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INTEGRAL REPRESENTATIONS OF RATIOS OF BINOMIAL COEFFICIENTS

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Abstract: Many series can be expressed in integral form. In this paper we develop integral representations of ratios of binomial coefficients, many of which can then be expressed in closed form.

The series under consideration depend on a number of parameters which in specific cases reduce to known series representations. The results obtained extend those published by previous authors.

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1. Introduction

The binomial coefficients are defined by

$$\binom{n}{m} = \begin{cases} \dfrac{n!}{m!\,(n-m)!}; & n \geq m, \\ 0; & n < m \end{cases}$$

for n and m positive integers, or, more generally,

$$\binom{z}{w} = \frac{\Gamma(z+1)}{\Gamma(w+1)\,\Gamma(z-w+1)}$$

for z and w non-negative integers, where $\Gamma(x)$ is the Gamma function.

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Binomial coefficients play an important role in many areas of mathematics, including number theory, statistics and probability.

Recently, Batir [2], [3] considered series of the form

$$S(a,k)=\sum_{n=1}^{\infty}\frac{1}{n^k\binom{an}{n}}$$

for $a=2$ and 3 and was able to give some closed form expressions through integral representations of $S(a,k)$.

Batir's motivation was aroused through the known Riemann-zeta series

$$\zeta(4)=\frac{36}{17}\sum_{n=1}^{\infty}\frac{1}{n^4\binom{2n}{n}}=\frac{36}{17}S(2,4)=\frac{\pi^4}{90},$$
$$\zeta(3)=\frac{5}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3\binom{2n}{n}},\quad \text{and}\quad \zeta(2)=\sum_{n=1}^{\infty}\frac{1}{n^2\binom{2n}{n}}=3S(2,2)=\frac{\pi^2}{6},$$

where

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s},\qquad \mathrm{Re}(s)>1$$

is the Riemann-zeta function.

Indeed, Apéry [1] used the series representation to prove the irrationality of $\zeta(2)$ and $\zeta(3)$.

Borwein, Bailey and Girgensohn [4] have also recently given closed forms, recursion formulas and experimental results for the series

$$\sum_{n=1}^{\infty}\frac{n^k}{\binom{2n}{n}}\quad \text{and}\quad \sum_{n=1}^{\infty}\frac{n^k}{2^n\binom{3n}{n}}.$$

Borwein, Bailey and Girgensohn have given

$$\sum_{n=1}^{\infty}\frac{n^k}{\binom{2n}{n}}=\frac{(-1)^{k+1}}{2}\sum_{j=1}^{k+1}j!S(k+1,j)\frac{\binom{2j}{j}}{3^j}\left[\frac{2\pi}{3\sqrt{3}}+\sum_{i=0}^{j-1}\frac{3^i}{(2i+1)\binom{2i}{i}}\right],\tag{1.1}$$

where $S(a,b)$ are Stirling numbers of the second kind. Any interested reader referring to the Wolfram internet site [16] should note that (1.1) is listed incorrectly.

Many particular cases are explicitly detailed, for example

$$\sum_{n=1}^{\infty} \frac{n^2}{\binom{2n}{n}} = \frac{4}{3} + \frac{10\pi}{27\sqrt{3}}.$$

It is also of some interest to note that the listing, on the Wolfram internet site,

$$\sum_{n=1}^{\infty} \frac{18-9n}{\binom{2n}{n}} = \frac{2\pi}{\sqrt{3}} = \frac{9}{2} - \frac{9}{20}\, {}_2F_1 \left[\begin{array}{c} 2,4 \\ \frac{7}{4} \end{array} \middle| \frac{1}{4} \right]$$

is incorrectly given.

Reciprocals of binomial coefficients are also prolific in the mathematical literature. Mansour and West [5] set up the problem: Let B_n be the hyperoctahedral group, the set of all signed permutations on n letters, and let $B_n(T)$ be the set of all signed permutations in B_n which avoid a set T of signed pattern. They then show that some of the cardinalities encountered involve reciprocals of binomial coefficients.

Weinzierl [14] states that in the calculation of higher order corrections to scattering processes in particle physics one encounters higher order transcendental functions. In the expansion of transcendental functions in a small parameter around rational numbers Weinzierl develops an algorithm which allows for the evaluation of particular sums of reciprocals of binomial coefficients. Many results on reciprocals of binomial coefficient identities may be seen in the papers of Mansour [6], Pla [7], Rockett [8], Sury [11], Sury, Wang and Zhao [12], Trif [13], and Zhao and Wang [17].

It is well known that it is difficult to compute the values of combinatorial sums involving reciprocals of binomial coefficients, so any closed form representations is of great benefit.

