n
\end{array}\right] \frac{1}{x(x-1) \cdots(x-n)},
$$

and observe that we can define

$$
\frac{1}{0}\left[\begin{array}{l}
n \\
0
\end{array}\right]=\lim _{x \rightarrow n}\left[\begin{array}{c}
x \\
x-n
\end{array}\right] \frac{1}{x-n}=\lim _{x \rightarrow n} x(x-1) \cdots(x-n+1) \sigma_{n}(x) .
$$

The last limit is well defined and equal to $n!\sigma_{n}(n)$. The problem is reduced to find the value of $\sigma_{n}(n)$, and this can be done by means of the generating function given in [3]:

$$
\left(\frac{t \mathrm{e}^{t}}{\mathrm{e}^{t}-1}\right)^{x}=\sum_{n=0}^{\infty} x \sigma_{n}(x) t^{n}
$$

For $x=n$, we immediately obtain

$$
n \sigma_{n}(n)=\left[t^{n}\right]\left(\frac{t \mathrm{e}^{t}}{\mathrm{e}^{t}-1}\right)^{n}
$$

We can now apply the LIF in the well-known form which gives the generating function of a sequence (see [4]): if $c_{n}=\left[t^{n}\right] F(t) \phi(t)^{n}$ then $\mathscr{G}\left\{c_{n}\right\}=C(t)=F(w)$ $\left(1-t \phi^{\prime}(w)\right)^{-1}$, where $w=w(t)$ is the solution of $w=t \phi(w)$ such that $w(0)=0$. In our case, we have $F(t)=1$ and by solving the equation $w=t w \mathrm{e}^{w} /\left(\mathrm{e}^{w}-1\right)$, we easily find $w=-\ln (1-t)$. By performing the remaining computations we eventually find

$$
\mathscr{G}\left\{n \sigma_{n}(n)\right\}=\left(\frac{1-t}{t} \ln \frac{1}{1-t}\right)^{-1} .
$$

This is the exponential generating function of the Cauchy numbers $C(t)$ (see $[1, \mathrm{p}$. 293]), and we then find

$$
\frac{1}{0}\left[\frac{n}{0}\right]=(n-1)!n \sigma_{n}(n)=\frac{C_{n}}{n} ;
$$

the identity obtained for $p=0, q=1, r=n$ is analogous to (6.3). It can be written as

$$
\sum_{k=0}^{n-1} \frac{n}{n-k}\binom{n}{k}^{-1}\left[\begin{array}{l}
n \\
k
\end{array}\right]\left[\begin{array}{c}
n \\
n-k
\end{array}\right]=\left[\begin{array}{c}
2 n \\
n
\end{array}\right]\binom{2 n}{n}^{-1}-C_{n}
$$

In a similar way, we can observe that with the substitution $y=x-n$, we obtain

$$
\frac{1}{y+n}\left[\begin{array}{c}
y+n \\
y
\end{array}\right]=(y+n-1) \cdots(y+1) y \sigma_{n}(y+n)
$$

However, since $\left[\begin{array}{c}-n \\ -k\end{array}\right]=\left\{\begin{array}{l}k \\ n\end{array}\right\}$ (see [3]), we have

$$
\frac{1}{0}\left\{\begin{array}{l}
n \\
0
\end{array}\right\}=\lim _{y \rightarrow-n}(n-1+y) \cdots(1+y) y \sigma_{n}(y)=(-1)^{n} n!\sigma_{n}(0)
$$

This time we use the generating function

$$
\left(\frac{1}{t} \ln \frac{1}{1-t}\right)^{x}=\sum_{n=0}^{\infty} x \sigma_{n}(x+n) t^{n}
$$

from which we obtain

$$
-n \sigma_{n}(0)=\left[t^{n}\right]\left(\frac{1}{t} \ln \frac{1}{1-t}\right)^{-n}
$$

We can now apply the LIF (see also [3, formula (6.101)]) finding $w=\left(\mathrm{e}^{t}-1\right) / \mathrm{e}^{t}$, and eventually

$$
\mathscr{G}\left\{-n \sigma_{n}(0)\right\}=\frac{t}{\mathrm{e}^{t}-1}
$$

This is the exponential generating function of the Bernoulli numbers and

$$
\frac{1}{0}\left\{\begin{array}{l}
n \\
0
\end{array}\right\}=(-1)^{n}(n-1)!n \sigma_{n}(0)=\frac{(-1)^{n-1} B_{n}}{n}
$$

The expression obtained for $p=0, q=1, r=n$ :

$$
\sum_{k=0}^{n-1} \frac{n}{n-k}\binom{n}{k}^{-1}\left\{\begin{array}{l}
n \\
k
\end{array}\right\}\left\{\begin{array}{c}
n \\
n-k
\end{array}\right\}=\left\{\begin{array}{c}
2 n \\
n
\end{array}\right\}\binom{2 n}{n}^{-1}-(-1)^{n} B_{n}
$$

is similar to (6.4) and quite interesting.

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