Sury [11] used the Beta function

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt \quad \text{for } p>0,\ q>0$$

to observe that

$$\frac{1}{\binom{n}{m}} = \frac{m!\,(n-m)!}{n!} = \frac{\Gamma(m+1)\Gamma(n-m+1)}{\Gamma(n+1)} = (n+1)\int_0^1 t^m (1-t)^{n-m}\,dt.$$

Utilising the integral identity for the inverse binomial coefficients, Sury and Trif further showed that

$$\sum_{m=0}^{n} \frac{1}{\binom{n}{m}} = \frac{n+1}{2^n} \sum_{m=0}^{n} \frac{2^m}{m+1} = \frac{n+1}{2^n} \sum_{j \text{ odd}} \frac{1}{j} \binom{n+1}{j}, \tag{1.2}$$

and similarly, it can be shown that

$$\sum_{m=0}^{2n} \frac{(-1)^m}{\binom{2n}{m}} = \frac{2n+1}{n+1}.$$

Sury, Wang and Zhao [12] proved the following theorem.

Theorem 1. *In the ring of $Q[T]$ of rational polynomials, the identity*

$$\begin{aligned} \sum_{r=m}^{n} \frac{T^r (1-T)^{n-r}}{\binom{n}{r}} \\ = (n+1) \sum_{r=m}^{n} \frac{T^{n+1} (1-T)^{n-r}}{r+1} + (n+1) \sum_{r=0}^{n-m} \frac{T^{n-r} (1-T)^{n-m+1}}{(m+r+1) \binom{m+r}{r}} \end{aligned} \tag{1.3}$$

holds for $m \le n$. An equivalent form is that for $\lambda \neq -1$

$$\begin{aligned} \sum_{r=m}^{n} \frac{\lambda^r}{\binom{n}{r}} = (n+1) \sum_{r=0}^{n-m} \frac{\lambda^{m+r}}{(\lambda+1)^{r+1}} \sum_{i=0}^{n-m-r} \binom{n-m-r}{i} \frac{(-1)^i}{m+1+i} \\ + (n+1) \frac{\lambda^{n+1}}{(\lambda+1)^{n+2}} \sum_{r=m}^{n} \frac{(\lambda+1)^{r+1}}{r+1}. \end{aligned} \tag{1.4}$$

By the use of Theorem 1 and noting that for $|x| < 1$,

$$\sum_{r=1}^{\infty} \frac{(2x)^{2r}}{r \binom{2r}{r}} = \frac{2x \arcsin(x)}{\sqrt{1-x^2}} = \int_0^1 \frac{4x^2 t}{1-4x^2 t (1-t)} dt.$$

Sury, Wang and Zhao [12] showed, among other results, that

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)\cdots(n+j)} = \frac{1}{(j-1)(j-1)!}, \tag{1.5}$$

$$\sum_{n=0}^{\infty} \frac{1}{(3n+1)(3n+2)(3n+3)} = \frac{\pi\sqrt{3} - 3\ln 3}{12}, \tag{1.6}$$

$$\sum_{n=0}^{\infty} \frac{1}{(4n+1)(4n+2)(4n+3)(4n+4)} = \frac{6\ln 2 - \pi}{24} \tag{1.7}$$

and

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\binom{n+j}{n}} = j2^{j-1}\left(\ln 2 - \sum_{r=1}^{j-1}\frac{1}{r}\right) - j\sum_{r=1}^{j-1}(-1)^r \binom{j-1}{r}\frac{2^{j-1-r}}{r}, \quad \text{for} \quad j = 1, 2, \ldots . \tag{1.8}$$

More general versions of identities (1.5) to (1.8) have been given by Sofo [10].

Identities (1.5) to (1.8) are reciprocal binomial identities of the form

$$S(a,j) = \frac{1}{j!}\sum_{n=0}^{\infty}\frac{1}{\binom{an+j}{an}} \tag{1.9}$$

and

$$T(a,j) = \frac{1}{j!}\sum_{n=0}^{\infty}\frac{(-1)^n}{\binom{an+j}{an}} \tag{1.10}$$

for $j = 1, 2, 3 \ldots,\ a \in \mathbb{R}^+ \setminus \{0\}$.

The identity (1.5)

$$S(1,j) = \frac{1}{(j-1)(j-1)!}$$

and

$$\begin{aligned} S(2,j) &= \frac{2^{j-2}}{(j-1)!}\left[\ln 2 + \sum_{r=1}^{j-2}(-1)^r\binom{j-2}{r}\left(\frac{2^r-1}{r\cdot 2^r}\right)\right] \\ &= \frac{1}{j!}\ {}_3F_2\left[\left.\begin{matrix} \frac{1}{2}, 1, 1, \\ \frac{1+j}{2}, \frac{2+j}{2} \end{matrix}\ \right|\ 1\right] \end{aligned}$$

and others were also previously given by Sofo [9].

Sofo [10] proved the following theorem.

Theorem 2. *For $m \geq 1$ and $a > 0$ and j a positive integer, then*

$$S(a,j,m) = \frac{1}{j!}\sum_{n=0}^{\infty}\frac{\binom{n+m-1}{n}}{\binom{an+j}{an}} = \frac{1}{j!}\sum_{n=0}^{\infty}\frac{\binom{n+m-1}{m-1}}{\binom{an+j}{j}} \tag{1.11}$$

$$= \frac{1}{(j-1)!} \int_0^1 \frac{(1-x)^{j-1}}{(1-x^a)^m} dx \tag{1.12}$$

$$= \frac{1}{j!} \, {}_{j+1}F_j \left[\begin{array}{c} m, \frac{1}{a}, \frac{2}{a}, \frac{3}{a}, \ldots, \frac{j}{a} \\ 1+\frac{1}{a}, 1+\frac{2}{a}, 1+\frac{3}{a}, \ldots, 1+\frac{j}{a} \end{array} \middle| 1 \right] \tag{1.13}$$

$$= \sum_{n=0}^{\infty} (m)_n \Bigg/ n! \prod_{k=1}^{j} (an+k) \tag{1.14}$$

and

$$T(a,j,m) = \frac{1}{j!} \sum_{n=0}^{\infty} \frac{(-1)^n \binom{n+m-1}{n}}{\binom{an+j}{an}} = \frac{1}{j!} \sum_{n=0}^{\infty} \frac{(-1)^n \binom{n+m-1}{m-1}}{\binom{an+j}{j}} \tag{1.15}$$

$$= \frac{1}{(j-1)!} \int_0^1 \frac{(1-x)^{j-1}}{(1+x^a)^m} dx \tag{1.16}$$

$$= \frac{1}{j!} \, {}_{j+1}F_j \left[\begin{array}{c} m, \frac{1}{a}, \frac{2}{a}, \frac{3}{a}, \ldots, \frac{j}{a} \\ 1+\frac{1}{a}, 1+\frac{2}{a}, 1+\frac{3}{a}, \ldots, 1+\frac{j}{a} \end{array} \middle| -1 \right] \tag{1.17}$$

$$= \sum_{n=0}^{\infty} (-1)^n (m)_n \Bigg/ n! \prod_{k=1}^{j} (an+k) \,. \tag{1.18}$$

2. Some Generalisations

The following identities can be established using the ideas of Theorem 2.

Theorem 3. *Let the conditions of Theorem 2 hold. Further, let*

$$W(a,j,m) = \frac{\binom{n+m-1}{n}}{\binom{an+j}{an}} \tag{2.1}$$

and s be a positive integer, then

$$S(a,j,m,s) = \sum_{n=1}^{\infty} n^s W(a,j,m) \tag{2.2}$$

$$= \frac{j}{a^s} \int_0^1 (1-x)^{j-1} (\rho_-(x))^s \, dx, \tag{2.3}$$

where $(\rho_-(x))^0 = \frac{1}{(1-x^a)^m}$ *and*

$$\begin{aligned}(\rho_-(x))^s &= \left(x\frac{d}{dx}\left\{\frac{1}{(1-x^a)^m}\right\}\right)^s \\ &= \underbrace{x\frac{d}{dx}\left(x\frac{d}{dx}\left(\cdots x\frac{d}{dx}\left(\frac{1}{(1-x^a)^m}\right)\right)\right)}_{s-times},\end{aligned}$$

is the consecutive derivative operator of the continuous function $(1-x^a)^{-m}$ *for* $x \in (0,1)$, *and similarly,*

$$T(a,j,m,s) = \sum_{n=1}^{\infty} (-1)^n n^s W(a,j,m) \tag{2.4}$$

$$= \frac{j}{a^s}\int_0^1 (1-x)^{j-1}(\rho_+(x))^s\,dx, \tag{2.5}$$

where

$$\begin{aligned}(\rho_+(x))^0 &= \frac{1}{(1+x^a)^m}, \\ (\rho_+(x))^s &= \left(x\frac{d}{dx}\left\{\frac{1}{(1+x^a)^m}\right\}\right)^s \\ &= \underbrace{x\frac{d}{dx}\left(x\frac{d}{dx}\left(\cdots x\frac{d}{dx}\left(\frac{1}{(1+x^a)^m}\right)\right)\right)}_{s-times},\end{aligned}$$

is the consecutive derivative operator of the continuous function $(1+x^a)^{-m}$ *for* $x \in [0,1]$.

Proof. Consider

$$\begin{aligned}S(a,j,m,s) &= \sum_{n=1}^{\infty} n^s \frac{\binom{n+m-1}{m-1}}{\binom{an+j}{an}} = j\sum_{n=1}^{\infty} n^s \binom{n+m-1}{m-1}\frac{\Gamma(j)\Gamma(an+1)}{\Gamma(an+1+j)} \\ &= j\sum_{n=1}^{\infty} n^s \binom{n+m-1}{m-1} B(an+1,j) \\ &= j\sum_{n=1}^{\infty} n^s \binom{n+m-1}{m-1}\int_0^1 x^{an}(1-x)^{j-1}\,dx,\end{aligned}$$

interchanging sum and integral we have

$$S\left(a,j,m,s\right)=j\int_0^1 \left(1-x\right)^{j-1}\sum_{n=1}^{\infty} n^s \binom{n+m-1}{m-1} x^{an}dx.$$

Now, consider

$$\begin{aligned}
\left(\rho_-\left(x\right)\right)^0 &= \sum_{n=0}^{\infty}\binom{n+m-1}{m-1} x^{an} = \frac{1}{\left(1-x^a\right)^m}\,, \\
\left(\rho_-\left(x\right)\right)^1 &= x\frac{d}{dx}\left\{\frac{1}{\left(1-x^a\right)^m}\right\} = a\sum_{n=0}^{\infty} n\binom{n+m-1}{m-1} x^{an}\,, \\
&\vdots \\
\left(\rho_-\left(x\right)\right)^s &= x\frac{d}{dx}\underbrace{\left(x\frac{d}{dx}\left(\cdots\left(x\frac{d}{dx}\left(\frac{1}{\left(1-x^a\right)^m}\right)\right)\right)\right)}_{s\ \text{times}} \\
&= a^s\sum_{n=0}^{\infty} n^s\binom{n+m-1}{m-1} x^{an},
\end{aligned}$$

so that

$$S\left(a,j,m,s\right)=\frac{j}{a^s}\int_0^1 \left(1-x\right)^{j-1}\left(\rho_-\left(x\right)\right)^s dx.$$

In a similar fashion, we have that

$$T\left(a,j,m,s\right)=j\int_0^1 \left(1-x\right)^{j-1}\sum_{n=1}^{\infty} \left(-1\right)^n n^s \binom{n+m-1}{m-1} x^{an}dx.$$

Now, consider

$$\begin{aligned}
\left(\rho_+\left(x\right)\right)^0 &= \sum_{n=0}^{\infty}\left(-1\right)^n\binom{n+m-1}{m-1} x^{an} = \frac{1}{\left(1+x^a\right)^m}\,, \\
\left(\rho_+\left(x\right)\right)^1 &= x\frac{d}{dx}\left\{\frac{1}{\left(1+x^a\right)^m}\right\} = a\sum_{n=0}^{\infty}\left(-1\right)^n n\binom{n+m-1}{m-1} x^{an}\,, \\
&\vdots \\
\left(\rho_+\left(x\right)\right)^s &= x\frac{d}{dx}\underbrace{\left(x\frac{d}{dx}\left(\cdots\left(x\frac{d}{dx}\left(\frac{1}{\left(1+x^a\right)^m}\right)\right)\right)\right)}_{s\ \text{times}},
\end{aligned}$$

$$= a^s \sum_{n=0}^{\infty} (-1)^n n^s \binom{n+m-1}{m-1} x^{an},$$

so that

$$T(a,j,m,s) = \frac{j}{a^s} \int_0^1 (1-x)^{j-1} (\rho_+(x))^s \, dx\,,$$

hence the theorem is proved. □

We list some cases as follows

$$S(a,j,m,1) = \sum_{n=1}^{\infty} nW(a,j,m) = mj \int_0^1 (1-x)^{j-1} \frac{x^a}{(1-x^a)^{m+1}} dx,$$

$$S(a,j,m,2) = \sum_{n=1}^{\infty} n^2 W(a,j,m) = mj \int_0^1 (1-x)^{j-1} \frac{x^a(1+mx^a)}{(1-x^a)^{m+2}} dx,$$

$$S(a,j,m,3) = \sum_{n=1}^{\infty} n^3 W(a,j,m)$$
$$= mj \int_0^1 (1-x)^{j-1} \frac{x^a\left(1+x^a(1+3m)+m^2x^{2a}\right)}{(1-x^a)^{m+3}} dx,$$

$$S(a,j,m,4) = \sum_{n=1}^{\infty} n^4 W(a,j,m)$$
$$= mj \int_0^1 (1-x)^{j-1} \frac{x^a\left(1+x^a(4+7m)+x^{2a}\left(1+4m+6m^2\right)+m^3x^{3a}\right)}{(1-x^a)^{m+4}} dx,$$

$$T(a,j,m,1) = \sum_{n=1}^{\infty} (-1)^n nW(a,j,m) = mj \int_0^1 (1-x)^{j-1} \frac{x^a}{(1+x^a)^{m+1}} dx,$$

$$T(a,j,m,2) = \sum_{n=1}^{\infty} (-1)^n n^2 W(a,j,m)$$
$$= mj \int_0^1 (1-x)^{j-1} \frac{x^a(-1+mx^a)}{(1+x^a)^{m+2}} dx,$$

$$T(a,j,m,3)=\sum_{n=1}^{\infty}(-1)^n n^3 W(a,j,m)$$
$$=mj\int_0^1 (1-x)^{j-1}\frac{x^a\left(-1+x^a(1+3m)-m^2x^{2a}\right)}{(1+x^a)^{m+3}}dx,$$

$$T(a,j,m,4)=\sum_{n=1}^{\infty}(-1)^n n^4 W(a,j,m)=mj\int_0^1 (1-x)^{j-1}$$
$$\times\frac{x^a\left(-1+x^a(4+7m)-x^{2a}\left(1+4m+6m^2\right)+m^3x^{3a}\right)}{(1+x^a)^{m+4}}dx.$$

A great number of simpler cases can be explicitly evaluated, we list a few:

$$S(1,j,m,1)=\sum_{n=1}^{\infty}\frac{n\binom{n+m-1}{n}}{\binom{n+j}{n}}=\frac{mj}{(j-m)(j-m-1)}$$
$$=\left(\frac{m}{j+1}\right)\ {}_2F_1\left[\begin{array}{c}2,m+1\\ j+2\end{array}\middle|1\right],\quad j\neq m,m+1,$$

$$S(2,m+3,m,2)=\sum_{n=1}^{\infty}n^2\frac{\binom{n+m-1}{n}}{\binom{2n+m+3}{2n}}=\frac{m+3}{2^{m+1}(m-3)(m-2)(m-1)}$$
$$\times\left[4\cdot 2^m(6+m)-m^4-2m^3-7m^2-22m-24\right],\quad m\neq 1,2,3.$$

Note the special cases

$$S(2,4,1,2)=28\ln 2-19,$$
$$S(2,5,2,2)=\frac{225}{4}-80\ln 2,$$
$$S(2,6,3,2)=54\ln 2-\frac{291}{8},$$
$$S(3,12,7,3)=\sum_{n=1}^{\infty}n^3\frac{\binom{n+6}{n}}{\binom{3n+12}{3n}}=\frac{13627262}{32805}-\frac{166936\pi}{729\sqrt{3}}.$$

Also,

$$S\left(2,\frac{9}{2},\frac{3}{2},2\right)=180-\frac{4059\sqrt{2}}{32},$$
$$T(2,4,3,2)=\sum_{n=1}^{\infty}(-1)^n n^2\frac{\binom{n+2}{n}}{\binom{2n+4}{4}}=\frac{15\pi}{16}-3,$$

$$T\left(2,8,4,2\right)=\sum_{n=1}^{\infty}\left(-1\right)^{n}n^{2}\frac{\binom{n+3}{n}}{\binom{2n+8}{2n}}=87\pi-64\ln 2-229.$$

The following theorem details a particular case of (2.2) and (2.4).

Theorem 4. *For $a=1/b$, $b=2,3,4,5,\ldots$, $s=2$ and $j\geq m+3$*

$$S\left(\frac{1}{b},j,m,2\right)=\sum_{n=1}^{\infty}n^{2}\frac{\binom{n+m-1}{m-1}}{\binom{n/b+j}{j}}$$

forms the rational numbers

$$S\left(\frac{1}{b},j,m,2\right)=mj\left(j-3-m\right)!\sum_{\mu=0}^{(m+2)(b-1)}\alpha_{\mu}^{b,m}\left[\frac{1}{\prod_{v=1}^{j-2-m}\left(v+\frac{\mu+1}{b}\right)}+\frac{m}{\prod_{v=1}^{j-2-m}\left(v+\frac{\mu+2}{b}\right)}\right],\quad (2.6)$$

where $\alpha_{\mu}^{b,m}$ are the coefficients of the expansion of the sums of the powers of $x^{\mu/b}$ in $\left(\sum_{\mu=0}^{b-1}x^{\mu/b}\right)^{m+2}$.

Similarly for b an even positive integer

$$T\left(\frac{1}{b},j,m,2\right)=\sum_{n=1}^{\infty}\left(-1\right)^{n}\frac{n^{2}\binom{n+m-1}{m-1}}{\binom{n/b+j}{j}}\quad (2.7)$$
$$=mj\left(j-3-m\right)!\sum_{\mu=0}^{(m+2)(b-1)}\left(-1\right)^{\mu}\alpha_{\mu}^{b,m}\left[\frac{m}{\prod_{v=1}^{j-2-m}\left(v+\frac{\mu+2}{b}\right)}-\frac{1}{\prod_{v=1}^{j-2-m}\left(v+\frac{\mu+1}{b}\right)}\right].$$

Proof. Consider, from (2.2)

$$S\left(\frac{1}{b},j,m,2\right)=\sum_{n=1}^{\infty}n^{2}\frac{\binom{n+m-1}{m-1}}{\binom{n/b+j}{j}}=mj\int_{0}^{1}\left(1-x\right)^{j-1}\frac{\left(x^{1/b}+mx^{2/b}\right)}{\left(1-x^{1/b}\right)^{m+2}}dx$$
$$=mj\int_{0}^{1}\left(1-x\right)^{j-3-m}\left(x^{1/b}+mx^{2/b}\right)\left(\frac{1-x}{1-x^{1/b}}\right)^{m+2}dx.$$

Now, note that

$$1-x=\left(1-x^{1/b}\right)\sum_{\mu=0}^{b-1}x^{\mu/b}$$

so that

$$\begin{aligned}
&S\left(\frac{1}{b},j,m,2\right)\\
&=mj\int_0^1(1-x)^{j-3-m}\left(x^{1/b}+mx^{2/b}\right)\left(\sum_{\mu=0}^{b-1}x^{\mu/b}\right)^{m+2}dx\\
&=mj\int_0^1(1-x)^{j-3-m}\left(x^{1/b}+mx^{2/b}\right)\sum_{\mu=0}^{(m+2)(b-1)}\alpha_\mu^{b,m}x^{\mu/b}dx\\
&=mj\int_0^1\sum_{r=0}^{j-3-m}(-1)^r\binom{j-3-m}{r}\sum_{\mu=0}^{(m+2)(b-1)}\alpha_\mu^{b,m}\left[x^{r+\frac{\mu+1}{b}}+mx^{r+\frac{\mu+2}{b}}\right]dx\\
&=mj\sum_{\mu=0}^{(m+2)(b-1)}\alpha_\mu^{b,m}\sum_{r=0}^{j-3-m}(-1)^r\binom{j-3-m}{r}\left[\frac{1}{r+1+\frac{\mu+1}{b}}+\frac{m}{r+1+\frac{\mu+2}{b}}\right]\\
&=mj\sum_{\mu=0}^{(m+2)(b-1)}\alpha_\mu^{b,m}\left[\frac{1}{\left(1+\frac{\mu+1}{b}\right)\binom{j-2-m+\frac{\mu+1}{b}}{j-3-m}}+\frac{m}{\left(1+\frac{\mu+2}{b}\right)\binom{j-2-m+\frac{\mu+2}{b}}{j-3-m}}\right]\\
&=mj\sum_{\mu=0}^{(m+2)(b-1)}\alpha_\mu^{b,m}\left[\frac{\Gamma(j-2-m)\,\Gamma\left(2+\frac{\mu+1}{b}\right)}{\left(1+\frac{\mu+1}{b}\right)\Gamma\left(j-1-m+\frac{\mu+1}{b}\right)}\right.\\
&\qquad\qquad\left.+\frac{m\Gamma(j-2-m)\,\Gamma\left(2+\frac{\mu+2}{b}\right)}{\left(1+\frac{\mu+2}{b}\right)\Gamma\left(j-1-m+\frac{\mu+2}{b}\right)}\right]\\
&=mj\,(j-3-m)!\sum_{\mu=0}^{(m+2)(b-1)}\alpha_\mu^{b,m}\left[\frac{1}{\prod_{v=1}^{j-2-m}\left(v+\frac{\mu+1}{b}\right)}+\frac{m}{\prod_{v=1}^{j-2-m}\left(v+\frac{\mu+2}{b}\right)}\right],
\end{aligned}$$

which is the result (2.6).

The proof of (2.7) follows a similar pattern and will not be recorded here. □

Note. In the case of $b=2$

$$\alpha_\mu^{2,m}=\binom{m+2}{\mu}\quad\text{for}\quad\mu=0,1,2,\ldots,m+2$$

and hence

$$S\left(\frac{1}{2},j,m,2\right)=\sum_{n=1}^{\infty}\frac{n^2\binom{n+m-1}{m-1}}{\binom{\frac{n}{2}+j}{j}}$$

$$= mj\,(j-3-m)!\sum_{\mu=0}^{m+2}\binom{m+2}{\mu}\left[\frac{1}{\prod_{v=1}^{j-2-m}\left(v+\frac{\mu+1}{2}\right)}\right.$$
$$\left.+\frac{m}{\prod_{v=1}^{j-2-m}\left(v+\frac{\mu+2}{2}\right)}\right].$$

In particular

$$S\left(\frac{1}{2},3,m,2\right)=\sum_{n=1}^{\infty}\frac{n^2\binom{n+m-1}{m-1}}{\binom{\frac{n}{2}+3}{3}}$$
$$=\frac{24m\left(m^3-18m^2+71m+186\right)}{(m-6)\,(m-5)\,(m-4)\,(m-3)\,(m-2)\,(m-1)},\quad m\neq 1,\ldots,6$$

and

$$S\left(\frac{1}{2},m+3,m,2\right)=\sum_{n=1}^{\infty}\frac{n^2\binom{n+m-1}{m-1}}{\binom{\frac{n}{2}+m+3}{m+3}}$$
$$=\frac{2^{4+m}\left(m^6+11m^5+53m^4+117m^3+122m^2+48m\right)+8\,(m-3)\,(m+1)\,m}{(m+1)\,(m+4)\,(m+5)\,(m+6)}.$$

Similarly

$$T\left(\frac{1}{2},j,m,2\right)=\sum_{n=1}^{\infty}(-1)^n\,\frac{n^2\binom{n+m-1}{m-1}}{\binom{\frac{n}{2}+j}{j}}$$
$$= mj\,(j-3-m)!\sum_{\mu=0}^{m+2}(-1)^{\mu}\binom{m+2}{\mu}\left[\frac{m}{\prod_{v=1}^{j-2-m}\left(v+\frac{\mu+2}{2}\right)}\right.$$
$$\left.-\frac{1}{\prod_{v=1}^{j-2-m}\left(v+\frac{\mu+1}{2}\right)}\right].$$

In the case when the exponents of the binomial coefficients in the denominator are unequal, we can state the following theorem.

Theorem 5. *Let a and b be real positive numbers such that $a\geq b$ and let*

$$V\,(a,b,j,m)=\frac{\binom{n+m-1}{n}}{\binom{an+j}{bn}} \tag{2.8}$$

for j and m positive real numbers where $j \geq m+1$, then

$$S(a,b,j,m)=\sum_{n=0}^{\infty}V(a,b,j,m)=j\int_0^1\frac{(1-x)^{j-1}}{\left(1-x^b(1-x)^{a-b}\right)^m}dx$$
$$+m(a-b)\int_0^1\frac{x^b(1-x)^{a-b+j-1}}{\left(1-x^b(1-x)^{a-b}\right)^{m+1}}dx \quad (2.9)$$

and

$$T(a,b,j,m)=\sum_{n=0}^{\infty}(-1)^nV(a,b,j,m)=j\int_0^1\frac{(1-x)^{j-1}}{\left(1+x^b(1-x)^{a-b}\right)^m}dx$$
$$-m(a-b)\int_0^1\frac{x^b(1-x)^{a-b+j-1}}{\left(1+x^b(1-x)^{a-b}\right)^{m+1}}dx. \quad (2.10)$$

Proof. Consider

$$\sum_{n=0}^{\infty}\frac{\binom{n+m-1}{n}}{\binom{an+j}{bn}}=\sum_{n=0}^{\infty}\binom{n+m-1}{n}\frac{\Gamma(bn+1)\Gamma((a-b)n+j+1)}{\Gamma(an+j+1)}$$
$$=\sum_{n=0}^{\infty}\binom{n+m-1}{n}(j+(a-b)n)\frac{\Gamma(bn+1)\Gamma((a-b)n+j+1)}{\Gamma(an+j+1)}$$
$$=\sum_{n=0}^{\infty}\binom{n+m-1}{n}(j+(a-b)n)B(bn+1,(a-b)n+j)$$
$$=\sum_{n=0}^{\infty}\binom{n+m-1}{n}(j+(a-b)n)\int_0^1x^{bn}(1-x)^{(a-b)n+j-1}dx$$
$$=\sum_{n=0}^{\infty}\binom{n+m-1}{n}\left[j\int_0^1x^{bn}(1-x)^{(a-b)n+j-1}dx\right.$$
$$\left.+(a-b)\int_0^1nx^{bn}(1-x)^{(a-b)n+j-1}dx\right],$$

Interchanging sum and integral, we have

$$=j\int_0^1(1-x)^{j-1}\sum_{n=0}^{\infty}\binom{n+m-1}{n}\left(x^b(1-x)^{(a-b)}\right)^ndx$$

$$+ (a-b) \int_0^1 (1-x)^{j-1} \sum_{n=0}^{\infty} n \binom{n+m-1}{n} \left(x^b (1-x)^{(a-b)}\right)^n dx$$
$$= j \int_0^1 \frac{(1-x)^{j-1}}{\left(1-x^b (1-x)^{a-b}\right)^m} dx + m (a-b) \int_0^1 \frac{x^b (1-x)^{a-b+j-1}}{\left(1-x^b (1-x)^{a-b}\right)^{m+1}} dx.$$

For $T(a,b,j,m)$, we have

$$T(a,b,j,m) = j \int_0^1 (1-x)^{j-1} \sum_{n=0}^{\infty} (-1)^n \binom{n+m-1}{n} \left(x^b (1-x)^{(a-b)}\right)^n dx$$
$$+ (a-b) \int_0^1 (1-x)^{j-1} \sum_{n=0}^{\infty} (-1)^n n \binom{n+m-1}{n} \left(x^b (1-x)^{(a-b)}\right)^n dx$$
$$= j \int_0^1 \frac{(1-x)^{j-1}}{\left(1+x^b (1-x)^{a-b}\right)^m} dx - m (a-b) \int_0^1 \frac{x^b (1-x)^{a-b+j-1}}{\left(1+x^b (1-x)^{a-b}\right)^{m+1}} dx,$$

and the theorem is proved. □

By consideration of the ratio of successive terms of $V(a,b,j,m)$ we may express $S(a,b,j,m)$ and $T(a,b,j,m)$, for $a > b$, in terms of generalised hypergeometric functions as follows:

$$S(a,b,j,m) = {}_{a+1}F_a \left[\begin{array}{c} m, \frac{1}{b}, \frac{2}{b}, \ldots, \frac{b}{b}, \frac{1+j}{a-b}, \frac{2+j}{a-b}, \ldots, \frac{a-b+j}{a-b} \\ \frac{1+j}{a}, \frac{2+j}{a}, \ldots, \frac{a+j}{a} \end{array} \middle| \frac{b^b}{a^a} (a-b)^{a-b} \right], \quad (2.11)$$

$$T(a,b,j,m) = {}_{a+1}F_a \left[\begin{array}{c} m, \frac{1}{b}, \frac{2}{b}, \ldots, \frac{b}{b}, \frac{1+j}{a-b}, \frac{2+j}{a-b}, \ldots, \frac{a-b+j}{a-b} \\ \frac{1+j}{a}, \frac{2+j}{a}, \ldots, \frac{a+j}{a} \end{array} \middle| -\frac{b^b}{a^a} (a-b)^{a-b} \right]. \quad (2.12)$$

In the degenerative case when $a = b$, (2.11) reduces to (1.13) and (2.12) reduces to (1.17).

Some examples can now be given

$$S(2,1,1,m) = \sum_{n=0}^{\infty} \frac{\binom{n+m-1}{n}}{\binom{2n+1}{n}} = {}_2F_1 \left[\begin{array}{c} 2, m \\ \frac{3}{2} \end{array} \middle| \frac{1}{4} \right]$$

$$= \frac{2}{3} + \frac{(m+1)}{3^m}\binom{2m-2}{m-1}\left[\frac{2\pi}{3\sqrt{3}} + \sum_{r=0}^{m-2}\frac{3^r}{(r+1)\binom{2r+1}{r}}\right],$$

$$S(2,1,1,2) = \sum_{n=0}^{\infty}\frac{n+1}{\binom{2n+1}{n}} = \frac{4\sqrt{3}\pi}{27} + \frac{4}{3}.$$

Similarly,

$$\begin{aligned} T(2,1,1,1) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{\binom{2n+1}{n}} = {}_2F_1\left[\left.\begin{matrix}2,1\\ \frac{3}{2}\end{matrix}\right| -\frac{1}{4}\right] \\ &= \frac{2}{5} + \frac{4}{5\sqrt{5}}\ln\left(\frac{\alpha}{\beta}\right), \end{aligned}$$

where $\alpha = \frac{\sqrt{5}+1}{2}$, the golden ratio and $(-\beta) = \frac{\sqrt{5}-1}{2}$ are the two zeros of the quadratic $x^2 - x - 1 = 0$.

$$T(2,1,1,3) = \sum_{n=0}^{\infty}(-1)^n\frac{\binom{n+2}{n}}{\binom{2n+1}{n}} = {}_2F_1\left[\left.\begin{matrix}2,3\\ \frac{3}{2}\end{matrix}\right| -\frac{1}{4}\right] = \frac{2}{5}$$

and in general

$$\begin{aligned} T(2,1,1,m) &= \sum_{n=0}^{\infty}(-1)^n\frac{\binom{n+m-1}{n}}{\binom{2n+1}{n}} = {}_2F_1\left[\left.\begin{matrix}2,m\\ \frac{3}{2}\end{matrix}\right| -\frac{1}{4}\right] \\ &= \frac{2}{5} + \frac{(3-m)}{5^m}\binom{2m-2}{m-1}\left[\sum_{r=0}^{m-2}\frac{5^r}{(r+1)\binom{2r+1}{r}} + \frac{2}{\sqrt{5}}\ln\left(\frac{\alpha}{\beta}\right)\right]. \end{aligned}$$

3. Conclusion

In this paper we have been able to consider series of the form

$$\sum_{n=1}^{\infty} n^s \frac{\binom{n+m-1}{n}}{\binom{an+j}{an}}, \quad \sum_{n=0}^{\infty}\frac{\binom{n+m-1}{n}}{\binom{an+j}{bn}},$$

and an alternating companion,

$$\sum_{n=0}^{\infty}(-1)^n\frac{\binom{n+m-1}{n}}{\binom{an+j}{bn}}$$

and develop their representations in integral form.

These series generalise a class of series of reciprocal binomial coefficient type, including the class of type $\sum_{n=1}^{\infty} \frac{n^k}{\binom{2n}{n}}$ developed by Borwein, Bailey and Girgensohn.

We can also develop integral identities for products of reciprocals of binomial coefficients of the form

$$\sum_{n\geq 0} \frac{\binom{n+m-1}{n}}{\binom{an+j}{j}\binom{bn+k}{k}},$$

this will be reported in another forum.

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